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Michał Korch, Institute of Mathematics, University of Warsaw, Poland.
email: `m_korch@mimuw.edu.pl`

A GENERALIZED EGOROV'S STATEMENT FOR IDEALS

Abstract

We consider the generalized Egorov's statement (Egorov's Theorem without the assumption on measurability of the functions, see [10]) in the case of an ideal convergence and a number of different types of ideal convergence notion. We prove that in those cases the generalized Egorov's statement is independent from ZFC.

1 Introduction

In this paper we consider various versions of the classic Egorov's Theorem. Let us recall (see e.g. [6]) that the classic Egorov's Theorem states that given a sequence of measurable functions (we restrict our attention to the real functions $[0,1] \to [0,1]$) which is pointwise convergent on $[0,1]$ and $\varepsilon > 0$, one can find a measurable set $A \subseteq [0,1]$ with $m(A) \geq 1 - \varepsilon$ such that the sequence converges uniformly on A (m denotes the Lebesgue measure).

It is interesting whether we can drop the assumption on measurability of the functions in the above theorem. A statement which says that given any sequence of functions $[0,1] \to [0,1]$ which is pointwise convergent and $\varepsilon > 0$, there exists a set $A \subseteq [0,1]$ with $m^*(A) \geq 1 - \varepsilon$ (m^* denotes the outer measure) such that the sequence converges uniformly on A, is called the generalized Egorov's statement. T. Weiss in his unpublished manuscript (see [10]) proved that it is independent from ZFC, and this fact was used in [3]. Then R. Pinciroli studied the method of T. Weiss more systematically (see [7]). For example, he related it to cardinal coefficients: non(N) (the lowest

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possible cardinality of a non-null set), $\mathfrak{b}$ (the lowest possible cardinality of a family of sequences of natural numbers unbounded in the sense of the order $\leq^*$ of eventual domination) and $\mathfrak{d}$ (the lowest possible cardinality of a family of sequences of natural numbers such that every sequence is dominated in the sense of $\leq^*$ by a sequence in the family). In particular, he proved that non(N) $< \mathfrak{b}$ implies that the generalized Egorov's statement holds, but if, for example, non(N) $= \mathfrak{d} = \mathfrak{c}$, then it fails.

We can also define a notion of convergence of a sequence of functions with respect to a given ideal I on ω. There are different types of convergence with respect to I, and pointwise and uniform convergence are the most common. Given two notions of convergence with respect to an ideal, we can ask whether the classic Egorov's Theorem (with the measurability assumption) holds for those two notions of convergence in the sense of whether the weaker convergence implies the stronger convergence on a subset of arbitrarily large measure. The answer may often be negative as in the case of uniform and pointwise convergence for many analytic P-ideals (see [4, Theorem 3.4]). But one can also define other types of convergence, e.g. equi-ideal convergence. And, for example, in the case of analytic P-ideal so called weak Egorov's Theorem for ideals (between equi-ideal and pointwise ideal convergence) was proved by N. Mrożek (see [4, Theorem 3.1]).

Therefore, we ask whether in the case of an ideal and two notions of convergence for which Egorov's theorem with measurability assumption holds, we can drop this assumption. This paper deals with this question in relation to different types of ideal convergence notions.

2 Using Pinciroli's method

We start by a generalization of the method presented by R. Pinciroli (see [7], and also [8]). The core of this method can be generalized to the following theorem.

Theorem 1. *Assume that non(N) $< \mathfrak{b}$. Let $\Phi \in (\omega^\omega)^{[0,1]}$. Then for any $\varepsilon > 0$, there exists $A \subseteq [0,1]$ such that $m^*(A) \geq 1 - \varepsilon$ and Φ is bounded on A.*

Proof: We follow the arguments of Pinciroli (see [7]).

Assume that non(N) $< \mathfrak{b}$. Notice that this statement holds for example in a model obtained by $\aleph_2$-iteration with countable support of Laver forcing (see e.g. [1]). Also it can be easily proven, that under this assumption there exists a set $Y \subseteq [0,1]$ of cardinality less that $\mathfrak{b}$ such that $m^*(Y) = 1$. Indeed, if $N \subseteq [0,1]$ is a set of positive outer measure with $|N| < \mathfrak{b}$, then let $Y =$

$\{x+y\colon x\in N, y\in\mathbb{Q}\}$, where $+$ denotes addition modulo 1. Then Y has outer measure 1 under the Zero-One Law.

Therefore, every function $\varphi\colon[0,1]\to\omega^\omega$ maps Y onto a K_σ-set, where K_σ denotes the σ-ideal of subsets of ω^ω generated by the compact (equivalently bounded) sets. We get that $\Phi[Y]\in K_\sigma$. Assume that $\Phi[Y]\subseteq\bigcup_{n\in\omega}B_n$ with each B_n bounded. Let $A_n=\Phi^{-1}[\bigcup_{i=0}^n B_i]$. Therefore, $\Phi[A_n]$ is bounded, and for any $\varepsilon>0$, there exists $n\in\omega$ such that $m^*(A_n)\geq 1-\varepsilon$. □

In the products of the form ω^S and $(\omega^S)^T$ we consider partial orderings, denoted by the same symbol $\leq$, given by $x\leq y$, if $x(s)\leq y(s)$ for $x,y\in\omega^S$, $s\in S$, and $\phi\leq\psi$, if $\phi(t)\leq\phi(t)$ for $\phi,\psi\in(\omega^S)^T$, where $\phi(t),\psi(t)\in\omega^S$. We say that a function $o: X\to P$ from a set X into a partially ordered set P is cofinal if for every $p\in P$ there exists $x\in X$ such that $p\leq o(x)$.

