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Gromov–Witten Invariants of the Hilbert Scheme of Two Points on a Hirzebruch Surface

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ABSTRACT. In Gromov–Witten theory the virtual localization method is used only when the invariant curves are isolated under a torus action. In this paper, we explore a strategy to apply the localization formula to compute the Gromov–Witten invariants by carefully choosing the related cycles to circumvent the continuous families of invariant curves when there are any. For the example of the two-pointed Hilbert scheme of Hirzebruch surface F_1, we manage to compute some Gromov–Witten invariants, and then by combining with the associativity law of (small) quantum cohomology ring, we succeed in computing all 1- and 2-pointed Gromov–Witten invariants of genus 0 of the Hilbert scheme with the help of [13].

1. Introduction

In the 1990s, enumerative geometry experienced a big impetus from the work by physicists, in which they calculated the number of rational curves of arbitrary degree in a quintic threefold. To explain their work and verify their results, Gromov–Witten theory was established. Now this subject finds applications in many areas of mathematics, for example, enumerative geometry, the theory of singularities, integrable systems, to name just a few.

One of the big problems in Gromov–Witten theory since its inception is how to compute the Gromov–Witten invariants. Generally, this is a very hard problem. Some techniques have been developed to attack the problem, for example, the degeneration method, finding a mirror model to reduce the problem to period calculations, and so on. When the spaces enjoy much symmetry, the computation problem can be dealt with more easily. Especially when there is a torus action on the target space, computations can be carried out by the virtual localization method [10], which is a modification of the usual topological localization formula. This is the main technique adopted in this work.

Gromov–Witten invariants can be wrapped up as the quantum product on the cohomology ring of the target space. See [4; 8] for references. This quantum product is associative and supercommutative, so that it makes the cohomology ring into a new ring called the quantum cohomology ring. This new ring structure is a deformation of the cohomology ring of the space. If unraveled properly, the associativity of the quantum product exhibits very strong relations among Gromov–Witten invariants. These relations can be more efficiently

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utilized when combined with other methods for the computations of the invariants.

From the early stage on, Gromov–Witten invariants and quantum cohomology have been computed for spaces with strong symmetry; see, for example, [1; 2; 3]. There are other cases where quantum cohomology is successfully decided for spaces with weaker symmetry. For example, Okounkov and Pandharipande [15] determine the ring structure of the equivariant quantum cohomology of the Hilbert schemes of points on the plane together with its relations to other areas of mathematics; Graber [9] and Pontoni [16] compute Gromov–Witten invariants and describe the quantum cohomology of the Hilbert scheme of two points on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ and study their enumerative applications.

In this work, we consider the Hilbert scheme of two points on a Hirzebruch surface. The Hilbert scheme is of dimension four, admitting a two-dimensional torus action, which is not regarded as sufficient amount of symmetry for standard application of the localization method. In fact, the condition for complete application of the localization formula is quite strict: the invariant curves under the torus action have to be isolated. This happens when the target space admits a torus action of high enough dimension. When the invariant curves occur as continuous families, it is not known how to carry out in general the localization process. The Hilbert scheme we study exhibits continuous families of invariant curves in addition to isolated invariant curves. To attack the problem, we adopt the following strategy. We make use of the property of this space that it admits different invariant subvarieties to represent relevant cohomology classes by the Poincaré duality and thus by carefully choosing representing cycles to avoid the continuous families of invariant curves we are able to implement the standard localization procedure. This strategy allows us to compute all one-pointed and some two-pointed Gromov–Witten invariants of the space with the help of [13]. These results suffice for us to compute the quantum product of generators of the cohomology ring of the Hilbert scheme. Then with the relations from the associativity of the (small) quantum cohomology ring, we can calculate all two-pointed GW-invariants. If we work harder, then we can even determine the (small) quantum cohomology ring of the Hilbert scheme [6].

The structure of this paper is as follows. In Section 2, we first introduce the necessary background for the Hilbert scheme. This includes its blowup construction and thus its cohomology ring. We determine the weights at fixed points and the invariant curves of the two-dimensional torus action. After these preparatory work, we start out to compute one-pointed Gromov–Witten invariants and some two-pointed invariants in Section 3. In Section 4, we compute the quantum product for the generators of the cohomology ring, and then with the associativity law of the quantum product we determine all other two-pointed Gromov–Witten invariants.

2. Background

2.1. The Hilbert Scheme of Two Points over an Algebraic Surface

For the definition and structure of Hilbert schemes of points over smooth algebraic surfaces, we refer the reader to [14]. Since the spaces we are going to deal with admit torus actions with only finitely many fixed points, their odd degree homology and cohomology groups vanish, so we adopt the convention $A_k(X) = H_{2k}(X)$ and $A^k(X) = H^{2k}(X)$. By the Poincaré duality, $A^k(X) = A_{n-k}(X)$ for $n = \dim_{\mathbb{C}} X$, and thus we go back and forth between homology and cohomology groups. To decide the cohomology groups of the two-pointed Hilbert scheme, we first recall its blowup construction.

Let $\Delta\colon S \to S \times S$ be the diagonal, and $\widetilde{S \times S}$ be the blowup of $S \times S$ along Δ with the exceptional divisor $P(TS)$, the projectivization of the tangent bundle of S. The group $\mathbb{Z}_2$ acts on $S \times S$ by switching the points, which fixes the diagonal, so automatically inducing an involution on the blowup, which fixes the exceptional divisor. The Hilbert scheme $S^{[2]}$ is the quotient of $\widetilde{S \times S}$ under this involution.

This blowup construction can be described in the fiber square

$$\begin{array}{ccc} P(TS) & \xrightarrow{\ j\ } & \widetilde{S \times S} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\ \Delta\ } & S \times S, \end{array}$$

where j is the inclusion, and f, g are the projections.

Let $\phi\colon \widetilde{S \times S} \to S^{[2]}$ be the quotient, and let $A_*(\widetilde{S \times S})^{\mathbb{Z}_2}$ be the fixed subgroup of $A_*(\widetilde{S \times S})$ of the involution. By Example 1.7.6 in [7], there is a canonical isomorphism $\phi^*\colon A_*(S^{[2]}) \to A_*(\widetilde{S \times S})^{\mathbb{Z}_2}$, where for a subvariety $V \subset S^{[2]}$, $\phi^*[V] = \sum e_W[W]$, the sum over all irreducible components of $\phi^{-1}(V)$, and $e_W = \#\{g \in \mathbb{Z}_2 : g|_W = \mathrm{id}_W\}$. We have the identities

$$\phi^*\phi_* = 2\,\mathrm{id}, \qquad \phi_*\phi^* = 2\,\mathrm{id}\,.$$

Furthermore, the intersection products are related by $x \cdot y = \frac{1}{2}\phi_*(\phi^*x \cdot \phi^*y)$ for $x, y \in A_*(S^{[2]})$. See Example 8.3.12 in [7].

Let T be the tautological bundle $\mathcal{O}(-1)$ on $P(TS)$, and let $E = g^*(TS)/T$. Because f is a local complete intersection morphism of relative dimension zero, the Gysin map $f^*\colon A_*(S \times S) \to A_*(\widetilde{S \times S})$ is well-defined. From Proposition 6.7 and Example 8.3.9 in [7] we have the following:

Proposition 2.1. *There are split exact sequences*

$$0 \to A_k(S) \xrightarrow{\ \alpha\ } A_k(P(TS)) \oplus A_k(S \times S) \xrightarrow{\ \beta\ } A_k(\widetilde{S \times S}) \to 0$$

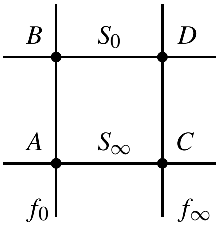


Figure 1 Special Divisors in F_a

with $\alpha(x) = (c_1(E) \cap g^*x, -\Delta_* x)$ *and* $\beta(\widetilde{x}, y) = j_*\widetilde{x} + f^*y$. *A left inverse for* α *is given by* $(\widetilde{x}, y) \mapsto g_*(\widetilde{x})$. *The ring structure of* $A^*(\widetilde{S \times S})$ *is determined by the following rules*:

(i) $f^*y \cdot f^*y' = f^*(y \cdot y')$;
(ii) $j_*\widetilde{x} \cdot j_*\widetilde{x'} = j_*(c_1(T) \cdot \widetilde{x} \cdot \widetilde{x'})$;
(iii) $f^*y \cdot j_*\widetilde{x} = j_*((g^*\Delta^*y) \cdot \widetilde{x})$.

This proposition tells us that $A_k(\widetilde{S \times S})^{\mathbb{Z}_2}$ consists of two parts, coming from $A_k(S \times S)$ and $A_k(P(TS))$. The former part is given by the Gysin map f^*. From Theorem 6.7 in [7], the relation between f^* and the proper transform is described in the following:

PROPOSITION 2.2. *Let* V *be a* k*-dimensional subvariety in* $S \times S$, *and let* $\widetilde{V}$ *be the proper transform of* V *in* $\widetilde{S \times S}$. *Then*

$$f^*[V] = [\widetilde{V}] + j_*\{c(E) \cap g^*s(V \cap \Delta, V)\}_k$$

in $A_k(\widetilde{S \times S})$, *where* $\{\cdot\}_k$ *means the degree* k *part of a class. In particular, when* $\dim(V \cap \Delta) \leq k - 2$, *we have* $f^*[V] = [\widetilde{V}]$.

The latter part is presented in the following proposition. See P.606 [11].

PROPOSITION 2.3. *Let* $\zeta = c_1(T)$. *Then, as graded rings,*

$$A^*(P(TS)) = A^*(S)[\zeta]/(\zeta^2 - c_1(TS)\zeta + c_2(TS)).$$

Now we specialize to the Hirzebruch surface $F_a = \operatorname{Proj} S^\bullet\mathcal{F}$ with projection $\pi : F_a \to \mathbb{P}^1$ for $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$, $a \geq 0$. The projection $\mathcal{F} \to \mathcal{O}_{\mathbb{P}^1}(-a)$ determines a section of π, which is denoted as S_∞; the projection $\mathcal{F} \to \mathcal{O}_{\mathbb{P}^1}$ determines another section of π, denoted as S_0. Note that the twisting sheaf $\mathcal{O}_{F_a}(1) \cong \mathcal{L}(S_\infty)$ [12]. We call the points on S_∞ over $(0, 1), (1, 0) \in \mathbb{P}^1$ as A and C, the points on S_0 over $(0, 1), (1, 0) \in \mathbb{P}^1$ as B and D, respectively. We denote the fibers over $(0, 1)$ and $(1, 0)$ by f_0 and f_∞, respectively. These special divisors and their geometric configuration are illustrated in Figure 1.

There are numerical equivalence relations $S_0 \equiv S_\infty + af$ and $K \equiv -2S_0 - (2-a)f$ for the canonical divisor class K. Also, any two fibers of π are numerically equivalent. So $A_0(F_a) = \mathbb{Z}pt$, $A_1(F_a) = \mathbb{Z}S \oplus \mathbb{Z}f$, and $A_2(F_a) = \mathbb{Z}F_a$, where pt is any point, S means either one of S_0 and S_∞, and f means either one of f_0 and f_∞. We keep the freedom to choose whichever one of them is needed later. The intersection products of the generators are as listed here:

$$S_0^2 = a, \qquad S_\infty^2 = -a, \qquad S_0 \cdot S_\infty = 0, \qquad f^2 = 0, \qquad S \cdot f = 1.$$

From Proposition 2.3, $A_*(P(TF_a))$ consist of two parts, the pullbacks of the homology classes of F_a under g and the cap products of ζ with these pullbacks, which can be simplified by the following:

LEMMA 2.4. *We have the following relations among various classes*:

(1) $P(TF_a|_{S_0}) = P(TF_a|_{S_\infty}) + aP(TF_a|_f)$;
(2) $\zeta \cap P(TF_a|_{S_\infty}) = (2-a)P(TF_a|_{pt}) - c_1(E) \cap P(TF_a|_{S_\infty})$;
(3) $\zeta \cap P(TF_a|_{S_0}) = (2+a)P(TF_a|_{pt}) - c_1(E) \cap P(TF_a|_{S_0})$;
(4) $\zeta \cap P(TF_a|_f) = 2P(TF_a|_{pt}) - c_1(E) \cap P(TF_a|_f)$;
(5) $\zeta \cap P(TF_a) = 2P(TF_a|_{S_\infty}) + (2+a)P(TF_a|_f) - c_1(E) \cap P(TF_a)$.

Proof. (1) is obtained by pulling back the relation $S_0 \equiv S_\infty + af$. To prove (2), note that $c_1(g^*(TF_a)) = c_1(T) + c_1(E) = \zeta + c_1(E)$, so

$$\zeta \cap P(TF_a|_{S_\infty}) = c_1(g^*(TF_a)) \cap P(TF_a|_{S_\infty}) - c_1(E) \cap P(TF_a|_{S_\infty}).$$

But $c_1(g^*(TF_a)) \cap P(TF_a|_{S_\infty}) = g^*(2S_0 + (2-a)f) \cap P(TF_a|_{S_\infty}) = (2-a)P(TF_a|_f) \cap P(TF_a|_{S_\infty}) = (2-a)P(TF_a|_{pt})$, noting that $c_1(TF_a) = -K \equiv 2S_0 + (2-a)f$. This finishes (2). The remaining cases can be proven similarly. □

From this lemma, the homology groups of $P(TF_a)$ can be generated as follows:

$$A_0(P(TF_a)) = \mathbb{Z}pt;$$
$$A_1(P(TF_a)) = \mathbb{Z}P(TF_a|_{pt}) \oplus \mathbb{Z}c_1(E) \cap P(TF_a|_{S_\infty}) \oplus \mathbb{Z}c_1(E) \cap P(TF_a|_f);$$
$$A_2(P(TF_a)) = \mathbb{Z}P(TF_a|_S) \oplus \mathbb{Z}P(TF_a|_f) \oplus \mathbb{Z}c_1(E) \cap P(TF_a);$$
$$A_3(P(TF_a)) = \mathbb{Z}P(TF_a).$$

Also, we take generators for $A_k(F_a \times F_a)^{\mathbb{Z}_2}$ in the following way:

$$\begin{aligned}
A_0(F_a \times F_a)^{\mathbb{Z}_2} &= \mathbb{Z}pt;\\
A_1(F_a \times F_a)^{\mathbb{Z}_2} &= \mathbb{Z}(S_\infty \times pt + pt \times S_\infty) \oplus \mathbb{Z}(f \times pt + pt \times f);\\
A_2(F_a \times F_a)^{\mathbb{Z}_2} &= \mathbb{Z}(S_0 \times S_\infty + S_\infty \times S_0) \oplus \mathbb{Z}(f_0 \times f_\infty + f_\infty \times f_0)\\
&\qquad \oplus \mathbb{Z}(S_\infty \times f + f \times S_\infty) \oplus \mathbb{Z}(F_a \times pt + pt \times F_a);\\
A_3(F_a \times F_a)^{\mathbb{Z}_2} &= \mathbb{Z}(F_a \times S_\infty + S_\infty \times F_a) \oplus \mathbb{Z}(F_a \times f + f \times F_a);\\
A_4(F_a \times F_a)^{\mathbb{Z}_2} &= \mathbb{Z}(F_a \times F_a).
\end{aligned}$$

Under the identification of α in Proposition 2.1, $c_1(E)\cap P(TF_a|_{S_\infty})$ is identified with $\Delta_*(S_\infty)$, $c_1(E)\cap P(TF_a|_f)$ is identified with $\Delta_*(f)$, and $c_1(E)\cap P(TF_a)$ is identified with $\Delta_*(F_a)$. However, in $A_k(F_a\times F_a)^{\mathbb{Z}_2}$,

$$\begin{aligned}\Delta_*(S_\infty)&=S_\infty\times pt+pt\times S_\infty,\\ \Delta_*(f)&=f\times pt+pt\times f,\\ \Delta_*(F_a)&=F_a\times pt+pt\times F_a+S_\infty\times f+f\times S_\infty+\frac{a}{2}(f_0\times f_\infty+f_\infty\times f_0),\end{aligned}$$

so we can take the generators in $A_*(\widetilde{F_a\times F_a})^{\mathbb{Z}_2}$ either from $A_*(P(TF_a))$ via j_* or from $A_*(F_a\times F_a)^{\mathbb{Z}_2}$ via f^* as follows:

$$\begin{aligned}\alpha_0&=2pt,\quad \forall pt\in P(TF_a);\\ \alpha_1&=2P(TF_a|pt),\\ \alpha_2&=S_\infty\times pt+pt\times S_\infty,\\ \alpha_3&=f\times pt+pt\times f;\\ \alpha_4&=2P(TF_a|_{S_\infty}),\\ \alpha_5&=2P(TF_a|_f),\\ \alpha_6&=S_0\times S_\infty+S_\infty\times S_0,\\ \alpha_7&=f_0\times f_\infty+f_\infty\times f_0,\\ \alpha_8&=S_\infty\times f+f\times S_\infty,\\ \alpha_9&=F_a\times pt+pt\times F_a;\\ \alpha_{10}&=2P(TF_a),\\ \alpha_{11}&=F_a\times S_\infty+S_\infty\times F_a,\\ \alpha_{12}&=F_a\times f+f\times F_a;\\ \alpha_{13}&=F_a\times F_a,\end{aligned}$$

where we omit the symbols j_* and f^*. Define $\beta_i=(\phi^*)^{-1}(\alpha_i)=\frac{1}{2}\phi_*(\alpha_i)$ for each i. Then the homology groups of $F_a^{[2]}$ are generated by $\beta_0,\beta_1,\dots,\beta_{13}$.

