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Families of Elliptic Curves in $\mathbb{P}^3$ and Bridgeland Stability

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Abstract. We study wall crossings in Bridgeland stability for the Hilbert scheme of elliptic quartic curves in the three-dimensional projective space. We provide a geometric description of each of the moduli spaces we encounter, including when the second component of this Hilbert scheme appears. Along the way, we prove that the principal component of this Hilbert scheme is a double blowup with smooth centers of a Grassmannian, exhibiting a completely different proof of this known result by Avritzer and Vainsencher. This description allows us to compute the cone of effective divisors of this component.

Introduction

The global geometry of a given Hilbert scheme is generally very difficult to study. Recently, the theory of Bridgeland stability has provided a new set of tools to study the geometry of these Hilbert schemes. For instance, the study of the Hilbert scheme of points on surfaces has benefited from these new tools (see [A+13; BM14; CHW17; LZ16; MM13; Nue16; YY14]). A sensible step forward is now to apply these tools to examine families of curves contained in threefolds. The first instance of this was carried out by the last author in [Sch15], where he studies the Hilbert scheme of twisted cubics. In this paper, we continue this investigation about curves in $\mathbb{P}^3$ and analyze the global geometry, as well as wall-crossing phenomena, of the Hilbert scheme $\mathrm{Hilb}^{4t}(\mathbb{P}^3)$, which parameterizes subschemes of $\mathbb{P}^3$ of genus 1 and degree 4.

A smooth curve of genus 1 and degree 4 in $\mathbb{P}^3$, which we refer to as an elliptic quartic, is the transversal intersection of two quadric surfaces. By considering the pencil that these quadrics generate, we realize the family of smooth elliptic quartics as an open subset of $\mathbb{G}(1,9)$, the Grassmannian of lines in the space $|\mathcal{O}_{\mathbb{P}^3}(2)|$ of quadric surfaces in $\mathbb{P}^3$. We show that the Hilbert scheme $\mathrm{Hilb}^{4t}(\mathbb{P}^3)$ is a moduli

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space of Bridgeland stable objects, and moreover, one of its components is related through birational transformations to the Grassmannian $\mathbb{G}(1,9)$ via wall-crossing.

Let us recall the notion of Bridgeland stability in order to state this result precisely. For classical slope stability with respect to a given polarization H on a smooth projective complex variety X, the number $\mu_H(E) = (H^{n-1} \cdot \mathrm{ch}_1(E))/(H^n \cdot \mathrm{ch}_0(E))$ is called the slope for any coherent sheaf $E \in \mathrm{Coh}(X)$. A coherent sheaf is then called slope semistable if all proper nontrivial subsheaves have smaller slope. For Bridgeland stability, one replaces the category of coherent sheaves with a different Abelian subcategory $\mathcal{A} \subset D^b(X)$ and replaces the slope with a homomorphism $Z : K_0(X) \to \mathbb{C}$, mapping $\mathcal{A}$ to the upper half-plane or the negative real line, where $K_0(X)$ is the Grothendieck group. The slope is then given by

$$\mu(E) = -\frac{\Re Z(E)}{\Im Z(E)}$$

for any $E \in \mathcal{A}$. In addition, one demands that every object in $D^b(X)$ has a canonical filtration into semistable factors called the Harder–Narasimhan filtration and the so-called support property, which ensures that the set of stability conditions $\mathrm{Stab}(X)$ can be naturally given the structure of a complex manifold.

We can now state our main result. Let us fix a class $v \in K_0(X)$. Then there is a locally finite wall and chamber structure in $\mathrm{Stab}(X)$ such that the set of semistable objects of class v is constant within each chamber. Our main result describes the wall and chamber structure of a subspace of $\mathrm{Stab}(\mathbb{P}^3)$ and the corresponding moduli spaces of semistable objects in the case of elliptic quartics in $\mathbb{P}^3$.

Theorem A. *Let $v = (1, 0, -4, 8) = \mathrm{ch}(\mathcal{I}_C)$, where $C \subset \mathbb{P}^3$ is an elliptic quartic curve. There is a path $\gamma : [0, 1] \to \mathbb{R}_{>0} \times \mathbb{R} \subset \mathrm{Stab}(\mathbb{P}^3)$ such that the moduli spaces of semistable objects with Chern character v in its image outside of walls are given in the following order.*

(0) *The empty space $M_0 = \emptyset$.*
(1) *The Grassmannian $M_1 = \mathbb{G}(1, 9)$ parameterizing pencils of quadrics. The only nonideal sheaves in the moduli space come from the case where a 2-plane is contained in the base locus of the pencil.*
(2) *The second moduli space M_2 is the blow up of $\mathbb{G}(1,9)$ along a smooth locus isomorphic to $\mathbb{G}(1, 3) \times (\mathbb{P}^3)^\vee$ parameterizing the nonideal sheaves in M_1. The exceptional divisor generically parameterizes unions of a line and a plane cubic intersecting themselves in a single point. The only nonideal sheaves in this moduli space come from the case where the line is contained in the plane.*
(3) *The third moduli space M_3 has two irreducible components M_3^1 and M_3^2. The first component M_3^1 is the blowup of M_2 along the smooth incidence variety parameterizing length two subschemes in a plane in $\mathbb{P}^3$. The second component M_3^2 is a $\mathbb{P}^{14}$-bundle over $\mathrm{Hilb}^2(\mathbb{P}^3) \times (\mathbb{P}^3)^\vee$. It generically parameterizes unions of plane quartics with two points, either outside the curve or embedded. The two components intersect transversally along the exceptional locus*

of the blowup. The only nonideal sheaves occur in the case where at least one of the two points is not scheme-theoretically contained in the plane.

(4) *The fourth moduli space M_4 has two irreducible components M_4^1 and M_4^2. The first component is equal to M_3^1. The second component is birational to M_3^2. The new locus parameterizes plane quartics with two points such that exactly one point is scheme-theoretically contained in the plane.*

(5) *The fifth moduli space is the Hilbert scheme* $\mathrm{Hilb}^{4t}(\mathbb{P}^3)$, *which has two components* Hilb_1^{4t} *and* Hilb_2^{4t}. *The principal component* Hilb_1^{4t} *contains an open subset of elliptic quartic curves and is equal to M_3^1. The second component is of dimension* 23 *and is birational to M_3^2. Moreover, the two components intersect transversally along a locus of dimension* 15. *The component* Hilb_2^{4t} *differs from M_4^2 in the locus of plane cubics together with two points scheme-theoretically contained in the plane.*

As a consequence of Theorem A, we obtain that the Hilbert scheme $\mathrm{Hilb}^{4t}(\mathbb{P}^3)$ has two components. This is a well-known fact (see [CN12; Got08]). More interestingly, the previous result describes what is called the principal component, which parameterizes smooth elliptic curves along with their flat limits. We will denote this component by Hilb_1^{4t}, and our next result describes its global geometry.

THEOREM B ([VA92]). *The closure of the family of smooth elliptic quartics in the Hilbert scheme* $\mathrm{Hilb}^{4t}(\mathbb{P}^3)$, *is a double blowup of the Grassmannian* $\mathbb{G}(1, 9)$ *along smooth centers.*

A comment is in order about the previous theorem. The description of $\mathrm{Hilb}_1^{4t}(\mathbb{P}^3)$ was proved in [VA92] by Vainsencher and Avritzer using classical methods. Our techniques to reprove their result are distinct, as we make use of the bounded derived category of coherent sheaves on $\mathbb{P}^3$ and Bridgeland stability.

Since the principal component $\mathrm{Hilb}_1^{4t}(\mathbb{P}^3)$ is a double blowup, it is natural to ask what are the subschemes of $\mathbb{P}^3$ that the exceptional divisors parameterize and whether they span extremal rays in the cone of effective divisors $\mathrm{Eff}(\mathrm{Hilb}_1^{4t})$. Proposition 4.11 and the following result answer these two questions. Consequently, we have a moduli interpretation for the generators of $\mathrm{Eff}(\mathrm{Hilb}_1^{4t})$, which is the following.

Let E_1 be the closure of the locus parameterizing subschemes of $\mathbb{P}^3$ that are the unions of a plane cubic and an incident line. By E_2 we denote the closure of the locus parameterizing plane quartics with two nodes and two embedded points at such nodes. Let Δ denote the closure of the locus of nodal elliptic curves.

THEOREM C. *The cone of effective divisors of* Hilb_1^{4t} *is generated by* $\mathrm{Eff}(\mathrm{Hilb}_1^{4t}) = \langle E_1, E_2, \Delta\rangle$.

Ingredients

The notion of tilt stability on a smooth projective threefold was introduced in [BMT14]. It is defined in a similar way one defines Bridgeland stability on a surface. Thus, these two notions of stability share computational properties. Tilt stability is intended as a stepping stone to Bridgeland stability. The proof of Theorem A is mostly based on this theory.

In contrast to the surface case, computing which objects destabilize at a given wall is difficult due to the lack of unique stable factors in the Jordan–Hölder filtration of a strictly semistable object. Computing the walls numerically in tilt stability is of similar difficulty as in the surface case and is often possible. On the other hand, although it is generally difficult to determine all walls in Bridgeland stability on a given path, it is not so difficult to determine which objects destabilize at a given wall. To resolve this issue, we apply a technique from [Sch15], which allows us to translate walls from tilt stability into Bridgeland stability.

To identify the global structure of the Bridgeland moduli spaces, a careful analysis of its singularities is necessary. We apply deformation theory to these problems, and large parts of it reduce to heavy Ext-computation. Even though this can be done by hand, computer calculations with [M2] turn out to be tremendously helpful. The situation is more involved when it comes to the intersection of the two components. We reduce the question to a single ideal in that case and apply the technique of [PS85]. We make use of the Macaulay2 implementation [Ilt12] of this technique.

The proof of Theorem C uses the description of the exceptional divisors provided in Proposition 4.11 and exhibits the dual curves to them in order to conclude.

Organization

In Section 1, we recall basic definitions about stability conditions. In Section 2, we carry out numerical computations in tilt stability needed to understand walls in Bridgeland stability. In Section 3, we describe the equations of some ideals depending on the exact sequences they fit in. We use this description to understand the local geometry of the intersection of the two components of our Hilbert scheme. In Section 4, we translate the computations in tilt stability to Bridgeland stability. Furthermore, we analyze singularities to provide proofs of Theorem A and Theorem B. In Section 5, we prove Theorem C. contains our Macaulay2 code.

Notation

We work over the field of the complex numbers throughout. We also use the following notation.