For a sequence of functions $f_n:[0,1]\to[0,1]$ and subset $A\subseteq[0,1]$ we consider a notion of convergence $f_n\looparrowright f$ on A. We assume that if $B\subseteq A$ and $f_n\looparrowright f$ on A, then $f_n\looparrowright f$ on B. We write $f_n\looparrowright f$ provided that $f_n\looparrowright f$ on $[0,1]$. Let $\mathcal{F}\subseteq\{\langle f_n\rangle_{n\in\omega}:\forall_{n\in\omega}f_n:[0,1]\to[0,1]\}$ be an arbitrary family of sequences of functions.

We consider two hypotheses between $\mathcal{F}$ and $\looparrowright$:

($H^{\Rightarrow}(\mathcal{F},\looparrowright)$) There exists $o:\mathcal{F}\to(\omega^\omega)^{[0,1]}$ such that for every $F\in\mathcal{F}$ and every $A\subseteq[0,1]$, if $o(F)[A]$ is bounded in $(\omega^\omega,\leq)$, then $F\looparrowright 0$ on A.

($H^{\Leftarrow}(\mathcal{F},\looparrowright)$) There exists cofinal $o:\mathcal{F}\to(\omega^\omega)^{[0,1]}$ such that for every $F\in\mathcal{F}$ and every $A\subseteq[0,1]$, if $F\looparrowright 0$ on A, then $o(F)[A]$ is bounded in $(\omega^\omega,\leq)$.

Theorem 2. *Assume that non(N) $<\mathfrak{b}$, and $H^{\Rightarrow}(\mathcal{F},\looparrowright)$. Then for any $\langle f_n\rangle_{n\in\omega}\in\mathcal{F}$ and any $\varepsilon>0$, there exists $A\subseteq[0,1]$ such that $m^*(A)\geq 1-\varepsilon$ and $f_n\looparrowright 0$ on A.*

Proof: Apply Theorem 1 for $o(\langle f_n\rangle_{n\in\omega})$ given by $H^{\Rightarrow}(\mathcal{F},\looparrowright)$. □

Recall that $Z\subseteq\omega^\omega$ is a $\mathfrak{c}$-Lusin set if it is of cardinality $\mathfrak{c}$, and if $A\subseteq Z$ is meagre, then $|A|<\mathfrak{c}$. The existence of such a set is independent from ZFC. Notice also that there exists a model of ZFC in which non(N) $=\mathfrak{c}$, and there exists $\mathfrak{c}$-Lusin set. To get this model it suffices to iterate $\aleph_2$-times Cohen forcing with finite supports over a model of GCH (see [1, Model 7.5.8 and Lemma 8.2.6]).

Theorem 3. *Assume that non(N) $=\mathfrak{c}$, and that there exists a $\mathfrak{c}$-Lusin set. If $H^{\Leftarrow}(\mathcal{F},\looparrowright)$ holds, then there exist $\langle f_n\rangle_{n\in\omega}\in\mathcal{F}$ and $\varepsilon>0$ such that for all $A\subseteq[0,1]$ with $m^*(A)\geq 1-\varepsilon$, $f_n\not\looparrowright 0$ on A.*

Proof: Again, we follow the arguments of Pinciroli (see [7]). Let $Z\subseteq\omega^\omega$ be a $\mathfrak{c}$-Lusin set. Since every compact set is meagre in ω^ω, every K_σ set is also meagre. Therefore, if $A \subseteq Z$ is a K_σ set, then $|A| < \mathfrak{c}$. Let $o : \mathcal{F} \to (\omega^\omega)^{[0,1]}$ be a cofinal function given by $H^{\Leftarrow}(\mathcal{F}, \looparrowright)$. Let φ be a bijection between $[0,1]$ and Z. Finally, let $\langle f_n \rangle_{n\in\omega} = F \in \mathcal{F}$ be such that $o(F) \geq \varphi$.

To get a contradiction, assume that for every $i \in \omega$, there exists $A_i \subseteq [0,1]$ such that $m^*(A_i) \geq 1 - 1/2^i$ and $f_n \looparrowright 0$ on A_i. Let $A = \bigcup_{i\in\omega} A_i$. For any $i \in \omega$, $o(F)[A_i]$ is bounded because $f_n \looparrowright 0$ on A_i, and so $\varphi[A_i]$ is bounded since $o(F) \geq \varphi$. Therefore, $\varphi[A] \in K_\sigma$ and $|A| = |\varphi[A]| < \mathfrak{c}$ because $\varphi[A] \subseteq Z$. This is a contradiction because $m^*(A) = 1$ and $\mathrm{non}(\mathrm{N}) = \mathfrak{c}$. □

The following theorem was proved by R. Pinciroli in [7].

Corollary 4. *Assume that non(N) $< \mathfrak{b}$. Then for any $\langle f_n \rangle_{n\in\omega}$ such that $f_n \colon [0,1] \to [0,1]$ for $n \in \omega$, and $f_n \to 0$, and any $\varepsilon > 0$, there exists $A \subseteq [0,1]$ such that $m^*(A) \geq 1 - \varepsilon$ and $f_n \rightrightarrows 0$ on A.*
On the other hand, assume that non(N) $= \mathfrak{c}$, and that there exists a $\mathfrak{c}$-Lusin set. Then there exist $\langle f_n \rangle_{n\in\omega}$ such that $f_n \colon [0,1] \to [0,1]$ for $n \in \omega$, and $f_n \to 0$, and $\varepsilon > 0$ such that for all $A \subseteq [0,1]$ with $m^*(A) \geq 1 - \varepsilon$, $f_n \not\rightrightarrows 0$ on A.

