Michigan Math. J. 67 (2018), 715–741

# Counting the Ideals of Given Codimension of the Algebra of Laurent Polynomials in Two Variables

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Abstract. We establish an explicit formula for the number $C_n(q)$ of ideals of codimension (colength) $n$ of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of Laurent polynomials in two variables over a finite field $\mathbb{F}_q$ of cardinality $q$. This number is a palindromic polynomial of degree $2n$ in $q$. Moreover, $C_n(q) = (q-1)^2 P_n(q)$, where $P_n(q)$ is another palindromic polynomial; the latter is a $q$-analogue of the sum of divisors of $n$, which happens to be the number of subgroups of $\mathbb{Z}^2$ of index $n$.

## 1. Introduction

Let $\mathbb{F}_q$ be a finite field of cardinality $q$, and let $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ be the algebra of Laurent polynomials in two variables with coefficients in $\mathbb{F}_q$.

Our main aim is to give a formula for the number $C_n(q)$ of ideals of codimension $n$ of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$. By codimension of an ideal $I$ we mean the dimension of the quotient vector space $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]/I$ over $\mathbb{F}_q$.

Our main result is the following.

Theorem 1.1. *For each integer $n \geq 1$, we have*

$$C_n(q) = \sum_{\lambda \vdash n} (q-1)^{2\nu(\lambda)} q^{n-\ell(\lambda)} \prod_{\substack{i=1,\dots,t \\ d_i \geq 1}} \frac{q^{2d_i} - 1}{q^2 - 1},$$

*where the sum runs over all partitions $\lambda$ of $n$. The expression $C_n(q)$ is a monic polynomial of degree $2n$ in the variable $q$ with integer coefficients. Moreover, the polynomial $C_n(q)$ is divisible by $(q-1)^2$.*

The notation $\ell(\lambda)$, $\nu(\lambda)$, $d_i$ appearing in the formula will be explained in Section 3.1. The proof of the theorem will be given in Section 5.3; it relies on a parameterization by Conca and Valla [8] of the affine cells in the Ellingsrud–Strømme decomposition of the Hilbert scheme of $n$ points on the affine plane.

Note that since $C_n(q)$ is divisible by $(q-1)^2$, for each $n \geq 1$, we may define a unique polynomial $P_n(q)$ by

$$C_n(q) = (q-1)^2 P_n(q), \tag{1.1}$$

which clearly implies $C_n(1) = 0$ for all $n \geq 1$. Table 1 (resp. Table 2) displays the polynomials $C_n(q)$ (resp. the polynomials $P_n(q)$) for $n \leq 12$.

Received November 23, 2016. Revision received February 2, 2018.

**Table 1** The polynomials $C_n(q)$

| $n$ | $C_n(q)$ |
|---|---|
| 1 | $q^2-2q+1$ |
| 2 | $q^4-q^3-q+1$ |
| 3 | $q^6-q^5-q^4+2q^3-q^2-q+1$ |
| 4 | $q^8-q^7-q+1$ |
| 5 | $q^{10}-q^9-q^7+q^6+q^4-q^3-q+1$ |
| 6 | $q^{12}-q^{11}+q^7-2q^6+q^5-q+1$ |
| 7 | $q^{14}-q^{13}-q^{10}+q^9+q^5-q^4-q+1$ |
| 8 | $q^{16}-q^{15}-q+1$ |
| 9 | $q^{18}-q^{17}-q^{13}+q^{12}+q^{11}-q^{10}-q^8+q^7+q^6-q^5-q+1$ |
| 10 | $q^{20}-q^{19}-q^{11}+2q^{10}-q^9-q+1$ |
| 11 | $q^{22}-q^{21}-q^{16}+q^{15}+q^7-q^6-q+1$ |
| 12 | $q^{24}-q^{23}+q^{15}-q^{14}-q^{10}+q^9-q+1$ |

