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Further Evaluation of Wahl Vanishing Theorems for Surface Singularities in Characteristic p

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Abstract. Let $(\operatorname{Spec} R, \mathfrak{m})$ be a rational double point defined over an algebraically closed field k of characteristic $p \geq 0$. We evaluate further the dimensions of the local cohomology groups, which were treated by Wahl in 1975 as vanishing theorem C (resp., D) under the assumption that p is a very good prime (resp., good prime) with respect to $(\operatorname{Spec} R, \mathfrak{m})$. We use Artin's classification of rational double points and completely determine the dimensions $\dim_k H^1_E(S_X)$ and $\dim_k H^1_E(S_X \otimes \mathcal{O}_X(E))$, supplementing Wahl's theorems. In the proof, we concretely construct derivations that do not lift to the minimal resolution $X \to \operatorname{Spec} R$ and an equisingular family that injects into a versal deformation of the rational double point $(\operatorname{Spec} R, \mathfrak{m})$.

1. Introduction

In 1975, Jonathan Wahl proved the Grauert–Riemenschneider vanishing theorem along with three other types of vanishing theorems for a surface singularity $(\operatorname{Spec} R, \mathfrak{m})$ defined over an algebraically closed field k of characteristic $p \geq 0$ [18]. Last decades witnessed that these theorems on local cohomology groups have played influential roles in the theory of surface singularities. Among them, there are what he calls Theorems C and D, which bear restrictions on the characteristic of the ground field k. The versions specialized for rational double points state the following:

Theorem C. *Let $\pi : X \to \operatorname{Spec} R$ be the minimal resolution of a rational double point. Then $H^1_E(S_X) = 0$, and in particular the resolution is equivariant, except possibly in the following cases*:

$$\begin{array}{ll} A_n, & p \mid (n+1), \\ D_n, & p = 2, \\ E_6, & p = 2, 3, \\ E_7, & p = 2, 3, \\ E_8, & p = 2, 3, 5. \end{array}$$

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THEOREM D. *Let $\pi : X \to \operatorname{Spec} R$ be the minimal resolution of a rational double point. Then $H^1_E(S_X(E)) = 0$, except possibly in the following cases*:

$$\begin{array}{ll} D_n, & p=2, \\ E_6, & p=2,3, \\ E_7, & p=2,3, \\ E_8, & p=2,3,5. \end{array}$$

In these theorems, $E = \bigcup_i E_i$ denotes the exceptional divisor in X and its irreducible decomposition, S_X is the sheaf of logarithmic derivations $S_X = \Theta_X(-\log E)$ (cf. [19]), which fits in an exact sequence

$$0 \to S_X \to \Theta_X \to \bigoplus_i \mathcal{N}_{E_i/X} \to 0.$$

We say that a resolution π is equivariant if the natural inclusion $\pi_*\Theta_X \hookrightarrow \Theta_R$ is an isomorphism (cf. [4, Section 1]).

In this paper, we completely evaluate the dimensions of k-vector spaces $H^1_E(S_X)$ and $H^1_E(S_X \otimes \mathcal{O}_X(E))$ for each isomorphism class $(\operatorname{Spec} R, \mathfrak{m})$ of rational double points and show that the following equality holds:

$$\#\{-2 \text{ curves in } E\} + \dim_k H^1_E(S_X) + \dim_k H^1(S_X) = \tau \text{ (Tjurina number)}.$$

Before we proceed, there may be two things we should be aware of. First, the classification of rational double points over an algebraically closed field k of arbitrary characteristic was completed by Artin [3] in 1977. Wahl's original vanishing theorems do not depend on it. Second, in the theory of root systems, there are notions of good primes [16, Ch. I, Section 4], very good primes [15, 3.13], which nicely coincide with Wahl's theorems. It follows that, for irreducible root systems, the bad (= not good) prime numbers are:

$$\begin{array}{ll} A_n, & \text{none}, \\ B_n, C_n, D_n, & p=2, \\ E_6, E_7, F_4, G_2, & p=2,3, \\ E_8, & p=2,3,5. \end{array}$$

For the root system of type A_n, a prime number p is defined to be very good if p does not divide $n+1$. This suggests that there are further relationships yet to be discovered between rational surface singularities and Lie algebras.

Our main theorem is the following:

THEOREM 1.1. *Let $X \to \operatorname{Spec} R$ be the minimal resolution of a rational double point defined over an algebraically closed field k of characteristic $p \geq 0$. Then the following assertions hold*:

(i) *The natural morphism* $H^1(S_X) \to H^1(X \setminus E, S_X)$ *is an inclusion.*
(ii) *The dimension of* $H^1_E(S_X \otimes \mathcal{O}_X(E))$ *is either zero or given as*

$$\begin{cases} 1 & \text{for } E_8^0 \text{ in } p=5;\ E_6^0, E_7^0, E_8^1 \text{ in } p=3;\ \text{and } E_6^0, E_7^2, E_8^3 \text{ in } p=2; \\ 2 & \text{for } E_8^0 \text{ in } p=3 \text{ and } E_7^1, E_8^2 \text{ in } p=2; \\ 3 & \text{for } E_7^0, E_8^1 \text{ in } p=2; \\ 4 & \text{for } E_8^0 \text{ in } p=2; \\ n-r-1 & \text{for } D_{2n}^r, D_{2n+1}^r \text{ in } p=2. \end{cases}$$

(iii) *We have an isomorphism*

$$H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X) \cong H^1_E(S_X),$$

whose dimension is zero with the following exceptions:

$$\begin{cases} 1 & \text{for } A_n \text{ with } p \mid (n+1); \\ 1 & \text{for } E_8^0 \text{ in } p=5;\ E_6^1, E_7^0, E_8^1 \text{ in } p=3;\ \text{and } E_6^0, E_7^3, E_8^3 \text{ in } p=2; \\ 2 & \text{for } E_6^0, E_8^0 \text{ in } p=3 \text{ and } E_7^2, E_8^2 \text{ in } p=2; \\ 3 & \text{for } E_7^1, E_8^1 \text{ in } p=2; \\ 4 & \text{for } E_7^0, E_8^0 \text{ in } p=2; \\ n+1-r & \text{for } D_{2n}^r \text{ in } p=2; \\ n-r & \text{for } D_{2n+1}^r \text{ in } p=2. \end{cases}$$

Statement (i) in the theorem gives an answer to the question raised in [18, (5.18.2)] for rational double points (cf. [20, Rem. 8.8]).