The intersection products of the generators of complementary dimensions can be computed by Proposition 2.1. For generators of $A_1(F_a^{[2]})$ and $A_3(F_a^{[2]})$, they are

$$\begin{aligned}&\beta_1\cdot\beta_{10}=-2,&&\beta_1\cdot\beta_{11}=0,&&\beta_1\cdot\beta_{12}=0;\\ &\beta_2\cdot\beta_{10}=0,&&\beta_2\cdot\beta_{11}=-a,&&\beta_2\cdot\beta_{12}=1;\\ &\beta_3\cdot\beta_{10}=0,&&\beta_3\cdot\beta_{11}=1,&&\beta_3\cdot\beta_{12}=0.\end{aligned}\tag{2.1}$$

We also compute the following intersection products for later use:

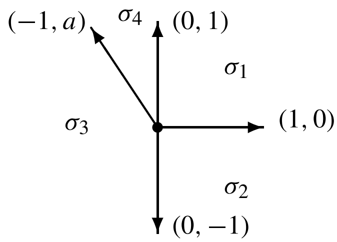
$$\begin{aligned}&\beta_9\cdot\beta_{10}=2\beta_1,\qquad \beta_9\cdot\beta_{11}=\beta_2,\qquad \beta_9\cdot\beta_{12}=\beta_3,\\ &\beta_{10}^2=4\beta_4+2(2+a)\beta_5-2a\beta_7-4\beta_8-4\beta_9,\\ &\beta_{10}\cdot\beta_{11}=2\beta_4,\quad \beta_{10}\cdot\beta_{12}=2\beta_5,\quad \beta_{11}^2=\beta_6-a\beta_8-a\beta_9,\\ &\beta_{11}\cdot\beta_{12}=\beta_8+\beta_9,\qquad \beta_{12}^2=\beta_7,\end{aligned}\tag{2.2}$$

from which we conclude that the cohomology ring of $F_a^{[2]}$ is generated by β_{10}, β_{11}, β_{12}, and the nondivisor class β_9, and the other basis elements are expressed as

$$\beta_1 = \frac{1}{2}\beta_9\beta_{10}, \qquad \beta_2 = \beta_9\beta_{11}, \qquad \beta_3 = \beta_9\beta_{12}, \qquad \beta_4 = \frac{1}{2}\beta_{10}\beta_{11},$$
$$\beta_5 = \frac{1}{2}\beta_{10}\beta_{12}, \qquad \beta_6 = \beta_{11}^2 + \beta_{11}\beta_{12}, \qquad \beta_7 = \beta_{12}^2, \qquad \beta_8 = \beta_{11}\beta_{12} - \beta_9.$$

2.2. *Torus Action on Hilbert Schemes*

As a toric variety, F_a can be constructed from the following fan:



The four affine varieties from the four two-dimensional cones are

$$U_{\sigma_1} = \operatorname{Spec}\mathbb{C}[x, y], \qquad U_{\sigma_2} = \operatorname{Spec}\mathbb{C}[x, y^{-1}],$$
$$U_{\sigma_3} = \operatorname{Spec}\mathbb{C}[x^{-1}, x^{-a}y^{-1}], \qquad U_{\sigma_4} = \operatorname{Spec}\mathbb{C}[x^{-1}, x^a y].$$

Note that the origins of the four affine planes correspond to the points we named A, B, D, and C, respectively, in Figure 1. Now the two-dimensional torus $T = \mathbb{C}^{*2}$ acts on F_a by acting on the variables $(\lambda, \mu) \cdot (x, y) = (\lambda^{-1}x, \mu^{-1}y)$ so that the weights at the fixed points are A: λ, μ; B: λ, $-\mu$; C: $-\lambda$, $a\lambda + \mu$; and D: $-\lambda$, $-a\lambda - \mu$. A moment of thinking shows that the representative cycles β_1 through β_{13} are invariant under the torus action, so they can be lifted to the equivariant cohomology groups. This is how they are used in localization procedure.

There are two types of fixed points of the torus action on the Hilbert scheme, the reduced ones and nonreduced ones. The reduced fixed points are unordered pairs of distinct fixed points on F_a, that is, A, B, C, and D. These points and their weights under the torus action are listed in the following:

LEMMA 2.5. *The weights of the T-action on the tangent spaces of $F_a^{[2]}$ at these six fixed points are*:

(1) (AB): λ, μ, λ, $-\mu$;
(2) (AC): λ, μ, $-\lambda$, $a\lambda + \mu$;
(3) (AD): λ, μ, $-\lambda$, $-a\lambda - \mu$;
(4) (BC): λ, $-\mu$, $-\lambda$, $a\lambda + \mu$;

(5) $(BD)\colon \lambda, -\mu, -\lambda, -a\lambda-\mu$;
(6) $(CD)\colon -\lambda, a\lambda+\mu, -\lambda, -a\lambda-\mu$.

The nonreduced fixed points are located at fixed points on F_a. It is easy to see that around each such fixed point, there are two fixed points of $F_a^{[2]}$, one corresponding to the direction of S_∞ or S_0 and one corresponding to the direction of the fiber through it. We denote them using subscripts "1" or "2", respectively. In total, we have eight of them, A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2.

To determine the weights of the torus action at these fixed points, we take A_1 as an example. It lives in the first affine plane U_{σ_1}, where A_1 is represented by the ideal (x^2, y). The full deformation of (x^2, y) in $F_a^{[2]}$ is described by $(x^2+\varepsilon_1 x+\varepsilon_2, y+\varepsilon_3 x+\varepsilon_4)$. So there are four curves passing through A_1 given by families of ideals in $\mathbb{C}[x,y]$: $I_1(\varepsilon)=(x^2+\varepsilon x, y)$, $I_2(\varepsilon)=(x^2+\varepsilon, y)$, $I_3(\varepsilon)=(x^2, y+\varepsilon x)$, and $I_4(\varepsilon)=(x^2, y+\varepsilon)$.

LEMMA 2.6. *The weights of the T-action on the tangent space of $F_a^{[2]}$ at A_1 are λ, 2λ, $\mu-\lambda$, and μ.*

Proof. We have

$$(\lambda,\mu)\cdot I_1(\varepsilon)=(\lambda^{-2}x^2+\varepsilon\lambda^{-1}x, \mu^{-1}y)=(x^2+\varepsilon\lambda x, y)=I_1(\varepsilon\lambda).$$

So the weight on the tangent direction of this curve is λ. Similarly,

$$\begin{aligned}
(\lambda,\mu)\cdot I_2(\varepsilon)&=(\lambda^{-2}x^2+\varepsilon, \mu^{-1}y)=I_2(\varepsilon\lambda^2),\\
(\lambda,\mu)\cdot I_3(\varepsilon)&=(\lambda^{-2}x^2, \mu^{-1}y+\varepsilon\lambda^{-1}x)=I_3(\varepsilon\lambda^{-1}\mu),\\
(\lambda,\mu)\cdot I_4(\varepsilon)&=(\lambda^{-2}x^2, \mu^{-1}y+\varepsilon)=I_4(\varepsilon\mu).
\end{aligned}$$

So the weights on the tangent directions of these curves are 2λ, $\mu-\lambda$, and μ. $\square$

The weights at the remaining seven fixed points can be determined similarly. We list all the weights here:

$$\begin{aligned}
&A_1\colon \lambda, 2\lambda, \mu-\lambda, \mu;\\
&A_2\colon \lambda, \mu, \lambda-\mu, 2\mu;\\
&B_1\colon 2\lambda, \lambda, -\mu, -\lambda-\mu;\\
&B_2\colon \lambda, -2\mu, -\mu, \lambda+\mu;\\
&C_1\colon -2\lambda, -\lambda, a\lambda+\mu, (a+1)\lambda+\mu;\\
&C_2\colon -\lambda, 2a\lambda+2\mu, a\lambda+\mu, -(a+1)\lambda-\mu;\\
&D_1\colon -\lambda, -2\lambda, (1-a)\lambda-\mu, -a\lambda-\mu;\\
&D_2\colon -\lambda, -a\lambda-\mu, (a-1)\lambda+\mu, -2a\lambda-2\mu.
\end{aligned}\tag{2.3}$$

$A_1(F_a^{[2]})$ is freely generated by β_1, β_2, β_3. For later use, we need the intersection numbers of the curve classes with its anticanonical bundle, which are used to decide the virtual dimensions of the moduli space of stable maps. They are worked out in the following

LEMMA 2.7. $\int_{\beta_1} c_1(TF_a^{[2]}) = 0$, $\int_{\beta_2} c_1(TF_a^{[2]}) = 2 - a$, $\int_{\beta_3} c_1(TF_a^{[2]}) = 2$.

Proof. Take the image of $P(TF_a|_A)$ in the Hilbert scheme under $(\phi^*)^{-1}$ in Subsection 2.1 to represent β_1. The two fixed points in this curve are A_1 and A_2, at which T acts with weights $\lambda - \mu$ and $\mu - \lambda$. Then by the localization formula,

$$\int_{\beta_1} c_1(TF_a^{[2]}) = \frac{\lambda + 2\lambda + \mu - \lambda + \mu}{\lambda - \mu} + \frac{\lambda + \mu + \lambda - \mu + 2\mu}{\mu - \lambda} = 0,$$

where the numerators are the respective sums of the weights at A_1 and A_2. Other two equalities can be verified similarly by taking *pt* to be B in the definition of β_2 and f to be f_0 and *pt* to be C in the definition of β_3. $\square$

2.3. Invariant Curves

Here an invariant curve means an irreducible curve of genus 0 in the Hilbert scheme invariant under the T-action. To apply virtual localization to calculate GW-invariants, we have to find all of them. To this end, we make use of the blowup construction of the Hilbert scheme.

Let us recall that $f\colon \widetilde{F_a \times F_a} \to F_a \times F_a$ is the blowup of $F_a \times F_a$ along the diagonal Δ, with the exceptional divisor $P(TF_a)$. Then $F_a^{[2]}$ is the $\mathbb{Z}_2$-quotient of the blowup. Since this quotient map is equivariant for T, it suffices to find the invariant curves in $\widetilde{F_a \times F_a}$. First of all, an invariant curve either is completely contained in $P(TF_a)$, or is disjoint from it, or intersects it at only points.

We first consider the case where the invariant curve is contained in $P(TF_a)$. Since the projection $P(TF_a) \to F_a$ is equivariant, this invariant curve is mapped to either a fixed point or an invariant curve in F_a. In the former case, it must be the fiber of $P(TF_a)$ over this fixed point, one for each of A, B, C, and D. They are isolated invariant curves. We assign names to these curves by listing their end points. For example, the invariant curve over A is denoted as $[A_1, A_2]$, connecting A_1 and A_2. Here and subsequently the symbol $[P, Q]$ means the invariant curve connecting two fixed points P and Q. Three other such invariant curves are $[B_1, B_2]$, $[C_1, C_2]$, and $[D_1, D_2]$.

In the latter case, the invariant curve is mapped onto an invariant curve in F_a. We only have four invariant curves in F_a, that is, S_∞, S_0, f_0, and f_∞. Take S_∞ through A and C as an example. Then the invariant curve must be contained in $P(TF_a|_{S_\infty})$, but $TF_a|_{S_\infty} = TS_\infty \oplus N_{S_\infty|F_a}$, where $N_{S_\infty|F_a}$ represents fiber directions at S_∞ in F_a, so $P(TF_a|_{S_\infty})$ is also a rational ruled surface. With the induced torus action on this surface, the two sections corresponding to the tangent directions and fiber directions of S_∞ in F_a are invariant. The section from tangent directions goes from A_1 to C_1, but from the further discussions it will follow that it is not isolated, so we put it aside for now. The other one $[A_2, C_2]$, representing the fiber directions, is isolated. Similarly, we have isolated invariant curves $[A_1, B_1]$, $[B_2, D_2]$, and $[C_1, D_1]$, corresponding to fiber or normal directions of f_0, S_0, f_∞.

Assume now that an invariant curve only intersects $P(TF_a)$ at points. Since the blowup map f composed with the two projections from $F_a \times F_a$ to F_a gives rise to two equivariant maps to F_a, the images of the invariant curve under these maps are either fixed points or invariant curves in F_a. They cannot both be points since the curve is connected. If they are a point and a curve not containing it, we get an isolated invariant curve in $F_a^{[2]}$. Considering the $\mathbb{Z}_2$-symmetry, we have eight of them $[AB, AD]$, $[AC, AD]$, $[AB, BC]$, $[BC, BD]$, $[AC, BC]$, $[BC, CD]$, $[AD, BD]$, and $[AD, CD]$. If the point is contained in the curve, then we get an invariant curve with a nonreduced point on it, which is also isolated. They are listed as $[A_1, AC]$, $[A_2, AB]$, $[B_2, AB]$, $[B_1, BD]$, $[C_1, AC]$, $[C_2, CD]$, $[D_1, BD]$, and $[D_2, CD]$.

Suppose both images are curves. When they are disjoint, they are either the pair S_∞, S_0 or the pair f_0, f_∞. In the first case, the T-action near $(A, B) \in S_\infty \times S_0$ is described by $(\lambda, \mu) \cdot (x, y) = (\lambda x, \lambda y)$, so we have a one-dimensional family of invariant curves connecting AB to CD. Near $(A, C) \in f_0 \times f_\infty$, the action is expressed as $(\lambda, \mu) \cdot (x, y) = (\mu x, \lambda^a \mu y)$. Since the two weights are independent, no invariant curve is generated from this action.

Now we turn to the case where the two image curves are distinct, thus intersecting at one fixed point, for example, S_∞, f_0. Then T-action near $(A, A) \in S_\infty \times f_0$ is expressed as $(\lambda, \mu) \cdot (x, y) = (\lambda x, \mu y)$. This action does not produce any invariant curve satisfying the condition. Other combinations neither produce anything new.

Finally, we are left with the case where the two image curves coincide. This situation is reduced to a concrete example. Let $\mathbb{C}^*$ act on $\mathbb{C}$ as $\lambda \cdot x = \lambda x$, inducing an action on $\mathbb{C}^{[2]} = \mathbb{C}^{(2)}$, the symmetric product of $\mathbb{C}$. We define a map $\tau : \mathbb{C}^{(2)} \to \mathbb{C}^2$ by $\tau(x, y) = (x + y, xy)$, which is an isomorphism. With this map, $\mathbb{C}^2$ inherits a $\mathbb{C}^*$-action by $\lambda \cdot (x, z) = (\lambda x, \lambda^2 z)$. This means that the weights of the torus action at the origin are λ and 2λ.

The map τ extends to an isomorphism from the symmetric product of $\mathbb{P}^1$ to $\mathbb{P}^2$, still denoted as $\tau : (\mathbb{P}^1)^{(2)} \to \mathbb{P}^2$ by $\tau((a, b), (x, y)) = (ay + bx, ax, by)$. The image of the diagonal of $(\mathbb{P}^1)^{(2)}$ is a conic curve, but it is not isolated as an invariant curve. In fact, there is a one-dimensional family of invariant conic curves in $\mathbb{P}^2$, which breaks up to two coordinate lines [16]. We summarize the conclusions in the following:

LEMMA 2.8. *Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ as $\lambda \cdot (x, y) = (\lambda x, y)$. Then it induces an action on $\mathbb{P}^2$ as $\lambda \cdot (x, y, z) = (\lambda x, \lambda^2 y, z)$ via τ. At $(0, 0, 1) \in \mathbb{P}^2$, this action has weights λ and 2λ; the three coordinate lines are isolated invariant lines, and there is a one-dimensional family of invariant curves defined by $x^2 = \mu yz$ with $\mu \in \mathbb{C}^* - 0$, whose class is twice the line class in $\mathbb{P}^2$. As $\mu \to 0$, this family approaches the double cover of the line $x = 0$; as $\mu \to \infty$, it degenerates to the nodal curve the union of lines $y = 0$ and $z = 0$.*

Now we apply this picture to S_∞, S_0, f_0, and f_∞ in F_a. Taking S_∞ as an example, we get an embedding $\mathbb{P}^2$ in $F_a^{[2]}$ and the three coordinate lines in $\mathbb{P}^2$ as the curves $[A_1, AC]$, $[AC, C_1]$ before and an isolated invariant curve $[A_1, C_1]$ from A_1 to C_1, which was postponed before, and a one-dimensional family of invariant curves from A_1 to C_1. It is easy to see that the invariant curves $[A_1, AC]$ and $[A_1, C_1]$ at A_1 agree with the curves I_1 and I_2 in Lemma 2.6. Similarly, we have the new isolated invariant lines $[A_2, B_2]$, $[B_1, D_1]$, and $[C_2, D_2]$ and the corresponding one-dimensional families of invariant curves. This lemma also explains that, at every nonreduced fixed point, there always appears a pair of a weight and its double in the list of (2.3).

Up till this point, we have found all the isolated and one-dimensional families of invariant curves in $F_a^{[2]}$. For the purpose of computing GW-invariants, their curve classes in $A_1(F_a^{[2]})$ should be decided. For brevity, we still use the symbol $[P, Q]$ to denote the homology class for the invariant curve from P to Q.

LEMMA 2.9. (1) $[A_1, A_2] = [B_1, B_2] = [C_1, C_2] = [D_1, D_2] = \beta_1$;
(2) $[A_1, AC] = [C_1, AC] = \beta_2 - \beta_1$, $[B_1, BD] = [D_1, BD] = \beta_2 + a\beta_3 - \beta_1$, $[A_2, AB] = [B_2, AB] = [C_2, CD] = [D_2, CD] = \beta_3 - \beta_1$;
(3) $[A_1, C_1] = \beta_2 - \beta_1$, $[A_2, B_2] = [C_2, D_2] = \beta_3 - \beta_1$, $[B_1, D_1] = \beta_2 + a\beta_3 - \beta_1$;
(4) $[A_2, C_2] = 2\beta_2 + a\beta_1$, $[A_1, B_1] = [C_1, D_1] = 2\beta_3$, $[B_2, D_2] = 2\beta_2 + 2a\beta_3 - a\beta_1$.

Proof. (1) This is by definition.

(2) Let L be the proper transform of $S_\infty \times A + A \times S_\infty$ in the blowup of $F_a \times F_a$ in Section 2.1. Then by Proposition 2.2 we have, as homology classes,

$$f^*(S_\infty \times A + A \times S_\infty) = L + P(TF_a|_A),$$

where $P(TF_a|_A)$ comes from the second term in the formula. However, $S_\infty \times A + A \times S_\infty$ is rationally equivalent to $S_\infty \times pt + pt \times S_\infty$ in $F_a \times F_a$, where pt is a point off S_∞. After projecting to $A_*(F_a^{[2]})$ by ϕ_*, L gives rise to the curve $[A_1, AC]$, so, as classes, $[A_1, AC] = \beta_2 - \beta_1$. Others can be proven similarly.

(3) We take $[A_1, C_1]$ as an example. By Lemma 2.8 the class $[A_1, C_1]$ is the same as a line class in $\mathbb{P}^2$, one of which is the curve $[A_1, AC]$, resulting in the conclusion.