**Table 2** The polynomials $P_n(q)$

| $n$ | $P_n(q)$ | $P_n(1)$ |
|---|---|---|
| 1 | $1$ | 1 |
| 2 | $q^2+q+1$ | 3 |
| 3 | $q^4+q^3+q+1$ | 4 |
| 4 | $q^6+q^5+q^4+q^3+q^2+q+1$ | 7 |
| 5 | $q^8+q^7+q^6+q^2+q+1$ | 6 |
| 6 | $q^{10}+q^9+q^8+q^7+q^6+2q^5+q^4+q^3+q^2+q+1$ | 12 |
| 7 | $q^{12}+q^{11}+q^{10}+q^9+q^3+q^2+q+1$ | 8 |
| 8 | $q^{14}+q^{13}+q^{12}+q^{11}+q^{10}+q^9+q^8$ $+q^7+q^6+q^5+q^4+q^3+q^2+q+1$ | 15 |
| 9 | $q^{16}+q^{15}+q^{14}+q^{13}+q^{12}+q^9$ $+q^8+q^7+q^4+q^3+q^2+q+1$ | 13 |
| 10 | $q^{18}+q^{17}+q^{16}+q^{15}+q^{14}+q^{13}+q^{12}+q^{11}+q^{10}$ $+q^8+q^7+q^6+q^5+q^4+q^3+q^2+q+1$ | 18 |
| 11 | $q^{20}+q^{19}+q^{18}+q^{17}+q^{16}+q^{15}$ $+q^5+q^4+q^3+q^2+q+1$ | 12 |
| 12 | $q^{22}+q^{21}+q^{20}+q^{19}+q^{18}+q^{17}+q^{16}+q^{15}$ $+q^{14}+2q^{13}+2q^{12}+2q^{11}+2q^{10}+2q^9+q^8$ $+q^7+q^6+q^5+q^4+q^3+q^2+q+1$ | 28 |

Theorem 1.1 has two interesting consequences. The first one concerns the polynomials $P_n(q)$. Let us state it.

COROLLARY 1.2. *For each* $n \geq 1$, *the polynomial* $P_n(q)$ *is a monic polynomial of degree* $2n-2$ *with integer coefficients*, *and we have*

$$P_n(1) = \sigma(n) = \sum_{d|n; d\geq 1} d.$$

As is well known, the sum $\sigma(n)$ of positive divisors of $n$ is equal to the number of subgroups of index $n$ of the free Abelian group $\mathbb{Z}^2$ of rank two. Thus Theorem 1.1 and Corollary 1.2 imply that the number of ideals of codimension $n$ of the Laurent polynomial algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$, that is, of the algebra of the group $\mathbb{Z}^2$, is, up to the factor $(q-1)^2$, a $q$-analogue[1] of the number of subgroups of index $n$ of $\mathbb{Z}^2$.

A similar phenomenon had been observed by Bacher and the second-named author in [4]: up to a power of $q-1$, the number of right ideals of codimension $n$ of the algebra $\mathbb{F}_q[F_2]$ of the rank two free group $F_2$ is a $q$-analogue of the number of subgroups of index $n$ of $F_2$. In fact, it was this observation that prompted us to compute the number of ideals of codimension $n$ of the algebra $\mathbb{F}_q[\mathbb{Z}^2]$ of the free Abelian group $\mathbb{Z}^2$, that is, of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$.

In a similar context, the following holds.

(a) By [11] (see also Section 3.1) the number of ideals of codimension $n$ of the polynomial algebra $\mathbb{F}_q[x, y]$, which is the algebra of the free Abelian monoid $\mathbb{N}^2$, is a $q$-analogue of the number $p(n)$ of partitions of $n$; as is well known, the latter is equal to the number of ideals of the monoid $\mathbb{N}^2$ whose complement is of cardinality $n$.

(b) In a noncommutative setting, by [24; 3] the number of right ideals of codimension $n$ of the free algebra $\mathbb{F}_q\langle x, y\rangle$ is a $q$-analogue of the number of right ideals of the free monoid $\langle x, y\rangle^*$ whose complement is of cardinality $n$.

(c) Similarly, by [23, Section 6.3] the number of right ideals of codimension $n$ of the algebra $\mathbb{F}_q[F_r]$ of the free group $F_r$ on $r$ generators is, up to a power of $q-1$, a $q$-analogue of the number of subgroups of index $n$ of $F_r$.

REMARK 1.3. The commutative algebra $L_r = \mathbb{F}_q[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$ of Laurent polynomials in $r$ variables ($r \geq 3$) provides a distinct contrast with the cases discussed. We can show that the number of right ideals of codimension 2 of $L_r$, which is the algebra of the free Abelian group $\mathbb{Z}^r$, is equal to $(q-1)^r R_r(q)$, where

$$R_r(q) = \frac{1}{2}((q+1)^r + (q-1)^r) + \frac{q^r - 1}{q-1} - 1.$$

The latter is a $q$-analogue of $R_r(1) = 2^{r-1} + r - 1$. Now the number of subgroups of index 2 of $\mathbb{Z}^r$ is equal to $2^r - 1$, which is different from $R_r(1)$ when $r \geq 3$.

[1] By a $q$-analogue of an integer $r$ we mean a polynomial $P(q)$ in the variable $q$ such that $P(1) = r$.

The second consequence of Theorem 1.1 expresses the generating function of the polynomials $C_n(q)$ as a nice infinite product.