As a corollary, we have a characterization of tame rational double points.

Corollary 1.2. *Let* (Spec $R, \mathfrak{m}$) *be a rational double point defined over an algebraically closed field* k *of characteristic* $p > 0$. *Then the following conditions are equivalent*:

(i) (Spec $R, \mathfrak{m}$) *is of type* A (*resp. of type* D *or* E) *and* p *is a very good prime* (*resp., a good prime*), *or* (Spec $R, \mathfrak{m}$) *is of type* E_8^1 *in* $p=5$, *or of type* E_7^1, E_8^2 *in* $p=3$, *or of type* E_6^1, E_8^4 *in* $p=2$.
(ii) $H^1_E(S_X) = 0$ *for the minimal resolution* $\pi : X \to \operatorname{Spec} R$.
(iii) $H^1_E(S_X) = H^1(S_X) = 0$ *for the minimal resolution* $\pi : X \to \operatorname{Spec} R$.
(iv) *In a versal deformation of* (Spec $R, \mathfrak{m}$), *the bundle of fibers that have the same rational double point as* (Spec $R, \mathfrak{m}$) *is trivial*, *that is*, *the closed fiber with the reduced base* Spec k *only.* (*The rational double point* (Spec $R, \mathfrak{m}$) *has no moduli.*)

Wahl vanishing theorems have come into focus in the study of three-dimensional canonical singularities in arbitrary characteristic, and we needed further evaluation of dimensions of $H^1_E(S_X)$ and $H^1_E(S_X \otimes \mathcal{O}_X(E))$. We have the smooth morphism from the simultaneous resolution functor of a versal deformation of a rational double point (Spec $R, \mathfrak{m}$) (cf. [2; 8]) to the deformation functor of its minimal resolution X:

$$\operatorname{Res}\mathcal{X}/S \to \mathrm{D}_X .$$

Nonzero elements of $H^1_E(S_X)$ correspond to families of resolutions that are nontrivial in $\operatorname{Res}\mathcal{X}/S$ but are trivial deformations of X. These are linked with phenomena of three-dimensional canonical singularities peculiar to positive characteristic p, which are recently observed in [6; 7; 9; 13].

As an application, we have a condition for a well-known property in characteristic zero to hold (cf. [12, Cor. (1.14)]).

COROLLARY 1.3. *Let* $(\operatorname{Spec} R, \mathfrak{m})$ *be a rational double point defined over an algebraically closed field k of characteristic* $p > 0$, *and let Y be a three-dimensional quasi-projective normal variety over k with at most canonical singularities whose general hyperplane section* $H \subset Y$ *has a rational double point of the same type as* $(\operatorname{Spec} R, \mathfrak{m})$. *If the equivalent conditions on* $(\operatorname{Spec} R, \mathfrak{m})$ *in the previous corollary are satisfied, then there exists a zero-dimensional subvariety* $Z_0 \subset Y$ *such that, for any point* $y \in Y \setminus Z_0$, *the complete local ring* $\hat{\mathcal{O}}_{Y,y}$ *is either regular or isomorphic to* $\hat{R}\hat{\otimes}k[[w]]$. (*Reid's rule continues to hold.*)

We cannot expect that the converse of the last corollary is true (cf. Remark 6.1).

2. Preliminaries

When we say that $(\operatorname{Spec} R, \mathfrak{m})$ is a surface singularity, it is understood that $(R, \mathfrak{m})$ is a two-dimensional excellent normal local ring with maximal ideal $\mathfrak{m}$. We identify surface singularities $(\operatorname{Spec} R_1, \mathfrak{m}_1)$ and $(\operatorname{Spec} R_2, \mathfrak{m}_2)$ if there exists an isomorphism between complete local rings $\hat{R}_1 \cong \hat{R}_2$. We use the term *equisingular deformations* of the resolution in the sense of Wahl [19].

The following proposition is in implicit form in Artin's work [2, Cor. 4.6]. Here $\mathfrak{R}$ denotes a locally quasi-separated algebraic space that represents the functor $\operatorname{Res}\mathcal{X}/S$ of simultaneous resolutions of flat families of a surface singularity $(\operatorname{Spec} R, \mathfrak{m})$ defined over an algebraically closed field k.

PROPOSITION 2.1. *Suppose* $\mathcal{X}/S$ *is a versal deformation of a rational surface singularity* $(\operatorname{Spec} R, \mathfrak{m})$ *at* $s_0 \in S$ *and has minimum tangent space dimension there. Then the minimal resolution* $X \to \operatorname{Spec} R$ *is equivariant if and only if the universal family* $\mathcal{X}'_{\mathfrak{R}}/\mathfrak{R}$ *is a versal deformation of X with minimum tangent space dimension.*

Proof. ($\Leftarrow$) Suppose that there exists a nonzero element $\theta \in H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X)$. Then we have the diagram

$$\begin{array}{ccc} X \times T & & X \times T \\ \downarrow_{\pi\times\mathrm{id}} & & \downarrow_{\pi\times\mathrm{id}} \\ \operatorname{Spec} R \times T & \xrightarrow{\varphi} & \operatorname{Spec} R \times T \end{array}$$

where $T := \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, and φ is the automorphism given by sending $f \in R$ to $f + \theta(f)\epsilon$. But no morphism $X \times T \to X \times T$ makes this diagram commutative.

We may consider $\varphi \circ (\pi \times \mathrm{id})$ as a resolution of $\operatorname{Spec} R \times T$. This gives a nontrivial extension of $T \to S$ to $T \to \mathfrak{R}$ although $X \times T$ is a trivial deformation of X. This contradicts the fact that $\mathcal{X}'_{\mathfrak{R}}/\mathfrak{R}$ has the minimum tangent space dimension as a versal deformation of X.

($\Rightarrow$) This is proved by Artin. □

The following is a refinement of the equality used by Shepherd-Barron in [14, Prop. 3.1].