(4) By Lemma 2.4(2), $\zeta \cap P(TF_a|_{S_\infty}) = (2 - a)\beta_1 - 2\beta_2$, where here and in the following the equality takes place as classes in the Hilbert scheme. Now $TF_a|_{S_\infty} = TS_\infty \oplus N_{S_\infty|F_a}$, where $TS_\infty = \mathcal{O}(2)$ for $S_\infty \simeq \mathbb{P}^1$, 7y and $N_{S_\infty|F_a} = \mathcal{O}(-a)$. This means that $P(TF_a|_{S_\infty}) = \mathrm{Proj}(\mathcal{O}(-2) \oplus \mathcal{O}(a)) \cong \mathrm{Proj}(\mathcal{O} \oplus \mathcal{O}(-2 - a))$ is also a rational ruled surface, where the isomorphism is produced by multiplying $\mathcal{O}(-a)$. Let η be its ∞-section in the standard convention. Then, by Lemma 7.9, Ch.2 in [12], $\zeta \cap P(TF_a|_{S_\infty}) = -\eta - a\beta_1$, so $\eta = 2\beta_2 - 2\beta_1$. Clearly, $[A_2, C_2]$ is the 0-section in the ruled surface, so $[A_2, C_2] = \eta + (2 + a)\beta_1 = 2\beta_2 + a\beta_1$.

Similarly, by Lemma 2.4(4), $\zeta \cap P(TF_a|_{f_0}) = 2\beta_1 - 2\beta_3$. Now $P(TF_a|_{f_0}) = Tf_0 \oplus N_{f_0|TF_a}$, where again $Tf_0 = T\mathbb{P}^1 = \mathcal{O}(2)$, and $N_{f_0|TF_a}$ is trivial since $c_1(N_{f_0|TF_a}) = c_1(TF_a|_{f_0}) - c_1(Tf_0) = f_0 \cdot (2S_0 + (2 + a)f_0) - 2 = 0$. So we get $P(TF_a|_{f_0}) = \mathrm{Proj}(\mathcal{O} \oplus \mathcal{O}(-2))$, which is again a ruled surface. Let η' be its ∞-section. Then $\eta' = -\zeta \cap P(TF_a|_{f_0}) = 2\beta_3 - 2\beta_1$ again by Lemma 7.9, Ch.2 [12]. As $[A_1, B_1]$ is the 0-section, we have $[A_1, B_1] = \eta' + 2\beta_1 = 2\beta_3$. It is obvious that $[C_1, D_1] = [A_1, B_1]$.

Finally, by Lemma 2.4(3) and (1) we have $\zeta \cap P(TF_a|_{S_0}) = (2+a)\beta_1 - 2\beta_2 - 2a\beta_3$. Because $TF_a|_{S_0} = TS_0 \oplus N_{S_0|F_a} = \mathcal{O}(2) \oplus \mathcal{O}(a)$, $P(TF_a|_{S_0}) = \operatorname{Proj}(\mathcal{O}(-2) \oplus \mathcal{O}(-a))$, which is isomorphic to $\operatorname{Proj}(\mathcal{O} \oplus \mathcal{O}(2-a))$ by multiplying $\mathcal{O}(2)$ if $a \geq 2$ and isomorphic to $\operatorname{Proj}(\mathcal{O} \oplus \mathcal{O}(-1))$ by multiplying $\mathcal{O}(1)$ if $a = 1$.

When $a \geq 2$, again by Lemma 7.9, Ch.2 in [12], we have $\zeta \cap P(TF_a|_{S_0}) = -\xi + 2\beta_1$, where ξ is the ∞-section, hence $\xi = 2\beta_2 + 2a\beta_3 - a\beta_1$, and so $[B_1, D_1] = \xi + (a-2)\beta_1 = 2\beta_2 + 2a\beta_3 - 2\beta_1$. When $a = 1$, we get $\zeta \cap P(TF_a|_{S_0}) = -\xi' + \beta_1$, where ξ' is the ∞-section, hence $\xi' = 2\beta_2 + 2\beta_3 - 2\beta_1$ and $[B_2, D_2] = \xi' + \beta_1 = 2\beta_2 + 2\beta_3 - \beta_1$. □

Results (2) and (3) of this lemma agree with the conclusion of Lemma 2.8, which shows that the two isolated invariant curves described there at a nonreduced fixed point share the same curve class. Also from this lemma, the generic invariant curve in the one-dimensional family connecting AB and CD is of class $2\beta_2 + a\beta_3$ since, as limits, it breaks up into two nodal curves composed of the curve from AB to AD of class $\beta_2 + a\beta_3$ intersecting the curve from AD to CD of class β_2 and the curve from AB to BC of class β_2 intersecting the curve from BC to CD of class $\beta_2 + a\beta_3$.

All the isolated invariant curves and one-dimensional families of invariant curves for $F_1^{[2]}$ are shown in Figure 2. In this diagram, the isolated invariant curves are depicted by straight or curved lines with their degrees on them; the one-dimensional families of invariant curves are shown by wavy lines with the degree of generic curves in the families attached. In Figure 3, we display weights at each fixed point for convenient reference along isolated invariant curves.

3. Computations of Gromov–Witten Invariants

3.1. Connected Components Analysis

When a torus T acts on a smooth projective variety with finitely many fixed points, a fixed point of the induced action on the moduli space of stable maps $\overline{\mathcal{M}}_{0,n}(X, \beta)$ has the following properties [10]:

(1) all marked points, nodes, contracted components, and ramification points on the domain curve are mapped to fixed points in X;

(2) a noncontracted irreducible component has to be rational and is mapped onto an invariant rational curve in X, ramified only over two fixed points in the rational curve.

When we fix a degree $\beta \in H_2(X, \mathbb{Z})$, the degrees of noncontracted components have to add up to β. In general, there are higher-dimensional families of invariant curves in X, but if for some β, all the invariant curves that appear in a connected component of the fixed point loci in the moduli space are isolated, then a map f in the connected component can be described by a decorated graph Γ in the following way: Γ has one vertex v for each connected component in the inverse

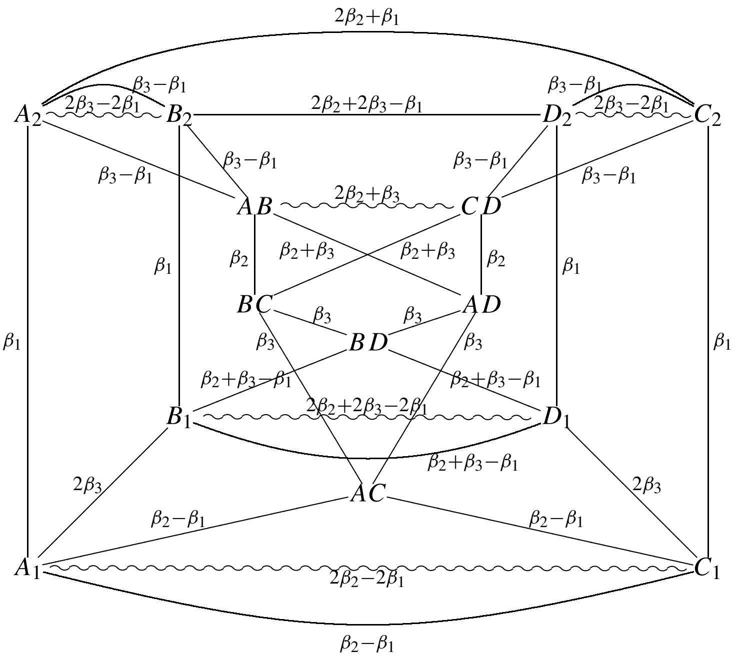


Figure 2 Invariant Curves of $F_1^{[2]}$

image under f of a fixed point in X, which is labeled with the name of that fixed point; Γ has one edge e for each noncontracted component, whose two vertices are labeled with two different fixed points and which is labeled with the degree d_e of the map from the component to its image. Also, we label each vertex v with a number for each marked point. Then the connected components are described by such graphs.

For each decorated graph Γ, we define $\overline{\mathcal{M}}_\Gamma = \prod_{v\in\Gamma} \overline{\mathcal{M}}_{0,\mathrm{val}(v)}$, where by convention $\overline{\mathcal{M}}_{0,1} = \overline{\mathcal{M}}_{0,2} = pt$. The universal family of T-fixed stable maps $\pi\colon \mathcal{C} \to \overline{\mathcal{M}}_\Gamma$ induces a morphism $\gamma\colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{0,n}(X,\beta)$. The automorphism group A of this family is filtered by an exact sequence

$$1 \to \prod_{e\in\Gamma} \mathbb{Z}/(d_e) \to A \to \mathrm{Aut}(\Gamma) \to 1,$$

where $\mathrm{Aut}(\Gamma)$ denotes the automorphism group of Γ. The induced morphism $\gamma/A\colon \overline{\mathcal{M}}_\Gamma/A \to \overline{\mathcal{M}}_{0,n}(X,\beta)$ is a closed immersion as a connected component of fixed points.

The tangent space T^1 and the obstruction space T^2 of $\overline{\mathcal{M}}_{0,n}(X,\beta)$ are related in the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^0(\Omega_C(D), \mathcal{O}) \to H^0(C, f^*TX) \\ &\to T^1 \to \mathrm{Ext}^1(\Omega_C(D), \mathcal{O}) \to H^1(C, f^*X) \to T^2 \to 0, \end{aligned}$$

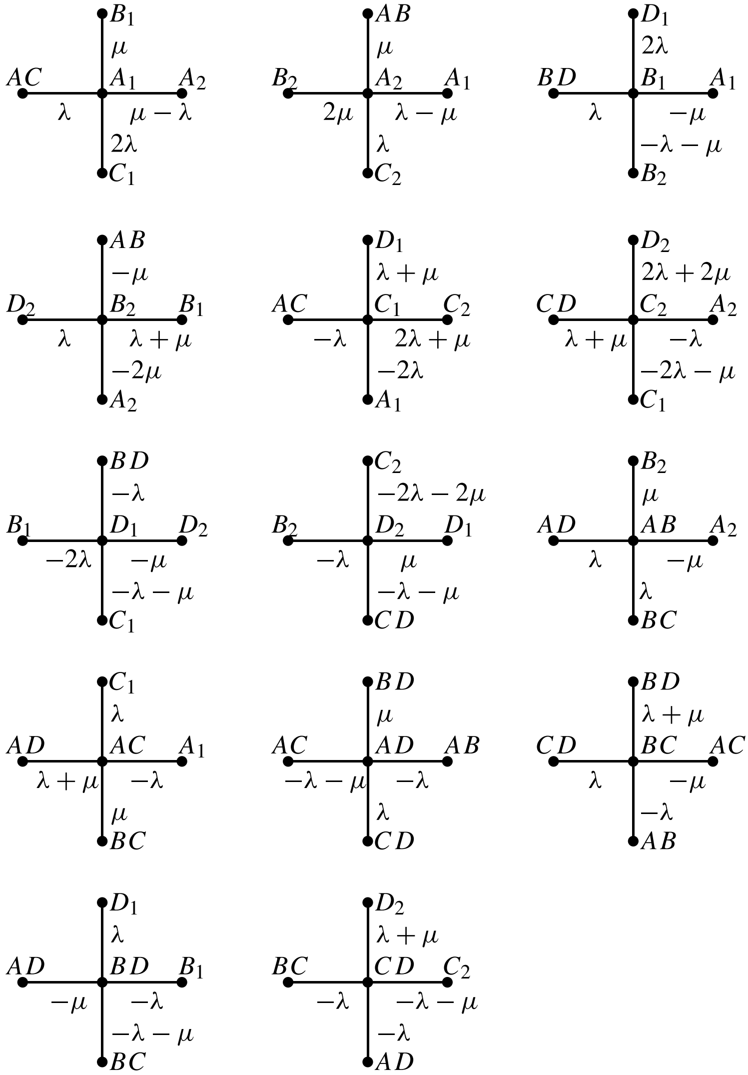


Figure 3 Weights at Fixed Points

in which D represents the divisor of marked points on C. When restricted to a connected component $\overline{\mathcal{M}}_\Gamma/A$, the four terms other than the sheaves T^1 and T^2 form vector bundles as fibres, denoted as B_1, B_2, B_4, and B_5 respectively, each decomposing as the direct sum of the fixed part B_i^f and the moving part B_i^m under the torus action. The moving parts inherit a natural T-action. Then the equivariant Euler class of the virtual normal bundle of the connected component is computed as

$$e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}}) = \frac{e^{\mathbb{C}^*}(B_2^m)e^{\mathbb{C}^*}(B_4^m)}{e^{\mathbb{C}^*}(B_1^m)e^{\mathbb{C}^*}(B_5^m)}.$$

This is the denominator in the virtual localization formula.

3.2. *One-Point Gromov–Witten Invariants*

From now on we only consider F_1 and use F to denote it. We first treat the curve class $d\beta_1$ for an integer $d > 0$. In general the virtual dimension of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is equal to $\operatorname{virdim}\overline{\mathcal{M}}_{g,n}(X, \beta) = c_1(X)\beta + (1-g)\dim X + 3g - 3 + n$. Since $\operatorname{virdim}\overline{\mathcal{M}}_{0,n}(F^{[2]}, d\beta_1) = n + 1$ by Lemma 2.7, to get nontrivial GW-invariants $\langle\alpha_1, \alpha_2, \ldots, \alpha_n\rangle_{0,n,d\beta_1}$ for $\alpha_i \in A^*(F^{[2]})$, the cohomological degrees of α_i should add up to $n + 1$. This happens only when one class has degree 2 and other classes all have degree 1. However, by the axiom of divisors of GW-invariants, when $\deg(\alpha_n) = 1$, $\langle\alpha_1, \alpha_2, \ldots, \alpha_n\rangle_{0,n,d\beta_1} = \int_{d\beta_1}\alpha_n\langle\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\rangle_{0,n-1,d\beta_1}$. So it suffices to compute $\langle\alpha\rangle_{0,1,d\beta_1}$ for $\alpha \in A_2(F^{[2]})$, noting that $\dim F^{[2]} = 4$. From [13] we have the following:

THEOREM 3.1. (i) $\langle\beta_j\rangle_{0,1,d\beta_1} = 0$ *for* $j = 6, 7, 8, 9$;
(ii) $\langle\beta_4\rangle_{0,1,d\beta_1} = -\frac{2}{d}$, $\langle\beta_5\rangle_{0,1,d\beta_1} = -\frac{4}{d}$.

Proof. (i) is clear from [13].

(ii) Again from [13], $\langle\beta_4\rangle_{0,1,d\beta_1} = 2(K \cdot S_\infty)/d = 2(-2S_\infty - 3f) \cdot S_\infty/d = -\frac{2}{d}$, and $\langle\beta_5\rangle_{0,1,d\beta_1} = 2(K \cdot f)/d = 2(-2S_\infty - 3f) \cdot f/d = -\frac{4}{d}$. □

$A_1(F^{[2]})$ is freely generated by β_1, $\beta_2 - \beta_1$, and $\beta_3 - \beta_1$, and from Figure 2 the invariant curves can all be expressed as linear combinations in these generators with nonnegative coefficients. By virtual localization formula, GW-invariants of any class β vanish except for $\beta = d\beta_1 + d_2(\beta_2 - \beta_1) + d_3(\beta_3 - \beta_1)$ for nonnegative integers d, d_2, d_3. With Theorem 3.1, we just need to assume that d_2 and d_3 are not simultaneously zero. To compute one-pointed GW-invariants, we first note that $\operatorname{virdim}\overline{\mathcal{M}}_{0,1}(F^{[2]}, \beta) = d_2 + 2d_3 + 2$.

Since $\dim F^{[2]} = 4$, we only need to consider $(d_2, d_3) = (1, 0)$, $(2, 0)$, or $(0, 1)$ to get nonzero invariants. Our strategy is to choose a suitable cycle to represent the homology class so that only finitely many nodal invariant curves of the given curve class intersect the cycle, since only these invariant curves have nontrivial contributions in the localization formula. In particular, if no such invariant curve intersects the cycle, then the GW-invariant vanishes. This prompts the idea that we purposely choose the representative of the curve class so that it either stays away from any such invariant curves or intersects with as few of them as possible.

For the pairs $(2, 0)$ and $(0, 1)$, $\operatorname{virdim}\overline{\mathcal{M}}_{0,1}(F^{[2]}, \beta) = 4$, so the insertion for nonzero GW-invariants must be a point class.

PROPOSITION 3.2. *For any curve class* $\beta \in A_1(F^{[2]})$, $\langle pt\rangle_{0,1,\beta} = 0$ *except that* $\langle pt\rangle_{0,1,\beta_3} = 2$.

Proof. We first remark that in this proof and throughout this chapter, we constantly refer to Figure 2 for configuration of fixed points and invariant curves and to Figure 3 for relevant weights at fixed points.

For the pair $(2, 0)$, we take the point BD for the point class. Then any connected invariant curve passing through BD has to contain β_3 from Figure 2, which is not allowed in $(2, 0)$. So the localization formula expansion does not have any nonzero term in it, that is, $\langle pt\rangle_{0,1,\beta} = 0$ in this case.

For the second pair $(0, 1)$, we take the point AC for the point class. When $d \neq 1$, it is away from any nodal invariant curve of degree β. So $\langle pt\rangle_{0,1,\beta} = 0$ in this case.

Now assume that $d = 1$, that is, $\beta = \beta_3$. Then there are two nonzero terms in the localization formula from the connected components described by the following graphs:

where here and in the following, the boldface points mean where the marked points are mapped to, and the numbers above the line segments mean the degrees of the maps.

Now we determine the equivariant Euler classes of their virtual normal bundles. For Γ_1, first of all, $e^{\mathbb{C}^*}(B_1^m) = -\mu, e^{\mathbb{C}^*}(B_4^m) = 1$. To compute $e^{\mathbb{C}^*}(B_2^m)/e^{\mathbb{C}^*}(B_5^m)$, we use the localization formula to $f^*TF^{[2]}$ in equivariant topological K-theory, which is used in [5], that is, the virtual bundle

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \frac{t^\lambda + t^\mu + t^{-\lambda} + t^{\lambda+\mu}}{1 - t^{-\mu}} + \frac{t^\lambda + t^{-\mu} + t^{-\lambda} + t^{\lambda+\mu}}{1 - t^\mu}\\ &= 1 + t^\lambda + t^{-\lambda} + t^{-\mu} + t^\mu + t^{\lambda+\mu},\end{aligned}$$

and then by taking weights, $e^{\mathbb{C}^*}(B_2^m)/e^{\mathbb{C}^*}(B_5^m) = \lambda^2\mu^2(\lambda + \mu)$. Hence $e^{\mathbb{C}^*}(N_{\Gamma_1}^{\mathrm{vir}}) = -\lambda^2\mu(\lambda + \mu)$.