COROLLARY 1.4. (a) *We have*

$$1+\sum_{n\geq 1}\frac{C_n(q)}{q^n}t^n=\prod_{i\geq 1}\frac{(1-t^i)^2}{1-(q+q^{-1})t^i+t^{2i}}.$$

(b) *The polynomials $C_n(q)$ and $P_n(q)$ are palindromic.*

The previous infinite product shows up in [12, p. 10] (see, e.g., equations (9.2) and (10.1)) and probably in other papers on basic hypergeometric series; in an algebraic geometry context it appears in [20, Thm. 4.1.3], where it is equal to the generating function of the $E$-polynomials of the punctual Hilbert schemes of the complex two-dimensional torus (see details in Section 6.3).

Using Corollary 1.4, we gave explicit expressions for the coefficients of the polynomials $C_n(q)$ and $P_n(q)$ in the companion paper [22] (see Thms. 1.1 and 1.2 loc. cit.). We obtained a rather striking positivity result, namely the coefficients of $P_n(q)$ are all *nonnegative* integers. For completeness, we recall our formulas for the coefficients of the polynomials $C_n(q)$ and $P_n(q)$ in the Appendix.

As pointed to us by Frank Garvan, the polynomials $C_n(q)$ are related to the *crank of partitions*. Recall that the crank is a function from partitions into the integers, which explains the Ramanujan congruences modulo 11 and whose existence was conjectured by Dyson and later proved by Garvan; see [14, Section 7]. Denote as in [1] the number of partitions of $n$ with crank $m$ by $N_V(m,n)$. We have the following relation between the integers $N_V(m,n)$, the number $p(n)$ of all partitions of $n$, and the polynomials $C_n(q)$.

COROLLARY 1.5. *For each $n$, we have*

$$\sum_{m\in\mathbb{Z}}N_V(m,n)q^m=p(n)+\sum_{i=1}^{n}p(n-i)\frac{C_i(q)}{q^i}.$$

The paper is organized as follows. Section 2 is devoted to some preliminaries: we first recall the one-to-one correspondence between the ideals of the localization $S^{-1}A$ of an algebra $A$ and certain ideals of $A$; we also count tuples of polynomials subject to certain constraints over a finite field.

In Section 3 we recall Conca and Valla's parameterization of the affine cells in a decomposition of the Hilbert scheme of $n$ points in the plane; these cells are indexed by the partitions of $n$. We show how to deduce a parameterization of the cells in the induced decomposition of the Hilbert scheme of $n$ points in a Zariski open subset of the plane.

In Section 4 we apply the techniques of the preceding section to compute the number of ideals of codimension $n$ of $\mathbb{F}_q[x,y,y^{-1}]$. In passing we give a criterion (Proposition 4.1), which will also be used in the proof of Theorem 1.1.

In Section 5 we define an invertible Gröbner cell, which is a Zariski open subset of the corresponding affine cell, and compute its cardinality over a finite field. We derive a proof of Theorem 1.1.

The proofs of Corollaries 1.2, 1.4, and 1.5 are given in Section 6.

In Appendix we briefly recall the results on the coefficients of $C_n(q)$ and $P_n(q)$ we obtained in [22].

## 2. Preliminaries

We fix a ground field $k$. By algebra we mean an associative unital $k$-algebra. In this paper all algebras are assumed to be *commutative.*

### *2.1. Ideals in Localizations*

Let $A$ be a (commutative) algebra, $S$ a multiplicative submonoid of $A$ not containing 0, and $S^{-1}A$ the corresponding localization of $A$. We assume that the canonical algebra map $i : A \to S^{-1}A$ is injective (this is the case, e.g., when $A$ is a domain).

Recall the well-known correspondence between the ideals of $S^{-1}A$ and those of $A$ (see [7, Chapter 2, Section 2, n$^{\text{o}}$ 4–5], [10, Prop. 2.2]).

(a) For any ideal $J$ of $S^{-1}A$, the set $i^{-1}(J) = J \cap A$ is an ideal of $A$, and we have $J = i^{-1}(J)S^{-1}A$. The map $J \mapsto i^{-1}(J)$ is an injection from the set of ideals of $S^{-1}A$ to the set of ideals of $A$.

(b) An ideal $I$ of $A$ is of the form $i^{-1}(J)$ for some ideal $J$ of $S^{-1}A$ if and only if for all $s \in S$, the endomorphism of $A/I$ induced by the multiplication by $s$ is injective.

Given an integer $n \geq 1$, an *$n$-codimensional* ideal of $A$ is an ideal such that $\dim_k A/I = n$. For such an ideal, the previous condition (b) is then equivalent to: for all $s \in S$, the endomorphism of $A/I$ induced by the multiplication by $s$ is a linear isomorphism.

We leave the proof of the following lemma to the reader.