PROPOSITION 2.2. *For a rational double or triple point* $(\operatorname{Spec} R, \mathfrak{m})$ *and its minimal resolution* $\pi : X \to \operatorname{Spec} R$ *with the exceptional divisor* E, *we have the equality*

$$\dim_k H^1(\Theta_X) + \dim_k H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X) = \tau,$$

where τ *is* $\dim_k \operatorname{Ext}^1_R(\Omega_R, R)$, *that is*, *the Tjurina number of* $(\operatorname{Spec} R, \mathfrak{m})$.

Proof. This is based on the fact that there is a smooth morphism of functors $\operatorname{Res} \mathcal{X}/S \to \mathrm{D}_X$ [2, Lemma 3.3] and that the tangent space $\operatorname{Res} \mathcal{X}/S(k[\epsilon]/(\epsilon^2))$ has dimension τ [2, Thm. 3]. The equality is the dimension formula for the surjective k-linear mapping $\operatorname{Res} \mathcal{X}/S(k[\epsilon]/(\epsilon^2)) \to \mathrm{D}_X(k[\epsilon]/(\epsilon^2))$. □

PROPOSITION 2.3. *Let* $(\operatorname{Spec} R, \mathfrak{m})$ *be a rational surface singularity defined over an algebraically closed field* k *of characteristic* $p \geq 0$, *and let* $\pi : X \to \operatorname{Spec} R$ *be its minimal resolution with the reduced exceptional divisor* $E = \bigcup_i E_i$. *Then we have the equalities*

$$\dim_k H^1_E(S_X) = \dim_k H^1(S_X \otimes \mathcal{O}_X(2K_X + E)),$$
$$\dim_k H^1(S_X) = \dim_k H^1_E(S_X \otimes \mathcal{O}_X(2K_X + E)),$$

where S_X *is the locally free sheaf defined as the kernel of the surjection from the tangent to normal sheaves* $\Theta_X \to \bigoplus_i \mathcal{N}_{E_i/X}$. *In particular, the local cohomology groups* $H^1_E(S_X)$, $H^1_E(S_X \otimes \mathcal{O}_X(2K_X + E))$ *are finite-dimensional* k*-vector spaces.*

Proof. We have $\wedge^2 S_X \cong \mathcal{O}_X(-K_X - E)$, and hence $S_X^\vee \cong S_X \otimes \mathcal{O}_X(K_X + E)$. We combine this with the Grothendieck local duality theorem to have the equalities $\dim_k H^1_E(S_X) = \dim_k H^1(S_X^\vee \otimes K_X) = \dim_k H^1(S_X \otimes \mathcal{O}_X(2K_X + E))$ and $\dim_k H^1_E(S_X \otimes \mathcal{O}_X(2K_X + E)) = \dim_k H^1(S_X^\vee \otimes \mathcal{O}_X(-K_X - E)) = \dim_k H^1(S_X)$. Then use the Leray spectral sequence $E_1^{i,j} := R^i \eta_* R^j \pi_* \mathcal{F} \Rightarrow R^{i+j}(\eta \circ \pi)_* \mathcal{F}$, where $\eta : \operatorname{Spec} R \to \operatorname{Spec} k$ is the structural morphism, and $\mathcal{F} := S_X$ or $S_X \otimes \mathcal{O}_X(2K_X + E)$. Then we find that the k-vector spaces in question are of finite dimension. This indeed follows from the exact sequence $H^1(\pi_* \mathcal{F}) \to H^1(\mathcal{F}) \to H^0(R^1 \pi_* \mathcal{F}) \to H^2(\pi_* \mathcal{F})$ with the first and last terms zero, so the remaining terms are of finite dimension. □

The following theorem was pointed out by Liedtke and Satriano [10, Prop. 4.6].

THEOREM 2.4. *For a rational double point* $(\operatorname{Spec} R, \mathfrak{m})$, *we have the equality*

$$\dim_k H^1(\Theta_X) = \dim_k H^1_E(\Theta_X) = \#\{-2 \text{ curves in } E\} + \dim_k H^1(S_X),$$

where $\pi : X \to \operatorname{Spec} R$ *is the minimal resolution with the exceptional divisor* E.

Proof. We repeat the argument in [18, Thm. 6.1]. We have an exact sequence

$$0 \to \Theta_X \to S_X(E) \to \Theta_E \otimes \mathcal{O}_X(E) \to 0,$$

which gives

$$0 \to H^0_E(\Theta_E \otimes \mathcal{O}_X(E)) \to H^1_E(\Theta_X) \to H^1_E(S_X(E)) \to 0.$$

Proposition 2.3 gives the second equality. For the first equality, we use the local duality theorem and the standard isomorphism $\wedge^2 \Omega_X \cong K_X$. □

3. Lower Estimates of $\dim_k H^1(S_X)$ and $\dim_k H^1_E(S_X)$

Observing Artin's list of rational double points [3] enables us to get the lower bounds of dimensions of $H^1(S_X)$ and $H^1_E(S_X)$. First, we restrict ourselves to the case $p \geq 3$ and illustrate how the proof works. We concentrate on the case $p = 2$ in Section 5.

PROPOSITION 3.1. *Let* $X \to \operatorname{Spec} R$ *be the minimal resolution of a rational double point* $(\operatorname{Spec} R, \mathfrak{m})$. *If* $(\operatorname{Spec} R, \mathfrak{m})$ *is of the following type, then we have an equisingular deformation that provides a linear subspace* $\mathfrak{E} \subset H^1(S_X)$ *whose dimension is*

$$\begin{cases} 1 & \text{for } E_8^0 \text{ in } p = 5 \text{ and } E_6^0, E_7^0, E_8^1 \text{ in } p = 3; \\ 2 & \text{for } E_8^0 \text{ in } p = 3. \end{cases}$$

Moreover, the linear subspace $\mathfrak{E}$ *does not collapse in the tangent map*

$$H^1(S_X) \to H^1(X \setminus E, S_X).$$

Proof. For each rational double point $(\operatorname{Spec} R, \mathfrak{m})$, we present its concrete deformation $\mathcal{X} \to \mathcal{S}$ whose fiber $\mathcal{X}_s$ $(s \in \mathcal{S})$ has a rational double point with the same Dynkin diagram as $(\operatorname{Spec} R, \mathfrak{m})$, but only the central fiber $\mathcal{X}_0$ admits an isomorphism to $(\operatorname{Spec} R, \mathfrak{m})$. This deformation has a simultaneous resolution without any base extension, which results in an equisingular deformation.