$\mathcal{I}_{Z/X}, \mathcal{I}_Z$ ideal sheaf of a closed subscheme $Z \subset X$
$D^b(X)$ bounded derived category of coherent sheaves on X
$\mathrm{ch}_X(E), \mathrm{ch}(E)$ Chern character of an object $E \in D^b(X)$
$\mathrm{ch}_{\leq l,X}(E), \mathrm{ch}_{\leq l}(E)$ $(\mathrm{ch}_{0,X}(E), \dots, \mathrm{ch}_{l,X}(E))$
$\mathbb{G}(r,k)$ the Grassmannian parameterizing subspaces $\mathbb{P}^r \subset \mathbb{P}^k$
Hilb_1^{4t} closure of the locus of elliptic quartic curves in $\mathrm{Hilb}(\mathbb{P}^3)$

Hilb^{4t}_2 closure in $\mathrm{Hilb}(\mathbb{P}^3)$ of the locus of unions of plane quartic curves with two points in $\mathbb{P}^3$

1. Preliminaries on Stability Conditions

In this section, we recall the construction of Bridgeland stability conditions on $\mathbb{P}^3$ due to [BMT14; Mac14b]. We refer the reader to [Bri07] for a detailed introduction to the theory of Bridgeland stability. Let X be a smooth projective threefold. A *Bridgeland stability condition* on $D^b(X)$ is a pair $(Z, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded t-structure, and $Z : K_0(X) = K_0(\mathcal{A}) \to \mathbb{C}$ is an additive homomorphism that maps any nontrivial object in $\mathcal{A}$ to the upper half-plane or the negative real line. Additionally, technical properties such as the existence of Harder–Narasimhan filtrations and the support property have to be fulfilled. Bridgeland's main result is that the set of stability condition can be given the structure of a complex manifold. We will denote this *stability manifold* by $\mathrm{Stab}(X)$.

Let H be the very ample generator of $\mathrm{Pic}(\mathbb{P}^3)$. Due to the simplicity of the cohomology of $\mathbb{P}^3$, we will abuse notation by writing $\mathrm{ch}_i(E) = H^{3-i}\,\mathrm{ch}_i(E)$ for any $E \in D^b(X)$. For $\beta \in \mathbb{R}$, we define the *twisted Chern character* by $\mathrm{ch}^{\beta} := e^{-\beta H} \cdot \mathrm{ch}$. In more detail, we have

$$\mathrm{ch}^{\beta}_0 = \mathrm{ch}_0, \qquad \mathrm{ch}^{\beta}_1 = \mathrm{ch}_1 - \beta\,\mathrm{ch}_0, \qquad \mathrm{ch}^{\beta}_2 = \mathrm{ch}_2 - \beta\,\mathrm{ch}_1 + \frac{\beta^2}{2}\,\mathrm{ch}_0,$$
$$\mathrm{ch}^{\beta}_3 = \mathrm{ch}_3 - \beta\,\mathrm{ch}_2 + \frac{\beta^2}{2}\,\mathrm{ch}_1 - \frac{\beta^3}{6}\,\mathrm{ch}_0\,.$$

We write a twisted version of the classical *slope function* as

$$\mu_{\beta}(\mathrm{ch}_0, \mathrm{ch}_1) := \frac{\mathrm{ch}^{\beta}_1}{\mathrm{ch}^{\beta}_0} = \frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \beta,$$

where division by 0 is interpreted as $+\infty$. In [BMT14] the notion of *tilt stability* has been introduced as an auxiliary notion in between classical slope stability and Bridgeland stability on threefolds. We will recall this construction and a few of its properties. Tilting is used to obtain a new heart of a bounded t-structure. For more information on the general theory of tilting, we refer to [HRS96]. A torsion pair is defined by

$$\mathcal{T}_{\beta} := \{E \in \mathrm{Coh}(\mathbb{P}^3) : \text{any quotient } E \twoheadrightarrow G \text{ satisfies } \mu_{\beta}(G) > 0\},$$
$$\mathcal{F}_{\beta} := \{E \in \mathrm{Coh}(\mathbb{P}^3) : \text{any subsheaf } F \subset E \text{ satisfies } \mu_{\beta}(F) \leq 0\}.$$

A new heart of a bounded t-structure is given by the extension closure $\mathrm{Coh}^{\beta}(\mathbb{P}^3) := \langle \mathcal{F}_{\beta}[1], \mathcal{T}_{\beta} \rangle$. Equivalently, the objects in $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$ are complexes $E \in D^b(X)$ satisfying $H^i(E) = 0$ for $i \neq 0, -1$, $H^{-1}(E) \in \mathcal{F}_{\beta}$, and $H^0(E) \in \mathcal{T}_{\beta}$. Let $\alpha > 0$ be a positive real number. The new slope function is

$$\nu_{\alpha,\beta}(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2) := \frac{\mathrm{ch}^{\beta}_2 - \frac{\alpha^2}{2}\,\mathrm{ch}^{\beta}_0}{\mathrm{ch}^{\beta}_1} = \frac{\mathrm{ch}_2 - \beta\,\mathrm{ch}_1 + \frac{\beta^2}{2}\,\mathrm{ch}_0 - \frac{\alpha^2}{2}\,\mathrm{ch}_0}{\mathrm{ch}_1 - \beta\,\mathrm{ch}_0}.$$

As in classical slope stability, an object $E \in \mathrm{Coh}^{\beta}(\mathbb{P}^3)$ is called $\nu_{\alpha,\beta}$-*(semi)stable* or *tilt (semi)stable* with respect to (α, β) if for all short exact sequences $0 \to F \to E \to G \to 0$ in $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$, the inequality $\nu_{\alpha,\beta}(F) < (\le)\nu_{\alpha,\beta}(G)$ holds. Note that in regard to [BMT14] this slope has been modified by switching α with $\sqrt{3}\alpha$. We prefer this point of view because it makes the walls semicircular. In concrete computations, it becomes relevant to restrict the Chern characters of semistable objects. One of the main tools to perform this restriction is the following inequality for semistable objects.

Theorem 1.1 (Bogomolov–Gieseker inequality for tilt stability, [BMT14, Corollary 7.3.2]). *Any $\nu_{\alpha,\beta}$-semistable object $E \in \mathrm{Coh}^{\beta}(\mathbb{P}^3)$ satisfies*

$$\begin{aligned} Q^{\mathrm{tilt}}(E) &:= (\mathrm{ch}_1^{\beta}(E))^2 - 2\,\mathrm{ch}_0^{\beta}(E)\,\mathrm{ch}_2^{\beta}(E) \\ &= (\mathrm{ch}_1(E))^2 - 2\,\mathrm{ch}_0(E)\,\mathrm{ch}_2(E) \ge 0. \end{aligned}$$

Let $v = \mathrm{ch}_{\le 2}(E) = (v_0, v_1, v_2)$ for some object $E \in D^b(\mathbb{P}^3)$. A *numerical wall* in tilt stability for v is by definition induced by a class $(r, c, d) \in \mathbb{Z}^2 \times \frac{1}{2}\mathbb{Z}$ as the set of solutions (α, β) to the equation $\nu_{\alpha,\beta}(v) = \nu_{\alpha,\beta}(r, c, d)$, where we assume that this is a nontrivial proper solution set. For example, throughout this article, we will always choose $v = \mathrm{ch}_{\le 2}(\mathcal{I}_C)$, where $C \subset \mathbb{P}^3$ is an elliptic quartic curve, and study moduli spaces involving these objects.

A subset of a numerical wall is an *actual wall* if the set of stable or semistable objects with class v changes at it. Numerical walls in tilt stability satisfy Bertram's nested wall theorem. For surfaces, it was proved in [Mac14a]. A proof in the threefold case can be found in [Sch15].

Theorem 1.2 (Structure theorem for walls in tilt stability). *All numerical walls in the following statements are for fixed $v = (v_0, v_1, v_2)$.*

(1) *Numerical walls in tilt stability are of the form*

$$x\alpha^2 + x\beta^2 + y\beta + z = 0$$

for $x = v_0 c - v_1 r$, $y = 2(v_2 r - v_0 d)$, and $z = 2(v_1 d - v_2 c)$. In particular, they are either semicircular walls with center on the β-axis or vertical rays.

(2) *If two numerical walls given by $\nu_{\alpha,\beta}(r, c, d) = \nu_{\alpha,\beta}(v)$ and $\nu_{\alpha,\beta}(r', c', d') = \nu_{\alpha,\beta}(v)$ intersect for any $\alpha \ge 0$, then (r, c, d), (r', c', d'), and v are linearly dependent. In particular, the two walls are completely identical.*

(3) *The curve $\nu_{\alpha,\beta}(v) = 0$ is given by the hyperbola*

$$v_0\alpha^2 - v_0\beta^2 + 2v_1\beta - 2v_2 = 0.$$

Moreover, this hyperbola intersects all semicircular walls at their top point.

(4) *If $v_0 \ne 0$, then there is exactly one vertical numerical wall given by $\beta = v_1/v_0$. If $v_0 = 0$, then there is no vertical wall.*

(5) *If a numerical wall has a single point at which it is an actual wall, then all of it is an actual wall.*

On smooth projective surfaces tilt stability is enough to get a Bridgeland stability condition (see [Bri08; AB13]). On threefolds, Bayer, Macrì, and Toda proposed another tilt to obtain a suitable category to define a Bridgeland stability condition as follows. Let

$$\mathcal{T}'_{\alpha,\beta} := \{E \in \mathrm{Coh}^{\beta}(\mathbb{P}^3) : \text{any quotient } E \twoheadrightarrow G \text{ satisfies } \nu_{\alpha,\beta}(G) > 0\},$$
$$\mathcal{F}'_{\alpha,\beta} := \{E \in \mathrm{Coh}^{\beta}(\mathbb{P}^3) : \text{any subobject } F \hookrightarrow E \text{ satisfies } \nu_{\alpha,\beta}(F) \leq 0\}$$

and set $\mathcal{A}^{\alpha,\beta} := \langle \mathcal{F}'_{\alpha,\beta}[1], \mathcal{T}'_{\alpha,\beta} \rangle$. For any $s > 0$, they define

$$Z_{\alpha,\beta,s} := -\mathrm{ch}_3^{\beta} + \left(s + \frac{1}{6}\right)\alpha^2 \mathrm{ch}_1^{\beta} + i\left(\mathrm{ch}_2^{\beta} - \frac{\alpha^2}{2}\mathrm{ch}_0^{\beta}\right),$$
$$\lambda_{\alpha,\beta,s} := -\frac{\Re(Z_{\alpha,\beta,s})}{\Im(Z_{\alpha,\beta,s})}.$$

To prove that this yields a Bridgeland stability condition, Bayer, Macrì, and Toda conjectured a generalized Bogomolov–Gieseker inequality involving third Chern characters for tilt semistable objects with $\nu_{\alpha,\beta} = 0$. In [BMS16], it was shown that the conjecture is equivalent to a more general inequality that drops the hypothesis $\nu_{\alpha,\beta} = 0$. In the case of $\mathbb{P}^3$ the inequality was proved in [Mac14b]. Recall the definition of Q^{tilt} from Theorem 1.1.

THEOREM 1.3 (BMT Inequality). *Any $\nu_{\alpha,\beta}$-stable object $E \in \mathrm{Coh}^{\beta}(\mathbb{P}^3)$ satisfies*

$$\alpha^2 Q^{\mathrm{tilt}}(E) + 4(\mathrm{ch}_2^{\beta}(E))^2 - 6\,\mathrm{ch}_1^{\beta}(E)\,\mathrm{ch}_3^{\beta}(E) \geq 0.$$

Similar inequalities were proved for the smooth quadric threefold [Sch14] and all Abelian threefolds [BMS16; MP13a; MP13b]. Recently, the inequality has also been generalized to all Fano threefolds of Picard rank 1 in [Li15]. By using the definition of $\mathrm{ch}^{\beta}(E)$, we find $x(E), y(E) \in \mathbb{R}$ such that the BMT inequality becomes

$$\alpha^2 Q^{\mathrm{tilt}}(E) + \beta^2 Q^{\mathrm{tilt}}(E) + x(E)\beta + y(E) \geq 0.$$

This means that the solution set is given by the complement of a semidisc with center on the β-axis or a quadrant to one side of a vertical line.