Proof: Let $\langle f_n \rangle_{n\in\omega}$ be such that $f_n \to 0$. Set $\varepsilon_n = 1/2^n$, $n \in \omega$. Consider $\mathcal{F} = \{\langle f_n \rangle_{n\in\omega} : \forall_{n\in\omega} f_n : [0,1] \to [0,1] \wedge f_n \to 0\}$ and $\looparrowright = \rightrightarrows$. Define

$$o : \mathcal{F} \to (\omega^\omega)^{[0,1]}$$

in the following way. Let

$$oF(x)(n) = \min\{m \in \omega \colon \forall_{l\geq m} f_l(x) \leq \varepsilon_n\}.$$

We get exactly the reasoning and the results of R. Pinciroli (see [7]). He proves that the above function o proves that both $H^{\Leftarrow}(\mathcal{F}_{\to}, \rightrightarrows)$ and $H^{\Rightarrow}(\mathcal{F}_{\to}, \rightrightarrows)$ hold, and then proves Theorems 2 and 3 in this particular case. □

In next sections we apply the method used in the proof of Corollary 4. Assume that we are given two notions of convergence of sequences of functions $f_n \rightsquigarrow f$ and $f_n \looparrowright f$ such that $f_n \looparrowright f$ implies $f_n \rightsquigarrow f$. We take

$$\mathcal{F}_{\rightsquigarrow} = \{\langle f_n \rangle_{n\in\omega} \colon \forall_{n\in\omega}\ f_n : [0,1] \to [0,1] \wedge f_n \rightsquigarrow 0\}$$

and we apply Theorem 2 and Theorem 3 with a suitable function

$$o : \mathcal{F}_{\rightsquigarrow} \to (\omega^\omega)^{[0,1]}$$

to get a conclusion on the stronger convergence $f_n \looparrowright 0$ of sequences from $\mathcal{F}_{\rightsquigarrow}$.

3 Pointwise and equi-ideal convergence (for analytic P-ideals)

Let I be an analytic P-ideal and $f_n\colon [0,1] \to [0,1]$, $n \in \omega$. By the well-known result of Solecki $I = Exh(\phi)$ ([9]), where ϕ is a lower semicontinuous submeasure (a function $\phi\colon 2^\omega \to [0,\infty]$ satisfying the following conditions: $\phi(\emptyset) = 0, \phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$ and $\phi(A) = \lim_{n\to\omega} \phi(A \cap n)$, for any $A, B \subseteq \omega$) and $Exh(\phi) = \{A \subseteq \omega\colon \lim_{n\to\infty} \phi(A \setminus n) = 0\}$ (see also [4]).

Fix a lower continuous submeasure ϕ such that $I = Exh(\phi)$. Recall that we have the following notion of convergence (see [4]) on a set $A \subseteq [0,1]$:

pointwise ideal, $f_n \to_I 0$ if and only if

$$\forall_{\varepsilon>0}\forall_{x\in A}\exists_{k\in\omega}\phi(\{n \in \omega\colon f_n(x) \geq \varepsilon\} \setminus k) < \varepsilon,$$

equi-ideal, $f_n \twoheadrightarrow_I 0$ if and only if

$$\forall_{\varepsilon>0}\exists_{k\in\omega}\forall_{x\in A}\phi(\{n \in \omega\colon f_n(x) \geq \varepsilon\} \setminus k) < \varepsilon,$$

uniform ideal, $f_n \rightrightarrows_I 0$ if and only if

$$\forall_{\varepsilon>0}\exists_{k\in\omega}\phi(\{n \in \omega\colon \sup_{x\in A} f_n(x) \geq \varepsilon\} \setminus k) < \varepsilon.$$

It was proved in [4] that these notion of convergence are independent from the submeasure representation of I. Moreover, the pointwise ideal and uniform ideal convergences can be expressed without the notion of a submeasure and they coincide with the notion of well-known ideal convergences defined for any ideal I on ω (see the next section and also [5]).

Obviously, $f_n \rightrightarrows_I 0 \Rightarrow f_n \twoheadrightarrow_I 0 \Rightarrow f_n \to_I 0$.

It was also proved in [4] that the ideal version of Egorov's Theorem holds (in the case of analytic P-ideals) between equi-ideal and pointwise ideal convergence, i.e. if $\langle f_n\rangle_{n\in\omega}$ is a sequence of measurable functions with $f_n \to_I 0$ on $[0,1]$ and $\varepsilon > 0$, then there exists $A \subseteq [0,1]$ such that $m(A) \geq 1-\varepsilon$ and $f_n \twoheadrightarrow_I 0$ on A. Moreover, it was proved that the ideal version of Egorov's Theorem (in the case of analytic P-ideals) does not hold between uniform ideal and pointwise ideal convergence except for the trivial and "pathological" cases (see also [5]).

Notice that since I is a proper ideal, $\lim_{i\to\infty} \phi(\omega \setminus i) > 0$. If $\lim_{i\to\infty} \phi(\omega \setminus i) < \infty$, let

$$\varepsilon_n = \frac{\lim_{i\to\infty} \phi(\omega \setminus i)}{2^{n+1}}$$

for $n \in \omega$. Otherwise set $\varepsilon_n = 1/2^{n+1}$. To use the method described in the previous section, we state the following definition. For a sequence of functions $F = \langle f_n \rangle_{n\in\omega}$, $f_n\colon [0,1] \to [0,1]$ such that $f_n \to_I 0$, let $o_\phi F \in (\omega^\omega)^{[0,1]}$, and

$$(o_\phi F)(x)(n) = \min\{k \in \omega\colon \phi(\{m \in \omega\colon f_m(x) \geq \varepsilon_n\} \setminus k) < \varepsilon_n\}.$$

The function $o_\phi\colon \mathcal{F}_{\to_I} \to (\omega^\omega)^{[0,1]}$ is well defined, because for each $n \in \omega$, $\{k \in \omega\colon \phi(\{m \in \omega\colon f_m(x) \geq \varepsilon_n\} \setminus k) < \varepsilon_n\}$ is not empty since $f_n \to_I 0$.