For Γ_2, $e^{\mathbb{C}^*}(B_1^m) = -\lambda - \mu$ and $e^{\mathbb{C}^*}(B_4^m) = 1$. Again by K-theoretic localization formula,

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \frac{t^\lambda + t^\mu + t^{-\lambda} + t^{-\lambda-\mu}}{1 - t^{\lambda+\mu}} + \frac{t^\lambda + t^\mu + t^{-\lambda} + t^{\lambda+\mu}}{1 - t^{-\lambda-\mu}}\\ &= 1 + t^\lambda + t^{-\lambda} + t^\mu + t^{\lambda+\mu} + t^{-\lambda-\mu},\end{aligned}$$

so $e^{\mathbb{C}^*}(B_2^m)/e^{\mathbb{C}^*}(B_5^m) = \lambda^2\mu(\lambda + \mu)^2$, and therefore $e^{\mathbb{C}^*}(N_{\Gamma_2}^{\mathrm{vir}}) = -\lambda^2\mu(\lambda + \mu)$. Putting these terms in the localization formula, we get

$$\langle pt\rangle_{0,1,\beta_3} = \frac{-\lambda^2\mu(\lambda + \mu)}{-\lambda^2\mu(\lambda + \mu)} + \frac{-\lambda^2\mu(\lambda + \mu)}{-\lambda^2\mu(\lambda + \mu)} = 1 + 1 = 2,$$

where the numerators are the equivariant class of the point AC, which is the product of all the weights at AC. □

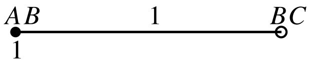
For the pair $(1, 0)$, $\operatorname{virdim} \overline{\mathcal{M}}_{0,1}(F^{[2]}, \beta) = 3$, so we need to feed a class in $A_1(F^{[2]})$ to get nonzero GW-invariants.

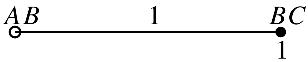
PROPOSITION 3.3. *For* $\beta = d\beta_1 + (\beta_2 - \beta_1)$,

(i) $\langle\beta_1\rangle_{0,1,\beta} = 0$ *for any* d;
(ii) $\langle\beta_2\rangle_{0,1,\beta} = 0$ *for* $d \neq 1$; -1 *for* $d = 1$;
(iii) $\langle\beta_3\rangle_{0,1,\beta} = 0$ *for* $d \neq 1$; 1 *for* $d = 1$.

Proof. (i) Let us take the invariant curve $[B_1, B_2]$ to be the representative of β_1. Since any nodal invariant curve touching this representative has to contain β_3, which is not allowed, $\langle\beta_1\rangle_{0,1,\beta} = 0$ for any $d \geq 0$.

(ii) For β_2, we take the invariant curve $[AB, BC]$ as a representative. When $d \neq 1$, for the same reason, this invariant curve does not intersect any nodal invariant curve of class β, so the GW-invariants are equal to zero. When $d = 1$, that is, $\beta = \beta_2$, there are two nonzero terms in the localization formula from connected components described by the following graphs:

Γ_1 

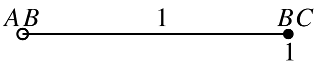
Γ_2 

Following the same procedure as in the previous proposition, we get $e^{\mathbb{C}^*}(N_{\Gamma_1}^{\text{vir}}) = -\lambda^2\mu$ and $e^{\mathbb{C}^*}(N_{\Gamma_2}^{\text{vir}}) = \lambda^2\mu$. Using the virtual localization formula, we have

$$\langle\beta_2\rangle_{0,1,\beta_2} = \frac{-\lambda\mu^2}{-\lambda^2\mu} + \frac{-\lambda\mu(\lambda+\mu)}{\lambda^2\mu} = -1,$$

where the numerators are the equivariant class of the curve $[AB, BC]$ restricted to AB and BC, which are the products of the weights of the normal bundle of the curve at these points.

(iii) Finally, for $\langle\beta_3\rangle_{0,1,\beta}$, we take the invariant curve from BC to BD to be the representative of β_3. Again, when $d \neq 1$, this invariant curve does not intersect any nodal invariant curve of class β, which implies that the GW-invariants vanish. When $d = 1$, only one component contributes a nonzero term, described by the following graph:

Γ 

Now $e^{\mathbb{C}^*}(N_{\Gamma}^{\text{vir}}) = \lambda^2\mu$. By the localization formula, $\langle\beta_3\rangle_{0,1,\beta_2} = \lambda^2\mu/(\lambda^2\mu) = 1$. □

Till this point, we have computed all one-pointed GW-invariants of $F^{[2]}$.

3.3. Two-Point Gromov–Witten Invariants

When $n = 2$, virdim $\overline{\mathcal{M}}_{0,2}(F^{[2]}, \beta)$ is equal to $d_2 + 2d_3 + 3$ for $\beta = d\beta_1 + d_2(\beta_2 - \beta_1) + d_3(\beta_3 - \beta_1)$. For the dimensional reason, we must have $d_2 + 2d_3 \leq 5$ to get nonzero invariants. The complete list of these pairs of (d_2, d_3) is (5, 0), (4, 0), (3, 0), (2, 0), (1, 0), (3, 1), (2, 1), (1, 1), (0, 1), (1, 2), and (0, 2).

In the following, we shall treat these cases one by one. The strategy is almost the same as for computing one-point invariants: if we can choose a cycle representing one class that never intersects any nodal invariant curve of the given curve class or if we can choose the representative cycles for both classes that do not intersect any nodal invariant curve of the curve class simultaneously, then the GW-invariant in question must vanish. Thus we carefully choose such representatives so that they intersect as few such curves as possible.

When one insertion is a point class, this can be carried out more readily. We are in such a situation for the pairs $(5,0)$, $(3,1)$, $(1,2)$, $(4,0)$, $(2,1)$, and $(0,2)$. For the former three pairs, $\operatorname{virdim} \overline{\mathcal{M}}_{0,2}(F^{[2]},\beta)=8$, so both insertions must be the point class.

PROPOSITION 3.4. *For $\beta = d\beta_1 + 5(\beta_2-\beta_1)$ and $d\beta_1+3(\beta_2-\beta_1)+(\beta_3-\beta_1)$, $\langle pt, pt\rangle_{0,2,\beta}=0$ for any d.*

Proof. For $\beta = d\beta_1 + 5(\beta_2-\beta_1)$, we choose the point BD to represent one point class. Then BD does not lie in any nodal invariant curve of degree β, so $\langle pt, pt\rangle_{0,2,\beta}=0$ in this case.

For $\beta = d\beta_1+3(\beta_2-\beta_1)+(\beta_3-\beta_1)$, we choose BD for one point class and AC for the other. Then any nodal invariant curve of degree β does not pass through both points simultaneously, so $\langle pt, pt\rangle_{0,2,\beta}=0$ in this case. □

Unfortunately, for $\beta = d\beta_1+(\beta_2-\beta_1)+2(\beta_3-\beta_1)$, the localization formula does not apply to $\langle pt, pt\rangle_{0,2,\beta}$ since every fixed point is traversed by a one-dimensional family of invariant curves of the given curve class. In the next section, we will take on this with the help of the associativity law of quantum product.

Now for the latter three pairs, the other insertion must be from $A_1(F^{[2]})$.

PROPOSITION 3.5. *For $\beta = d\beta_1+4(\beta_2-\beta_1)$, $d\beta_1+2(\beta_3-\beta_1)$, and $d\beta_1+2(\beta_2-\beta_1)+(\beta_3-\beta_1)$, $\langle pt, \beta_i\rangle_{0,2,\beta}=0$ for $i=1,2,3$ and any d.*

Proof. First, for $\beta = d\beta_1+4(\beta_2-\beta_1)$, if we take the point BD for the point class, then any nodal invariant curve of the designated degree cannot pass through this point, so $\langle pt, \beta_i\rangle_{0,2,\beta}=0$ for $i=1,2,3$.

Then for $\beta = d\beta_1+2(\beta_3-\beta_1)$, we choose the point D_1 for the point class, the invariant curve $[A_1, A_2]$ for β_1, $[AB, BC]$ for β_2, and $[BC, AC]$ for β_3. Then we see that $\langle pt, \beta_i\rangle_{0,2,\beta}=0$ for $i=1,2,3$.

Finally, assume that $\beta = d\beta_1+2(\beta_2-\beta_1)+(\beta_3-\beta_1)$. If we take the point BD for the point class and keep the representative for β_1 as before, then we see that $\langle pt, \beta_1\rangle_{0,2,\beta}=0$; if we take the point D_1 for the point class and keep the representative for β_2 as before, then we see that $\langle pt, \beta_2\rangle_{0,2,\beta}=0$.

To consider $\langle pt, \beta_3\rangle_{0,2,\beta}$, we take D_2 for the point class and $[B_2, AB]$ for the representative of $\beta_3-\beta_1$. Then we see that no nodal invariant curve of this degree connects the two cycles. So $\langle pt, \beta_3-\beta_1\rangle_{0,2,\beta}=0$. By the preceding, $\langle pt, \beta_1\rangle_{0,2,\beta}=0$, so we have $\langle pt, \beta_3\rangle_{0,2,\beta}=0$. □

For the pairs $(d_2,d_3)=(3,0)$, $(1,1)$, we have $\operatorname{virdim} \overline{\mathcal{M}}_{0,2}(F^{[2]},\beta)=6$. Then the degree decomposition of the two insertions is either $2+4$ or $3+3$. The first type is dealt with in the following:

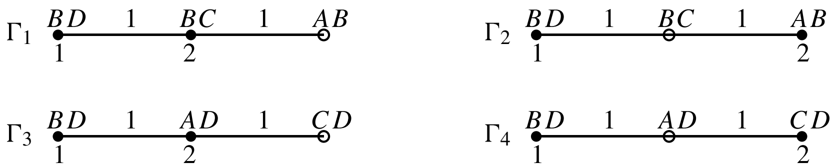
PROPOSITION 3.6. (i) *For* $\beta = d\beta_1 + 3(\beta_2 - \beta_1)$, $\langle pt, \beta_i\rangle_{0,2,\beta} = 0$ *for* $i = 4, 5, 6, 7, 8, 9$ *and any* d;

(ii) *For* $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$, $\langle pt, \beta_4\rangle_{0,2,\beta} = \langle pt, \beta_6\rangle_{0,2,\beta} = \langle pt, \beta_8\rangle_{0,2,\beta} = 0$ *for any* d. *Also,* $\langle pt, \beta_9\rangle_{0,2,\beta} = 0$ *for* $d \neq 2$ *and* 2 *for* $d = 2$.

Proof. (i) We take the point BD for the point class and the standard representatives for β_i listed in Section 2.1, where f_0 is assigned to f, and the point A assigned to pt in those expressions. Then we see that any nodal invariant curve has to contain β_3, which is excluded by the given curve class, so $\langle pt, \beta_i\rangle_{0,2,\beta} = 0$ for $i = 4, \dots, 9$.

(ii) For $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$, if we take the point BD for the point class and the standard representative for β_4, then, for the same reason, $\langle pt, \beta_4\rangle_{0,2,\beta} = 0$ for any d.

Now we compute $\langle pt, \beta_6\rangle_{0,2,\beta}$. We choose BD for the point class and the standard representative for β_6. Then we see that $\langle pt, \beta_6\rangle_{0,2,\beta} = 0$ if $d \neq 2$. When $d = 2$, there are nonzero terms from the fixed point loci described by the following graphs:



For Γ_1, $e^{\mathbb{C}^*}(B_1^m) = \lambda$ and $e^{\mathbb{C}^*}(B_4^m) = -\lambda(\lambda+\mu)$. To compute $e^{\mathbb{C}^*}(B_2^m)/e^{\mathbb{C}^*}(B_5^m)$, let us use e_1 to denote the invariant curve from BC to BD and e_2 to denote the invariant curve from AB to BC. Then, by K-theoretic localization,

$$\begin{aligned}\chi(f^*TF^{[2]}|_{e_1}) &= \frac{t^{\lambda} + t^{-\mu} + t^{-\lambda} + t^{\lambda+\mu}}{1 - t^{-\lambda-\mu}} \\ &\quad + \frac{t^{\lambda} + t^{-\mu} + t^{-\lambda} + t^{-\lambda-\mu}}{1 - t^{\lambda+\mu}} \\ &= t^{\lambda} + t^{-\lambda} + t^{-\mu} + t^{\lambda+\mu} + t^{-\lambda-\mu} + 1.\end{aligned}$$

Also, $\chi(f^*TF^{[2]}|_{e_2}) = 2t^{\lambda} + t^{-\lambda} + t^{-\mu} + 1$ by the preceding. Then, applying a normalization sequence, we have

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \chi(f^*TF^{[2]}|_{e_1}) + \chi(f^*TF^{[2]}|_{e_2}) - TF^{[2]}|_{BC} \\ &= t^{-\lambda-\mu} + 2t^{\lambda} + t^{-\lambda} + t^{-\mu} + 2.\end{aligned}$$

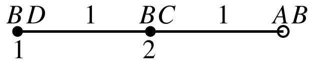
From this we get $e^{\mathbb{C}^*}(B_2^m)/e^{\mathbb{C}^*}(B_5^m) = -\lambda^3\mu(\lambda+\mu)$, so $e^{\mathbb{C}^*}(N_{\Gamma_1}^{\mathrm{vir}}) = \lambda^3\mu(\lambda+\mu)^2$. Similarly,

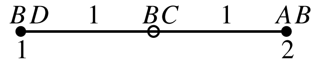
$$\begin{aligned}e^{\mathbb{C}^*}(N_{\Gamma_2}^{\mathrm{vir}}) &= -\lambda^3\mu^2(\lambda+\mu), \\ e^{\mathbb{C}^*}(N_{\Gamma_3}^{\mathrm{vir}}) &= -\lambda^3\mu^2(\lambda+\mu), \\ e^{\mathbb{C}^*}(N_{\Gamma_4}^{\mathrm{vir}}) &= \lambda^3\mu(\lambda+\mu)^2.\end{aligned}$$

Using the localization formula, we have

$$\begin{aligned}\langle pt, \beta_6\rangle_{0,2,\beta_2+\beta_3} &= \frac{\lambda^2\mu(\lambda+\mu)\mu(\lambda+\mu)}{\lambda^3\mu(\lambda+\mu)^2} + \frac{\lambda^2\mu(\lambda+\mu)\mu^2}{-\lambda^3\mu^2(\lambda+\mu)} \\ &\quad + \frac{\lambda^2\mu(\lambda+\mu)\mu(\lambda+\mu)}{-\lambda^3\mu^2(\lambda+\mu)} + \frac{\lambda^2\mu(\lambda+\mu)(\lambda+\mu)^2}{\lambda^3\mu(\lambda+\mu)^2} = 0.\end{aligned}$$

To compute $\langle pt, \beta_8\rangle_{0,2,\beta}$, we take BD for point class and the standard representative for β_8, where f is taken to be f_0. Then $\langle pt, \beta_8\rangle_{0,2,\beta} = 0$ for $d \neq 2$. For $d = 2$, there are two connected components contributing to localization described by the following graphs:

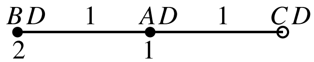
Γ_1 

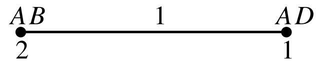
Γ_2 

The equivariant Euler classes of the normal bundles are the same as those of Γ_1 and Γ_2 for $\langle pt, \beta_6\rangle_{0,2,\beta_2+\beta_3}$, that is, $e^{\mathbb{C}^*}(N_{\Gamma_1}^{\mathrm{vir}}) = \lambda^3\mu(\lambda+\mu)^2$ and $e^{\mathbb{C}^*}(N_{\Gamma_2}^{\mathrm{vir}}) = -\lambda^3\mu^2(\lambda+\mu)$. So

$$\langle pt, \beta_8\rangle_{0,2,\beta_2+\beta_3} = \frac{-\lambda^2\mu(\lambda+\mu)\lambda(\lambda+\mu)}{\lambda^3\mu(\lambda+\mu)^2} + \frac{-\lambda^2\mu(\lambda+\mu)\lambda\mu}{-\lambda^3\mu^2(\lambda+\mu)} = 0.$$

If we take AD for point class and the point B in the standard representative for β_9, then we see that $\langle pt, \beta_9\rangle_{0,2,\beta} = 0$ for $d \neq 2$. For $d = 2$, the nonzero terms in the localization formula come from the connected components described by the following graphs:

Γ_1 

Γ_2 

The virtual normal bundles are $e^{\mathbb{C}^*}(N_{\Gamma_1}^{\mathrm{vir}}) = -\lambda^3\mu^2(\lambda+\mu)$ and $e^{\mathbb{C}^*}(N_{\Gamma_2}^{\mathrm{vir}}) = -\lambda^3\mu^2(\lambda+\mu)$. So, by the localization formula,

$$\langle pt, \beta_9\rangle_{0,2,\beta_2+\beta_3} = \frac{-\lambda^2\mu(\lambda+\mu)\lambda\mu}{-\lambda^3\mu^2(\lambda+\mu)} + \frac{-\lambda^2\mu(\lambda+\mu)\lambda\mu}{-\lambda^3\mu^2(\lambda+\mu)} = 2. \qquad \square$$

Here the invariants $\langle pt, \beta_5\rangle_{0,2,\beta}$ and $\langle pt, \beta_7\rangle_{0,2,\beta}$ when $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$ are not treated because of families of invariant curves. We will come back to these in the last section.

For the pairs $(d_2, d_3) = (3, 0), (1, 1)$, the second-type decomposition $3 + 3$ is dealt with in the following:

PROPOSITION 3.7. (i) *For* $\beta = d\beta_1 + 3(\beta_2 - \beta_1)$, $\langle \beta_i, \beta_j\rangle_{0,2,\beta} = 0$ *for* $i, j = 1, 2, 3$ *and any* d;

(ii) *For* $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$, $\langle \beta_1, \beta_2\rangle_{0,2,\beta} = 0$ *for any* d, $\langle \beta_2, \beta_2\rangle_{0,2,\beta} = \langle \beta_2, \beta_3\rangle_{0,2,\beta} = 0$ *for any* $d \neq 2$, *but for* $d = 2$, $\langle \beta_2, \beta_2\rangle_{0,2,\beta} = -1$ *and* $\langle \beta_2, \beta_3\rangle_{0,2,\beta} = 1$.