Using the same proof as in the surface case in [Bri08, Proposition 14.1] leads to the following lemma. It allows us to identify the moduli space of slope stable sheaves as a moduli space of tilt stable objects.

LEMMA 1.4. *Let $v = (v_0, v_1, v_2, v_3) \in K_0(\mathbb{P}^3)$ be such that $\beta < \mu(v)$ and (v_0, v_1) is primitive. Then an object E with $\mathrm{ch}(E) = v$ is $\nu_{\alpha,\beta}$-stable for all $\alpha \gg 0$ if and only if E is a slope stable sheaf.*

An important question is how moduli spaces change set theoretically at walls in Bridgeland stability. In case the destabilizing subobject and quotient are both stable, this has a satisfactory answer, and a proof can, for example, be found in [Sch15, Lemma 3.10]. Note that this does not work in the case of tilt stability due to the lack of unique Jordan–Hölder filtrations.

Lemma 1.5. *Let $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}(\mathbb{P}^3)$ be such that there are stable objects $F, G \in \mathcal{A}$ with $\mu_\sigma(F) = \mu_\sigma(G)$. Then there is an open neighborhood U around σ where nontrivial extensions $0 \to F \to E \to G \to 0$ are stable for all $\sigma' \in U$, where $F \hookrightarrow E$ does not destabilize E.*

Another crucial issue is the construction of reasonably behaved moduli spaces of Bridgeland stable objects. A recent result by Piyaratne and Toda is a major step toward this. It applies in particular to the case of $\mathbb{P}^3$, since the conjectural BMT-inequality is known.

Theorem 1.6 ([PT15]). *Let X be a smooth projective threefold such that the conjectural construction of Bridgeland stability from* [BMT14] *works. Then any moduli space of semistable objects for such a Bridgeland stability condition is a universally closed algebraic stack of finite type over $\mathbb{C}$.*

If there are no strictly semistable objects, then the moduli space becomes a proper algebraic space of finite type over $\mathbb{C}$.

Our strategy to compute concrete wall crossing follows that of [Sch15]. We do numerical computations in tilt stability and then translate them into Bridgeland stability. Let $v = (v_0, v_1, v_2, v_3)$ be the Chern character of an object in $D^b(X)$. For any $\alpha > 0$, $\beta \in \mathbb{R}$, and $s > 0$, we denote the set of $\lambda_{\alpha,\beta,s}$-semistable objects with Chern character v by $M_{\alpha,\beta,s}(v)$ and the set of $\nu_{\alpha,\beta}$-semistable objects with Chern character v by $M^{\mathrm{tilt}}_{\alpha,\beta,s}(v)$. Analogously to our notation for twisted Chern characters, we write $v^\beta := (v_0^\beta, v_1^\beta, v_2^\beta, v_3^\beta) = v \cdot e^{-\beta H}$. We also write

$$P_v := \{(\alpha, \beta) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : \nu_{\alpha,\beta}(v) > 0\}.$$

We need the following technical statement. Under mild hypotheses, it says that on one side of the hyperbola $\{\nu_{\alpha,\beta}(v) = 0\}$ all the chambers and walls of tilt stability occur in Bridgeland stability. Note that $\nu_{\alpha,\beta}(v) = 0$ implies $\lambda_{\alpha,\beta,s}(v) = \infty$. This is a crucial fact in establishing the following relation between walls in tilt stability and walls in Bridgeland stability.

Theorem 1.7 ([Sch15, Theorem 6.1]). *Let $\alpha_0 > 0$, $\beta_0 \in \mathbb{R}$, and $s > 0$, be such that $\nu_{\alpha_0,\beta_0}(v) = 0$ and $v_1^{\beta_0} > 0$.*

(1) *Assume there is an actual wall in Bridgeland stability for v at (α_0, β_0) given by*

$$0 \to F \to E \to G \to 0.$$

That means $\lambda_{\alpha_0,\beta_0,s}(F) = \lambda_{\alpha_0,\beta_0,s}(G)$ and $\mathrm{ch}(E) = -v$ for semistable $E, F, G \in \mathcal{A}^{\alpha_0,\beta_0}(\mathbb{P}^3)$. Further, assume that there is a neighborhood U of (α_0, β_0) such that the same sequence also defines an actual wall in $U \cap P_v$, that is, E, F, G remain semistable in $U \cap P_v \cap \{\lambda_{\alpha,\beta,s}(F) = \lambda_{\alpha,\beta,s}(G)\}$. Then $E[-1], F[-1], G[-1] \in \mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$ are ν_{α_0,β_0}-semistable. In particular, there is an actual wall in tilt stability at (α_0, β_0).

(2) *Assume that all ν_{α_0,β_0}-semistable objects with class v are stable. Then there is a neighborhood U of (α_0, β_0) such that*

$$M_{\alpha,\beta,s}(v) = M^{\text{tilt}}_{\alpha,\beta}(v)$$

for all $(\alpha, \beta) \in U \cap P_v$. Moreover, in this case, all objects in $M_{\alpha,\beta,s}(v)$ are $\lambda_{\alpha,\beta,s}$-stable.

(3) *Assume that there is an actual wall in tilt stability for v at (α_0, β_0) given by*

$$0 \to F^n \to E \to G^m \to 0$$

such that $F, G \in \mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$ are ν_{α_0,β_0}-stable objects, $\mathrm{ch}(E) = v$, and $\nu_{\alpha_0,\beta_0}(F) = \nu_{\alpha_0,\beta_0}(G)$. Assume further that the set

$$P_v \cap P_{\mathrm{ch}(F)} \cap P_{\mathrm{ch}(G)} \cap \{\lambda_{\alpha,\beta,s}(F) = \lambda_{\alpha,\beta,s}(G)\}$$

is nonempty. Then there is a neighborhood U of (α_0, β_0) such that F, G are $\lambda_{\alpha,\beta,s}$-stable for all $(\alpha, \beta) \in U \cap P_v \cap \{\lambda_{\alpha,\beta,s}(F) = \lambda_{\alpha,\beta,s}(G)\}$. In particular, there is an actual wall in Bridgeland stability in $U \cap P_v$ defined by the same sequence.

This theorem will be used as follows in the the remainder of the article. Assume that we have determined all exact sequences that give walls in tilt stability for objects with a fixed Chern character v. By part (1) of the theorem, we know that on one side of the hyperbola $\nu_{\alpha,\beta}(v) = 0$ the only walls in Bridgeland stability have to be defined by an exact sequence giving a wall in tilt stability. We will then use part (3) to show that every such sequence does indeed define a wall in Bridgeland stability. At this point, we know all exact sequences defining walls on a path close to one side of the hyperbola $\nu_{\alpha,\beta}(v) = 0$. Finally, we have to use part (2) to show that all the moduli spaces of tilt stable objects actually occur in Bridgeland stability on this path.

By doing this, we can translate simple computations in tilt stability into the more complicated framework of Bridgeland stability. Sometimes there are exact sequences giving identical numerical walls in tilt stability, but different numerical walls in Bridgeland stability. Therefore, this translation allows us to observe additional chambers that are hidden in tilt stability.

2. Tilt Stability for Elliptic Quartics

Let C be the complete intersection of two quadrics in $\mathbb{P}^3$, that is, an elliptic quartic curve. We will compute all walls in tilt stability for $\beta < 0$ with respect to $v = \mathrm{ch}(\mathcal{I}_C)$. There is a locally free resolution $0 \to \mathcal{O}(-4) \to \mathcal{O}(-2)^{\oplus 2} \to \mathcal{I}_C \to 0$. This leads to

$$\mathrm{ch}^{\beta}(\mathcal{I}_C) = \left(1, -\beta, \frac{\beta^2}{2} - 4, -\frac{\beta^3}{6} + 4\beta + 8\right).$$

We denote the set of tilt semistable objects with respect to (α, β) and class v by $M^{\text{tilt}}_{\alpha,\beta}(v)$.

Theorem 2.1. *There are three walls for* $M^{\mathrm{tilt}}_{\alpha,\beta}(1,0,-4,8)$ *for* $\alpha > 0$ *and* $\beta < 0$. *Moreover, the following table lists pairs of tilt semistable objects whose extensions completely describe all strictly semistable objects at each of the corresponding walls. Let* L *be a line in* $\mathbb{P}^3$, V *a plane in* $\mathbb{P}^3$, $Z \subset \mathbb{P}^3$ *a length two zero-dimensional subscheme,* $Z' \subset V$ *a length two zero-dimensional subscheme, and let* $P \in \mathbb{P}^3$ *and,* $Q \in V$ *be points.*

$\alpha^2 + (\beta+3)^2 = 1$	$\mathcal{O}(-2)^{\oplus 2}, \mathcal{O}(-4)[1]$
$\alpha^2 + (\beta+\frac{7}{2})^2 = \frac{17}{4}$	$\mathcal{I}_L(-1), \mathcal{O}_V(-3)$
$\alpha^2 + (\beta+\frac{9}{2})^2 = (\frac{7}{2})^2$	$\mathcal{I}_Z(-1), \mathcal{O}_V(-4)$ $\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4)$ $\mathcal{O}(-1), \mathcal{I}_{Z'/V}(-4)$

The hyperbola $\nu_{\alpha,\beta}(1,0,-4) = 0$ *is given by the equation*

$$\beta^2 - \alpha^2 = 8.$$

Moreover, there are no semistable objects for (α, β) *inside the smallest semicircle.*

It is interesting to note that all relevant objects in this theorem are sheaves and not actual 2-term complexes. The key difference to the classical picture, as we will see later, is that some sheaves of positive rank with torsion will turn out to be stable and replace ideal sheaves of heavily singular curves in some chambers.

The fact that the smallest wall is given by the equation $\alpha^2 + (\beta+3)^2 = 1$ was already proved in [Sch15, Theorem 5.1] in more generality. Moreover, it was shown there that all semistable objects E at the wall are given by extensions of the form $0 \to \mathcal{O}(-2)^{\oplus 2} \to E \to \mathcal{O}(4)[1] \to 0$ and that there are no tilt semistable objects inside this semicircle.

To prove the remainder of Theorem 2.1, we need to put numerical restrictions on potentially destabilizing objects. This can be done by the following two lemmas.

Lemma 2.2 ([Sch14, Lemma 5.4]). *Let* $E \in \mathrm{Coh}^{\beta}(\mathbb{P}^3)$ *be tilt semistable with respect to some* $\beta \in \mathbb{Z}$ *and* $\alpha \in \mathbb{R}_{>0}$.

(1) *If* $\mathrm{ch}^{\beta}(E) = (1,1,d,e)$, *then* $d - 1/2 \in \mathbb{Z}_{\leq 0}$. *In the case* $d = -1/2$, *we get* $E \cong \mathcal{I}_L(\beta+1)$ *where* L *is a line plus* $1/6 - e$ *(possibly embedded) points in* $\mathbb{P}^3$. *If* $d = 1/2$, *then* $E \cong \mathcal{I}_Z(\beta+1)$ *for a zero-dimensional subscheme* $Z \subset \mathbb{P}^3$ *of length* $1/6 - e$.