Lemma 5. *Let $F = \langle f_n \rangle_{n\in\omega}$ be a sequence of functions with $f_n\colon [0,1] \to [0,1]$. Then $f_n \twoheadrightarrow_I 0$ on $A \subseteq [0,1]$ if and only if $(o_\phi(\langle f_n \rangle_{n\in\omega}))[A]$ is bounded in ω^ω. In particular, $H^\Rightarrow(\mathcal{F}_{\to_I}, \twoheadrightarrow_I)$ holds.*

Proof: By definition, $f_n \twoheadrightarrow_I 0$ on A if and only if for any $n \in \omega$, there exists $k \in \omega$ such that for all $x \in A$, $\phi(\{m \in \omega\colon f_m(x) \geq \varepsilon_n\} \setminus k) < \varepsilon_n$. This is true if and only if there exists a sequence $\langle k_n \rangle_{n\in\omega}$ of natural numbers such that for any $n \in \omega$ and $x \in A$, $\phi(\{m \in \omega\colon f_m(x) \geq \varepsilon_n\} \setminus k_n) < \varepsilon_n$, which holds if and only if for all $x \in A$, $(o_\phi F)(x)(n) \leq k_n$. $\square$

Corollary 6. *Assume that non$(N) < \mathfrak{b}$. Let I be any analytic P-ideal, $\varepsilon > 0$, and let $F = \langle f_n \rangle_{n\in\omega}$, $f_n\colon [0,1] \to [0,1]$ for $n \in \omega$, be such that $f_n \to_I 0$. Then there exists $A \subseteq [0,1]$ with $m^*(A) \geq 1 - \varepsilon$ such that $f_n \twoheadrightarrow_I 0$ on A (the ideal version of the generalized Egorov's statement between equi-ideal and pointwise ideal convergence for analytic P-ideals is consistent with ZFC).*

Proof: Apply Theorem 2 and Lemma 5. $\square$

Lemma 7. *For any $\varphi\colon [0,1] \to \omega^\omega$, there exists $F = \langle f_n \rangle_{n\in\omega}$,*

$$f_n\colon [0,1] \to [0,1]$$

for $n \in \omega$ with $f_n \to_I 0$ such that $o_\phi F \geq \varphi$. In particular, $H^\Leftarrow(\mathcal{F}_{\to_I}, \twoheadrightarrow_I)$ holds.

Proof: Fix $x \in [0,1]$. Notice that $\phi(\omega \setminus n)$ is a decreasing sequence with limit greater or equal to $2\varepsilon_0 > 0$, so $\phi(\omega \setminus n) \geq 2\varepsilon_0 > 0$ for any $n \in \omega$. Therefore, for each $m, n \in \omega$, there exists $k > n$ such that $\phi(k \setminus n) > \varepsilon_m$. Let $\langle k_i \rangle_{i\in\omega}$, be an increasing sequence such that $k_0 = 0$ and $\phi(k_{i+1} \setminus \varphi(x)(i)) > \varepsilon_i$, $i \in \omega$. Set $f_j(x) = \varepsilon_i$ if $k_i \leq j < k_{i+1}$. Then $f_m(x) \geq \varepsilon_n$ if and only if $m < k_{n+1}$. Therefore, if $\phi(\{m \in \omega\colon f_m(x) \geq \varepsilon_n\} \setminus k) < \varepsilon_n$, then $k \geq \varphi(x)(n)$, so $(o_\phi F)(x)(n) \geq \varphi(x)(n)$ for any $n \in \omega$.

This proves that o is a cofinal function. Therefore, by Lemma 5, the property $H^\Leftarrow(\mathcal{F}_{\to_I}, \twoheadrightarrow_I)$ holds. $\square$

Corollary 8. *Assume that non(N) = $\mathfrak{c}$, and that there exists a $\mathfrak{c}$-Lusin set. Let I be any analytic P-ideal. Then there exists $F = \langle f_n \rangle_{n\in\omega}$, $f_n \colon [0,1] \to [0,1]$ for $n \in \omega$ with $f_n \to_I 0$ and $\varepsilon > 0$ such that for every $A \subseteq [0,1]$ with $m^*(A) \geq 1 - \varepsilon$, $f_n \not\rightrightarrows_I 0$ on A (the negation of the ideal version of the generalized Egorov's statement between equi-ideal and pointwise ideal convergence for analytic P-ideals is consistent with ZFC).*

Proof: We use Theorem 3 and Lemma 7. □

4 Countably generated ideals

Recall that an ideal I over ω is countably generated (satisfies the chain condition) if there exists a sequence $\langle C_i \rangle_{i\in\omega}$ of elements of I such that $C_i \subseteq C_{i+1}$ for all $i \in \omega$ and for every $A \in I$, there exists $k \in \omega$ such that $A \subseteq C_k$.

Let $\langle f_n \rangle_{n\in\omega}$, $f_n \colon [0,1] \to [0,1]$, and let I be an ideal on ω. Recall the classic notion of ideal convergence on $A \subseteq [0,1]$:

pointwise ideal, $f_n \to_I 0$ if and only if $\forall_{\varepsilon>0}\forall_{x\in A}\{n \in \omega \colon f_n(x) \geq \varepsilon\} \in I$,

quasinormal ideal, $f_n \xrightarrow{QN}_I 0$ if and only if there exists a sequence of positive reals $\langle \varepsilon_n \rangle_{n\in\omega}$ such that $\varepsilon_n \to_I 0$ and $\forall_{x\in A}\{n \in \omega \colon f_n(x) \geq \varepsilon_n\} \in I$,

uniform ideal, $f_n \rightrightarrows_I 0$ if and only if

$$\forall_{\varepsilon>0}\exists_{B\in I}\forall_{x\in A}\{n \in \omega \colon f_n(x) \geq \varepsilon\} \subseteq B.$$

The quasinormal convergence with respect to an ideal I is also sometimes called I-equal convergence. Notice that in the case of countably generated ideals the generalized Egorov's statement holds between uniform ideal and quasinormal ideal convergence (see [2, Theorem 3.2]).