Proof. (i) Let $\beta = d\beta_1 + 3(\beta_2 - \beta_1)$. If we take the invariant curve $[B_1, B_2]$ for β_1, then it stays away from any nodal invariant curve of degree β. So $\langle \beta_1, \beta_j\rangle_{0,2,\beta} = 0$ for $j = 1, 2, 3$.

If we take $[AB, BC]$ for one representative of β_2 and $[AD, CD]$ for another representative of β_2, then the two representatives do not touch any invariant curve of degree β simultaneously. So $\langle \beta_2, \beta_2 \rangle_{0,2,\beta} = 0$.

If we still take $[AB, BC]$ for the representative of β_2 and $[AD, BD]$ for the representative of β_3, then again, for the same reason, $\langle \beta_2, \beta_3 \rangle_{0,2,\beta} = 0$.

If we take $[BC, BD]$ for one representative of β_3 and $[AC, AD]$ for another representative of β_3, then , for the same reason, $\langle \beta_3, \beta_3 \rangle_{0,2,\beta} = 0$.

(ii) Then we consider the case where $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$. If we take $[B_1, B_2]$ for β_1 and $[AD, CD]$ for β_2, then we see that $\langle \beta_1, \beta_2 \rangle_{0,2,\beta} = 0$.

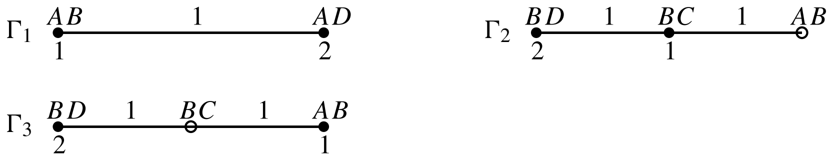
When $d \neq 2$, we take $[AB, BC]$ and $[AD, CD]$ for two representatives of β_2 and $[AD, BD]$ for β_3. Then we see that $\langle \beta_2, \beta_2 \rangle_{0,2,\beta} = \langle \beta_2, \beta_3 \rangle_{0,2,\beta} = 0$.

Let us now assume that $d = 2$. For $\langle \beta_2, \beta_2 \rangle_{0,2,\beta_2+\beta_3}$, the nonzero terms in the localization formula come from the components described by the following graphs:

We have $e^{\mathbb{C}^*}(N_{\Gamma_1}^{\mathrm{vir}}) = -\lambda^3\mu^2(\lambda+\mu)$ and $e^{\mathbb{C}^*}(N_{\Gamma_2}^{\mathrm{vir}}) = \lambda^3\mu(\lambda+\mu)^2$. With localization,

$$\langle \beta_2, \beta_2 \rangle_{0,2,\beta_2+\beta_3} = \frac{-\lambda\mu^2\lambda\mu(\lambda+\mu)}{-\lambda^3\mu^2(\lambda+\mu)} + \frac{-\lambda\mu(\lambda+\mu)\lambda(\lambda+\mu)^2}{\lambda^3\mu(\lambda+\mu)^2} = -1.$$

To compute $\langle \beta_2, \beta_3 \rangle_{0,2,\beta_2+\beta_3}$, we take $[AB, BC]$ for β_2 and $[AD, BD]$ for β_3. Three nonzero terms appear in the localization formula from the following components:



From the work done before, $e^{\mathbb{C}^*}(N_{\Gamma_1}^{\mathrm{vir}}) = -\lambda^3\mu^2(\lambda+\mu)$, $e^{\mathbb{C}^*}(N_{\Gamma_2}^{\mathrm{vir}}) = \lambda^3\mu(\lambda+\mu)^2$, and $e^{\mathbb{C}^*}(N_{\Gamma_3}^{\mathrm{vir}}) = -\lambda^3\mu^2(\lambda+\mu)$. So, by the localization formula,

$$\begin{aligned}\langle \beta_2, \beta_3 \rangle_{0,2,\beta_2+\beta_3} &= \frac{-\lambda\mu^2\lambda^2(\lambda+\mu)}{-\lambda^3\mu^2(\lambda+\mu)} \\ &\quad + \frac{-\lambda\mu(\lambda+\mu)\lambda^2(\lambda+\mu)}{\lambda^3\mu(\lambda+\mu)^2} + \frac{-\lambda\mu^2\lambda^2(\lambda+\mu)}{-\lambda^3\mu^2(\lambda+\mu)} = 1. \qquad \square\end{aligned}$$

Note that for $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$, $\langle \beta_1, \beta_1 \rangle$, $\langle \beta_1, \beta_3 \rangle$, and $\langle \beta_3, \beta_3 \rangle$ are not computable by this method.

For the pairs $(d_2, d_3) = (2, 0), (0, 1)$, we have $\operatorname{virdim} \overline{\mathcal{M}}_{0,2}(F^{[2]}, \beta) = 5$, which can be decomposed as either $1 + 4$ or $2 + 3$. For the first type, one insertion has to be a point class.

Proposition 3.8. (i) *For* $\beta = d\beta_1 + 2(\beta_2 - \beta_1)$, $\langle pt, \beta_j \rangle_{0,2,\beta} = 0$ *for* $j = 10, 11, 12$ *and any* d;

(ii) *For* $\beta = d\beta_1 + (\beta_3 - \beta_1)$, $\langle pt, \beta_j\rangle_{0,2,\beta} = 0$, *for* $j = 10, 11, 12$ *and* $d \neq 1$; *for* $d = 1$, $\langle pt, \beta_{10}\rangle_{0,2,\beta_3} = 0$, $\langle pt, \beta_{11}\rangle_{0,2,\beta_3} = 2$, *and* $\langle pt, \beta_{12}\rangle_{0,2,\beta_3} = 0$.

Proof. By the axiom of divisors, $\langle pt, \beta_j\rangle_{0,2,\beta} = \beta_j \cdot \beta\langle pt\rangle_{0,1,\beta}$, where the intersection product of β_j with β can be determined by the results in (2.1), and $\langle pt\rangle_{0,1,\beta}$ is determined in Proposition 3.2. □

For the second type, we have the following:

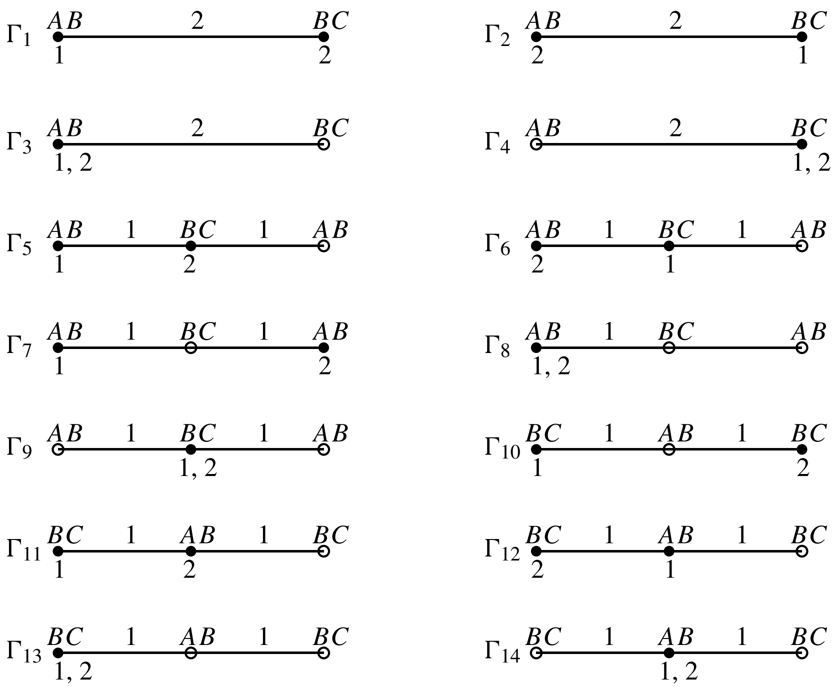
PROPOSITION 3.9. (i) *For* $\beta = d\beta_1 + 2(\beta_2 - \beta_1)$, $\langle \beta_i, \beta_j\rangle_{0,2,\beta} = 0$ *for* $i = 1, 2, 3$, $j = 4, \dots, 9$, *and any* d;

(ii) *For* $\beta = d\beta_1 + (\beta_3 - \beta_1)$, $\langle \beta_1, \beta_j\rangle_{0,2,\beta} = 0$ *for* $j = 5, 7, 8, 9$ *and any* d; $\langle \beta_2, \beta_j\rangle_{0,2,\beta} = 0$ *for* $j = 5, 7$ *and any* d; $\langle \beta_2, \beta_j\rangle_{0,2,\beta} = 0$ *for* $j = 8, 9$ *and* $d \neq 1$, *but* $\langle \beta_2, \beta_8\rangle_{0,2,\beta_3} = \langle \beta_2, \beta_9\rangle_{0,2,\beta_3} = 1$; $\langle \beta_3, \beta_j\rangle_{0,2,\beta} = 0$ *for* $j = 4, 5, 7, 8, 9$ *and any* d; $\langle \beta_3, \beta_6\rangle_{0,2,\beta} = 0$ *for* $d \neq 1$, *but* $\langle \beta_3, \beta_6\rangle_{0,2,\beta_3} = 2$.

Proof. (i) If we take $[B_1, B_2]$ as the representative for β_1 and the standard representatives of β_j, where we make free choices for f and pt, then we see that $\langle \beta_1, \beta_j\rangle_{0,2,\beta} = 0$ for $j = 4, \dots, 9$ and any d.

If we take $[AB, BC]$ for β_2, then $\langle \beta_2, \beta_j\rangle_{0,2,\beta} = 0$ for $j = 4, 5, 8, 9$ and any d, where in the standard representatives of β_5, β_8, and β_9, we take f to be f_∞ and pt to be the point D. Also, $\langle \beta_2, \beta_6\rangle_{0,2,\beta} = \langle \beta_2, \beta_7\rangle_{0,2,\beta} = 0$ when $d \neq 2$.

So we need to consider the cases where $d = 2$. For $\langle \beta_2, \beta_6\rangle_{0,2,2\beta_2}$, the nonzero terms in the localization formula come from the following components:



Here as examples we only work out the equivariant Euler classes of the normal bundles for Γ_1 and Γ_9. The other components can be dealt with either similarly or as before.

For Γ_1, $e^{\mathbb{C}^*}(B_1^m) = 1$ and $e^{\mathbb{C}^*}(B_4^m) = 1$. Since the induced action on the invariant line from AB to BC has weights $\frac{1}{2}\lambda$ and $-\frac{1}{2}\lambda$ at the two ends because of the double cover, using K-theoretic localization, we have

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \frac{t^\lambda + t^\mu + t^\lambda + t^{-\mu}}{1 - t^{(-1/2)\lambda}} + \frac{t^\lambda + t^{-\mu} + t^{-\lambda} + t^{\lambda+\mu}}{1 -^{(-1/2)\lambda}}\\ &= t^\lambda + t^{-\lambda} + t^{-\mu} +^{(-1/2)\lambda} + t^{(-1/2)\lambda} - t^{(1/2)\lambda+\mu} + 1.\end{aligned}$$

From this we have $e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_1}) = -\frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu}$.

For Γ_9, $e^{\mathbb{C}^*}(B_1^m) = \lambda^2$ and $e^{\mathbb{C}^*}(B_4^m) = (-\lambda - e_3)(-\lambda - e_4) = (\lambda + e_3)(\lambda + e_4)$, where e_3 and e_4 are the Euler classes of the respective cotangent line bundles over $\overline{\mathcal{M}}_{0,4}$, which correspond to the nodal points of the component represented by the vertex BC with the components represented by two edges from AB to BC. By K-theoretic localization and using normalization sequence as before, we have

$$\chi(f^*TF^{[2]}) = 3t^\lambda + t^{-\lambda} + t^{-\mu} - t^{\lambda+\mu} + 2.$$

So $e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_9}) = \lambda^2\mu(\lambda + e_3)(\lambda + e_4)/(\lambda + \mu)$.

All the equivariant Euler classes of the normal bundles are listed as follows:

$$\begin{aligned}
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_1}) &= -\frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu},\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_2}) &= -\frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu},\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_3}) &= \frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu},\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_4}) &= \frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu},\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_5}) &= \frac{\lambda^5\mu}{\lambda+\mu},\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_6}) &= \frac{\lambda^5\mu}{\lambda+\mu},\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_7}) &= -2\frac{\lambda^5\mu}{\lambda+\mu},\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_8}) &= -2\frac{\lambda^5\mu}{\lambda+\mu},\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_9}) &= \frac{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)}{\lambda+\mu},\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_{10}}) &= -2\lambda^5,\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_{11}}) &= \lambda^5,
\end{aligned}$$

$$
\begin{aligned}
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_{12}}) &= \lambda^5,\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_{13}}) &= -2\lambda^5,\\
e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma_{14}}) &= -\lambda^2(\lambda-e_3)(\lambda-e_4).
\end{aligned}
$$

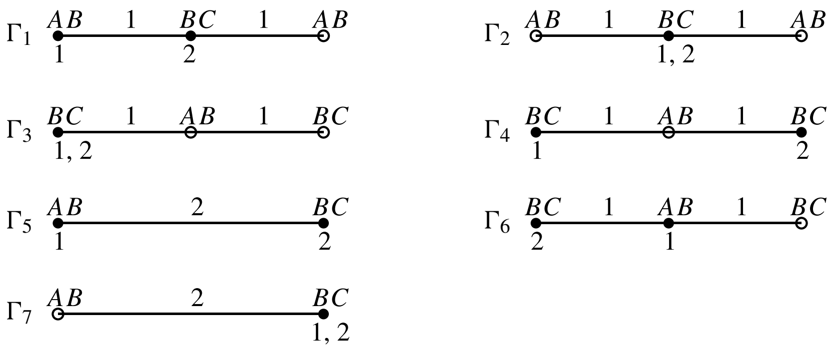
Using the localization formula, we have

$$
\begin{aligned}
\langle\beta_2,\beta_6\rangle_{0,1,2\beta_2} = &-2\frac{\lambda\mu^3(\lambda+\mu)(\lambda+2\mu)}{\lambda^5\mu}+\frac{\lambda\mu^4(\lambda+2\mu)}{\lambda^5\mu}\\
&+\frac{\lambda\mu^2(\lambda+\mu)^2(\lambda+2\mu)}{\lambda^5\mu}+\frac{\lambda\mu^3(\lambda+\mu)^2}{\lambda^5\mu}\\
&+\frac{\lambda\mu^3(\lambda+\mu)^2}{\lambda^5\mu}+\frac{\lambda\mu^4(\lambda+\mu)}{-2\lambda^5\mu}\\
&+\frac{\lambda\mu^4(\lambda+\mu)}{-2\lambda^5\mu}+\frac{1}{2}\int_{\overline{\mathcal{M}}_{0,4}}\frac{\lambda\mu^2(\lambda+\mu)^3}{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)}\\
&+\frac{\lambda\mu^2(\lambda+\mu)^2}{-2\lambda^5}+\frac{\lambda\mu^3(\lambda+\mu)}{\lambda^5}\\
&+\frac{\lambda\mu^3(\lambda+\mu)}{\lambda^5}+\frac{\lambda\mu^2(\lambda+\mu)^2}{-2\lambda^5}\\
&-\frac{1}{2}\int_{\overline{\mathcal{M}}_{0,4}}\frac{\lambda\mu^4}{\lambda^2(\lambda-e_3)(\lambda-e_4)}=0,
\end{aligned}
$$

where the factor 2 in front of the first term in the sum takes care of terms from Γ_1 and Γ_2. Here we used the fact that $\int_{\overline{\mathcal{M}}_{0,4}} e_3=\int_{\overline{\mathcal{M}}_{0,4}} e_4=1$.

From now on, since all the computational steps can be carried out similarly as before, we omit the details for evaluating the invariants, just being content with listing the graphs for the connected components of the fixed loci.

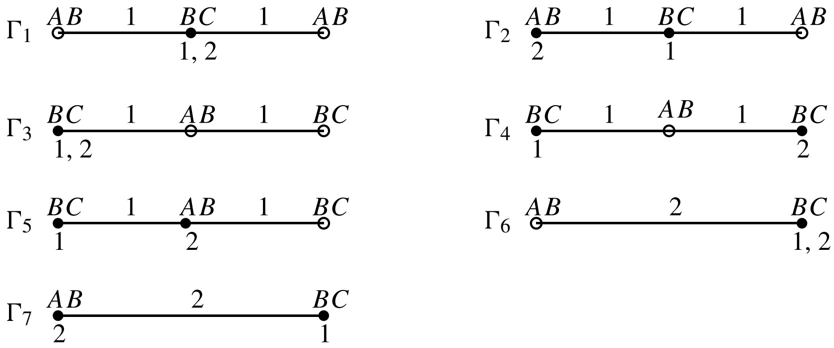
Let us turn to $\langle\beta_2,\beta_7\rangle_{0,2,2\beta_2}$. We still take $[AB, BC]$ for β_2. The nonzero terms in the localization formula are from the following components:



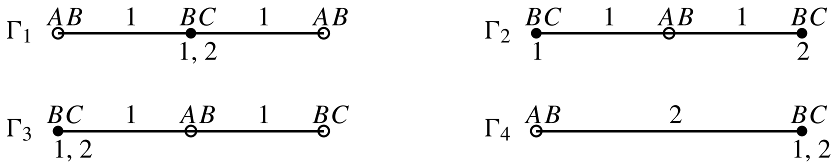
If we take $[BC, BD]$ for β_3 and the standard representatives for β_4, β_5, and β_8, where we take f to be f_∞, then $\langle\beta_3,\beta_j\rangle_{0,2,\beta}=0$ for $j=4,5,8$ and any d; if we take $[AC, BC]$ for β_3 and the standard representative for β_9 where we assign D to pt, then we see that $\langle\beta_3,\beta_9\rangle_{0,2,\beta}=0$. Now we keep the invariant line between

BC and BD for β_3. It is not hard to see that $\langle\beta_3, \beta_6\rangle_{0,2,\beta} = \langle\beta_3, \beta_7\rangle_{0,2,\beta} = 0$ when $d \neq 2$.