If $(\operatorname{Spec} R, \mathfrak{m})$ is of type E_8^0 in $p = 5$, then the deformation is $z^2 + x^3 + y^5 + s_1 x y^4 = 0$ over $\operatorname{Spec} k[s_1]$. This has an E_8^0 singularity on the central fiber $\mathcal{X}_0$ and an E_8^1 singularity on a noncentral fiber $\mathcal{X}_s$ $(s \neq 0)$. We obtain a simultaneous resolution by blowing up the singular loci consecutively. The flatness follows, for example, from [11, Thm. 23.1].

We similarly present deformations. If $(\operatorname{Spec} R, \mathfrak{m})$ is of type E_6^0 in $p = 3$, then $z^2 + x^3 + y^4 + s_1 x^2 y^2 = 0$. If $(\operatorname{Spec} R, \mathfrak{m})$ is of type E_7^0 in $p = 3$, then $z^2 + x^3 + xy^3 + s_1 x^2 y^2 = 0$. If $(\operatorname{Spec} R, \mathfrak{m})$ is of type E_8^1 in $p = 3$, then $z^2 + x^3 + y^5 + x^2 y^3 + s_1 x^2 y^2 = 0$ over $\operatorname{Spec} k[s_1]$. If $(\operatorname{Spec} R, \mathfrak{m})$ is of type E_8^0 in $p = 3$, then $z^2 + x^3 + y^5 + s_1 x^2 y^3 + s_2 x^2 y^2 = 0$ over $\operatorname{Spec} k[s_1, s_2]$.

Each family is locally induced by an injection to a versal deformation of the rational double point $(\operatorname{Spec} R, \mathfrak{m})$, which corresponds to a subspace of $\mathrm{D}_R(k[\epsilon]/(\epsilon^2))$. The k-linear morphism in question is the composition (cf. [19, (1.6)]) $\mathrm{ES}(k[\epsilon]/(\epsilon^2)) \to \mathrm{D}_R(k[\epsilon]/(\epsilon^2)) \subset H^1(X \setminus E, S_X)$, so we have the last assertion. □

In the following, we consider the quotient $H^0(X \setminus E, S_X)/H^0(X, S_X)$. Note that because of an isomorphism $\mathcal{N}_{E_i/X} \cong \mathcal{O}_{E_i}(-2)$, we have $H^0(\bigoplus_i \mathcal{N}_{E_i/X}) = 0$, which induces an isomorphism $H^0(X \setminus E, S_X)/H^0(X, S_X) \cong H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X)$.

Proposition 3.2. *Suppose $p \geq 3$, and let $X \to \operatorname{Spec} R$ be the minimal resolution of a rational double point $(\operatorname{Spec} R, \mathfrak{m})$ of the following type. Then we calculate the dimension of $H^0(X \setminus E, S_X)/H^0(X, S_X)$ as*

$$\begin{cases} 1 & \text{for } A_n \text{ with } p \mid (n+1), \\ 1 & \text{for } E_8^0 \text{ in } p = 5 \text{ and } E_7^0 \text{ in } p = 3, \\ 2 & \text{for } E_6^0, E_8^0 \text{ in } p = 3. \end{cases}$$

For E_8^1, E_6^1 in $p = 3$, we have

$$\dim_k H^0(X \setminus E, S_X)/H^0(X, S_X) \geq 1.$$

Proof. A quadratic transformation $x' = x/y$, $y' = y$, $z' = z/y$ gives equalities of derivations $\partial/\partial x = (1/y')\partial/\partial x'$, $\partial/\partial y = \partial/\partial y' - (x'/y')\partial/\partial x' - (z'/y')\partial/\partial z'$, $\partial/\partial z = (1/y')\partial/\partial z'$. As was pointed out by Burns and Wahl [4, Prop. 1.2], any derivation D of the local ring $(R, \mathfrak{m})$ that satisfies $D(\mathfrak{m}) \subset \mathfrak{m}$ can be extended to a regular derivation on $\operatorname{Proj} \bigoplus_{i \geq 0} \mathfrak{m}^i$ (the point blowup of $\mathfrak{m}$).

For A_n: $R \cong k[[x, y, z]]/(xy + z^{n+1})$ with $p \mid (n+1)$, we have the exact sequence with the defining ideal $\mathcal{I} = (xy + z^{n+1})$ (cf. [11, Thm. 25.2])

$$0 \to \left(\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) \to T_{\mathbf{A}_k^3} \otimes R \overset{(y\,x\,0)}{\longrightarrow} \operatorname{Hom}_R(\mathcal{I}/\mathcal{I}^2, R).$$

This is the Koszul complex associated with the regular sequence $x, y \in R$. The derivation $\frac{\partial}{\partial z} \in \operatorname{Der}_k(R)$ does not lift to $\operatorname{Proj} \bigoplus_{i \geq 0} \mathfrak{m}^i$ (the blowup of $\mathfrak{m}$).

For E_8^0: $R \cong k[[x, y, z]]/(z^2 + x^3 + y^5)$ in $p = 5$, we have the exact sequence with the defining ideal $\mathcal{I} = (z^2 + x^3 + y^5)$,

$$0 \to \left(\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial z}\right) \to T_{\mathbf{A}_k^3} \otimes R \overset{(x^2\,0\,-z)}{\longrightarrow} \operatorname{Hom}_R(\mathcal{I}/\mathcal{I}^2, R).$$

The derivation $\frac{\partial}{\partial y} \in \operatorname{Der}_k(R)$ does not lift to the point blowup $\operatorname{Proj} \bigoplus_{i \geq 0} \mathfrak{m}^i$.

For E_7^0: $R \cong k[[x, y, z]]/(z^2 + x^3 + xy^3)$ in $p = 3$, we have the exact sequence

$$0 \to \left(\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial z}\right) \to T_{\mathbf{A}_k^3} \otimes R \overset{(y^3\,0\,2z)}{\longrightarrow} \operatorname{Hom}_R(\mathcal{I}/\mathcal{I}^2, R).$$

The derivation $\frac{\partial}{\partial y}$ does not lift to the point blowup.