Let us therefore compare the pointwise and uniform ideal convergences. First, we show that the classic version (for measurable functions) of Egorov's Theorem holds in the case of convergence with respect to a countably generated ideal.

Theorem 9. *If $I \subseteq 2^\omega$ is a countably generated ideal and $f_n \colon [0,1] \to [0,1]$, $n \in \omega$ are measurable functions such that $f_n \to_I 0$ and $\varepsilon > 0$, then there exists a measurable set $B \subseteq [0,1]$ such that $m(B) \leq \varepsilon$ and $f_n \rightrightarrows_I 0$ on $[0,1] \setminus B$.*

Proof: Assume that I is countably generated and fix sets $\langle C_i \rangle_{i\in\omega}$ such that $C_i \subseteq C_{i+1}$ for all $i \in \omega$ and for every $A \in I$, there exists $k \in \omega$ such that $A \subseteq C_k$. For $n, k \in \omega$, let

$$E_{n,k} = \left\{x \in [0,1] \colon \left\{m \in \omega \colon f_m(x) > \frac{1}{2^k}\right\} \setminus C_n \neq \emptyset\right\}.$$

Notice that

$$E_{n,k} = \bigcup_{m \in \omega \setminus C_n} \left\{ x \in [0,1] \colon f_m(x) > \frac{1}{2^k} \right\}$$

is measurable for each $n, k \in \omega$. Moreover, $E_{n+1,k} \subseteq E_{n,k}$ and $\bigcap_{n \in \omega} E_{n,k} = \emptyset$ for all $k \in \omega$. Let $\varepsilon > 0$. For each $k \in \omega$, there exists $n_k \in \omega$ such that

$$m(E_{n_k,k}) \leq \frac{\varepsilon}{2^{k+1}}.$$

Let $B = \bigcup_{k \in \omega} E_{n_k,k}$. So $m(B) \leq \varepsilon$, and if $x \notin B$, then

$$\left\{ m \in \omega \colon f_m(x) > \frac{1}{2^k} \right\} \subseteq C_{n_k},$$

for any $k \in \omega$, so $f_n \rightrightarrows_I 0$ on $[0,1] \setminus B$. □

Let us consider the generalized Egorov's statement in this setting. The results presented below were proved by Joanna Jureczko using the method of T. Weiss (see [10]) directly. We continue to apply the generalization of Pinciroli's method as presented above.

Assume that I is countably generated, and fix sets $\langle C_i \rangle_{i \in \omega}$ such that $C_i \subseteq C_{i+1}$ for all $i \in \omega$ and for every $A \in I$, there exists $k \in \omega$ such that $A \subseteq C_k$. We can assume that $C_{i+1} \setminus C_i \neq \emptyset$ for all $i \in \omega$.

If $F = \langle f_n \rangle_{n \in \omega}$, $f_n \to_I 0$, we define

$$(o_{\langle C_i \rangle} F)(x)(n) = \min \left\{ k \in \omega \colon \left\{ m \in \omega \colon f_m(x) > \frac{1}{2^n} \right\} \subseteq C_k \right\}.$$

Notice that if $A \subseteq [0,1]$, then $f_n \rightrightarrows_I 0$ on A if and only if $(o_{\langle C_i \rangle} F)[A]$ is bounded, and so $H^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$ holds. Therefore, we get the following theorem.

Corollary 10. *Assume that non$(N) < \mathfrak{b}$. Let I be any countably generated ideal, and let $\varepsilon > 0$. Let $F = \langle f_n \rangle_{n \in \omega}$, $f_n \colon [0,1] \to [0,1]$, for $n \in \omega$ be such that $f_n \to_I 0$. Then there exists $A \subseteq [0,1]$ with $m^*(A) \geq 1 - \varepsilon$ such that $f_n \rightrightarrows_I 0$ on A (the ideal version of the generalized Egorov's statement between uniform ideal and pointwise ideal convergence for countably generated ideals is consistent with ZFC).*

Proof: Apply Theorem 2. □

Lemma 11. *For any $\varphi \colon [0,1] \to \omega^\omega$ there exists*

$$F = \langle f_n \rangle_{n \in \omega}, f_n \colon [0,1] \to [0,1], f_n \to_I 0$$

such that $o_{\langle C_i \rangle} F = \varphi$. In particular, $H^{\Leftarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$ holds.

Proof: Without a loss of generality we can assume that $\varphi(x)$ is increasing for all $x \in [0,1]$. Let $x \in [0,1]$. Let $f_j(x) = 1/2^n$ if and only if $j \in C_{\varphi(x)(n+1)} \setminus C_{\varphi(x)(n)}$. □

Corollary 12. *Assume that non(N) = $\mathfrak{c}$, and that there exists a $\mathfrak{c}$-Lusin set. Let I be any countably generated ideal. Then there exists $F = \langle f_n \rangle_{n\in\omega}$, $f_n\colon [0,1] \to [0,1]$ for $n \in \omega$ with $f_n \to_I 0$, and $\varepsilon > 0$ such that for all $A \subseteq [0,1]$ with $m^*(A) \geq 1-\varepsilon$, $f_n \not\rightrightarrows_I 0$ on A (the negation of the ideal version of the generalized Egorov's statement between uniform ideal and pointwise ideals convergence for countably generated ideal is consistent with ZFC).*

Proof: Apply Theorem 3 and Lemma 11. □

5 I^* convergence for countably generated ideals

As before, let $\langle f_n \rangle_{n\in\omega}$, $f_n\colon [0,1] \to [0,1]$, and let I be an ideal on ω. We have the following notion of convergence $A \subseteq [0,1]$ (see [2]):

I^*-pointwise, $f_n \to_{I^*} 0$ if and only if for all $x \in A$, there exists $M = \{m_i\colon i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i}(x) \to 0$,

I^*-quasinormal, $f_n \xrightarrow{QN}_{I^*} 0$ if and only if there exists $M = \{m_i\colon i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i} \xrightarrow{QN} 0$ on A,

I^*-uniform, $f_n \rightrightarrows_{I^*} 0$ if and only if there exists $M = \{m_i\colon i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i} \rightrightarrows 0$ on A.