(2) *If* $\mathrm{ch}^{\beta}(E) = (0,1,d,e)$, *then* $d - 1/2 \in \mathbb{Z}$ *and* $E \cong I_{Z/V}(\beta + d + 1/2)$, *where* Z *is a dimension zero subscheme of length* $1/24 + d^2/2 - e$.

The next lemma determines the Chern characters of possibly destabilizing objects for $\beta = -2$.

LEMMA 2.3. *If an exact sequence* $0 \to F \to E \to G \to 0$ *in* $\mathrm{Coh}^{-2}(\mathbb{P}^3)$ *defines a wall for* $\beta = -2$ *with* $\mathrm{ch}_{\leq 2}(E) = (1, 0, -4)$, *then*

$$\mathrm{ch}^{-2}_{\leq 2}(F), \mathrm{ch}^{-2}_{\leq 2}(G) \in \left\{ \left(1, 1, -\frac{1}{2}\right), \left(0, 1, -\frac{3}{2}\right), \left(1, 1, \frac{1}{2}\right), \left(0, 1, -\frac{5}{2}\right) \right\}.$$

Proof. The four possible Chern characters group into two cases that add up to $\mathrm{ch}^{-2}_{\leq 2}(E) = (1, 2, -2)$.

Let $\mathrm{ch}^{-2}_{\leq 2}(F) = (r, c, d)$. By the definition of $\mathrm{Coh}^{-2}(\mathbb{P}^3)$, we have $0 \leq c \leq 2$. If $c = 0$, then $\nu_{\alpha,-2}(F) = \infty$, and this is in fact not wall for any $\alpha > 0$. If $c = 2$, then the same argument for the quotient G shows that there is no wall. Therefore, $c = 1$ must hold. We can compute

$$\nu_{\alpha,-2}(E) = -1 - \frac{\alpha^2}{4}, \qquad \nu_{\alpha,-2}(F) = d - \frac{r\alpha^2}{2}.$$

The wall is defined by $\nu_{\alpha,-2}(E) = \nu_{\alpha,-2}(F)$. This leads to

$$\alpha^2 = \frac{4d + 4}{2r - 1} > 0. \tag{1}$$

The next step is to rule out the cases $r \geq 2$ and $r \leq -1$. If $r \geq 2$, then $\mathrm{ch}_0(G) \leq -1$. By exchanging the roles of F and G in the following argument, it is enough to deal with the situation $r \leq -1$. In that case, we use (1) and the Bogomolov–Gieseker inequality to get the contradiction $2rd \leq 1$, $d < -1$, and $r \leq -1$.

Therefore, we know $r = 0$ or $r = 1$. By again interchanging the roles of F and G if necessary we only have to handle the case $r = 1$. Equation (1) implies $d > -1$. By Lemma 2.2 we get $d - 1/2 \in \mathbb{Z}_{\leq 0}$. Therefore, we are left with the cases claimed. □

Proof of Theorem 2.1. By assumption we are only dealing with walls that intersect the branch of the hyperbola with $\beta < 0$. As explained before, we already know the smallest wall. This semicircle intersects the β-axis at $\beta = -4$ and $\beta = -2$. Therefore, all other walls intersecting this branch of the hyperbola also have to intersect the ray $\beta = -2$. By Lemma 2.3 there are at most two walls intersecting the line $\beta = -2$. They correspond to the two solutions claimed to exist.

Let $0 \to F \to E \to G \to 0$ define a wall in $\mathrm{Coh}^{-2}(\mathbb{P}^3)$ with $\mathrm{ch}(E) = (1, 0, -4, 8)$. We can compute $\mathrm{ch}^{-2}(E) = (1, 2, -2, \frac{4}{3})$. A direct computation shows that the middle wall is given by $\mathrm{ch}^{-2}(F) = (1, 1, -1/2, e)$ and $\mathrm{ch}^{-2}(G) = (0, 1, -3/2, 4/3 - e)$. By Lemma 2.2 we get $F \cong \mathcal{I}_L(-1)$, where L is a line plus $1/6 - e$ (possibly embedded) points in $\mathbb{P}^3$. In particular, the inequality $e \leq 1/6$ holds. The same lemma also implies that $G \cong I_{Z/V}(-3)$, where Z is a dimension zero subscheme of length $e - 1/6$. Overall, this shows that $e = 1/6$. Therefore, L is a just a line, and $E \cong \mathcal{O}_V(-3)$.

The outermost wall is given by $\mathrm{ch}^{-2}(F) = (1, 1, 1/2, e)$ and $\mathrm{ch}^{-2}(G) = (0, 1, -5/2, 4/3 - e)$. We use again Lemma 2.2 to get $F \cong \mathcal{I}_Z(-1)$ for a zero-dimensional subscheme $Z \subset \mathbb{P}^3$ of length $1/6 - e$. Therefore, we have $e - 1/6 \in \mathbb{Z}_{\geq 0}$. The lemma also shows that $G \cong I_{Z/V}(-4)$, where Z is a dimension zero subscheme of length $e + 11/6$. Overall, we get $e \in \{-11/6, -5/6, 1/6\}$. That corresponds exactly to the three cases in the theorem. □

3. Curves on the Intersection of the Two Components

Let $\mathrm{Hilb}_1^{4t} \subset \mathrm{Hilb}^{4t}(\mathbb{P}^3)$ be the closure of the locus of smooth elliptic quartic curves. By $\mathrm{Hilb}_2^{4t} \subset \mathrm{Hilb}^{4t}(\mathbb{P}^3)$ we denote the closure of the locus of plane quartics curves plus two disjoint points. A straightforward dimension count shows that $\dim \mathrm{Hilb}_1^{4t} = 16$ and $\dim \mathrm{Hilb}_2^{4t} = 23$. In this section, will prove some preliminary results about the intersection of the two components. We will do this following the approach of Piene and Schlessinger [PS85], which requires a careful analysis of the equations of the curves along this intersection.

PROPOSITION 3.1. *Let I_C be the ideal of a subscheme $C \subset \mathbb{P}^3$ of dimension* 1 *that fits into an exact sequence of the form* $0 \to \mathcal{I}_{Z'}(-1) \to I_C \to \mathcal{O}_V(-4) \to 0$, *where V is a plane in $\mathbb{P}^3$, and $Z' \subset V$ is a zero-dimensional subscheme of length two.*

(1) *The ideal I_C is projectively equivalent to one of the ideals*

$$(x^2, xy, xzw, f_4(x, y, z, w)),$$
$$(x^2, xy, xz^2, g_4(x, y, z, w)),$$

where $f_4 \in (x, y, zw)$, respectively, $g_4 \in (x, y, z^2)$ is of degree 4.

(2) *The ideal*

$$(x^2, xy, xz^2, y^4)$$

lies in the closure of the orbit of $\mathcal{I}_C$ under the action of PGL(4) *for any $\mathcal{I}_C$ as above.*

Proof. Up to the action of PGL(4), we can assume that either $I_{Z'} = (x, y, zw)$ or $I_{Z'} = (x, y, z^2)$ and $I_V = (x)$. The exact sequence $0 \to \mathcal{I}_{Z'}(-1) \to I_C \to \mathcal{O}_V(-4) \to 0$ implies that either $l(x, y, z, w) \cdot (x, y, zw) \subset I_C$ or $l(x, y, z, w) \cdot (x, y, z^2) \subset I_C$ for a linear polynomial $l(x, y, z, w) \in \mathbb{C}[x, y, z, w]$. Since the quotient is supported on V, we must have $l = x$. Therefore, either $(x^2, xy, xzw) \subset I_C$ or $(x^2, xy, xz^2) \subset I_C$. Since the quotient is $\mathcal{O}_V(-4)$, there has to be another degree 4 generator $f_4(x, y, z, w)$ with $xf_4(x, y, z, w) \in (x^2, xy, xzw)$, respectively, $g_4(x, y, z, w)$ with $xg_4(x, y, z, w) \in (x^2, xy, xz^2)$. That proves (1).

By (1) we can assume that either $I_C = (x^2, xy, xzw, f_4(x, y, z, w))$ for $f_4 \in (x, y, zw)$ or $I_C = (x^2, xy, xz^2, g_4(x, y, z, w))$ for $g_4 \in (x, y, z^2)$. We can take the limit as $t \to 0$ for the action of the element $g_t \in \mathrm{PGL}(4)$ that fixes x, y, z and maps $w \mapsto (1-t)z + tw$. Thus, we can assume that $I_C = (x^2, xy, xz^2, g_4(y, z))$, where $g_4 \in \mathbb{C}[y, z]$. Pick $\lambda \in \mathbb{C}\backslash\{0\}$ such that $g_4(\lambda, 1) \neq 0$. We analyze the action of $g_t \in \mathrm{PGL}(4)$ that fixes x, w, maps $y \mapsto \lambda y$, and maps $z \mapsto (1-t)y + tz$. We get

$$\begin{aligned}
&g_t \cdot (x^2, xy, xz^2, g_4(y, z)) \\
&\quad = (x^2, \lambda xy, (1-t)^2 xy^2 + 2(1-t)txyz + t^2 xz^2, g_4(\lambda y, (1-t)y + tz)) \\
&\quad = (x^2, xy, xz^2, g_4(\lambda y, (1-t)y + tz)).
\end{aligned}$$

Since $g_4(\lambda, 1) \neq 0$, we have $g_4(\lambda y, y) \neq 0$, and we can finish the proof of (2) by taking the limit as $t \to 0$. □

Next, we want to analyze the singularities of the point on the Hilbert scheme corresponding to (x^2, xy, xz^2, y^4). We will use [M2] and the techniques developed in [PS85].

PROPOSITION 3.2. *If $I_C = (x^2, xy, xz^2, y^4)$, then I_C lies on the intersection of two irreducible components of* $\mathrm{Hilb}(\mathbb{P}^3)$ *and is a smooth point on each of them. Moreover, the intersection is locally of dimension* 15 *and transversal.*

Proof. Let $p_C \in \mathrm{Hilb}(\mathbb{P}^3)$ be the point parameterizing C. Next, we use the comparison theorem [PS85, p. 764], which claims that the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^3)$ and the universal deformation space that parameterizes all homogeneous ideals with Hilbert function equal to that of I_C are isomorphic in an étale neighborhood of the point p_C if

$$\left(\frac{\mathbb{C}[x, y, z, w]}{I_C}\right)_d \cong H^0(C, \mathcal{O}_C(d))$$

for $d = \deg(f_i)$, where f_i are generators of I_C. For our particular ideal, this equality can, for example, directly be checked with help of Macaulay2 or by hand. The comparison theorem allows us to find local equations of the Hilbert scheme near p_C by using the same strategy as the proof of [PS85, Lemma 6]. In fact, this procedure has been implemented in the Macaulay2 package "VersalDeformations" (see [Ilt12]). In particular, the routine localHilbertScheme generates an ideal of the form (see Appendix A)

$$\begin{aligned}(-t_5t_{24}, -t_6t_{24}, -t_7t_{24}, -t_8t_{24}, t_{15}t_{24}, t_{16}t_{24}, t_{17}t_{24} - 2t_{22}t_{24}, t_{18}t_{24} - 2t_{23}t_{24})\\ \in \mathbb{C}[t_1, \dots, t_{24}].\end{aligned}$$

Then, étale locally at p_C, the Hilbert scheme is the transversal intersection of the hyperplane $(t_{24} = 0)$ and a 16-dimensional linear subspace. □

It is not hard to see that the two components (x^2, xy, xz^2, y^4) is lying on are Hilb^{4t}_1 and Hilb^{4t}_2 by giving explicit degenerations. However, it is also a direct consequence of the results in the next section.