For E_6^0: $R \cong k[[x,y,z]]/(z^2+x^3+y^4)$ in $p=3$, we have the exact sequence

$$0 \to \left(\frac{\partial}{\partial x}, z\frac{\partial}{\partial y} + y^3\frac{\partial}{\partial z}\right) \to T_{\mathbf{A}_k^3} \otimes R \overset{(0\, y^3\, 2z)}{\longrightarrow} \mathrm{Hom}_R(\mathcal{I}/\mathcal{I}^2, R).$$

Two derivations $\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial x}$ do not lift to the minimal resolution X. These form a two-dimensional subspace of $H^0(X \setminus E, S_X)/H^0(X, S_X)$.

For E_8^0: $R \cong k[[x,y,z]]/(z^2+x^3+y^5)$ in $p=3$, one has the exact sequence with the defining ideal $\mathcal{I} = (z^2+x^3+y^5)$,

$$0 \to \left(\frac{\partial}{\partial x}, z\frac{\partial}{\partial y} - y^4\frac{\partial}{\partial z}\right) \to T_{\mathbf{A}_k^3} \otimes R \overset{(0\, y^4\, z)}{\longrightarrow} \mathrm{Hom}_R(\mathcal{I}/\mathcal{I}^2, R).$$

Two derivations $\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial x}$ do not lift to the minimal resolution X.

By chasing the derivations a little further and using the original Theorem C, we know the dimension $\dim_k H^0(X \setminus E, S_X)/H^0(X, S_X)$ as stated. However, the following two types are not of this kind, and we only have lower bounds.

For E_8^1: $R \cong k[[x,y,z]]/(z^2+x^3+y^5+x^2y^3)$ in $p=3$, we have the derivation $D := y\partial/\partial x - x\partial/\partial y \in \mathrm{Der}_k(R)$. This satisfies $D(\mathfrak{m}) \subset \mathfrak{m}$, so this D lifts to a point blowup $\mathrm{Proj} \bigoplus_{i \geq 0} \mathfrak{m}^i$, on which there lies a rational double point of type E_7^0. But this D does not lift to the minimal resolution, because it is the very element considered in E_7^0.

For E_6^1 in $p=3$, we have $R \cong k[[x,y,z]]/(z^2+x^3+y^4+x^2y^2)$. The derivation $D := (y-xy)\partial/\partial x + (x-y^2)\partial/\partial y + yz\partial/\partial z \in \mathrm{Der}_k(R)$ does not lift to the minimal resolution. □

4. Proof of Main Theorem in $p \geq 3$

In this section, we give a proof to the following theorem.

THEOREM 4.1. *Let $X \to \mathrm{Spec}\, R$ be the minimal resolution of a rational double point defined over an algebraically closed field k of characteristic $p \neq 2$. Then the following assertions hold*:

(i) *The natural morphism $H^1(S_X) \to H^1(X \setminus E, S_X)$ is an inclusion.*

(ii) *The dimension of $H^1_E(S_X \otimes \mathcal{O}_X(E))$ is zero except*

$$\begin{cases} 1 & \text{for } E_8^0 \text{ in } p=5 \text{ and } E_6^0, E_7^0, E_8^1 \text{ in } p=3, \\ 2 & \text{for } E_8^0 \text{ in } p=3. \end{cases}$$

(iii) *We have an isomorphism*

$$H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X) \cong H^1_E(S_X),$$

whose dimension is zero with the following exceptions:

$$\begin{cases} 1 & \text{for } A_n \text{ with } p \mid (n+1), \\ 1 & \text{for } E_8^0 \text{ in } p=5 \text{ and } E_6^1, E_7^0, E_8^1 \text{ in } p=3, \\ 2 & \text{for } E_6^0, E_8^0 \text{ in } p=3. \end{cases}$$

Proof. If the characteristic p is either a good prime for the rational double point $(\operatorname{Spec} R, \mathfrak{m})$ of type D or E or a very good prime if $(\operatorname{Spec} R, \mathfrak{m})$ is of type A, then Wahl's Theorem D (resp., Theorem C) provides assertions (i) and (ii) (resp., (iii)). If p is none of these, then we need to show that the lower estimates given in the previous section attain indeed the actual values. This is trivially verified, since we have the equality coming from Proposition 2.2 and Theorem 2.4,

$$\#\{-2 \text{ curves in } E\} + \dim_k H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X) + \dim_k H^1(S_X) = \tau.$$

The Tjurina numbers in Artin's list [3] and Proposition 3.1 say that we have the desired equalities (cf. Remarks 5.4 for concrete values). □

REMARK 4.2. The pro-representable hull of equisingular deformations of X injects into a versal deformation of the rational double point $(\operatorname{Spec} R, \mathfrak{m})$. This forms a nonsingular subvariety whose dimension is as prescribed in Theorem 4.1(ii). The concrete equisingular families in the proof turn out to be the entire equisingular versal families. For E_8^0 in $p = 3$, we have a two-parameter family $z^2 + x^3 + y^5 + s_1x^2y^3 + s_2x^2y^2 = 0$ over $\operatorname{Spec} k[[s_1, s_2]]$. Two strata of dimensions one and zero, respectively, can be observed in it. The derivations $\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial x}$ of the central fiber do not lift to the minimal resolution. The latter extends to the derivation $y\frac{\partial}{\partial x} - s_1x\frac{\partial}{\partial y}$ of the family over $\operatorname{Spec} k[[s_1, s_2]]/(s_2)$. Note that the same rational double point as $(\operatorname{Spec} R, \mathfrak{m})$ always lies on the central fiber only.

5. Characteristic 2

As is often the case with characteristic 2, computation becomes more involved and demanding. However, we can complete the proof of our main theorem essentially in the same way as before.

PROPOSITION 5.1. *Let $\pi : X \to \operatorname{Spec} R$ be the minimal resolution of a rational double point $(\operatorname{Spec} R, \mathfrak{m})$ defined over an algebraically closed field k of characteristic 2. Then we have a linear subspace $\mathfrak{E} \subset H^1(S_X)$ corresponding to an equisingular family whose dimension is*

$$\begin{cases} 1 - r & \text{for } E_6^r, \\ 3 - r & \text{for } E_7^r, \\ 4 - r & \text{for } E_8^r, \\ n - r - 1 & \text{for } D_{2n}^r, D_{2n+1}^r. \end{cases}$$

The natural morphism $H^1(S_X) \to H^1(X \setminus E, S_X)$ induces an inclusion $\mathfrak{E} \to H^1(X \setminus E, S_X)$.