Notice that for any ideal I, the generalized Egorov's statement holds between I^*-uniform and I^*-quasinormal convergence (see [2, Theorem 3.3]).

Let us therefore compare the pointwise and uniform ideal convergences. First, we show that the classic version (for measurable functions) of Egorov's Theorem holds in the case of I^*-convergence with respect to a countably generated ideal I.

Theorem 13. *If $I \subseteq 2^\omega$ is a countably generated ideal and $f_n\colon [0,1] \to [0,1]$, $n \in \omega$ are measurable functions such that $f_n \to_{I^*} 0$ and $\varepsilon > 0$, then there exists a measurable set $B \subseteq [0,1]$ such that $m(B) \leq \varepsilon$ and $f_n \rightrightarrows_{I^*} 0$ on $[0,1] \setminus B$.*

Proof: Assume that I is countably generated and fix $\langle C_n \rangle_{n\in\omega}$ such that for all $A \in I$, there exists $n \in \omega$ with $A \subseteq C_n$. Let $\omega \setminus C_n = \{m_{i,n}\colon i \in \omega\}$, $m_{i+1,n} > m_{i,n}$, $i,n \in \omega$, and

$$F_n = \left\{ x \in [0,1]\colon \lim_{i\in\omega} f_{m_{i,n}}(x) = 0 \right\}$$

Obviously, $F_n \subseteq F_{n+1}$ for $n \in \omega$ and $\bigcup_{n\in\omega} F_n = [0,1]$. Moreover,

$$F_n = \bigcap_{i\in\omega} \bigcup_{j\in\omega} \bigcap_{k\geq j} \left\{ x \in [0,1] \colon f_{m_{k,n}}(x) < \frac{1}{2^i} \right\}$$

is measurable. Therefore, there exists $N \in \omega$ such that $m(F_N) \geq 1-\varepsilon/2$. Now apply the classic Egorov's Theorem for the set F_N, $\left\langle f_{m_{i,N}} \right\rangle_{i\in\omega}$ and $\varepsilon/2$ to get a set $A \subseteq F_N$ such that $f_{m_{i,N}}$ converges uniformly on $F_N \setminus A$ and $m(A) < \varepsilon/2$. Let $B = A \cup ([0,1] \setminus F_N)$. We get that $f_n \rightrightarrows_{I^*} 0$ on $[0,1] \setminus B$ and $m(B) \leq \varepsilon$. □

Let us consider the generalized Egorov's statement in this setting. Assume that I is countably generated and fix $\langle C_n \rangle_{n\in\omega}$ such that for all $A \in I$, there exists $n \in \omega$ such that $A \subseteq C_n$. Let $F = \langle f_n \rangle_{n\in\omega}$ be such that $f_n \to_{I^*} 0$. For $x \in [0,1]$ define $o_{\langle C_i \rangle}(F)(x) = \psi \in \omega^\omega$ by

$$\psi(0) = \min \left\{ n \in \omega : \langle f_m \rangle_{m\in\omega\setminus C_n} \to 0 \right\},$$
$$\psi(n) = \min \left\{ m \in \omega : \forall_{\substack{l\in\omega\setminus C_{\psi(0)} \\ l>m}} \ f_l(x) < \frac{1}{2^n} \right\}, \quad n > 0.$$

Obviously, $o_{\langle C_i \rangle} F$ is bounded if and only if $f_n \rightrightarrows_{I^*} 0$, and so the property $H^{\Rightarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$ holds.

Therefore, we get the following theorem.

Corollary 14. *Assume that non$(N) < \mathfrak{b}$. Let I be any countably generated ideal, and let $\varepsilon > 0$ and $F = \langle f_n \rangle_{n\in\omega}$, $f_n \colon [0,1] \to [0,1]$ for $n \in \omega$, with $f_n \to_{I^*} 0$. Then there exists $A \subseteq [0,1]$ with $m^*(A) \geq 1-\varepsilon$ such that $f_n \rightrightarrows_{I^*} 0$ on A (the ideal version of the generalized Egorov's statement between uniform I^* and pointwise I^* convergence for countably generated ideals is consistent with ZFC).*

Proof: Apply Theorem 2. □

Lemma 15. *For any $\varphi \colon [0,1] \to \omega^\omega$, there exists*

$$F = \langle f_n \rangle_{n\in\omega} \text{ with } f_n \colon [0,1] \to [0,1] \text{ and } f_n \to_{I^*} 0$$

such that $o_{\langle C_i \rangle} F \geq \varphi$. In particular, the condition $H^{\Leftarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$ holds.

Proof: It is enough to prove the lemma for φ such that $\varphi(x)$ is increasing for all $x \in [0,1]$. Let $x \in [0,1]$. Let $\omega \setminus C_{\varphi(x)(0)} = \{m_i \colon i \in \omega\}$, $m_{i+1} > m_i$ for $i \in \omega$. Let $f_j(x) = 1$ for $j \in C_{\varphi(x)(0)}$ and let $f_j(x) = 1/2^n$ if $j \in (\omega \setminus C_{\varphi(x)(0)}) \cap \{i \in \omega \colon \varphi(x)(n) \leq i < \varphi(x)(n+1)\}$. □