4. Bridgeland Stability

The goal of this section is to translate the computations in tilt stability to actual wall crossings in Bridgeland stability. We will analyze the singular loci of the occurring moduli spaces and use this to reprove the global description of the main component of the Hilbert scheme as in [VA92].

As a consequence of Theorem 1.7 and Theorem 2.1, we obtain the following corollary. In this application of Theorem 1.7, all exact sequences giving walls in tilt stability to the left-hand side of the unique vertical wall are of the form in (3). Therefore, we do not have more sequences giving walls in tilt stability than in Bridgeland stability to the left-hand side of the left branch of the hyperbola.

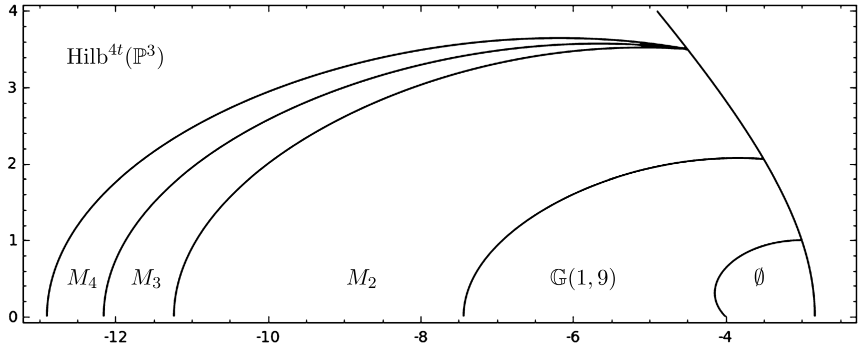

Figure 1 Wall and chamber structure in a subspace of Stab($\mathbb{P}^3$) for Hilb$^{4t}(\mathbb{P}^3)$ and their associated models according to Theorem A

Corollary 4.1. *There is a path $\gamma : [0, 1] \to \mathbb{R}_{>0} \times \mathbb{R} \subset \mathrm{Stab}(\mathbb{P}^3)$ that crosses the following walls for $v = (1, 0, -4, 8)$ in the given order. The walls are defined by the two given objects having the same slope. Moreover, all strictly semistable objects at each of the walls are extensions of those two objects. Let L be a line in $\mathbb{P}^3$, V a plane in $\mathbb{P}^3$, $Z \subset \mathbb{P}^3$ a length two zero dimensional subscheme, $Z' \subset V$ a length two zero dimensional subscheme, and let $P \in \mathbb{P}^3$ and $Q \in V$ be points.*

(1) $\mathcal{O}(-2)^{\oplus 2}, \mathcal{O}(-4)[1]$.
(2) $\mathcal{I}_L(-1), \mathcal{O}_V(-3)$.
(3) $\mathcal{I}_Z(-1), \mathcal{O}_V(-4)$.
(4) $\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4)$.
(5) $\mathcal{O}(-1), \mathcal{I}_{Z'/V}(-4)$.

We denote the moduli space of Bridgeland stable objects with Chern character $(1, 0, -4, 8)$ in the chambers from inside the smallest wall to outside the largest wall by $M_0, \dots, M_5$. The goal of this section is to give some description of these spaces. By Theorem 2.1 we have $M_0 = \emptyset$. After the largest wall, we must have $M_5 = \mathrm{Hilb}^{4t}(\mathbb{P}^3)$. More precisely, it is the moduli of ideal sheaves, which is the same as the Hilbert scheme due to [M+06, p. 1265]. See Figure 1 for a visualization of the walls.

Proposition 4.2. *The first moduli space M_1 is isomorphic to the Grassmannian $\mathbb{G}(1, 9)$.*

Proof. All extensions in $\mathrm{Ext}^1(\mathcal{O}(-4)[1], \mathcal{O}(-2)^{\oplus 2})$ are cokernels of morphisms $\mathcal{O}(-4) \to \mathcal{O}(-2)^{\oplus 2}$. The stability condition ensures that the two quadrics defining it are not collinear. Therefore, these extensions parameterize pencils of quadrics, and the moduli space is the Grassmannian $\mathbb{G}(1, 9)$. □

The tangent space of a moduli space of Bridgeland stable objects at any stable complex E is given by $\mathrm{Ext}^1(E, E)$ (see [Ina02] and [Lie06] for the deformation theory of moduli spaces of complexes). Obtaining these groups requires a substantial amount of diagram chasing and computations. To minimize the distress on the reader and the authors, we will prove the following lemma with heavy usage of [M2].

LEMMA 4.3. *Let the notation be as in Theorem 4.1. We have the following equalities*:

$$\mathrm{Ext}^1(\mathcal{I}_L(-1), \mathcal{O}_V(-3)) = \mathbb{C}, \qquad \mathrm{Ext}^1(\mathcal{O}_V(-3), \mathcal{I}_L(-1)) = \mathbb{C}^9,$$
$$\mathrm{Ext}^1(\mathcal{I}_L(-1), \mathcal{I}_L(-1)) = \mathbb{C}^4, \qquad \mathrm{Ext}^1(\mathcal{O}_V(-3), \mathcal{O}_V(-3)) = \mathbb{C}^3,$$
$$\mathrm{Ext}^1(\mathcal{I}_Z(-1), \mathcal{O}_V(-4)) = \begin{cases} \mathbb{C}, & Z \subset V, \\ 0 & \textit{otherwise}, \end{cases}$$
$$\mathrm{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_Z(-1)) = \mathbb{C}^{15},$$
$$\mathrm{Ext}^1(\mathcal{I}_Z(-1), \mathcal{I}_Z(-1)) = \mathbb{C}^6, \qquad \mathrm{Ext}^1(\mathcal{O}_V(-4), \mathcal{O}_V(-4)) = \mathbb{C}^3,$$
$$\mathrm{Ext}^1(\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4)) = \begin{cases} \mathbb{C}^3, & P = Q, \\ \mathbb{C}, & P \neq Q, \end{cases}$$
$$\mathrm{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_P(-1)) = \begin{cases} \mathbb{C}^{17}, & P = Q, \\ \mathbb{C}^{15}, & P \neq Q, \end{cases}$$
$$\mathrm{Ext}^1(\mathcal{I}_P(-1), \mathcal{I}_P(-1)) = \mathbb{C}^3, \qquad \mathrm{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_{Q/V}(-4)) = \mathbb{C}^5,$$
$$\mathrm{Ext}^1(\mathcal{O}(-1), \mathcal{I}_{Z'/V}(-4)) = \mathbb{C}^2, \qquad \mathrm{Ext}^1(\mathcal{I}_{Z'/V}(-4), \mathcal{O}(-1)) = \mathbb{C}^{15},$$
$$\mathrm{Ext}^1(\mathcal{O}(-1), \mathcal{O}(-1)) = 0, \qquad \mathrm{Ext}^1(\mathcal{I}_{Z'/V}(-4), \mathcal{I}_{Z'/V}(-4)) = \mathbb{C}^7.$$

If $Z \subset V$ is a double point supported at P, then

$$\mathrm{Ext}^1(\mathcal{I}_Z(-1), \mathcal{I}_{P/V}(-4)) = \mathbb{C}^3, \qquad \mathrm{Ext}^1(\mathcal{O}_V(-4)), \mathcal{I}_{P/V}(-4)) = \mathbb{C}^2,$$
$$\mathrm{Ext}^1(\mathcal{I}_Z(-1), \mathcal{I}_P(-1)) = \mathbb{C}^3, \qquad \mathrm{Ext}^1(\mathcal{O}_V(-4)), \mathcal{I}_P(-1)) = \mathbb{C}^{15}.$$

Proof. Under the action of PGL(4), there are two orbits of pairs of a line and a plane (L, V). Either we have $L \subset V$ or not. By choosing representatives defined over $\mathbb{Q}$, we can use [M2] to compute $\mathrm{Ext}^1(\mathcal{I}_L(-1), \mathcal{O}_V(-3)) = \mathbb{C}$, $\mathrm{Ext}^1(\mathcal{O}_V(-3), \mathcal{I}_L(-1)) = \mathbb{C}^9$, $\mathrm{Ext}^1(\mathcal{O}_V(-3), \mathcal{O}_V(-3)) = \mathbb{C}^3$, and $\mathrm{Ext}^1(\mathcal{I}_L(-1), \mathcal{I}_L(-1)) = \mathbb{C}^4$. All other equalities follow in the same way. The Macaulay2 code can be found in Appendix A. □

Since the dimension of tangent spaces is bounded from below by the dimension of the space, the following lemma can sometimes simplify computations.

LEMMA 4.4. *Let $0 \to F^n \to E \to G^m \to 0$ be an exact sequence at a wall in Bridgeland stability, where F and G are distinct stable objects of the same Bridgeland slope, and E is semistable to one side of the wall. Then*

$$\begin{aligned} \mathrm{ext}^1(E, E) \leq\, & n^2 \,\mathrm{ext}^1(F, F) + m^2 \,\mathrm{ext}^1(G, G) \\ & + nm \,\mathrm{ext}^1(F, G) + nm \,\mathrm{ext}^1(G, F) - n^2. \end{aligned}$$

Proof. The stability to one side of the wall implies $\mathrm{Hom}(E, F) = 0$. Since F is stable, we also know that $\mathrm{Hom}(F, F) = \mathbb{C}$. By the long exact sequence coming from applying $\mathrm{Hom}(\cdot, F)$ to the above exact sequence, we get $\mathrm{ext}^1(E, F) \leq m\,\mathrm{ext}^1(G, F) + n\,\mathrm{ext}^1(F, F) - n$. Moreover, we can use $\mathrm{Hom}(\cdot, G)$ to get $\mathrm{ext}^1(E, G) \leq m\,\mathrm{ext}^1(G, G) + n\,\mathrm{ext}^1(F, G)$. These two inequalities, together with application of $\mathrm{Hom}(E, \cdot)$, lead to the claim. □

We also have to handle the issue of potentially new components after crossing a wall. The following result will solve this issue in some cases.

LEMMA 4.5. *Let M and N be two moduli spaces of Bridgeland semistable objects separated by a single wall. Assume that $A \subset M$ and $B \subset N$ are the loci destabilized at the wall. If A intersects an irreducible component H of M nontrivially and H is not contained in A, then B must intersect the closure of $H \backslash A$ inside N.*

Proof. This follows from the fact that moduli spaces of Bridgeland semistable objects are universally closed. If B would not intersect the closure of $H \backslash A$ inside N, then this would correspond to a component in N that is not universally closed. □

To identify the global structure of some of the moduli spaces as blowups, we need the following classical result by Moishezon. Recall that the analytification of a smooth proper algebraic spaces of finite type over $\mathbb{C}$ of dimension n is a complex manifold with n independent meromorphic functions.