Proof. For each rational double point $(\operatorname{Spec} R, \mathfrak{m})$, we present its deformation $\mathcal{X} \to \mathcal{S}$ that has the property required in the proof of Proposition 3.1.

If $(\operatorname{Spec} R, \mathfrak{m})$ is of type D_{2n}^0, then the family is $z^2+x^2y+xy^n+s_1xy^{n-1}z+s_2xy^{n-2}z+\cdots+s_{n-1}xyz=0$ over $\operatorname{Spec} k[s_1,s_2,\ldots,s_{n-1}]$, which has the singularity D_{2n}^0 on the central fiber $\mathcal{X}_0$. A noncentral fiber $\mathcal{X}_s$ $(s\neq 0)$ has a singularity D_{2n}^q with some $q\geq 1$.

If D_{2n}^r with $1\leq r\leq n-2$ is the type of $(\operatorname{Spec} R, \mathfrak{m})$, then the family is $z^2+x^2y+xy^n+xy^{n-r}z+s_1xy^{n-r-1}z+\cdots+s_{n-r-1}xyz=0$ over $\operatorname{Spec} k[s_1,s_2,\ldots,s_{n-r-1}]$, which has D_{2n}^r on the central fiber $\mathcal{X}_0$. A noncentral fiber $\mathcal{X}_s$ $(s\neq 0)$ has D_{2n}^q with some $q\geq r+1$.

If D_{2n+1}^0 is the type of $(\operatorname{Spec} R, \mathfrak{m})$, then the family is $z^2+x^2y+y^nz+s_1xy^{n-1}z+s_2xy^{n-2}z+\cdots+s_{n-1}xyz=0$ over $\operatorname{Spec} k[s_1,s_2,\ldots,s_{n-1}]$, which has D_{2n+1}^0 on the central fiber $\mathcal{X}_0$. A noncentral fiber $\mathcal{X}_s$ $(s\neq 0)$ has D_{2n+1}^q with some $q\geq 1$.

If D_{2n+1}^r with $1\leq r\leq n-2$ is the type of $(\operatorname{Spec} R, \mathfrak{m})$, then the family is $z^2+x^2y+y^nz+xy^{n-r}z+s_1xy^{n-r-1}z+s_2xy^{n-r-2}z+\cdots+s_{n-r-1}xyz=0$ over $\operatorname{Spec} k[s_1,s_2,\ldots,s_{n-r-1}]$, which has D_{2n+1}^r on $\mathcal{X}_0$. A noncentral fiber $\mathcal{X}_s$ $(s\neq 0)$ has D_{2n+1}^q with some $q\geq r+1$.

If E_6^0 is the type of $(\operatorname{Spec} R, \mathfrak{m})$, then we choose $z^2+x^3+y^2z+s_1xyz=0$ over $\operatorname{Spec} k[s_1]$, which has the singularity E_6^0 on $\mathcal{X}_0$ and E_6^1 on $\mathcal{X}_s$ $(s\neq 0)$.

If E_7^r with $r=0,1,2$ is the type of $(\operatorname{Spec} R, \mathfrak{m})$, then the deformation is given by $z^2+x^3+xy^3+\eta_r+s_1\eta_{r+1}+\cdots+s_{3-r}\eta_3=0$ over $\operatorname{Spec} k[s_1,\ldots,s_{3-r}]$, where $\eta_0:=0$, $\eta_1:=x^2yz$, $\eta_2:=y^3z$, and $\eta_3:=xyz$. This has the singularity E_7^r on the central fiber $\mathcal{X}_0$. A noncentral fiber $\mathcal{X}_s$ $(s\neq 0)$ has E_7^q with some $q\geq r+1$.

If E_8^r with $r=0,1,2,3$ is the type of $(\operatorname{Spec} R, \mathfrak{m})$, then the deformation is $z^2+x^3+y^5+\theta_r+s_1\theta_{r+1}+\cdots+s_{4-r}\theta_4=0$ over $\operatorname{Spec} k[s_1,\ldots,s_{4-r}]$, where $\theta_0:=0$, $\theta_1:=xy^3z$, $\theta_2:=xy^2z$, $\theta_3:=y^3z$, and $\theta_4:=xyz$. This has the singularity E_8^r on the central fiber $\mathcal{X}_0$. A noncentral fiber $\mathcal{X}_s$ $(s\neq 0)$ has a singularity E_8^q with some $q\geq r+1$.

As before, we obtain a simultaneous resolution of $\mathcal{X}\to\mathcal{S}$ by consecutively blowing up the singular loci without any base extension. $\square$

In characteristic 2, special care is needed for rational double points of type D.

LEMMA 5.2. *For a rational double point of type D in characteristic* 2, *the following derivations do not lift to the minimal resolution*:

D_{2n}^0: $R\cong k[[x,y,z]]/(z^2+x^2y+xy^n)$,

$$x\frac{\partial}{\partial z},\ \frac{\partial}{\partial z},\ y\frac{\partial}{\partial z},\ y^2\frac{\partial}{\partial z},\ \ldots,\ y^{n-1}\frac{\partial}{\partial z};$$

D_{2n}^r: $R\cong k[[x,y,z]]/(z^2+x^2y+xy^n+xy^{n-r}z)$ *with* $r=1,2,\ldots,n-1$,

$$y^{n-r}\frac{\partial}{\partial y}+(x+ny^{n-1}+(n+r)y^{n-r-1}z)\frac{\partial}{\partial z},$$

$$\mathfrak{d}_1,\ y\mathfrak{d}_1,\ y^2\mathfrak{d}_1,\ \ldots,\ y^{n-r-1}\mathfrak{d}_1 \quad \textit{with}\ \mathfrak{d}_1:=x\frac{\partial}{\partial x}+(y^r+z)\frac{\partial}{\partial z};$$

D^0_{2n+1}: $R \cong k[[x,y,z]]/(z^2+x^2y+y^nz)$,

$$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial x}, \dots, y^{n-1}\frac{\partial}{\partial x};$$