Corollary 16. *Assume that non(N) = $\mathfrak{c}$, and that there exists a $\mathfrak{c}$-Lusin set. Let I be any countably generated ideal. Then there exists $F = \langle f_n \rangle_{n\in\omega}$, $f_n\colon [0,1] \to [0,1]$ for $n \in \omega$, with $f_n \to_{I^*} 0$, and $\varepsilon > 0$ such that for all $A \subseteq [0,1]$ with $m^*(A) \geq 1-\varepsilon$, $f_n \not\rightrightarrows_{I^*} 0$ on A (the negation of the ideal version of the generalized Egorov's statement between uniform I^* and pointwise I^* convergence for countably generated ideals is consistent with ZFC).*

Proof: Apply Theorem 3 and Lemma 15. □

6 Ideals Fin^α

Given an ideal $I \subseteq \omega$ and a sequence $\langle I_n \rangle_{n\in\omega}$ of ideals of ω, we can consider an ideal $I\text{-}\prod_{n\in\omega} I_n$ on ω^2 called the I-product of the sequence of ideals $\langle I_n \rangle_{n\in\omega}$ and define it in the following way. For any $A \subseteq \omega^2$,

$$A \in I\text{-}\prod_{n\in\omega} I_n \leftrightarrow \{n \in \omega\colon A_{(n)} \notin I_n\} \in I,$$

where $A_{(n)} = \{m \in \omega\colon \langle n, m\rangle \in A\}$ (see [5]). If $I_n = J$ for any $n \in \omega$, we usually denote $I\text{-}\prod_{n\in\omega} I_n$ as $I \times J$.

Fix a bijection $b\colon \omega^2 \to \omega$ and a bijection $a_\beta\colon \omega \setminus \{0\} \to \beta$ for any limit $\beta < \omega_1$. The ideals Fin^α, $\alpha < \omega_1$, are defined inductively (see [5]) in the following way. Let $\mathrm{Fin}^1 = \mathrm{Fin}$ be the ideal of finite subsets of ω. We set $\mathrm{Fin}^{\alpha+1} = \{b[A]\colon A \in \mathrm{Fin} \times \mathrm{Fin}^\alpha\}$ and for limit $\beta < \omega_1$, let $\mathrm{Fin}^\beta = \{b[A]\colon A \in \mathrm{Fin}\text{-}\prod_{i\in\omega} \mathrm{Fin}^{a_\beta(i+1)}\}$.

In [5, Theorem 3.25], N. Mrożek proves that ideal Fin^α for any $\alpha < \omega_1$ satisfies the Egorov's theorem for ideals (between uniform ideal and pointwise ideal convergences).

Let $\mathcal{F}_\alpha = \mathcal{F}_{\to_{\mathrm{Fin}^\alpha}}$. We get the following theorem.

Theorem 17. *Assume that non(N) $< \mathfrak{b}$. Let $0 < \alpha < \omega_1$, and let $\varepsilon > 0$ and $F = \langle f_n \rangle_{n\in\omega}$, $f_n\colon [0,1] \to [0,1]$ for $n \in \omega$, with $f_n \to_{Fin^\alpha} 0$. Then there exists $A \subseteq [0,1]$ with $m^*(A) \geq 1-\varepsilon$ such that $f_n \rightrightarrows_{Fin^\alpha} 0$ on A (the ideal version of the generalized Egorov's statement between uniform Fin^α and pointwise Fin^α convergence is consistent with ZFC).*

Proof: We define $o_\alpha\colon \mathcal{F}_\alpha \to (\omega^\omega)^{[0,1]}$ in the following way. Let $\varepsilon_n = \frac{1}{2^n}$ for $n \in \omega$, and let

$$\mathcal{F}^n_\alpha = \{\langle f_k \rangle_{k\in\omega}\colon \forall_{k\in\omega} f_k\colon [0,1] \to [0,1] \land \forall_{x\in[0,1]} \{q \in \omega\colon f_q(x) \geq \varepsilon_n\} \in \mathrm{Fin}^\alpha\}.$$

First, define $o^n_\alpha\colon \mathcal{F}^n_\alpha \to (\omega^\omega)^{[0,1]}$, $n \in \omega, 0 < \alpha < \omega_1$, by induction on α. Let

$$M_{1,n,x} = \min\{p \in \omega\colon \forall_{q\geq p} f_q(x) < \varepsilon_n\},$$

and let

$$(o_1^n F)(x)(k) = M_{1,n,x}$$

be a constant sequence. Given o_α^n, let

$$M_{\alpha+1,n,x} = \min\left\{p \in \omega\colon \forall_{q\geq p}\{m\in\omega\colon f_{b(q,m)}(x) \geq \varepsilon_n\} \in \mathrm{Fin}^\alpha\right\},$$

and

$$(o_{\alpha+1}^n F)(x)(k) = \begin{cases} M_{\alpha+1,n,x} & \text{for } k = b(p,q), \\ & p < M_{\alpha+1,n,x}+1, q\in\omega, \\ (o_\alpha^n \left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega})(x)(q) & \text{for } k = b(p,q), \\ & p \geq M_{\alpha+1,n,x}+1, q\in\omega. \end{cases}$$

This definition is correct, since $\left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega} \in \mathcal{F}_\alpha^n$ for $p \geq M_{\alpha+1,n,x}+1$.

Moreover, for limit $\beta < \omega_1$, let

$$M_{\beta,n,x} = \min\left\{p\in\omega\colon \forall_{q\geq p}\{m\in\omega\colon f_{b(q,m)}(x)\geq\varepsilon_n\} \in \mathrm{Fin}_{a_\beta(q)}\right\}$$

and

$$(o_\beta^n F)(x)(k) = \begin{cases} M_{\beta,n,x} & \text{for } k = b(p,q), \\ & p < M_{\beta,n,x}+1, q\in\omega, \\ (o_{a_\beta(p-1)}^n \left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega})(x)(q) & \text{for } k = b(p,q), \\ & p \geq M_{\beta,n,x}+1, q\in\omega. \end{cases}$$

This definition is correct, since, for each $p \geq M_{\beta,n,x}+1$, $\left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega} \in \mathcal{F}_{a_\beta(p-1)}^n$.