When $d = 2$, for $\langle\beta_3, \beta_6\rangle_{0,2,\beta}$, the nonzero terms in the localization formula come from the fixed point loci described by the following graphs:



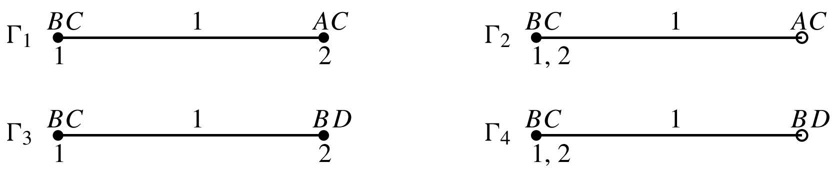
For $\langle\beta_3, \beta_7\rangle_{0,2,2\beta_2}$, we keep the representative for β_3. Then the connected components with nonzero terms in the localization formula are described by the following graphs:



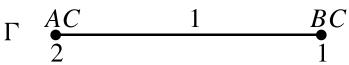
(ii) When $\beta = d\beta_1 + (\beta_3 - \beta_1)$, we take the invariant line between C_1 and C_2 for β_1 and the standard representatives for β_j, where we take f to be f_0 and pt to be A, then we see that $\langle\beta_1, \beta_j\rangle_{0,2,\beta} = 0$ for $j = 5, 7, 8, 9$ and any d.

Also, if we take the invariant line between AD and CD for β_2 and the standard representative for β_5, where f is taken to be f_0, then $\langle\beta_2, \beta_5\rangle_{0,2,\beta} = 0$ for any d.

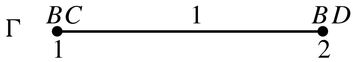
Now we take the invariant line from AB to BC for the representative for β_2. Then $\langle\beta_2, \beta_7\rangle_{0,2,\beta} = 0$ if $d \neq 1$. When $d = 1$, the nonzero terms appearing in the localization formula are given by the connected components described by the following graphs:



To compute $\langle\beta_2, \beta_8\rangle_{0,2,\beta}$, we still take the invariant line from AB to BC for the representative for β_2 and the standard representative for β_8, where f is taken to be f_∞. Then we see that $\langle\beta_2, \beta_8\rangle_{0,2,\beta} = 0$ if $d \neq 1$. When $d = 1$, there is only one nonzero term in the localization from the component described by the graph

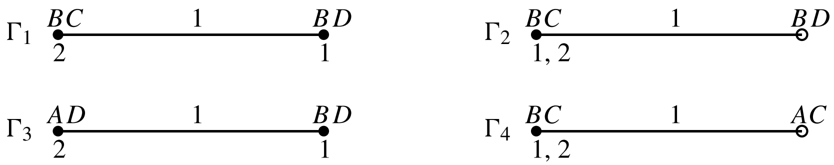


To compute $\langle \beta_2, \beta_9 \rangle_{0,2,\beta}$, we still take the invariant line from AB to BC for the representative for β_2 and the standard representative for β_9, where *pt* is taken to be D. Then we see that $\langle \beta_2, \beta_9 \rangle_{0,2,\beta} = 0$ if $d \neq 1$. When $d = 1$, there is only one nonzero term in the localization formula given by the connected component described by the graph



If we take the invariant line between BC and BD for β_3, we see that $\langle \beta_3, \beta_j \rangle_{0,2,\beta} = 0$ for $j = 4, 5$ and all d, where we can take either f_0 or f_∞ for f in the representative of β_5.

Now we fix the representative for β_3 to be the invariant line between BC and BD. Then $\langle \beta_3, \beta_6 \rangle_{0,2,\beta} = 0$ for all $d \neq 1$. For $d = 1$, there are nonzero terms in the localization from the connected components described by the following graphs:



To compute $\langle \beta_3, \beta_j \rangle_{0,2,\beta}$ for $j = 7, 8, 9$, we take the invariant line from CD to D_2 for $\beta_3 - \beta_1$ and the standard representatives for β_j, where f is taken to be f_0 and *pt* to be B. Then we see that $\langle \beta_3 - \beta_1, \beta_j \rangle_{0,2,\beta} = 0$ for any d. However, $\langle \beta_1, \beta_j \rangle_{0,2,\beta} = 0$, so $\langle \beta_3, \beta_j \rangle_{0,2,\beta} = 0$ for $j = 7, 8, 9$. □

In this proposition, four sequences of invariants are left untreated, which are $\langle \beta_1, \beta_4 \rangle_{0,2,\beta}$, $\langle \beta_2, \beta_4 \rangle_{0,2,\beta}$, $\langle \beta_1, \beta_6 \rangle_{0,2,\beta}$, and $\langle \beta_2, \beta_6 \rangle_{0,2,\beta}$ for $\beta = d\beta_1 + (\beta_3 - \beta_1)$, because they involve higher degrees on β_1 and thus encounter the problem of the families of invariant curves. They will be determined in the next section.

For the last pair $(d_2, d_3) = (1, 0)$, the virtual dimension of the moduli space is equal to 4. The degree decomposition of the two insertions has to be $1 + 3$ or $2 + 2$. For the first type, we have the following:

PROPOSITION 3.10. *For* $\beta = d\beta_1 + (\beta_2 - \beta_1)$,

(i) $\langle \beta_1, \beta_j \rangle_{0,2,\beta} = 0$ *for* $j = 10, 11, 12$ *and all* d;

(ii) $\langle \beta_i, \beta_j \rangle_{0,2,\beta} = 0$ *for* $i = 2, 3$, $j = 10, 11, 12$, *and* $d \neq 1$; *when* $d = 1$, $\langle \beta_2, \beta_{10} \rangle_{0,2,\beta_2} = 0$, $\langle \beta_2, \beta_{11} \rangle_{0,2,\beta_2} = 1$, $\langle \beta_2, \beta_{12} \rangle_{0,2,\beta_2} = -1$, $\langle \beta_3, \beta_{10} \rangle_{0,2,\beta_2} = 0$, $\langle \beta_3, \beta_{11} \rangle_{0,2,\beta_2} = -1$, *and* $\langle \beta_3, \beta_{12} \rangle_{0,2,\beta_2} = 1$.

Proof. By the axiom of divisors and Proposition 3.3. □

When $(d_2, d_3) = (1, 0)$, we have the second type of degree decomposition of the two insertions $2 + 2$.

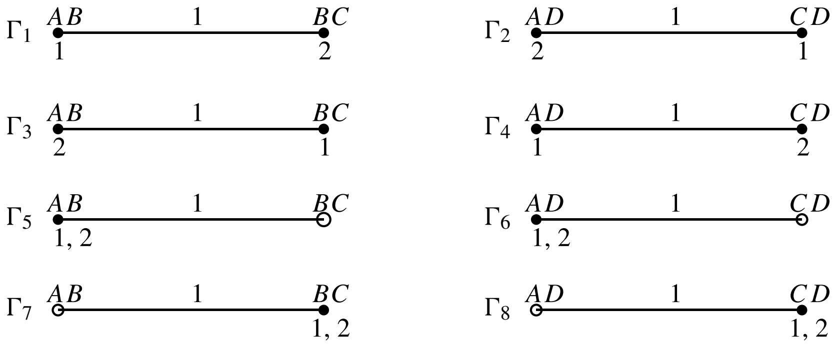
PROPOSITION 3.11. *For* $\beta = d\beta_1 + (\beta_2 - \beta_1)$, *we have*

(i) $\langle \beta_4, \beta_6 \rangle_{0,2,\beta} = \langle \beta_5, \beta_6 \rangle_{0,2,\beta} = 0$ *for any* d;

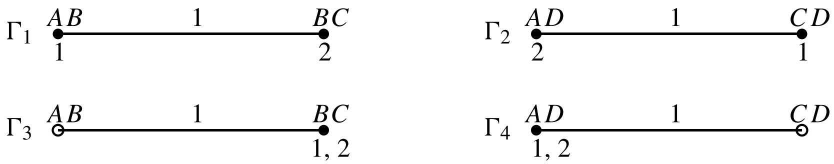
(ii) $\langle\beta_6, \beta_k\rangle_{0,2,\beta} = 0$ *for* $k = 6, 7, 8$ *and* $d \neq 1$, *but* $\langle\beta_6, \beta_6\rangle_{0,2,\beta_2} = 1$, $\langle\beta_6, \beta_7\rangle_{0,2,\beta_2} = -2$, *and* $\langle\beta_6, \beta_8\rangle_{0,2,\beta_2} = 1$;
(iii) $\langle\beta_k, \beta_9\rangle_{0,2,\beta} = 0$ *for* $k = 4, \ldots, 9$ *and any* d.

Proof. (i) If we take the standard representatives for β_4, β_5, and β_6, where f is taken to be f_0 for β_5, then we see that $\langle\beta_4, \beta_6\rangle_{0,2,\beta} = \langle\beta_5, \beta_6\rangle_{0,2,\beta} = 0$.

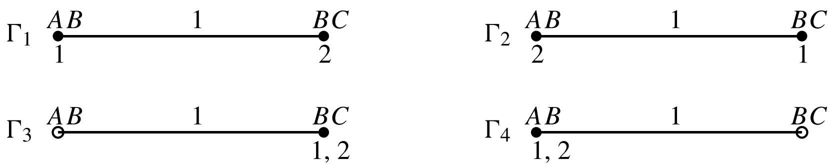
(ii) Also, we can see that when $d \neq 1$, $\langle\beta_6, \beta_6\rangle_{0,2,\beta} = 0$. Now we compute $\langle\beta_6, \beta_6\rangle_{0,2,\beta}$ when $d = 1$. There are eight nonzero terms in the localization formula from fixed point loci described by the following graphs:



Using the standard representative for β_7, we see that when $d \neq 1$, $\langle\beta_6, \beta_7\rangle_{0,2,\beta} = 0$. When $d = 1$, nonzero terms in the localization formula are given by the fixed point locus described by the following graphs:



We take f_0 for f in the representative for β_8. Then $\langle\beta_6, \beta_8\rangle_{0,2,\beta} = 0$ for $d \neq 1$. When $d = 1$, there are nonzero terms in the localization formula from fixed point loci given by the following graphs:



(iii) If we use the above representative for β_8 and the standard representative for β_9, where *pt* is taken to be D, then we see that $\langle\beta_8, \beta_9\rangle_{0,2,\beta} = 0$ for any d. If we take a different representative for β_9, where *pt* is taken to be B, then we see that $\langle\beta_9, \beta_9\rangle_{0,2,\beta} = 0$ for any d. Also, $\langle\beta_4, \beta_9\rangle_{0,2,\beta} = \langle\beta_5, \beta_9\rangle_{0,2,\beta} = 0$ for any d, where we use f_∞ for f and B for *pt* in the standard representatives for β_5 and β_9.

We keep the representative for β_9, where *pt* is taken to B. Then we see that when $d \neq 1$, $\langle\beta_6, \beta_9\rangle_{0,2,\beta} = \langle\beta_7, \beta_9\rangle_{0,2,\beta} = 0$. When $d = 1$, for $\langle\beta_6, \beta_9\rangle_{0,2,\beta}$, the connected components appearing as nonzero terms in the localization formula are the same as for $\langle \beta_6, \beta_8\rangle_{0,2,\beta}$.

For $\langle \beta_7, \beta_9\rangle_{0,2,\beta}$, we have the following connected components:

□

The invariants $\langle \beta_4, \beta_4\rangle_{0,2,\beta}$, $\langle \beta_4, \beta_5\rangle_{0,2,\beta}$, $\langle \beta_4, \beta_7\rangle_{0,2,\beta}$, $\langle \beta_4, \beta_8\rangle_{0,2,\beta}$, $\langle \beta_5, \beta_5\rangle_{0,2,\beta}$, $\langle \beta_5, \beta_7\rangle_{0,2,\beta}$, $\langle \beta_5, \beta_8\rangle_{0,2,\beta}$, $\langle \beta_7, \beta_7\rangle_{0,2,\beta}$, $\langle \beta_7, \beta_8\rangle_{0,2,\beta}$, and $\langle \beta_8, \beta_8\rangle_{0,2,\beta}$ are left untouched because of the existence of continuous families of invariant curves. They will be dealt with in the next section.

4. Other Two-Pointed Gromov–Witten Invariants

In this section, we first compute the quantum product of generators of the cohomology ring of the Hilbert scheme and then make use of the associativity law of the quantum product to determine other two-pointed Gromov–Witten invariants remaining from the localization method.

4.1. Quantum Product of Generators

The dual basis of our standard basis $\beta_0, \beta_1, \ldots, \beta_{12}, \beta_{13}$ is $\beta_{13}, -\frac{1}{2}\beta_{10}, \beta_{12}, \beta_{11}+\beta_{12}, -\frac{1}{2}\beta_5, -\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5, \frac{1}{2}\beta_7, \frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8, \frac{1}{2}\beta_7 + \beta_8, \beta_9, -\frac{1}{2}\beta_1, \beta_3, \beta_2 + \beta_3, \beta_0$.

With the computational results in Subsections 3.2 and 3.3, the quantum product of $\beta_{10}, \beta_{11}, \beta_{12}$ can be computed. First, by definition,

$$\begin{aligned}
\beta_{10} * \beta_{10} &= \beta_{10}^2 + \sum_{\beta\neq 0}\sum_i \langle \beta_{10}, \beta_{10}, T_i\rangle_\beta q^\beta T^i \\
&= \beta_{10}^2 + (\beta_3\cdot\beta_{10})^2\langle pt\rangle_{\beta_3} q_1 q_3 + (\beta_2\cdot\beta_{10})^2\langle \beta_2\rangle_{\beta_2} q_1 q_2 \beta_{12} \\
&\quad + (\beta_2\cdot\beta_{10})^2\langle\beta_3\rangle_{\beta_2} q_1 q_2(\beta_{11}+\beta_{12}) \\
&\quad + \sum_{d\neq 0}(d\beta_1\cdot\beta_{10})^2\langle\beta_4\rangle_{d\beta_1} q_1^d\left(-\frac{1}{2}\beta_5\right) \\
&\quad + \sum_{d\neq 0}(d\beta_1\cdot\beta_{10})^2\langle\beta_5\rangle_{d\beta_1} q_1^d\left(-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5\right) \\
&= \beta_{10}^2 + \sum_{d\neq 0}(-2d)^2\frac{-2}{d}q_1^d\left(-\frac{1}{2}\beta_5\right) \\
&\quad + \sum_{d\neq 0}(-2d)^2\frac{-4}{d}q_1^d\left(-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5\right) \\
&= \beta_{10}^2 + 8\sum_{d\neq 0} d q_1^d \beta_4 + 12\sum_{d\neq 0} d q_1^d \beta_5,
\end{aligned}$$

where $\langle\beta_{10}, \beta_{10}, T_i\rangle_\beta$ means GW-invariant of genus 0 and curve class β, and so on. We omit the number of insertions in the notation from now on. Also, here and subsequently, summations over $d \neq 0$ or just d in fact mean over $d > 0$ or $d \geq 0$. In these equalities, we use the axiom of divisors to reduce the 3-pointed GW-invariants to 1-pointed ones and omit trivial terms because either the invariants involved are zero or the intersections $\beta_2 \cdot \beta_{10} = \beta_3 \cdot \beta_{10} = 0$. The fact that $\beta_1 \cdot \beta_{10} = -2$ is also used. Similarly, other products can also be calculated and are summarized here:

$$\begin{aligned}
\beta_{10} * \beta_{10} &= \beta_{10}^2 + 8\sum_{d\neq 0} dq_1^d\beta_4 + 12\sum_{d\neq 0} dq_1^d\beta_5,\\
\beta_{10} * \beta_{11} &= \beta_{10}\beta_{11},\\
\beta_{10} * \beta_{12} &= \beta_{10}\beta_{12},\\
\beta_{11} * \beta_{11} &= \beta_{11}^2 + q_1q_2\beta_{11} + 2q_1q_3,\\
\beta_{11} * \beta_{12} &= \beta_{11}\beta_{12} - q_1q_2\beta_{11},\\
\beta_{12} * \beta_{12} &= \beta_{12}^2 + q_1q_2\beta_{11}.
\end{aligned} \tag{4.1}$$

Furthermore,

$$\begin{aligned}
\beta_9 * \beta_{10} &= \beta_9\beta_{10} + \sum_{\beta\neq 0}\sum_i \langle\beta_9, \beta_{10}, T_i\rangle_\beta q^\beta T^i\\
&= \beta_9\beta_{10} + \sum_d \langle\beta_9, \beta_{10}, pt\rangle_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} q_1^d q_2 q_3\\
&\quad + \sum_d \langle\beta_9, \beta_{10}, \beta_2\rangle_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3\beta_{12} = \beta_9\beta_{10}.
\end{aligned}$$

Here $\langle\beta_9, pt\rangle_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)}$ and $\langle\beta_9, \beta_2\rangle_{d\beta_1+(\beta_3-\beta_1)}$ are nontrivial only when $d = 2$ and $d = 1$ by Propositions 3.6 and 3.9, respectively, but the terms involving them also vanish by the axiom of divisors since $(\beta_2 + \beta_3) \cdot \beta_{10} = \beta_3 \cdot \beta_{10} = 0$. Similarly, $\beta_9 * \beta_{11}$ and $\beta_9 * \beta_{12}$ can be computed. The results are listed as follows:

$$\begin{aligned}
\beta_9 * \beta_{10} &= \beta_9\beta_{10},\\
\beta_9 * \beta_{11} &= \beta_9\beta_{11} + q_1q_3\beta_{12},\\
\beta_9 * \beta_{12} &= \beta_9\beta_{12} + 2q_1^2q_2q_3.
\end{aligned} \tag{4.2}$$

4.2. Associativity of Quantum Product

Gromov–Witten invariants enjoy strong relations arising from the associativity of the quantum product. In this section, we use the associativity relations among generators of the cohomology ring to derive all the remaining two-pointed invariants.