LEMMA 2.1. *If $J$ is a finite-codimensional ideal of $S^{-1}A$, then the canonical algebra map $i : A \to S^{-1}A$ induces an algebra isomorphism*

$$A/i^{-1}(J) \cong (S^{-1}A)/J.$$

It follows that there is a bijection between the set of $n$-codimensional ideals of $S^{-1}A$ and the set of $n$-codimensional ideals $I$ of $A$ such that, for all $s \in S$, the endomorphism of $A/I$ induced by the multiplication by $s$ is a linear isomorphism. The latter assertion is equivalent to $s$ being invertible modulo $I$, that is, the image of $s$ in $A/I$ being invertible.

The following criterion will be used in Sections 4 and 5.

LEMMA 2.2. *Let $A$ be a commutative algebra. For any $s \in A$, let $p : A \to A/(s)$ be the natural projection onto the quotient algebra of $A$ by the ideal generated by $s$. If $I$ is an ideal of $A$, then $s$ is invertible modulo $I$ if and only if $p(I) = A/(s)$.*

*Proof.* If $s$ is invertible modulo $I$, then there exists $t \in A$ such that $st - 1 \in I$. Hence $p(1)$ belongs to $p(I)$, which implies $p(I) = A/(s)$. Conversely, if $p(I) = A/(s)$, then $p(1) = p(u)$ for some $u \in I$. Hence $1 - u \in (s)$, which means that there is $t \in A$ such that $1 - u = st$. Thus $st \equiv 1 \pmod{I}$. □

### 2.2. *Counting Polynomials over a Finite Field*

In this subsection we assume that $k = \mathbb{F}_q$ is a finite field of cardinality $q$. We will need the following in Section 5.

PROPOSITION 2.3. *Let $d$, $h$ be integers $\geq 1$, and let $Q_1, \dots, Q_h \in \mathbb{F}_q[y]$ be coprime polynomials. The number of $(h+1)$-tuples $(P, P_1, \dots, P_h)$ satisfying the three conditions*

(i) *$P$ is a degree $d$ monic polynomial with $P(0) \neq 0$,*
(ii) *$P_1, \dots, P_h$ are polynomials of degree $< d$, and*
(iii) *$P$ and $P_1 Q_1 + \cdots + P_h Q_h$ are coprime*

*is equal to*

$$(q-1)^2 q^{(h-1)d} \frac{q^{2d}-1}{q^2-1}.$$

Before giving the proof, we state and prove two auxiliary lemmas.

LEMMA 2.4. *Let $R$ be a finite commutative ring, and let $a_1, \dots, a_h \in R$ be such that $a_1 R + \cdots + a_h R = R$. For any $b \in R$, the number of $h$-tuples $(x_1, \dots, x_h) \in R^h$ such that $a_1 x_1 + \cdots + a_h x_h = b$ is equal to* $(\operatorname{card} R)^{h-1}$.

*Proof.* The map $(x_1, \dots, x_h) \mapsto a_1 x_1 + \cdots + a_h x_h$ is a homomorphism $R^h \to R$ of additive groups. Since it is surjective, the number of $h$-tuples satisfying the assumed condition is equal to the cardinality of its kernel, which is equal to $\operatorname{card} R^h / \operatorname{card} R = (\operatorname{card} R)^{h-1}$. □

LEMMA 2.5. *Let $d \geq 1$ be an integer. The number of couples $(P, Q) \in \mathbb{F}_q[y]^2$ such that $P$ is a degree $d$ monic polynomial with $P(0) \neq 0$, $Q$ is of degree $< d$, and $P$ and $Q$ are coprime is equal to*

$$c_d = (q-1)^2 \frac{q^{2d}-1}{q^2-1}.$$

*Proof.* This amounts to counting the number of couples $(P, z)$, where $P \in \mathbb{F}_q[y]$ is a degree $d$ monic polynomial not divisible by $y$, and $z$ is an invertible element of the quotient ring $\mathbb{F}_q[y]/(P)$.

Expanding $P$ into a product of irreducible polynomials and using the Chinese remainder lemma, we have

$$1 + \sum_{d \geq 1} c_d t^d = \prod_{\substack{P \text{irreducible} \\ P \neq y}} \left( 1 + \sum_{k \geq 1} \operatorname{card}(\mathbb{F}_q[y]/(P))^{\times} t^{k \deg(P)} \right),$$

where the product is taken over all irreducible polynomials of $\mathbb{F}_q[y]$ different from $y$, and where $\deg(P)$ denotes the degree of $P$. First, observe that, for any irreducible polynomial $P \in \mathbb{F}_q[y]$, the group $(\mathbb{F}_q[y]/(P))^\times$ of invertible elements of $\mathbb{F}_q[y]/(P)$ is of cardinality $q^{k\deg(P)} - q^{(k-1)\deg(P)}$: indeed, there are $q^{k\deg(P)}$ polynomials of degree $< k\deg(P)$, and $q^{(k-1)\deg(P)}$ of them are divisible by $P$ and hence not invertible in $\mathbb{F}_q[y]/(P)$. Consequently,