THEOREM 4.6 ([Moi67]). *Any birational morphism $f : X \to Y$ between smooth proper algebraic spaces of finite type over $\mathbb{C}$ such that the contracted locus E is irreducible and the image $f(E)$ is smooth is the blowup of Y in $f(E)$.*

PROPOSITION 4.7. *The second moduli space M_2 is the blow up of $\mathbb{G}(1, 9)$ along the smooth locus $\mathbb{G}(1, 3) \times (\mathbb{P}^3)^\vee$ parameterizing pairs $(\mathcal{I}_L(-1), \mathcal{O}_V(-3))$. The center of the blow up parameterizes pencils whose base locus is not of dimension one. A generic point of the exceptional divisor parameterizes the union of a line and a plane cubic that intersect themselves at a point. The only nonideal sheaves in the moduli space come from the case where the line is contained in the plane.*

Proof. We know that M_1 is smooth. The wall separating M_1 and M_2 has strictly semistable objects given by extensions between $\mathcal{I}_L(-1)$ and $\mathcal{O}_V(-3)$. By Lemma 4.3 we have $\mathrm{Ext}^1(\mathcal{I}_L(-1), \mathcal{O}_V(-3)) = \mathbb{C}$, $\mathrm{Ext}^1(\mathcal{O}_V(-3), \mathcal{I}_L(-1)) = \mathbb{C}^9$, $\mathrm{Ext}^1(\mathcal{O}_V(-3), \mathcal{O}_V(-3)) = \mathbb{C}^3$, and $\mathrm{Ext}^1(\mathcal{I}_L(-1), \mathcal{I}_L(-1)) = \mathbb{C}^4$.

This means the locus of semistable objects occurring as extensions in $\mathrm{Ext}^1(\mathcal{I}_L(-1), \mathcal{O}_V(-3))$ for any L and V is isomorphic to $\mathbb{G}(1, 3) \times (\mathbb{P}^3)^\vee$, that is, it is smooth and irreducible. By Lemma 1.5 this is the locus destabilized at the wall in $\mathbb{G}(1, 9)$. By Lemma 4.4 any extension E in $\mathrm{Ext}^1(\mathcal{O}_V(-3), \mathcal{I}_L(-1))$ satisfies $\mathrm{ext}^1(E, E) \leq 16$. Lemma 4.5 shows that M_2 has to be connected, that is, it is smooth and irreducible. The locus of semistable objects that can be written as extensions in $\mathrm{Ext}^1(\mathcal{O}_V(-3), \mathcal{I}_L(-1))$ for any L, and V is irreducible of dimension 15, that is, it is a divisor in M_2. An immediate application of Theorem 4.6 implies the fact that M_2 is the blowup of $\mathbb{G}(1,9)$ in the smooth locus $\mathbb{G}(1,3) \times (\mathbb{P}^3)^\vee$.

The description of the exceptional divisor is immediate from the fact that curves C with ideal sheaves fitting into an exact sequence

$$0 \to \mathcal{I}_L(-1) \to \mathcal{I}_C \to \mathcal{O}_V(-3) \to 0$$

have to be unions of lines with a plane cubic intersecting in one point. If $L \subset V$, then no such extension can be an ideal sheaf, since the line would intersect the cubic in three points giving the wrong genus. □

The next moduli space will acquire a second component.

PROPOSITION 4.8. *The third moduli space M_3 has two irreducible components M_3^1 and M_3^2. The first component M_3^1 is the blowup of M_2 in the smooth incidence variety parameterizing length two subschemes in a plane in $\mathbb{P}^3$. The second component M_3^2 is a $\mathbb{P}^{14}$-bundle over* $\mathrm{Hilb}^2(\mathbb{P}^3) \times (\mathbb{P}^3)^\vee$ *parameterizing pairs $(\mathcal{I}_Z(-1), \mathcal{O}_V(-4))$. It generically parameterizes unions of plane quartics with two generic points in $\mathbb{P}^3$. The two components intersect transversally along the exceptional locus of the blowup. The only nonideal sheaves occur in the case where at least one of the two points is not scheme-theoretically contained in the plane.*

Proof. By Lemma 4.3 we have

$$\mathrm{Ext}^1(\mathcal{I}_Z(-1), \mathcal{O}_V(-4)) = \begin{cases} \mathbb{C}, & Z \subset V, \\ 0 & \text{otherwise}, \end{cases}$$
$$\mathrm{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_Z(-1)) = \mathbb{C}^{15}.$$

This means that the locus destabilized in M_2 is of dimension 7 and that the new locus appearing in M_3 is of dimension 23. Since M_2 is of dimension 16, the locus appearing in M_3 must be a new component M_3^2. The closure of what is left of M_2 is denoted by M_3^1. If M_3^2 is reduced, then it is a $\mathbb{P}^{14}$-bundle over $\mathrm{Hilb}^2(\mathbb{P}^3) \times (\mathbb{P}^3)^\vee$ parameterizing pairs $(\mathcal{I}_Z(-1), \mathcal{O}_V(-4))$. We will more strongly show that it is smooth.

Suppose Z is not scheme theoretically contained in V. Then Lemma 4.4 implies that any nontrivial extension E in $\mathrm{Ext}^1(\mathcal{O}_V(-4), \mathcal{I}_Z(-1))$ satisfies $\mathrm{ext}^1(E,E) \leq 23$. Therefore, it is a smooth point and can in particular not lie on M_3^1. Let E be an extension of the form $0 \to \mathcal{I}_Z(-1) \to E \to \mathcal{O}_V(-4) \to 0$, where $Z \subset V$. Any point on the intersection must satisfy $\mathrm{ext}^1(E,E) \geq 24$. Suppose E is not an ideal sheaf. If E fits into an exact sequence $0 \to \mathcal{I}_{Z/V}(-4) \to E \to \mathcal{O}(-1) \to 0$ or $0 \to \mathcal{I}_{Q/V}(-4) \to E \to \mathcal{I}_P(-1) \to 0$ for $P \neq Q$, then a direct application of Lemma 4.4 to these sequences shows $\mathrm{ext}^1(E,E) \leq 23$, a contradiction. Therefore, E must fit into an exact sequence $0 \to \mathcal{I}_{P/V}(-4) \to E \to \mathcal{I}_P(-1) \to 0$. Then we have the following commutative diagram with short exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & \mathcal{I}_{P/V}(-4) \arrow[r, two heads] \arrow[d, hook] & \mathcal{I}_{P/V}(-4) \arrow[d, hook] \\
\mathcal{I}_Z(-1) \arrow[r, hook] \arrow[d] & E \arrow[r, two heads] \arrow[d] & \mathcal{O}_V(-4) \arrow[d] \\
\mathcal{I}_Z(-1) \arrow[r, hook] & \mathcal{I}_P(-1) \arrow[r, two heads] & \mathcal{O}_P
\end{tikzcd}$$

Therefore, Z has to be a double point supported at P. By Lemma 4.3 we have

$$\operatorname{Ext}^1(\mathcal{I}_Z(-1), \mathcal{I}_{P/V}(-4)) = \mathbb{C}^3, \qquad \operatorname{Ext}^1(\mathcal{O}_V(-4)), \mathcal{I}_{P/V}(-4)) = \mathbb{C}^2,$$
$$\operatorname{Ext}^1(\mathcal{I}_Z(-1), \mathcal{I}_P(-1)) = \mathbb{C}^3, \qquad \operatorname{Ext}^1(\mathcal{O}_V(-4)), \mathcal{I}_P(-1)) = \mathbb{C}^{15}.$$

Next, we apply $\operatorname{Hom}(\cdot, \mathcal{I}_{P/V}(-4))$ to $0 \to \mathcal{I}_Z(-1) \to E \to \mathcal{O}_V(-4) \to 0$ to get $\operatorname{ext}^1(E, \mathcal{I}_{P/V}(-4)) \leq 5$. By applying $\operatorname{Hom}(\cdot, \mathcal{I}_P(-1))$ to the same sequence, we get $\operatorname{ext}^1(E, \mathcal{I}_P(-1)) \leq 18$. Finally, we can apply $\operatorname{Hom}(E, \cdot)$ to $0 \to \mathcal{I}_{P/V} \to E \to \mathcal{I}_P(-1) \to 0$ to get $\operatorname{ext}^1(E, E) \leq 23$.

Therefore, the intersection of M_3^1 and M_3^2 parameterizes some of the ideals fitting into an exact sequence $0 \to \mathcal{I}_Z(-1) \to I_C \to \mathcal{O}_V(-4) \to 0$, where $Z \subset V$. The intersection must have a closed orbit. By Proposition 3.1, there is precisely one such closed orbit. If the intersection was disconnected, then it would have at least two closed orbits. If it is reducible, then the closed orbit must lie on the intersection of all irreducible components. By Proposition 3.2 the intersection along the closed orbit is transversal of dimension 15, and its points are smooth on both components. That would be impossible if the intersection is not irreducible at the closed orbit. The singular locus on either component is closed and must therefore contain a closed orbit. Thus, the whole intersection must consist of points that are smooth on each of the two components individually. The induced map $M_3^1 \to M_2$ contracts the intersection, which is an irreducible divisor, onto a locus isomorphic to the smooth incidence variety parameterizing length two subschemes in a plane in $\mathbb{P}^3$. Theorem 4.6 implies the description of M_3^1.

The description of the curves parameterized by M_3^2 is again a consequence of the exact sequence that the ideal sheaves fit into. □

To reprove the description of the main component of the Hilbert scheme from [VA92], we have to make sure that none of the remaining walls modifies the first component.

Proposition 4.9. *The fourth moduli space M_4 has two irreducible components M_4^1 and M_4^2. The first component is equal to M_3^1. The second component is birational to M_3^2.*

Proof. Lemma 4.3 says that

$$\operatorname{Ext}^1(\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4)) = \begin{cases} \mathbb{C}^3, & P = Q, \\ \mathbb{C}, & P \neq Q, \end{cases}$$

$$\mathrm{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_P(-1)) = \begin{cases} \mathbb{C}^{17}, & P = Q, \\ \mathbb{C}^{15}, & P \neq Q. \end{cases}$$

Moreover, the moduli space of pairs $(\mathcal{I}_P(-1), \mathcal{I}_{Q/V}(-4))$ is irreducible of dimension 8, whereas the sublocus where $P = Q$ is of dimension 5. Therefore, the closure of the locus of extensions in $\mathrm{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_P(-1))$ for $P \neq Q$ is irreducible of dimension 22. The locus of extensions in $\mathrm{Ext}^1(\mathcal{I}_{P/V}(-4), \mathcal{I}_P(-1))$ for $P \in V$ is irreducible of dimension 21. Let M_4^1 be the closure of what is left from M_3^1 in M_4, and let M_4^2 be the closure of what is left from M_3^2.