First we have the following:

LEMMA 4.1. *There are identities of quantum product associated to divisor classes*:

(1) $2\beta_4 * \beta_{12} + 3\beta_5 * \beta_{12} - \beta_7 * \beta_{12} - 2\beta_8 * \beta_{12} - 2\beta_9 * \beta_{12} + 4\sum_{d\neq 0} dq_1^d \beta_4 * \beta_{12} + 6\sum_{d\neq 0} dq_1^d \beta_5 * \beta_{12} = \beta_5 * \beta_{10}$;
(2) $\beta_7 * \beta_{10} + 2q_1q_2\beta_4 = 2\beta_5 * \beta_{12}$;
(3) $\beta_8 * \beta_{10} + \beta_9 * \beta_{10} - 2q_1q_2\beta_4 = 2\beta_4 * \beta_{12}$;
(4) $\beta_4 * \beta_{12} = \beta_5 * \beta_{11}$.

Proof. For (1), we begin with the identity $\beta_{10} * (\beta_{10} * \beta_{12}) = (\beta_{10} * \beta_{10}) * \beta_{12}$. From the quantum product equalities in (4.1) in Subsection 4.1 and intersection product results (2.2) in Subsection 2.1, the left-hand side is equal to $\beta_{10} * (\beta_{10}\beta_{12}) = \beta_{10} * (2\beta_5) = 2\beta_5 * \beta_{10}$; the right-hand side is equal to

$$\begin{aligned}\Big(\beta_{10}^2 &+ 8\sum_{d\neq 0} dq_1^d\beta_4 + 12\sum_{d\neq 0} dq_1^d\beta_5\Big) * \beta_{12}\\ &= \Big(4\beta_4 + 6\beta_5 - 2\beta_7 - 4\beta_8 - 4\beta_9 + 8\sum_{d\neq 0} dq_1^d\beta_4 + 12\sum_{d\neq 0} dq_1^d\beta_5\Big) * \beta_{12}\\ &= 4\beta_4 * \beta_{12} + 6\beta_5 * \beta_{12} - 2\beta_7 * \beta_{12} - 4\beta_8 * \beta_{12} - 4\beta_9 * \beta_{12}\\ &\quad + 8\sum_{d\neq 0} dq_1^d\beta_4 * \beta_{12} + 12\sum_{d\neq 0} dq_1^d\beta_5 * \beta_{12},\end{aligned}$$

where we use the identity $\beta_{10}^2 = 4\beta_4 + 6\beta_5 - 2\beta_7 - 4\beta_8 - 4\beta_9$ from (2.2). Then equating the two sides, we obtain the first identity.

Then we look at the associativity identity $\beta_{10} * (\beta_{12} * \beta_{12}) = (\beta_{10} * \beta_{12}) * \beta_{12}$. Then, again by (4.1) and (2.2), the left-hand side is equal to $\beta_{10} * (\beta_{12}^2 + q_1q_2\beta_{11}) = \beta_{10} * \beta_7 + q_1q_2\beta_{10} * \beta_{11} = \beta_7 * \beta_{10} + q_1q_2\beta_{10}\beta_{11} = \beta_7 * \beta_{10} + 2q_1q_2\beta_4$; and the right-hand side is equal to $(\beta_{10}\beta_{12}) * \beta_{12} = 2\beta_5 * \beta_{12}$. This gives rise to identity (2).

Similarly, from the associativity identity $\beta_{10} * (\beta_{11} * \beta_{12}) = (\beta_{10} * \beta_{11}) * \beta_{12}$ we get the left-hand side to be equal to $\beta_{10} * (\beta_{11}\beta_{12} - q_1q_2\beta_{11}) = \beta_{10} * (\beta_8 + \beta_9) - q_1q_2\beta_{10} * \beta_{11} = \beta_8 * \beta_{10} + \beta_9 * \beta_{10} - 2q_1q_2\beta_4$, the right-hand side to be equal to $(\beta_{10}\beta_{11}) * \beta_{12} = 2\beta_4 * \beta_{12}$, and thus the third identity.

Finally, in the identity $(\beta_{10} * \beta_{11}) * \beta_{12} = \beta_{11} * (\beta_{10} * \beta_{12})$, the left hand-side has been worked out before; the right-hand side equals $\beta_{11} * (\beta_{10}\beta_{12}) = \beta_{11} * (2\beta_5) = 2\beta_5 * \beta_{11}$, and thus we get the fourth identity. □

For each identity, when expanded by the definition of quantum product, the terms at the two sides corresponding to the same cohomology class and the same power of the parameters should be equal to each other, so that we get relations among GW-invariants. We first consider the terms corresponding to the cohomology class 1 and $q_1^d q_2 q_3$.

Beginning with identity (2) in Lemma 4.1, we get the equation

$$\langle\beta_7, \beta_{10}, pt\rangle_d q_1^d q_2 q_3 = 2\langle\beta_5, \beta_{12}, pt\rangle_d q_1^d q_2 q_3,$$

where $\langle\beta_5, \beta_{12}, pt\rangle_d$ means the genus 0 invariants at the curve class $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$, and so on. However, by the axiom of divisors,

$\langle\beta_7,\beta_{10},pt\rangle_d=-2(d-2)\langle\beta_7,pt\rangle_d$, $\langle\beta_5,\beta_{12},pt\rangle_d=\langle\beta_5,pt\rangle_d$, so we have

$$\langle\beta_5,pt\rangle_d=-(d-2)\langle\beta_7,pt\rangle_d \quad \forall d. \tag{4.3}$$

From Lemma 4.1(1) we get

$$\begin{aligned}
&2\sum_d\langle\beta_4,\beta_{12},pt\rangle_d q_1^d q_2q_3+3\sum_d\langle\beta_5,\beta_{12},pt\rangle_d q_1^d q_2q_3\\
&\quad-\sum_d\langle\beta_7,\beta_{12},pt\rangle_d q_1^d q_2q_3-2\sum_d\langle\beta_8,\beta_{12},pt\rangle_d q_1^d q_2q_3\\
&\quad-2\sum_d\langle\beta_9,\beta_{12},pt\rangle_d q_1^d q_2q_3+4\sum_{l\neq 0} l q_1^l\sum_k\langle\beta_4,\beta_{12},pt\rangle_k q_1^k q_2q_3\\
&\quad+6\sum_{l\neq 0} l q_1^l\sum_k\langle\beta_5,\beta_{12},pt\rangle_k q_1^k q_2q_3=\sum_d\langle\beta_5,\beta_{10},pt\rangle_d q_1^d q_2q_3.
\end{aligned}$$

However, for any d, $\langle\beta_4,\beta_{12},pt\rangle_d=\langle\beta_8,\beta_{12},pt\rangle_d=0$ since, by Proposition 3.6, $\langle\beta_4,pt\rangle_d=\langle\beta_8,pt\rangle_d=0$ and, for any $d\neq 2$, $\langle\beta_9,\beta_{12},pt\rangle_d=\langle\beta_9,pt\rangle_d=0$, but when $d=2$, $\langle\beta_9,\beta_{12},pt\rangle_d=2$. With these in place, the last equation simplifies to

$$\begin{aligned}
&\sum_d(2d-1)\langle\beta_5,pt\rangle_d q_1^d q_2q_3-\sum_d\langle\beta_7,pt\rangle_d q_1^d q_2q_3\\
&\qquad+6\sum_{l\neq 0} l q_1^l\sum_k\langle\beta_5,pt\rangle_k q_1^k q_2q_3-4q_1^2q_2q_3=0.
\end{aligned}$$

Let $a_d=\langle\beta_5,pt\rangle_d$. Then

$$\sum_{l\neq 0} l q_1^l\sum_k\langle\beta_5,pt\rangle_k q_1^k q_2q_3=\sum_d(a_{d-1}+2a_{d-2}+\cdots+da_0)q_1^d q_2q_3.$$

Substituting this in the above equation, making use of relation (4.3), equating the terms in front of the monomial $q_1^d q_2q_3$, with $d>2$, and simplifying, we obtain the recursive relation

$$a_d=-\frac{6(d-2)}{(2d-3)(d-1)}(a_{d-1}+2a_{d-2}+\cdots+da_0). \tag{4.4}$$

This is summarized in the following:

PROPOSITION 4.2. *Let $\beta=d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)$, and let $\langle\cdot,\cdot\rangle_d$ denote the GW-invariants with respect to the curve class β.*

(1) *Let $\langle\beta_5,pt\rangle_d=a_d$. Then, for $d>2$, $\langle\beta_5,pt\rangle_d$ can be recursively calculated by* (4.4) *with the initial data $\langle\beta_5,pt\rangle_0=0$, $\langle\beta_5,pt\rangle_1=1$, $\langle\beta_5,pt\rangle_2=0$;*
(2) *$\langle\beta_7,pt\rangle_d=-\frac{1}{d-2}a_d$ for $d\neq 2$ with $\langle\beta_7,pt\rangle_2=2$.*

Proof. In (1), the initial data can be obtained directly by the localization formula. (2) follows from (4.3) with $\langle\beta_7,pt\rangle_2=2$ directly calculated. □

This finishes the computations left out in Proposition 3.6.

Next, we consider the terms corresponding to the second-degree cohomology classes and monomials $q_1^d q_2$ from the associative identities. First, we equate the two sides of identity (1) in Lemma 4.1 with the class β_4 inserted:

$$\begin{aligned}2\sum_d\langle\beta_4,\beta_{12},\beta_4\rangle_d q_1^d q_2+3\sum_d\langle\beta_5,\beta_{12},\beta_4\rangle_d q_1^d q_2\\ -\sum_d\langle\beta_7,\beta_{12},\beta_4\rangle_d q_1^d q_2-2\sum_d\langle\beta_8,\beta_{12},\beta_4\rangle_d q_1^d q_2\\ -2\sum_d\langle\beta_9,\beta_{12},\beta_4\rangle_d q_1^d q_2+4\sum_{l\neq 0} l q_1^l\sum_k\langle\beta_4,\beta_{12},\beta_4\rangle_k q_1^k q_2\\ +6\sum_{l\neq 0} l q_1^l\sum_k\langle\beta_5,\beta_{12},\beta_4\rangle_k q_1^k q_2=\sum_d\langle\beta_5,\beta_{10},\beta_4\rangle_d q_1^d q_2,\end{aligned}$$

where $\langle\beta_4,\beta_{12},\beta_4\rangle_d$, means the genus 0 invariants at the curve class $\beta=d\beta_1+(\beta_2-\beta_1)$, and so on. With $\beta=d\beta_1+(\beta_2-\beta_1)$, $\beta\cdot\beta_{12}=1$, $\beta\cdot\beta_{10}=-2(d-1)$, and $\langle\beta_9,\beta_{12},\beta_4\rangle_d=\langle\beta_9,\beta_4\rangle_d=0$ by Proposition 3.11, we can simplify the expression as

$$\begin{aligned}2\sum_d\langle\beta_4,\beta_4\rangle_d q_1^d q_2+\sum_d(2d+1)\langle\beta_5,\beta_4\rangle_d q_1^d q_2\\ -\sum_d\langle\beta_7,\beta_4\rangle_d q_1^d q_2-2\sum_d\langle\beta_8,\beta_4\rangle_d q_1^d q_2\\ +4\sum_{l\neq 0} l q_1^l\sum_k\langle\beta_4,\beta_4\rangle_k q_1^k q_2+6\sum_{l\neq 0} l q_1^l\sum_k\langle\beta_5,\beta_4\rangle_k q_1^k q_2=0.\end{aligned}\tag{4.5}$$

From identity (4) in Lemma 4.1 we get

$$\sum_d\langle\beta_4,\beta_{12},\beta_4\rangle_d q_1^d q_2=\sum_d\langle\beta_5,\beta_{11},\beta_4\rangle_d q_1^d q_2,$$

which implies that, for any d, $\langle\beta_5,\beta_4\rangle_d=-\langle\beta_4,\beta_4\rangle_d$.

From Lemma 4.1(2), noting that $\beta_4=2(-\frac{1}{2}\beta_5)-2(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5)$, we have

$$\sum_d\langle\beta_7,\beta_{10},\beta_4\rangle_d q_1^d q_2+4q_1q_2=2\sum_d\langle\beta_5,\beta_{12},\beta_4\rangle_d q_1^d q_2$$

or

$$-\sum_d(d-1)\langle\beta_7,\beta_4\rangle_d q_1^d q_2+2q_1q_2=\sum_d\langle\beta_5,\beta_4\rangle_d q_1^d q_2,$$

from which we learn that when $d>1$, $\langle\beta_7,\beta_4\rangle_d=\frac{1}{d-1}\langle\beta_4,\beta_4\rangle_d$.

From Lemma 4.1(3) we get

$$\begin{aligned}\sum_d\langle\beta_8,\beta_{10},\beta_4\rangle_d q_1^d q_2+\sum_d\langle\beta_9,\beta_{10},\beta_4\rangle_d q_1^d q_2-4q_1q_2\\ =2\sum_d\langle\beta_4,\beta_{12},\beta_4\rangle_d q_1^d q_2\end{aligned}$$

or

$$\sum_d (d-1)\langle \beta_8, \beta_4\rangle_d q_1^d q_2 + 2q_1q_2 = -\sum_d \langle \beta_4, \beta_4\rangle_d q_1^d q_2.$$

From this we know that when $d > 1$, $\langle \beta_8, \beta_4\rangle_d = -\frac{1}{d-1}\langle \beta_4, \beta_4\rangle_d$.

Substituting all these into (4.5), we obtain an equation for $\langle \beta_4, \beta_4\rangle_d$. Let $b_d = \langle \beta_4, \beta_4\rangle_d$. Simplification of this equation gives rise to the recursive relation

$$b_d = -\frac{2(d-1)}{d(2d-3)}(b_{d-1} + 2b_{d-2} + \cdots + db_0), \quad d > 1. \tag{4.6}$$

These results are included in the following:

PROPOSITION 4.3. *Let $\beta = d\beta_1 + (\beta_2 - \beta_1)$, and let $\langle \cdot, \cdot\rangle_d$ denote the GW-invariants of curve class β.*

(1) *Let $\langle \beta_4, \beta_4\rangle_d = b_d$. Then, for $d > 1$, $\langle \beta_4, \beta_4\rangle_d$ can be recursively calculated by (4.6) with the initial data $\langle \beta_4, \beta_4\rangle_0 = 1$, $\langle \beta_4, \beta_4\rangle_1 = -2$;*
(2) *$\langle \beta_4, \beta_5\rangle_d = -b_d$ and $\langle \beta_5, \beta_5\rangle_d = b_d$ for any d;*
(3) *$\langle \beta_4, \beta_7\rangle_d = \frac{1}{d-1}b_d$ for $d > 1$ with $\langle \beta_4, \beta_7\rangle_0 = -1$ and $\langle \beta_4, \beta_7\rangle_1 = 0$;*
(4) *$\langle \beta_4, \beta_8\rangle_d = -\frac{1}{d-1}b_d$ for $d > 1$ with $\langle \beta_4, \beta_8\rangle_0 = 1$ and $\langle \beta_4, \beta_8\rangle_1 = 0$;*
(5) *$\langle \beta_5, \beta_7\rangle_d = -\langle \beta_4, \beta_7\rangle_d$ and $\langle \beta_5, \beta_8\rangle_d = -\langle \beta_4, \beta_8\rangle_d$ for any d;*
(6) *$\langle \beta_7, \beta_7\rangle_d = \langle \beta_8, \beta_8\rangle_d = 1/(d-1)^2 b_d$ for $d > 1$ with $\langle \beta_7, \beta_7\rangle_0 = 1$, $\langle \beta_7, \beta_7\rangle_1 = 2$, $\langle \beta_8, \beta_8\rangle_0 = 1$, and $\langle \beta_8, \beta_8\rangle_1 = -1$;*
(7) *$\langle \beta_7, \beta_8\rangle_d = -1/(d-1)^2 b_d$ for $d > 1$ with $\langle \beta_7, \beta_8\rangle_0 = -1$ and $\langle \beta_7, \beta_8\rangle_1 = 0$.*

Proof. From identity (4) in Lemma 4.1 we get, for any d,

$$\langle \beta_4, \beta_{12}, \beta_5\rangle_d = \langle \beta_5, \beta_{11}, \beta_5\rangle_d,$$
$$\langle \beta_4, \beta_{12}, \beta_7\rangle_d = \langle \beta_5, \beta_{11}, \beta_7\rangle_d,$$
$$\langle \beta_4, \beta_{12}, \beta_8\rangle_d = \langle \beta_5, \beta_{11}, \beta_8\rangle_d,$$

or

$$\langle \beta_4, \beta_5\rangle_d = -\langle \beta_5, \beta_5\rangle_d,$$
$$\langle \beta_4, \beta_7\rangle_d = -\langle \beta_5, \beta_7\rangle_d,$$
$$\langle \beta_4, \beta_8\rangle_d = -\langle \beta_5, \beta_8\rangle_d.$$

Making use of identity (2) of the lemma, we get, for any d,

$$\langle \beta_7, \beta_{10}, \beta_7\rangle_d = 2\langle \beta_5, \beta_{12}, \beta_7\rangle_d,$$
$$\langle \beta_7, \beta_{10}, \beta_8\rangle_d = 2\langle \beta_5, \beta_{12}, \beta_8\rangle_d,$$

or

$$-(d-1)\langle \beta_7, \beta_7\rangle_d = \langle \beta_5, \beta_7\rangle_d,$$
$$-(d-1)\langle \beta_7, \beta_8\rangle_d = \langle \beta_5, \beta_8\rangle_d,$$

so, when $d > 1$,

$$\langle \beta_7, \beta_7 \rangle_d = \frac{1}{(d-1)^2} b_d,$$
$$\langle \beta_7, \beta_8 \rangle_d = -\frac{1}{(d-1)^2} b_d,$$

with initial values

$$\langle \beta_7, \beta_7 \rangle_0 = 1, \qquad \langle \beta_7, \beta_7 \rangle_1 = 2; \qquad \langle \beta_7, \beta_8 \rangle_0 = -1, \qquad \langle \beta_7, \beta_8 \rangle_1 = 0.$$

From the associativity identity (3) we get

$$\langle \beta_8, \beta_{10}, \beta_8 \rangle_d + \langle \beta_9, \beta_{10}, \beta_8 \rangle_d = 2\langle \beta_4, \beta_{12}, \beta_8 \rangle_d,$$

or for any d, $-(d-1)\langle \beta_8, \beta_8 \rangle_d = \langle \beta_4, \beta_8 \rangle_d$, so, for $d > 1$, $\langle \beta_8, \beta_8 \rangle_d = 1/(d-1)^2 b_d$ with the initial values $\langle \beta_8, \beta_8 \rangle_0 = 1$ and $\langle \beta_8, \beta_8 \rangle_1 = -1$. □

This completes the computations of Proposition 3.11.