$$\begin{aligned} 1 + \sum_{d\geq 1} c_d t^d &= \prod_{\substack{P \text{irreducible} \\ P\neq y}} \left(1 + (1 - q^{-\deg(P)}) \sum_{k\geq 1} (qt)^{k\deg(P)}\right) \\ &= \prod_{\substack{P \text{irreducible} \\ P\neq y}} \left(1 + (1 - q^{-\deg(P)}) \frac{(qt)^{\deg(P)}}{1-(qt)^{\deg(P)}}\right) \\ &= \prod_{\substack{P \text{irreducible} \\ P\neq y}} \frac{1 - t^{\deg(P)}}{1-(qt)^{\deg(P)}}. \end{aligned}$$

On one hand, the infinite product $\prod_{\substack{P \text{irreducible} \\ P\neq y}} (1 - t^{\deg(P)})^{-1}$ is equal to the zeta function $Z_{\mathbb{A}^1\setminus\{0\}}(t)$ of the affine line minus a point. On the other hand,

$$Z_{\mathbb{A}^1\setminus\{0\}}(t) = \frac{Z_{\mathbb{A}^1}(t)}{Z_{\{0\}}(t)} = \frac{1-t}{1-qt}.$$

Therefore

$$1 + \sum_{d\geq 1} c_d t^d = \frac{1-qt}{1-q^2 t} \Big/ \frac{1-t}{1-qt} = \frac{(1-qt)^2}{(1-t)(1-q^2 t)}.$$

Subtracting 1 from both sides, we obtain

$$\sum_{d\geq 1} c_d t^d = (q-1)^2 \frac{t}{(1-t)(1-q^2 t)},$$

from which it is easy to derive the desired formula for $c_d$. □

*Proof of Proposition 2.3.* We have to count the number of the $(h+2)$-tuples $(P, Q, P_1, \ldots, P_h)$ such that $P$ is a degree $d$ monic polynomial with $P(0) \neq 0$, $Q$ is a polynomial of degree $< d$ and coprime to $P$, each polynomial $P_i$ is of degree $< d$, and $\sum_{i=1}^h P_i Q_i \equiv Q$ modulo $P$.

By Lemma 2.5 the number of couples $(P, Q)$ satisfying these conditions is equal to $(q-1)^2(q^{2d}-1)/(q^2-1)$. Since $\operatorname{card} \mathbb{F}_q[y]/(P) = q^d$, by Lemma 2.4 we have $q^{d(h-1)}$ choices for the $h$-tuples $(P_1, \ldots, P_h)$. The number we wish to count is the product of the two previous ones. □

## 3. The Hilbert Scheme of Points in a Zariski Open Subset of the Plane

Let $k$ be a field. As is well known, the ideals of codimension $n$ of an affine $k$-algebra $A$ are in bijection with the $k$-points of the Hilbert scheme parameterizing finite subschemes of colength $n$ of the spectrum of $A$. For instance, the ideals of codimension $n$ of the polynomial algebra $k[x,y]$ are in bijection with the $k$-points of the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2_k)$ of $n$ points on the affine plane. Similarly, the ideals of codimension $n$ of the Laurent polynomial algebra $k[x,y,x^{-1},y^{-1}]$ are in bijection with the $k$-points of the Hilbert scheme $\mathrm{Hilb}^n((\mathbb{A}^1_k\setminus\{0\})\times(\mathbb{A}^1_k\setminus\{0\}))$ of $n$ points on the two-dimensional torus, which is a Zariski open subset of the plane.

In this paragraph we prove that the Hilbert scheme of $n$ points in a Zariski open subset of the plane is an open subscheme of the Hilbert scheme of $n$ points in the plane and show how to determine it explicitly.

### *3.1. Parameterizing the Finite-Codimensional Ideals of* $k[x,y]$

Computing the homology of the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2_k)$, Ellingsrud and Strømme [11] showed that it has a cellular decomposition indexed by the partitions $\lambda$ of $n$, each cell $C_\lambda$ being an affine space of dimension $n+\ell(\lambda)$, where $\ell(\lambda)$ is the length of $\lambda$. Earlier results by Białynicki-Birula [5; 6] on smooth varieties with $k^\times$-actions imply the same decomposition; the cells $C_\lambda$ are sometimes called "Białynicki-Birula cells".

It follows that, in the particular case where $k=\mathbb{F}_q$ is a finite field of cardinality $q$, the number $A_n(q)$ of ideals of $\mathbb{F}_q[x,y]$ of codimension $n$ is finite and given by the polynomial

$$A_n(q)=\sum_{\lambda\vdash n} q^{n+\ell(\lambda)}, \tag{3.1}$$

where the sum runs over all partitions $\lambda$ of $n$ (we indicate this by the notation $\lambda\vdash n$ or by $|\lambda|=n$). The polynomial $A_n(q)$ clearly has nonnegative integer coefficients, its degree is $2n$, and $A_n(1)=p(n)$ is equal to the number of partitions of $n$ (for more on the polynomials $A_n(q)$, see Remark 4.7).