If $P \neq Q$, then Lemma 4.4 implies smoothness. In particular, we can use Lemma 4.5 to show that all points in $\mathrm{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_P(-1))$ for $P \neq Q$ are in M_4^2 and no other component. Suppose we have a general nontrivial extension $0 \to \mathcal{I}_P(-1) \to E \to \mathcal{I}_{P/V}(-4) \to 0$. Then $E = I_C$ is an ideal sheaf of a plane quartic curve plus a double point in the plane. We can assume that the double point is not an embedded point due to the fact that E is general. Clearly, I_C is the flat limit of elements in $\mathrm{Ext}^1(\mathcal{I}_{Q/V}(-4), \mathcal{I}_P(-1))$ by choosing $P \notin V$ and regarding the limit as $P \to Q$. Therefore, E is a part of M_4^2.

We showed that $M_4 = M_4^1 \cup M_4^1$ and that M_4^2 is birational to M_3^2. We are left to show that $M_4^1 = M_4^2$. If not, there is an object E with a nontrivial exact sequence $0 \to \mathcal{I}_P(-1) \to E \to \mathcal{I}_{P/V}(-4) \to 0$ in M_4^1. By Lemma 4.5 this implies that there is also an object E' with nontrivial exact sequence $0 \to \mathcal{I}_{P/V}(-4) \to E' \to \mathcal{I}_P(-1) \to 0$ lying on M_3^1. But we already established that all those extensions are smooth points on M_3^2 in the previous proof. $\square$

We can now prove the following theorem.

THEOREM 4.10. *The Hilbert scheme* $\mathrm{Hilb}^{4t}(\mathbb{P}^3)$ *has two components* Hilb_1^{4t} *and* Hilb_2^{4t}. *The main component* Hilb_1^{4t} *contains an open subset of elliptic quartic curves and is a smooth double blowup of the Grassmannian* $\mathbb{G}(1,9)$. *The second component is of dimension* 23. *Moreover, the two components intersect transversally in a locus of dimension* 15.

Proof. By Lemma 4.3 we have

$$\mathrm{Ext}^1(\mathcal{O}(-1), \mathcal{I}_{Z'/V}(-4)) = \mathbb{C}^2,$$
$$\mathrm{Ext}^1(\mathcal{I}_{Z'/V}(-4), \mathcal{O}(-1)) = \mathbb{C}^{15},$$
$$\mathrm{Ext}^1(\mathcal{I}_{Z'/V}(-4), \mathcal{I}_{Z'/V}(-4)) = \mathbb{C}^7.$$

The moduli space of objects $\mathcal{I}_{Z'/V}$ is irreducible of dimension 5. Lemma 4.4 implies that all strictly semistable objects at the largest wall are smooth points on either M_4 or $M_5 = \mathrm{Hilb}^{4t}(\mathbb{P}^3)$. Therefore, we can again use Lemma 4.5 to see that $\mathrm{Hilb}^{4t}(\mathbb{P}^3)$ has exactly two components birational to M_4^1 and M_4^2. Moreover, this argument shows that the ideals that destabilize at the largest wall cannot lie on the intersection of the two components, and we have $M_5^1 = M_4^1$. □

We denote the exceptional divisor of the first blowup of the main component by E_1 and the exceptional divisor of the second blow up by E_2. We finish this section by describing which curves in fact lie in E_1 and E_2.

Proposition 4.11. *The general point in E_1 parameterizes subschemes of $\mathbb{P}^3$ that are the union of a plane cubic and an incident line. The general point in E_2 parameterizes subschemes of $\mathbb{P}^3$ that are plane quartics with two nodes and two embedded points at such nodes.*

Proof. By Corollary 4.1, any ideal sheaf I_C of a scheme in E_1 fits into an exact sequence of the form $0 \to \mathcal{I}_L(-1) \to \mathcal{I}_C \to \mathcal{O}_V(-3) \to 0$, where $L \subset \mathbb{P}^3$ is a line, and $V \subset \mathbb{P}^3$ is a plane. By Proposition 4.7 the reverse holds, that is, all ideal sheaves fitting into this sequence correspond to curves in E_1. For the general element in E_1, the line L is not contained V. Then the above sequence implies that $C \subset L \cup V$. If $C \subset V$, then there would be a morphism $\mathcal{O}(-1) \to \mathcal{I}_C$ destabilizing the curve earlier, a contradiction. Thus, L is an irreducible component of C, and another component of degree 3 lies in V.

By Theorem 4.10, the last two walls do not modify the main component. Therefore, Corollary 4.1 implies that any ideal sheaf I_C of a scheme in E_2 fits into an exact sequence of the form $0 \to \mathcal{I}_Z(-1) \to \mathcal{I}_C \to \mathcal{O}_V(-4) \to 0$, where $Z \subset \mathbb{P}^3$ is a zero-dimensional subscheme of length 2, and $V \subset \mathbb{P}^3$ is a plane. This implies that C is plane quartic curve plus two points. The two points have to be embedded, since otherwise the curve cannot be smoothened. Moreover, a classical result by Hironaka [Hir58, p. 360] implies that the two embedded points must occur at singularities of the plane quartic. □

5. Effective Divisors of the Principal Component Hilb_1^{4t}

In this section, we compute the cone of effective divisors $\mathrm{Eff}(\mathrm{Hilb}_1^{4t})$, where Hilb_1^{4t} denotes the principal component of the Hilbert scheme $\mathrm{Hilb}^{4t}(\mathbb{P}^3)$. By Theorem B, there is an isomorphism $\mathrm{Pic}(\mathrm{Hilb}_1^{4t}) \cong \mathbb{Z}^3$ with generators H, E_1, and E_2. Here, H denotes the pullback of the class $\sigma_1 \in A^1(\mathbb{G}(1,9))$, whereas E_1 is the exceptional divisor of the first blowup, and E_2 is the exceptional divisor of the second blowup. Set-theoretically, E_1 is the closure in Hilb_1^{4t} of the locus parameterizing subschemes of $\mathbb{P}^3$ that consist of a smooth plane cubic with an incident line. Moreover, E_2 is the closure in Hilb_1^{4t} of the locus parameterizing plane quartics with two nodes and two embedded points at such nodes.

As a consequence of Theorem B, we also have that $\mathrm{Pic}(\mathrm{Hilb}_1^{4t}) \otimes \mathbb{Q} \cong N^1(\mathrm{Hilb}_1^{4t}) \otimes \mathbb{Q}$, where $N^1(\mathrm{Hilb}_1^{4t}) \otimes \mathbb{Q}$ denotes the Néron–Severi group of Cartier divisors with rational coefficients up to numerical equivalence.

To describe the cone of effective divisors $\mathrm{Eff}(\mathrm{Hilb}_1^{4t})$, we need an additional divisor Δ defined as the closure of the locus of irreducible nodal elliptic quartics.

THEOREM C. *The cone of effective divisors of* Hilb_1^{4t} *is generated by* $\mathrm{Eff}(\mathrm{Hilb}_1^{4t}) = \langle \Delta, E_1, E_2 \rangle$.

The strategy of the proof is to construct a dual basis of curves to Δ, E_1, and E_2 in $N_1(\mathrm{Hilb}_1^{4t})$, the space of 1-cycles up to numerical equivalence. Since the closure of the convex cone of movable curves is dual to the effective cone, we will then observe that these curves are movable, which allows us to conclude the proof. In our context, a curve in a smooth algebraic variety X is called *movable* if it lies in a family that covers a dense open subset of X. We refer the reader to [B+13] for a detailed exposition on movable curves.

Before proceeding with the proof, we will define and describe some properties of our movable curves. Let $Q \subset \mathbb{P}^3$ be a a fixed smooth quadric. Then, the curve γ_1 is a general pencil in $|\mathcal{O}_Q(2)|$. This curve is movable because a generic curve parameterized by Hilb_1^{4t} is the transversal intersection of two quadric hypersurfaces Q_1, Q_2 where we can assume that one of these quadrics is smooth. Moreover, by construction $\gamma_1 \cdot E_1 = \gamma_1 \cdot E_2 = 0$.

On the other hand, the intersection $\gamma_1 \cdot \Delta = 12$ holds. This follows from the fact that the parameter space of plane curves of degree d in $\mathbb{P}^2$ contains a divisor of degree $3(d-1)^2$ of singular curves (see [GKZ08, Ch 13.D]). The following geometric argument is self-contained.

The base locus of a general pencil in $|\mathcal{O}_Q(2)|$, where Q stands for a smooth quadric, consists of 8 points. This means that the total space of this pencil $\mathcal{X}$ is the blowup of Q on these 8 points, and this implies that its topological Euler characteristic $\chi_{\mathrm{top}}(\mathcal{X}) = 12$. Observe that the pencil $\mathcal{X}$ is not a fibration over $\mathbb{P}^1$ due to the presence of singular fibers: if $\mathcal{X}$ were a fibration over $\mathbb{P}^1$, then the topological Euler characteristic $\chi_{\mathrm{top}}(\mathcal{X})$ would be zero. This means that we may count the singular fibers of $\mathcal{X}$ (which are the singular members of the pencil) by computing the topological Euler characteristic $\chi_{\mathrm{top}}(\mathcal{X})$. Since we are considering a general pencil, Bertini's theorem guarantees that the singular fibers of $\mathcal{X}$ are all nodal curves.

We now define two more curves γ_2 and γ_3. Let Λ_1 and Λ_2 be two 3-planes in $\mathbb{P}^7$. Let $s : \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^7$ be the Segre embedding, and for any $t \in \mathbb{P}^1$, we write $s_t : \mathbb{P}^3 \to \mathbb{P}^7$ for the restriction of s to $\mathbb{P}^3 \times \{t\}$. We have a projection $\pi : \mathbb{P}^7 \backslash \Lambda_1 \to \Lambda_2$. To summarize, we have the following diagram of maps with vertical projections:

$$\begin{tikzcd}
\mathbb{P}^3 \times \mathbb{P}^1 \arrow[r, "s \times \mathrm{id}"] \arrow[d] & \mathbb{P}^7 \times \mathbb{P}^1 \arrow[r, dashed, "\pi \times \mathrm{id}"] \arrow[d] & \Lambda_2 \times \mathbb{P}^1 \arrow[d] \\
\mathbb{P}^3 \arrow[r, "s_t"] & \mathbb{P}^7 \arrow[r, dashed, "\pi"] & \Lambda_2 \cong \mathbb{P}^3.
\end{tikzcd}$$

Observe that both s_t and π are linear maps.

LEMMA 5.1. *Let $t \in \mathbb{P}^1$, and let Λ_2 be general. If $\Lambda_1 \cap s_t(\mathbb{P}^3) = \emptyset$, then $\pi \circ s_t$ is an isomorphism. If $\Lambda_1 \cap s_t(\mathbb{P}^3)$ is a point, then the image of $\pi \circ s_t$ is a plane in Λ_2.*

Proof. The image of $\pi \circ s_t$ is the intersection of Λ_2 with the linear subspace generated by Λ_1 and $s_t(\mathbb{P}^3)$. □

The image of the Segre embedding $s(\mathbb{P}^3 \times \mathbb{P}^1)$ has degree four. Hence, Λ_1 can be chosen general such that it intersects the Segre embedding in exactly four points. If we also choose Λ_2 general, then by Lemma 5.1 we have that $\pi \circ s_t : \mathbb{P}^3 \to \Lambda_2 \cong \mathbb{P}^3$ is an isomorphism except for four values.