Notice that $\mathcal{F}_\alpha \subseteq \mathcal{F}_\alpha^n$, for any $n\in\omega$. Therefore, finally let

$$(o_\alpha F)(x)(k) = (o_\alpha^n F)(x)(m),$$

for $k = b(n,m)$, $n,m\in\omega$.

Now, notice that if $F = \langle f_r\rangle_{r\in\omega} \in \mathcal{F}_\alpha$, and $o_\alpha F$ is bounded on a set $A\subseteq[0,1]$, then $f_r \rightrightarrows_{\mathrm{Fin}^\alpha} 0$ on A. Indeed, if $o_\alpha F$ is bounded, then for each $n\in\omega$, $o_\alpha^n F$ is bounded. If so, $\{m\in\omega\colon \sup_{x\in A} f_m(x)\geq\varepsilon_n\}\in\mathrm{Fin}^\alpha$, for all $n\in\omega$. We fix $n\in\omega$ and prove this statement by induction on $\alpha<\omega_1$. Let $(o_\alpha^n F)(x)(k) < a_{k,n}$ for all $x\in A, k\in\omega$ and some $\langle a_{k,n}\rangle_{k\in\omega}\in\omega^\omega$. If $\alpha = 1$, we get $f_q(x) < \varepsilon_n$ for all $x\in A$ and all $q\geq a_0$, so $\{m\in\omega\colon \sup_{x\in A} f_m(x)\geq\varepsilon_n\}\in\mathrm{Fin}$. Now, assume that the statement holds for some $\alpha<\omega_1$. Then for all $x\in A$, $M_{\alpha+1,n,x} < a_{b(0,0)}$, so for all $p\geq a_{b(0,0)}$, $o_\alpha^n\left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega}$ is bounded by $\left\langle a_{b(p,q)}\right\rangle_{q\in\omega}$, and thus by the induction hypothesis, $\{r\in\omega\colon \sup_{x\in A} f_{b(p-1,r)} \geq$

$\varepsilon_n\} \in \mathrm{Fin}^\alpha$ for all $p \geq a_{b(0,0)}$. Therefore, $\{m \in \omega \colon \sup_{x\in A} f_m(x) \geq \varepsilon_n\} \in \mathrm{Fin}^{\alpha+1}$. Analogous reasoning can be easily applied for limit $\beta < \omega_1$. This proves that $H^{\Rightarrow}(\mathcal{F}_\alpha, \rightrightarrows_{\mathrm{Fin}^\alpha})$ holds.

Therefore, by Theorem 2, there exists $A \subseteq [0,1]$ with $m^*(A) \geq 1-\varepsilon$ such that $f_n \rightrightarrows_{\mathrm{Fin}^\alpha} 0$ on A. □

Theorem 18. *Assume that non(N) = $\mathfrak{c}$, and that there exists a $\mathfrak{c}$-Lusin set. Let $0 < \alpha < \omega_1$. Then there exist $\langle f_n\rangle_{n\in\omega} \in \mathcal{F}_\alpha$ and $\varepsilon > 0$ such that for all $A \subseteq [0,1]$ with $m^*(A) \geq 1-\varepsilon$, $f_n \not\rightrightarrows_{Fin^\alpha} 0$ on A (the negation of the ideal version of the generalized Egorov's statement between uniform Fin$^\alpha$ and pointwise Fin$^\alpha$ convergence for countably generated ideals is consistent with ZFC).*

Proof: As before, let $\varepsilon_n = 1/2^n$, $n \in \omega$. This time, we define o_α in a different way then in the previous proof. Namely, let

$$(o_\alpha F)(x)(n) = M_{\alpha,n,x},$$

where $M_{\alpha,n,x}$ is defined as in the previous proof. Notice that if $F = \langle f_n\rangle_{n\in\omega}$ is such that $f_n \rightrightarrows_{\mathrm{Fin}^\alpha} 0$ on a set $A \subseteq [0,1]$, then $\{m \in \omega \colon \sup_{x\in A} f_m(x) \geq \varepsilon_n\} \in \mathrm{Fin}^\alpha$ for all $n \in \omega$. If $\alpha = 1$, this means that $\min\{p \in \omega \colon \forall_{q\geq p} f_q(x) < \varepsilon_n\} = M_{1,n,x} = o_1F(x)(n)$ is bounded on A. If α is a limit ordinal, then for all $n \in \omega$, there exists M_n such that for all $q \geq M_n$, $\{m \in \omega \colon f_{b(q,m)}(x) \geq \varepsilon_n\} \in \mathrm{Fin}_{a_\alpha(q)}$. In other words, $M_{\alpha,n,x} = o_\alpha F(x)$ is bounded on A. Similar argument can be used in the case of a successor ordinal $\alpha > 1$.

Moreover, fix any $\varphi\colon [0,1] \to \omega^\omega$. Without a loss of generality, assume that for $x \in [0,1]$, $\varphi(x)$ is increasing. There exists $F = \langle f_n\rangle_{n\in\omega} \in \mathcal{F}$ such that $o_\alpha(F) \geq \varphi$. It is obvious for $\alpha = 1$. For $\alpha > 1$, let $f_n(x) = \varepsilon_k$ for $k = b(i,j)$, $\varphi(x)(k) \leq n < \varphi(x)(k+1)$. Therefore $H^{\Leftarrow}(\mathcal{F}_\alpha, \rightrightarrows_{\mathrm{Fin}^\alpha})$ holds.

In conclusion, by Theorem 3, there exist $\langle f_n\rangle_{n\in\omega} \in \mathcal{F}$ and $\varepsilon > 0$ such that for all $A \subseteq [0,1]$ with $m^*(A) \geq 1-\varepsilon$, $f_n \not\rightrightarrows_{\mathrm{Fin}^\alpha} 0$ on A. □

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