For our purposes, we need an explicit description of the affine cells $C_\lambda$. Let us recall a parameterization by Conca and Valla [8].

Given a positive integer $n$, there is a well-known bijection between the partitions of $n$ and the monomial ideals of codimension $n$ of $k[x,y]$. The correspondence is as follows: with a partition $\lambda$ of $n$, we associate the sequence

$$0=m_0<m_1\le\cdots\le m_t$$

of integers counting from right to left the boxes in each column of the Ferrers diagram of $\lambda$; we have $m_1+\cdots+m_t=n$. Then the associated monomial ideal $I^0_\lambda$ is given by

$$I^0_\lambda=(x^t,x^{t-1}y^{m_1},\dots,xy^{m_{t-1}},y^{m_t}). \tag{3.2}$$

(Note that the generating set in the right-hand side of (3.2) is in general not minimal.) The set $\mathcal{B}_\lambda=\{x^iy^j \mid 0\le i<t, 0\le j<m_i\}$ induces a linear basis of the $n$-dimensional quotient algebra $k[x,y]/I^0_\lambda$.

Consider the lexicographic ordering on the monomials $x^iy^j$ given by

$$1<y<y^2<\cdots<x<xy<xy^2<\cdots<x^2<x^2y<x^2y^2<\cdots.$$

Then the cell $C_\lambda$, called *Gröbner cell* in [8], is by definition the set of ideals $I$ of $k[x, y]$ such that the dominating terms (for this ordering) of the elements of $I$ generate the monomial ideal $I_\lambda^0$. It was proved in [11] that $C_\lambda$ is an affine space.

Here is how Conca and Valla explicitly parameterize $C_\lambda$. Given a partition $\lambda$ of $n$ and the associated sequence $0 = m_0 < m_1 \leq \cdots \leq m_t$, they first define the sequence of integers $d_1, \ldots, d_t$ by

$$d_i = m_i - m_{i-1} \geq 0. \tag{3.3}$$

We have $d_1 = m_1 > 0$.

Later we shall also need the integer

$$v(\lambda) = \operatorname{card}\{i = 1, \ldots, t \mid d_i \geq 1\}; \tag{3.4}$$

it is equal to the number of distinct values of the sequence $m_1 \leq \cdots \leq m_t$. Note that $v(\lambda) \geq 1$; moreover, $v(\lambda) = 1$ if and only if the partition is "rectangular", that is, $m_1 = \cdots = m_t$ $(> 0)$.

Let $T_\lambda$ be the set of $(t+1) \times t$-matrices $(p_{i,j})$ with entries in the one-variable polynomial algebra $k[y]$ satisfying the following conditions: $p_{i,j} = 0$ if $i < j$, the degree of $p_{i,j}$ is less than $d_j$ if $i \geq j$ and $d_j \geq 1$, and $p_{i,j} = 0$ for all $i$ if $d_j = 0$. The set $T_\lambda$ is an affine space of dimension $n + \ell(\lambda)$.

Now consider the $(t+1) \times t$-matrix

$$M_\lambda = \begin{pmatrix} y^{d_1} + p_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ p_{2,1} - x & y^{d_2} + p_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ p_{3,1} & p_{3,2} - x & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ p_{i-1,1} & p_{i-1,2} & \cdots & y^{d_{i-1}} + p_{i-1} & 0 & 0 & \cdots & 0 \\ p_{i,1} & p_{i,2} & \cdots & p_{i,i-1} - x & y^{d_i} + p_i & 0 & \cdots & 0 \\ p_{i+1,1} & p_{i+1,2} & \cdots & p_{i+1,i-1} & p_{i+1,i} - x & y^{d_{i+1}} + p_{i+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{t,1} & p_{t,2} & \cdots & p_{t,i-1} & p_{t,i} & p_{t,i+1} & \cdots & y^{d_t} + p_t \\ p_{t+1,1} & p_{t+1,2} & \cdots & p_{t+1,i-1} & p_{t+1,i} & p_{t+1,i+1} & \cdots & p_{t+1,t} - x \end{pmatrix}, \tag{3.5}$$

where for simplicity we set $p_i = p_{i,i}$.

By [8, Thm. 3.3] the map sending the polynomial matrix $(p_{i,j}) \in T_\lambda$ to the ideal $I_\lambda$ of $k[x, y]$ generated by all $t$-minors (the maximal minors) of the matrix $M_\lambda$ is a bijection of $T_\lambda$ onto $C_\lambda$. These minors are polynomial expressions with integer coefficients in the coefficients of the $p_{i,j}$ and in the variables $x, y$.