D^r_{2n+1}: $R \cong k[[x,y,z]]/(z^2+x^2y+y^nz+xy^{n-r}z)$ *with* $r = 1, 2, \dots, n-1$,

$$\mathfrak{d}_2, y\mathfrak{d}_2, y^2\mathfrak{d}_2, \dots, y^{n-r-1}\mathfrak{d}_2 \quad \textit{with } \mathfrak{d}_2 := (x+y^r)\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}.$$

Proof. Local calculation based on induction on $n \geq 2$. □

PROPOSITION 5.3. *Let* $\pi : X \to \operatorname{Spec} R$ *be the minimal resolution of a rational double point* $(\operatorname{Spec} R, \mathfrak{m})$ *defined over an algebraically closed field* k *of characteristic* 2. *We have a lower bound of* $\dim_k H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X)$ *as*

$$\begin{cases} 1 & \textit{for } A_n \textit{ with } 2 \mid (n+1), \\ 1 & \textit{for } E^0_6, \\ 4-r & \textit{for } E^r_7, E^r_8, \\ n+1-r & \textit{for } D^r_{2n}, \\ n-r & \textit{for } D^r_{2n+1}. \end{cases}$$

Proof. We give derivations of the local ring $(R, \mathfrak{m})$ that do not lift to the minimal resolution. For A_n, the derivation is exactly of the same form as before, so we omit it.

For E^0_6: $R \cong k[[x,y,z]]/(z^2+x^3+y^2z)$, we have the exact sequence with $\mathcal{I} = (z^2+x^3+y^2z)$,

$$0 \to \left(y^2\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right) \to T_{\mathbf{A}^3_k} \otimes R \overset{(x^2\,0\,y^2)}{\longrightarrow} \operatorname{Hom}_R(\mathcal{I}/\mathcal{I}^2, R).$$

The derivation $\frac{\partial}{\partial y} \in \operatorname{Der}_k(R)$ does not lift to the point blowup $\operatorname{Proj} \bigoplus_{i \geq 0} \mathfrak{m}^i$.

For E^0_7: $R \cong k[[x,y,z]]/(z^2+x^3+xy^3)$, we have the exact sequence with $\mathcal{I} = (z^2+x^3+xy^3)$,

$$0 \to \left(xy^2\frac{\partial}{\partial x} + (x^2+y^3)\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \to T_{\mathbf{A}^3_k} \otimes R \overset{(x^2+y^3\;xy^2\;0)}{\longrightarrow} \operatorname{Hom}_R(\mathcal{I}/\mathcal{I}^2, R).$$

Four derivations $\frac{\partial}{\partial z}, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial z} \in \operatorname{Der}_k(R)$ do not lift to the minimal resolution X.

For E^0_8: $R \cong k[[x,y,z]]/(z^2+x^3+y^5)$, we have the exact sequence

$$0 \to \left(y^4\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \to T_{\mathbf{A}^3_k} \otimes R \overset{(x^2\,y^4\,0)}{\longrightarrow} \operatorname{Hom}_R(\mathcal{I}/\mathcal{I}^2, R).$$

Four derivations $\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial z}, x\frac{\partial}{\partial z}$ do not lift to the minimal resolution X.

For E_7^1: $R \cong k[[x, y, z]]/(z^2 + x^3 + xy^3 + x^2yz)$, the following three derivations do not lift to the minimal resolution:

$$xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (yz + x)\frac{\partial}{\partial z},$$

$$\mathfrak{d}_3,\, y\mathfrak{d}_3 \quad \text{with } \mathfrak{d}_3 = xz\frac{\partial}{\partial x} + (yz + x)\frac{\partial}{\partial y} + (z^2 + y)\frac{\partial}{\partial z}.$$

For E_7^2: $R \cong k[[x, y, z]]/(z^2 + x^3 + xy^3 + y^3z)$, the following two derivations do not lift to the minimal resolution:

$$y\frac{\partial}{\partial y} + (z + x)\frac{\partial}{\partial z}, \qquad (xy^2 + y^2)\frac{\partial}{\partial x} + (x + z)\frac{\partial}{\partial y} + (y^2z + y^2)\frac{\partial}{\partial z}.$$

For E_7^3: $R \cong k[[x, y, z]]/(z^2 + x^3 + xy^3 + xyz)$, the following derivation does not lift to the minimal resolution:

$$y\frac{\partial}{\partial y} + (y^2 + z)\frac{\partial}{\partial z}.$$

For E_8^1: $R \cong k[[x, y, z]]/(z^2 + x^3 + y^5 + xy^3z)$, the following three derivations do not lift to the minimal resolution:

$$y^3\frac{\partial}{\partial x} + y^2z\frac{\partial}{\partial y} + (yz^2 + x)\frac{\partial}{\partial z},$$

$$\mathfrak{d}_4,\, y\mathfrak{d}_4 \quad \text{with } \mathfrak{d}_4 := y^2z\frac{\partial}{\partial x} + (yz^2 + x)\frac{\partial}{\partial y} + (z^3 + y)\frac{\partial}{\partial z}.$$

For E_8^2: $R \cong k[[x, y, z]]/(z^2 + x^3 + y^5 + xy^2z)$, the following two derivations do not lift to the minimal resolution:

$$x\frac{\partial}{\partial y} + y^2\frac{\partial}{\partial z}, \qquad y^2\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

For E_8^3: $R \cong k[[x, y, z]]/(z^2 + x^3 + y^5 + y^3z)$, the following derivation does not lift to the minimal resolution:

$$y\frac{\partial}{\partial y} + (y^2 + z)\frac{\partial}{\partial z}.$$

By Lemma 5.2 these altogether complete the proof. □

Proof of Theorem 1.1. The case $p \geq 3$ is already settled in Theorem 4.1. For $p = 2$, we combine Propositions 5.1 and 5.3 with the equality

$$\#\{-2 \text{ curves in } E\} + \dim_k H^0(X \setminus E, \Theta_X)/H^0(X, \Theta_X) + \dim_k H^1(S_X) = \tau.$$

The Tjurina numbers in Artin's list give the statements (cf. Table 1 for concrete values). □

REMARKS 5.4. (i) We summarize in Table 1 the invariants of rational double point $(\operatorname{Spec} R, \mathfrak{m})$ in bad prime p. For F-purity, we use Fedder's criterion (cf. [5]).