DEFINITION 5.2. Let E be a smooth elliptic quartic in $\mathbb{P}^3$. Let Λ_2 be a general 3-plane in $\mathbb{P}^7$.

(1) Let Λ_1 be another general 3-plane in $\mathbb{P}^7$. Then γ_2 is the image $(\pi \times \mathrm{id}) \circ (s \times \mathrm{id})(E \times \mathbb{P}^1)$. It is a flat family of smooth curves isomorphic to E everywhere, except for four special fibers.
(2) Consider four general points in $s(E \times \mathbb{P}^1)$ and let Λ_1' be the unique 3-plane generated by them. Then γ_3 is the image $(\pi \times \mathrm{id}) \circ (s \times \mathrm{id})(E \times \mathbb{P}^1)$. It is a flat family of smooth curves isomorphic to E everywhere except for four special fibers.

LEMMA 5.3. *The four singular fibers for γ_2 are plane quartic curves with only two nodes and embedded points at them. For γ_3, these four fibers are smooth plane cubic curves together with an incident line. Both γ_2 and γ_3 are movable.*

Proof. Let $t \in \mathbb{P}^1$ correspond to one of the four singular fibers of γ_2. Since Λ_1 is chosen general, it will not intersect $s(E \times \mathbb{P}^1)$. Therefore, Lemma 5.1 implies that the image $\pi(s_t(E))$ is a plane curve. Since $\pi \circ s_t$ is defined on all of E, the set-theoretic support of γ_2 at t is a plane curve of degree four with 2 nodes and no other singularities. Hence, we get a plane quartic with two embedded points at the only 2 nodes.

Let $t \in \mathbb{P}^1$ correspond to one of the four singular fibers γ_3. By definition the intersection of Λ_1' with $E \times \mathbb{P}^1$ contains four points one of which is of the form (x, t). Choose a plane $\mathbb{P}^2 \subset \Lambda_1'$ that does not intersect the Segre embedding $s(\mathbb{P}^3 \times \mathbb{P}^1)$ and a general $\mathbb{P}^4 \subset \mathbb{P}^7$. Then the projection of $s_t(\mathbb{P}^3)$ away from $\mathbb{P}^2$ onto $\mathbb{P}^4$ is the intersection of this $\mathbb{P}^4$ with the linear span of $s_t(\mathbb{P}^3)$ and $\mathbb{P}^2$, which is $\mathbb{P}^6$. In particular, it is of dimension 3, that is, E is projected isomorphically onto $\mathbb{P}^3 \subset \mathbb{P}^4$. Let $P \in \mathbb{P}^4$ be the image of (x, t) via this projection. Then we project from this point onto a general $\Lambda_2 \subset \mathbb{P}^4$. The image is isomorphic to E. Hence, we get in Hilb_1^{4t} a smooth plane cubic together with an incident line.

Both curve classes γ_2 and γ_3 are movable. Indeed, every smooth curve parameterized in Hilb_1^{4t} is contained in some representative of γ_2 and γ_3 by varying the curve E used to define them. □

Proof of Theorem C. Since E_1, E_2, and Δ are effective, we only need to show the containment $\mathrm{Eff}(\mathrm{Hilb}_1^{4t}) \subset \langle E_1, E_2, \Delta\rangle$. Observe that this latter containment is equivalent to the containment $\langle E_1, E_2, \Delta\rangle^\vee \subset \mathrm{Eff}(\mathrm{Hilb}_1^{4t})^\vee$ of dual cones. Since $\mathrm{Eff}(\mathrm{Hilb}_1^{4t})^\vee$ is the cone of movable curves, it suffices to exhibit that the dual cone $\langle E_1, E_2, \Delta\rangle^\vee$ is generated by movable curves. We already proved that $\gamma_1, \gamma_2, \gamma_3$ are movable curves. This means we are left to show that they generate the dual cone $\langle E_1, E_2, \Delta\rangle^\vee$. It suffices to check that the following intersection numbers hold. Note that, for our purposes, it is enough to show that the intersections are zero or positive, and therefore, we will skip proving that the intersections are transversal.

$$\begin{array}{lll} \gamma_1 \cdot E_1 = 0, & \gamma_2 \cdot E_1 = 0, & \gamma_3 \cdot E_1 = 4, \\ \gamma_1 \cdot E_2 = 0, & \gamma_2 \cdot E_2 = 4, & \gamma_3 \cdot E_2 = 0, \\ \gamma_1 \cdot \Delta = 12, & \gamma_2 \cdot \Delta = 0, & \gamma_3 \cdot \Delta = 0. \end{array}$$

The intersections with E_1 and E_2 follow directly from the definitions and Lemma 5.3. The intersection number $\gamma_1 \cdot \Delta = 12$ is also discussed previously. We are left to show that $\gamma_2 \cdot \Delta = \gamma_3 \cdot \Delta = 0$.

Suppose $\gamma_2 \cdot \Delta \neq 0$. Then there is a flat family $\pi : S \to Z$ for a smooth curve Z such that, for general $z \in Z$, the fiber S_z is a nodal complete intersection in Δ and that the special fiber S_0 is a curve in $\gamma_2 \cap E_2$. Therefore, S_0 is a plane quartic curve with exactly two nodes and simple embedded points at both nodes. The normalization $\tilde{S}$ smooths out the nodes in the general fibers by making them into $\mathbb{P}^1$. By [Bea96, Theorem III.7] this means that $\tilde{S}$ is birational over Z to $\mathbb{P}^1 \times Z$. We can resolve the rational map from $\mathbb{P}^1 \times Z$ to S by successively blowing up points. That leads to a family $X \to Z$ factoring through $S \to Z$ such that every fiber is a union of rational curves $\mathbb{P}^1$. That means that the special fiber S_0 is the set-theoretic image of such a union of rational curves. Every $\mathbb{P}^1$ must map to the normalization of the reduced structure of S_0. But the normalization of the reduced structure of S_0 is a smooth curve of genus 1, and $\mathbb{P}^1$ has no nontrivial maps to an elliptic curve.

Suppose $\gamma_3 \cdot \Delta \neq 0$. Then there is a flat family $\pi : S \to Z$ for a smooth curve Z such that, for general $z \in Z$, the fiber S_z is a nodal complete intersection in Δ, and the special fiber S_0 is a curve in $\gamma_3 \cap E_1$. This means S_0 is the union of a smooth plane cubic with an incident line. With the exact same argument as for γ_2, we can create a family $X \to Z$ factoring through $S \to Z$ such that every fiber is a union of rational curves $\mathbb{P}^1$. As previously, the special fiber S_0 is the image of such a union of rational curves. Since there is no nontrivial map from $\mathbb{P}^1$ to any elliptic curve, they must all map to the incident line, a contradiction. $\square$

A. Macaulay2 Code

This appendix contains all Macaulay2 code used in Proposition 3.2 and Lemma 4.3.

```
----------------------------------------
-- Computation for Proposition 3.2 --
----------------------------------------
needsPackage "VersalDeformations";
S = QQ[x,y,z,w];
F0 = matrix {{x^2,x*y,x*z^2,y^4}};
(F,R,G,C) = localHilbertScheme(F0,Verbose=>2);
T = ring first G;
sum G
----------------------------------
-- Computation for Lemma 4.3 --
----------------------------------
-- A = O_V(-3)
-- B = I_L(-1) L is not contained in V
-- C = I_L(-1) L is contained in V
X = Proj(QQ[x,y,z,w]);
A = (OO_X(0) / sheaf module ideal(x))**OO_X(-3);
B = (sheaf module ideal (y,z))**OO_X(-1);
C = (sheaf module ideal (x,y))**OO_X(-1);
Ext^1(B,A)
Ext^1(C,A)
Ext^1(A,B)
Ext^1(A,C)
Ext^1(A,A)
Ext^1(B,B)
Ext^1(C,C)
----------------------------------
-- A = O_V(-4)
-- B = I_Z(-1) Two separate points
--              outside V
-- C = I_Z(-1) Double point outside V
-- D = I_Z(-1) One point inside,
--              one point outside V
-- E = I_Z(-1) Two separate points
--              inside V
-- F = I_Z(-1) Double point scheme
--              theoretically in V
-- G = I_Z(-1) Double point set but
--              not scheme theoretically in V
X = Proj(QQ[x,y,z,w]);
A = (OO_X(0) / sheaf module ideal(x))**OO_X(-4);
B = (sheaf module ideal (y*(x-y),z,w))**OO_X(-1);
C = (sheaf module ideal (y^2,z,w))**OO_X(-1);
D = (sheaf module ideal (x*y,z,w))**OO_X(-1);
E = (sheaf module ideal (x,y*z,w))**OO_X(-1);
F = (sheaf module ideal (x,y,z^2))**OO_X(-1);
G = (sheaf module ideal (x^2,y,z))**OO_X(-1);
Ext^1(A,B)
Ext^1(A,C)
Ext^1(A,D)
Ext^1(A,E)
Ext^1(A,F)
Ext^1(A,G)
Ext^1(B,A)
Ext^1(C,A)
Ext^1(D,A)
Ext^1(E,A)
Ext^1(F,A)
Ext^1(G,A)
Ext^1(A,A)
Ext^1(B,B)
Ext^1(C,C)
----------------------------------
-- A = I_{Q/V}(-4)
-- B = I_P(-1) P \notin V
-- C = I_P(-1) P \in V, P \neq Q
-- D = I_P(-1) P = Q
X = Proj(QQ[x,y,z,w]);
A = (sheaf module ideal (x,y,z)
    / sheaf module ideal(x))**OO_X(-4);
B = (sheaf module ideal (y,z,w))**OO_X(-1);
C = (sheaf module ideal (x,y,w))**OO_X(-1);
D = (sheaf module ideal (x,y,z))**OO_X(-1);
Ext^1(A,B)
Ext^1(A,C)
Ext^1(A,D)
Ext^1(B,A)
Ext^1(C,A)
Ext^1(D,A)
Ext^1(A,A)
Ext^1(B,B)
Ext^1(D,D)
----------------------------------
-- A = O(-1)
-- B = I_{Z'/V}(-4) Two separate points
-- C = I_{Z'/V}(-4) Double point
X = Proj(QQ[x,y,z,w]);
A = OO_X(-1);
B = (sheaf module ideal (x,y,z^2)
    / sheaf module ideal(x))**OO_X(-4);
C = (sheaf module ideal (x,y,w*z)
    / sheaf module ideal(x))**OO_X(-4);
Ext^1(B,A)
Ext^1(C,A)
Ext^1(A,B)
Ext^1(A,C)
Ext^1(B,B)
Ext^1(C,C)
----------------------------------
-- A = I_Z(-1), Z \subset V double point at P
-- B = O_V(-4)
-- C = I_{P/V}(-4)
-- D = I_P(-1)
X = Proj(QQ[x,y,z,w]);
A = (sheaf module ideal(x,y,z^2))**OO_X(-1);
B = (OO_X(0) / sheaf module ideal(x))**OO_X(-4);
C = (sheaf module ideal(x,y,z)
    / sheaf module ideal(x))**OO_X(-4);
D = (sheaf module ideal(x,y,z))**OO_X(-1);
Ext^1(A,C)
Ext^1(B,C)
Ext^1(A,D)
Ext^1(B,D)
```

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