### 3.2. *Localizing*

Let $S$ be a multiplicative submonoid of $k[x, y]$ not containing 0. We assume that $S$ has a finite generating set $\Sigma$. We further concentrate on two cases, $\Sigma = \{y\}$ (in Section 4) and $\Sigma = \{x, y\}$ (in Section 5).

It follows from Section 2 that the set of $n$-codimensional ideals of the localization $S^{-1}k[x, y]$ can be identified with the subset of $\mathrm{Hilb}^n(\mathbb{A}_k^2)$ consisting of the $n$-codimensional ideals $I$ of $k[x, y]$ such that, for all $s \in S$, the endomorphism $\mu_s$ of $k[x, y]/I$ induced by the multiplication by $s$ is a linear isomorphism. The latter is equivalent to $\det \mu_s \neq 0$ for all $s \in \Sigma$.

By the considerations of Section 3.1, the set of $n$-codimensional ideals of the algebra $S^{-1}k[x,y]$ is the disjoint union

$$\coprod_{\lambda \vdash n} C_\lambda^\Sigma,$$

where $C_\lambda^\Sigma$ is the Zariski open subset of the affine Gröbner cell $C_\lambda$ consisting of the points satisfying $\det \mu_s \neq 0$ for all $s \in \Sigma$.

Consequently, the Hilbert scheme $\mathrm{Hilb}^n(\mathrm{Spec}(S^{-1}k[x,y]))$ parameterizing subschemes of colength $n$ in $\mathrm{Spec}(S^{-1}k[x,y])$ is an open subscheme of $\mathrm{Hilb}^n(\mathbb{A}_k^2)$ and hence an open subscheme of $\mathrm{Hilb}^n(\mathbb{P}_k^2)$. Since by [13; 16] the latter is smooth and projective, $\mathrm{Hilb}^n(\mathrm{Spec}(S^{-1}k[x,y]))$ is a smooth quasi-projective variety.

The endomorphism $\mu_x$ (resp. $\mu_y$) of $k[x,y]/I$ induced by the multiplication by $x$ (resp. by $y$) can be expressed as a matrix in the basis $\mathcal{B}_\lambda$. Observe that the entries of such a matrix are polynomial expressions with integer coefficients in the coefficients of the $p_{i,j}$. Therefore, if any $s \in \Sigma$ is a linear combination with integer coefficients of monomials in the variables $x$, $y$, then the Hilbert scheme $\mathrm{Hilb}^n(\mathrm{Spec}(S^{-1}k[x,y]))$ is defined over $\mathbb{Z}$ as a variety.

In particular, the schemes $\mathrm{Hilb}^n(\mathbb{A}_k^1 \times (\mathbb{A}_k^1 \setminus \{0\}))$ and $\mathrm{Hilb}^n((\mathbb{A}_k^1 \setminus \{0\})^2)$ are smooth quasi-projective varieties defined over $\mathbb{Z}$.

Example 3.1. Let $\lambda$ be the unique self-conjugate partition of 3. In this case, $t=2$, $m_1=1$, $m_2=2$, and hence $d_1=d_2=1$. The corresponding matrix $M_\lambda$, as in (3.5), is

$$M_\lambda = \begin{pmatrix} y+a & 0 \\ b-x & y+d \\ c & e-x \end{pmatrix},$$

where $a$, $b$, $c$, $d$, $e$ are scalars. The associated Gröbner cell $C_\lambda$ is a five-dimensional affine space parameterized by these five scalars. The ideal $I_\lambda$ is generated by the maximal minors of the matrix, namely by $(b-x)(e-x)-c(y+d)$, $(e-x)(y+a)$, and $(y+a)(y+d)$. It follows that modulo $I_\lambda$ we have the relations

$$\begin{aligned} x^2 &\equiv (b+e)x + cy + (cd-be), \\ xy &\equiv -ax + ey + ae, \\ y^2 &\equiv -(a+d)y - ad. \end{aligned}$$

In the basis $\mathcal{B}_\lambda = \{x, y, 1\}$ the multiplication endomorphisms $\mu_x$ and $\mu_y$ can be expressed as the matrices

$$\mu_x = \begin{pmatrix} b+e & -a & 1 \\ c & e & 0 \\ cd-be & ae & 0 \end{pmatrix} \quad \text{and} \quad \mu_y = \begin{pmatrix} -a & 0 & 0 \\ e & -(a+d) & 1 \\ ae & -ad & 0 \end{pmatrix}.$$

We have $\det \mu_x = e(ac-cd+be)$ and $\det \mu_y = -ad^2$.

It follows from these computations that if, for instance, $\Sigma = \{x, y\}$, then $C_\lambda^\Sigma$ is the complement in the affine space $\mathbb{A}_k^5$ of the union of the three hyperplanes $a=0$, $d=0$, and $e=0$ and of the quadric hypersurface $ac-cd+be=0$.