Table 1 Invariants of rational double points in bad prime p

p	Type	τ	$\dim_k H^1(X \setminus E, S_X)$ $/H^1(X, S_X)$	$\dim_k H^1(S_X)$	F-purity	On Proj $\bigoplus_{j\geq 0} \mathfrak{m}^j$
2	D_4^r	$8-2r$	$3-r$	$1-r$	(**)	A_1, A_1, A_1
	D_{2n}^r	$4n-2r$	$n+1-r$	$n-r-1$	(**)	$A_1, D_{2n-2}^{\max\{r-1,0\}}$ (*)
	D_5^r	$8-2r$	$2-r$	$1-r$	(**)	A_3, A_1
	D_{2n+1}^r	$4n-2r$	$n-r$	$n-r-1$	(**)	$A_1, D_{2n-1}^{\max\{r-1,0\}}$ (*)
	E_6^0	8	1	1	–	A_5
	E_6^1	6	0	0	F-pure	A_5
	E_7^0	14	4	3	–	D_6^0
	E_7^1	12	3	2	–	D_6^0
	E_7^2	10	2	1	–	D_6^1
	E_7^3	8	1	0	F-pure	D_6^2
	E_8^0	16	4	4	–	E_7^0
	E_8^1	14	3	3	–	E_7^0
	E_8^2	12	2	2	–	E_7^1
	E_8^3	10	1	1	–	E_7^2
	E_8^4	8	0	0	F-pure	E_7^3
3	E_6^0	9	2	1	–	A_5
	E_6^1	7	1	0	F-pure	A_5
	E_7^0	9	1	1	–	D_6
	E_7^1	7	0	0	F-pure	D_6
	E_8^0	12	2	2	–	E_7^0
	E_8^1	10	1	1	–	E_7^0
	E_8^2	8	0	0	F-pure	E_7^1
5	E_8^0	10	1	1	–	E_7
	E_8^1	8	0	0	F-pure	E_7

(*) $n \geq 3$. (**) F-pure if and only if $r = n-1$, that is, the type is either D_{2n}^{n-1} or D_{2n+1}^{n-1}.

(ii) It is possible to unify the derivations presented in the proof of Proposition 5.3 as derivations of families. For example, in the equisingular family of E_7^0,

$$z^2 + x^3 + xy^3 + s_1x^2yz + s_2y^3z + s_3xyz = 0 \quad \text{over } \operatorname{Spec} k[[s_1, s_2, s_3]],$$

we have the derivation

$$(s_2^2s_3x^2 + s_2^3xy^2 + s_1s_2xz + s_2y^2 + s_1s_2^2z^2 + s_2s_3z)\frac{\partial}{\partial x} + (s_1s_2s_3xy^2$$
$$+ s_1xy + s_2x + s_1s_2^2y^4 + s_2^3y^3 + s_1s_2^2s_3y^2z$$

$$+ s_2 s_3^2 y^2 + s_1 s_2 yz + s_3 y + s_2^2 z)\frac{\partial}{\partial y}$$

$$+ (s_2 s_3 x^2 + s_2^2 xy^2 + s_1 xz + y^2 + s_1 s_2 z^2 + s_3 z)\frac{\partial}{\partial z}.$$

For the family over $\operatorname{Spec} k[[s_1, s_2, s_3]]/(s_3)$, we have another derivation

$$s_1 xy\frac{\partial}{\partial x} + (s_1 y^2 + s_2 y)\frac{\partial}{\partial y} + (x + s_1 yz + s_2 z)\frac{\partial}{\partial z}.$$

(iii) As Artin observed in [3], by knowing resolution graphs and Tjurina numbers we can classify rational double points defined over an algebraically closed field k up to isomorphism. It turns out that we could also distinguish isomorphism classes of rational double points with the information of the Dynkin diagram and $\dim_k H^1(S_X)$ (or of the Dynkin diagram and $\dim_k H^1_E(S_X)$). But this observation does not generalize well to other rational surface singularities over k. Rational triple points in characteristic 2, for example, defined respectively by

$$\frac{w^2}{x + z^2 + wy} = \frac{x}{y} = \frac{y}{z}, \qquad \frac{w^2 + wy}{x + z^2} = \frac{x}{y} = \frac{y}{z},$$

have an identical resolution graph ($\mathbf{x}^2 = -3$, $\mathbf{o}^2 = -2$, cf. [1])

$$\begin{array}{ccccccccccccc} & & & & \mathbf{o} & & & & & & & & \\ & & & & | & & & & & & & & \\ \mathbf{x} & — & \mathbf{o} & — & \mathbf{o} & — & \mathbf{o} & — & \mathbf{o} & — & \mathbf{o}. \end{array}$$

After modification (cf. [17, Section 2]), we have rational double points of type D_6^1 and D_6^2, respectively, so these triple points cannot be isomorphic. But they both have $\tau = 10$, $\dim_k H^1(S_X) = 1$, $\dim_k H^1_E(S_X) = 1$, so we cannot distinguish these by such numerical invariants.

6. Proof of Corollary 1.2 and Corollary 1.3

Proof of Corollary 1.2. The equivalence of (i), (ii), (iii) immediately follows from Theorem 1.1. (iii) $\Rightarrow$ (iv) follows from the facts that the smooth morphism $\operatorname{Res} \mathcal{X}/S \to \mathrm{D}_X$ is an isomorphism (cf. [2, Cor. 4.6]) and that the equisingular functor $\mathrm{ES} \subset \mathrm{D}_X$ is trivial (cf. [19, Prop. 2.5]). (iv) $\Rightarrow$ (ii) follows from Proposition 2.1. □

Proof of Corollary 1.3. Use the same argument as in characteristic zero (cf. [12]). □

REMARK 6.1. As stated in Introduction, the converse of Corollary 1.3 cannot be true. This is obvious if we consider rational double points of the following types:

$$\begin{array}{ll} A_n, & p \mid (n+1), \\ E_8^0, & p = 5, \end{array}$$

$$E_7^0, \qquad p = 3,$$
$$E_8^0, E_8^1, \qquad p = 3.$$

In these cases the same statement as (iv) in Corollary 1.2 fails, but Corollary 1.3 continues to hold (cf. [7, Cor. 23, Thm. 3]).

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