To finish the computations of the remaining invariants in part (ii) in Proposition 3.7, we continue to work on the associativity law of quantum product, making use of the equalities in (4.1) and (4.2).

LEMMA 4.4. *There are identities*

(1) $2\beta_4 * \beta_9 = 2\beta_1 * \beta_{11} = \beta_2 * \beta_{10} + 2q_1q_3\beta_5$,
(2) $2\beta_5 * \beta_9 = 2\beta_1 * \beta_{12} = \beta_3 * \beta_{10} + 2q_1^2q_2q_3\beta_{10}$,
(3) $\beta_9 * (\beta_8 + \beta_9) - q_1q_2\beta_2 - q_1^2q_2q_3\beta_{12} = \beta_3 * \beta_{11} + 2q_1^2q_2q_3\beta_{11}$,
(4) $\beta_7 * \beta_9 + q_1q_2\beta_2 = \beta_3 * \beta_{12} + q_1^2q_2q_3\beta_{12}\beta_6 * \beta_9 - \beta_9 * (\beta_8 + \beta_9) + q_1q_2\beta_2 + q_1^2q_2q_3\beta_{12}$,
(5) $\beta_1 * \beta_{10} = 2\beta_4 * \beta_9 + 3\beta_5 * \beta_9 - \beta_7 * \beta_9 - 2\beta_9 * (\beta_8 + \beta_9) + 4\sum_{d\neq 0} dq_1^d \beta_4 * \beta_9 + 6\sum_{d\neq 0} dq_1^d \beta_5 * \beta_9$.

Proof. First, we look at the identity $\beta_9 * (\beta_{10} * \beta_{11}) = (\beta_9 * \beta_{10}) * \beta_{11} = (\beta_9 * \beta_{11}) * \beta_{10}$. By the computational results before, these terms are, respectively,

$$\begin{aligned}
\beta_9 * (\beta_{10} * \beta_{11}) &= \beta_9 * (\beta_{10}\beta_{11}) \\
&= \beta_9 * (2\beta_4) = 2\beta_4 * \beta_9, \\
(\beta_9 * \beta_{10}) * \beta_{11} &= (\beta_9\beta_{10}) * \beta_{11} = 2\beta_1 * \beta_{11}, \\
(\beta_9 * \beta_{11}) * \beta_{10} &= (\beta_9\beta_{11} + q_1q_3\beta_{12}) * \beta_{10} \\
&= \beta_2 * \beta_{10} + q_1q_3\beta_{10}\beta_{12} \\
&= \beta_2 * \beta_{10} + 2q_1q_3\beta_5,
\end{aligned}$$

thus giving rise to (1).

Now we take the identity $\beta_9 * (\beta_{10} * \beta_{12}) = \beta_{10} * (\beta_9 * \beta_{12}) = (\beta_{10} * \beta_9) * \beta_{12}$. The three sides are $\beta_9 * (\beta_{10} * \beta_{12}) = \beta_9 * (\beta_{10}\beta_{12}) = 2\beta_5 * \beta_9$, $\beta_{10} * (\beta_9 * \beta_{12}) = \beta_{10} * (\beta_9\beta_{12} + 2q_1^2q_2q_3) = \beta_3 * \beta_{10} + 2q_1^2q_2q_3\beta_{10}$, $(\beta_{10} * \beta_9) * \beta_{12} = (\beta_9\beta_{10}) * \beta_{12} = 2\beta_1 * \beta_{12}$. This gives (2).

In the identity $\beta_9 * (\beta_{11} * \beta_{12}) = \beta_{11} * (\beta_9 * \beta_{12})$, the two sides are $\beta_9 * (\beta_{11} * \beta_{12}) = \beta_9 * (\beta_{11}\beta_{12} - q_1q_2\beta_{11}) = \beta_9 * (\beta_8 + \beta_9) - q_1q_2\beta_9 * \beta_{11} = \beta_9 * (\beta_8 +$

$\beta_9) - q_1q_2(\beta_9\beta_{11} + q_1q_3\beta_{12}) = \beta_9 * (\beta_8 + \beta_9) - q_1q_2\beta_2 - q_1^2q_2q_3\beta_{12}$ and $\beta_{11} * (\beta_9 * \beta_{12}) = \beta_{11} * (\beta_9\beta_{12} + 2q_1^2q_2q_3) = \beta_3 * \beta_{11} + 2q_1^2q_2q_3\beta_{11}$. From this we get (3).

Now we consider the identity $\beta_9 * (\beta_{12} * \beta_{12}) = (\beta_9 * \beta_{12}) * \beta_{12}$. The two sides are equal to

$$\begin{aligned}
\beta_9 * (\beta_{12} * \beta_{12}) &= \beta_9 * (\beta_{12}^2 + q_1q_2\beta_{11}) \\
&= \beta_7 * \beta_9 + q_1q_2\beta_9 * \beta_{11} \\
&= \beta_7 * \beta_9 + q_1q_2(\beta_9\beta_{11} + q_1q_3\beta_{12}) \\
&= \beta_7 * \beta_9 + q_1q_2\beta_2 + q_1^2q_2q_3\beta_{12}, \\
(\beta_9 * \beta_{12}) * \beta_{12} &= (\beta_9\beta_{12} + 2q_1^2q_2q_3) * \beta_{12} \\
&= \beta_3 * \beta_{12} + 2q_1^2q_2q_3\beta_{12},
\end{aligned}$$

verifying (4).

Finally, we look at $\beta_9 * (\beta_{10} * \beta_{10}) = (\beta_9 * \beta_{10}) * \beta_{10}$. The right-hand side is equal to $(\beta_9\beta_{10}) * \beta_{10} = 2\beta_1 * \beta_{10}$, and the left-hand side is equal to

$$\begin{aligned}
&\Bigg(\beta_{10}^2 + 8\sum_{d\neq 0} dq_1^d\beta_4 + 12\sum_{d\neq 0} dq_1^d\beta_5\Bigg) * \beta_9 \\
&\quad = \Bigg(4\beta_4 + 6\beta_5 - 2\beta_7 - 4\beta_8 - 4\beta_9 + 8\sum_{d\neq 0} dq_1^d\beta_4 + 12\sum_{d\neq 0} dq_1^d\beta_5\Bigg) * \beta_9 \\
&\quad = 4\beta_4 * \beta_9 + 6\beta_5 * \beta_9 - 2\beta_7 * \beta_9 - 4\beta_9 * (\beta_8 + \beta_9) \\
&\qquad + 8\sum_{d\neq 0} dq_1^d\beta_4 * \beta_9 + 12\sum_{d\neq 0} dq_1^d\beta_5 * \beta_9,
\end{aligned}$$

proving (5). □

From this lemma we collect the following identities:

$$\begin{gathered}
\beta_4 * \beta_9 = \beta_1 * \beta_{11}, \qquad \beta_5 * \beta_9 = \beta_1 * \beta_{12}, \\
\beta_7 * \beta_9 = \beta_3 * \beta_{12} - q_1q_2\beta_2 + q_1^2q_2q_3\beta_{12}, \\
\beta_9 * (\beta_8 + \beta_9) = \beta_3 * \beta_{11} + q_1q_2\beta_2 + 2q_1^2q_2q_3\beta_{11} + q_1^2q_2q_3\beta_{12}.
\end{gathered}$$

Substituting all these into the right-hand side of identity (5) and simplifying, we get

$$\begin{aligned}
&2\beta_1 * \beta_{11} + 3\beta_1 * \beta_{12} - \beta_3 * \beta_{12} - 2\beta_3 * \beta_{11} - q_1q_2\beta_2 - 4q_1^2q_2q_3\beta_{11} \\
&\qquad - 3q_1^2q_2q_3\beta_{12} + 4\sum_{d\neq 0} dq_1^d\beta_1 * \beta_{11} + 6\sum_{d\neq 0} dq_1^d\beta_1 * \beta_{12} \\
&\quad = \beta_1 * \beta_{10}. \qquad (4.7)
\end{aligned}$$

Now we equate the corresponding terms at the two sides in front of the same cohomology class $-\frac{1}{2}\beta_{10}$ to get

$$\begin{aligned}2\sum_d \langle\beta_1,\beta_{11},\beta_1\rangle_d q_1^d q_2 q_3 + 3\sum_d \langle\beta_1,\beta_{12},\beta_1\rangle_d q_1^d q_2 q_3 \\ -\sum_d \langle\beta_3,\beta_{12},\beta_1\rangle_d q_1^d q_2 q_3 - 2\sum_d \langle\beta_3,\beta_{11},\beta_1\rangle_d q_1^d q_2 q_3 \\ +4\sum_{l\neq 0} l q_1^l \sum_k \langle\beta_1,\beta_{11},\beta_1\rangle_k q_1^k q_2 q_3 + 6\sum_{l\neq 0} l q_1^l \sum_k \langle\beta_1,\beta_{12},\beta_1\rangle_k q_1^k q_2 q_3 \\ = \sum_d \langle\beta_1,\beta_{10},\beta_1\rangle_d q_1^d q_2 q_3,\end{aligned}$$

where $\langle\beta_1,\beta_{11},\beta_1\rangle_d$ means the genus 0 invariants at the curve class $\beta = d\beta_1 + (\beta_2-\beta_1)+(\beta_3-\beta_1)$, and so on. Let $\langle\beta_1,\beta_1\rangle_d$ be denoted as c_d. Then from this equation, integrating out β_{11}, β_{12} and equating the coefficients for monomial $q_1^d q_2 q_3$, we get

$$\begin{aligned}(2d-1)\langle\beta_1,\beta_1\rangle_d - \langle\beta_3,\beta_1\rangle_d \\ + 6(c_{d-1}+2c_{d-2}+\cdots+(d-1)c_1+dc_0) = 0.\end{aligned} \tag{4.8}$$

Carrying out the same process for the identity $2\beta_1 * \beta_{12} = \beta_3 * \beta_{10} + 2q_1^2 q_2 q_3 \beta_{10}$ in Lemma 4.4(2), we obtain

$$2\sum_d \langle\beta_1,\beta_{12},\beta_1\rangle_d q_1^d q_2 q_3 = \sum_d \langle\beta_3,\beta_{10},\beta_1\rangle_d q_1^d q_2 q_3 - 4q_1^2 q_2 q_3,$$

which implies that, for any d, $(d-2)\langle\beta_3,\beta_1\rangle_d = -\langle\beta_1,\beta_1\rangle_d$, that is, for $d\neq 2$,

$$\langle\beta_3,\beta_1\rangle_d = -\frac{1}{d-2}\langle\beta_1,\beta_1\rangle_d.$$

Putting this back into equation (4.8) and simplifying, we get the recursive relation

$$c_d = -\frac{6(d-2)}{(d-1)(2d-3)}(c_{d-1}+2c_{d-2}+\cdots+(d-1)c_1+dc_0). \tag{4.9}$$

PROPOSITION 4.5. *Let $\beta = d\beta_1 + (\beta_2-\beta_1)+(\beta_3-\beta_1)$, and let $\langle\cdot,\cdot\rangle_d$ denote the GW-invariants of curve class β.*

(1) *Let $\langle\beta_1,\beta_1\rangle_d = c_d$. Then, for $d>2$, $\langle\beta_1,\beta_1\rangle_d$ can be recursively calculated by* (4.9) *with the initial data $\langle\beta_1,\beta_1\rangle_0 = 0$, $\langle\beta_1,\beta_1\rangle_1 = 1$, $\langle\beta_1,\beta_1\rangle_2 = -2$;*
(2) *$\langle\beta_1,\beta_3\rangle_d = -\frac{1}{d-2}c_d$ for $d\neq 2$ with $\langle\beta_1,\beta_3\rangle_2 = 0$;*
(3) *$\langle\beta_3,\beta_3\rangle_d = 1/(d-2)^2 c_d$ for $d\neq 2$ with $\langle\beta_3,\beta_3\rangle_2 = 2$.*

Proof. (1) and (2) have been proven before with the initial values computed directly. For (3), using the identity $2\beta_1 * \beta_{12} = \beta_3 * \beta_{10} + 2q_1^2 q_2 q_3 \beta_{10}$, we get $2\langle\beta_1,\beta_{12},\beta_3\rangle_d = \langle\beta_3,\beta_{10},\beta_3\rangle_d$, so for any d, $\langle\beta_1,\beta_3\rangle_d = -(d-2)\langle\beta_3,\beta_3\rangle_d$. Plugging this into (2) induces (3). $\square$

These are the complements to the results of Proposition 3.7.

Finally, we equate the terms at two sides in front of classes $-\frac{1}{2}\beta_5$ and $\frac{1}{2}\beta_7$, respectively, in (4.7) to get

$$\begin{aligned} &2\sum_d \langle \beta_1, \beta_{11}, \tilde{\beta}\rangle_d q_1^d q_3 + 3\sum_d \langle \beta_1, \beta_{12}, \tilde{\beta}\rangle_d q_1^d q_3 \\ &\quad - \sum_d \langle \beta_3, \beta_{12}, \tilde{\beta}\rangle_d q_1^d q_3 - 2\sum_d \langle \beta_3, \beta_{11}, \tilde{\beta}\rangle_d q_1^d q_3 \\ &\quad + 4\sum_{l\neq 0} l q_1^l \sum_k \langle \beta_1, \beta_{11}, \tilde{\beta}\rangle_k q_1^k q_3 + 6\sum_{l\neq 0} l q_1^l \sum_k \langle \beta_1, \beta_{12}, \tilde{\beta}\rangle_k q_1^k q_3 \\ &= \sum_d \langle \beta_1, \beta_{10}, \tilde{\beta}\rangle_d q_1^d q_3, \end{aligned}$$

where $\tilde{\beta}$ is equal to either β_4 or β_6, and the curve class $\beta = d\beta_1 + (\beta_3 - \beta_1)$.

After simplifying, this gives rise to two respective equations for β_4 and β_6:

$$\begin{aligned} &\sum_d \langle \beta_1, \beta_4\rangle_d q_1^d q_3 + 2\sum_{l\neq 0} l q_1^l \sum_k \langle \beta_1, \beta_4\rangle_k q_1^k q_3 \\ &\qquad = -\sum_d (d-1)\langle \beta_1, \beta_4\rangle_d q_1^d q_3, \\ &\sum_d \langle \beta_1, \beta_6\rangle_d q_1^d q_3 + 2\sum_{l\neq 0} l q_1^l \sum_k \langle \beta_1, \beta_6\rangle_k q_1^k q_3 - 2q_1q_3 \\ &\qquad = -\sum_d (d-1)\langle \beta_1, \beta_6\rangle_d q_1^d q_3, \end{aligned}$$

noting that, for $\beta = d\beta_1 + (\beta_3 - \beta_1)$,

$$\beta \cdot \beta_{10} = -2(d-1), \qquad \beta \cdot \beta_{11} = 1, \qquad \beta \cdot \beta_{12} = 0$$

and, by Proposition 3.9, $\langle \beta_3, \beta_4\rangle_d = 0$ for all d, $\langle \beta_3, \beta_6\rangle_d = 0$ for $d \neq 1$ and 2 for $d = 1$.

Let $f_d = \langle \beta_1, \beta_4\rangle_d$ and $g_d = \langle \beta_1, \beta_6\rangle_d$. Then solving these equations, we get

$$\langle \beta_1, \beta_4\rangle_d = f_d = -\frac{2}{d}(f_{d-1} + 2f_{d-2} + \cdots + (d-1)f_1 + df_0), \quad \forall d,$$

$$\langle \beta_1, \beta_6\rangle_d = g_d = -\frac{2}{d}(g_{d-1} + 2g_{d-2} + \cdots + (d-1)g_1 + dg_0), \quad d > 1.$$

Their initial values by localization are

$$\langle \beta_1, \beta_4\rangle_0 = 1, \qquad \langle \beta_1, \beta_4\rangle_1 = -2; \qquad \langle \beta_1, \beta_6\rangle_0 = 1, \qquad \langle \beta_1, \beta_6\rangle_1 = 0.$$

From the identity $2\beta_1 * \beta_{11} = \beta_2 * \beta_{10} + 2q_1q_3\beta_5$ in Lemma 4.4(1), we again equate the terms at two sides in front of classes $-\frac{1}{2}\beta_5$ and $\frac{1}{2}\beta_7$, respectively, to get

$$2\sum_d \langle \beta_1, \beta_{11}, \beta_4\rangle_d q_1^d q_3 = \sum_d \langle \beta_2, \beta_{10}, \beta_4\rangle_d q_1^d q_3 - 4q_1q_3,$$

$$2\sum_d \langle \beta_1, \beta_{11}, \beta_6\rangle_d q_1^d q_3 = \sum_d \langle \beta_2, \beta_{10}, \beta_6\rangle_d q_1^d q_3,$$

or

$$\sum_d \langle \beta_1, \beta_4 \rangle_d q_1^d q_3 = -\sum_d (d-1)\langle \beta_2, \beta_4 \rangle_d q_1^d q_3 - 2q_1 q_3,$$
$$\sum_d \langle \beta_1, \beta_6 \rangle_d q_1^d q_3 = -\sum_d (d-1)\langle \beta_2, \beta_6 \rangle_d q_1^d q_3.$$

So, for any $d > 1$,

$$\langle \beta_1, \beta_4 \rangle_d = -(d-1)\langle \beta_2, \beta_4 \rangle_d,$$
$$\langle \beta_1, \beta_6 \rangle_d = -(d-1)\langle \beta_2, \beta_6 \rangle_d,$$

and hence

$$\langle \beta_2, \beta_4 \rangle_d = -\frac{1}{d-1} f_d,$$
$$\langle \beta_2, \beta_6 \rangle_d = -\frac{1}{d-1} g_d.$$

Their initial values are

$$\langle \beta_2, \beta_4 \rangle_0 = 1, \qquad \langle \beta_2, \beta_4 \rangle_1 = 0; \qquad \langle \beta_2, \beta_6 \rangle_0 = 1, \qquad \langle \beta_2, \beta_6 \rangle_1 = 1.$$

Thus the computations in Proposition 3.9 are completed. So, at this point, we have computed all two-pointed Gromov–Witten invariants of the Hilbert scheme.

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