## 4. The Punctual Hilbert Scheme of the Complement of a Line in an Affine Plane

In this section we apply the considerations of the previous section to the case $\Sigma = \{y\}$. Here $S$ is the multiplicative submonoid of $k[x, y]$ generated by $y$ and $S^{-1}k[x, y] = k[x, y, y^{-1}] = k[x][y, y^{-1}]$.

By Section 3.2, the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^1_k \times (\mathbb{A}^1_k \setminus \{0\}))$, that is, the set of $n$-codimensional ideals of $k[x, y, y^{-1}]$, is the disjoint union over the partitions $\lambda$ of $n$ of the sets $C^y_\lambda$, where $C^y_\lambda$ consists of the ideals $I \in C_\lambda$ such that $y$ is invertible in $k[x, y]/I$. We call $C^y_\lambda$ the *semi-invertible Gröbner cell* associated with the partition $\lambda$.

### *4.1. A Criterion for the Invertibility of y*

Let $p_y : k[x, y] \to k[x]$ be the algebra map sending $x$ to itself and $y$ to 0. Then by Lemma 2.2, the set $C^y_\lambda$ consists of the ideals $I \in C_\lambda$ such that $p_y(I) = k[x]$.

Recall from Section 3.1 that $I_\lambda$ is generated by the maximal minors of the matrix $M_\lambda$ of (3.5), namely by the polynomials $f_0(x, y), \dots, f_t(x, y)$, where we define $f_i(x, y)$ to be the determinant of the $t \times t$-matrix obtained from $M_\lambda$ by deleting its $(i + 1)$th row. Then the ideal $p_y(I_\lambda)$ can be identified with the ideal of $k[x]$ generated by the polynomials $f_0(x, 0), \dots, f_t(x, 0) \in k[x]$ obtained by setting $y = 0$. We need to determine under what conditions this ideal is equal to the whole algebra $k[x]$.

Recall the entries of the matrix $M_\lambda$ and particularly the polynomials $p_{i,j}$ and $p_i = p_{i,i} \in k[y]$. Let $a_{i,j} = p_{i,j}(0)$ be the constant term of $p_{i,j}$. As before, we set $a_i = a_{i,i} = p_i(0)$. Note that $a_j = 1$ and $a_{i,j} = 0$ for all $i \neq j$ whenever $d_j = 0$.

Then $f_0(x, 0), \dots, f_t(x, 0)$ are the maximal minors of the matrix

$$M^y_\lambda = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{2,1} - x & a_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} - x & a_3 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1} & 0 & 0 & \cdots & 0 \\ a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,i-1} - x & a_i & 0 & \cdots & 0 \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,i-1} & a_{i+1,i} - x & a_{i+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{t,1} & a_{t,2} & a_{t,3} & \cdots & a_{t,i-1} & a_{t,i} & a_{t,i+1} & \cdots & a_t \\ a_{t+1,1} & a_{t+1,2} & a_{t+1,3} & \cdots & a_{t+1,i-1} & a_{t+1,i} & a_{t+1,i+1} & \cdots & a_{t+1,t} - x \end{pmatrix}.$$

To be precise, $f_i(x, 0)$ is the determinant of the square matrix obtained from $M^y_\lambda$ by deleting its $(i + 1)$th row.

The criterion we need is the following.

PROPOSITION 4.1. *We have $p_y(I_\lambda) = k[x]$ if and only if $a_i \neq 0$ for all $i = 1, \dots, t$ such that $d_i \geq 1$ (equivalently, for all $i = 1, \dots, t$).*

*Proof.* Since $a_i = 1$ when $d_i = 0$, it is equivalent to prove that $p_y(I_\lambda) = k[x]$ if and only if $a_1 a_2 \cdots a_t \neq 0$.

Set $I_x = p_y(I_\lambda) \subset k[x]$. The condition $a_1 a_2 \cdots a_t \neq 0$ is sufficient. Indeed, the last polynomial $f_t(x,0)$ is the determinant of a lower triangular matrix whose diagonal entries are the scalars $a_i$; hence, $f_t(x,0) = a_1 a_2 \cdots a_t$. Thus, if $f_t(x,0)$ is nonzero, then $I_x = k[x]$.

To check the necessity of the condition, we will prove that, for each $i = 1, \ldots, t$, the vanishing of the scalar $a_i$ implies that the ideal $I_x$ is contained in a proper ideal generated by a minor of $M_\lambda^y$.

If $a_1 = 0$, then $f_1(x,0) = \cdots = f_t(x,0) = 0$ since these are determinants of matrices whose first row is zero. It follows that $I_x$ is the principal ideal generated by the characteristic polynomial $f_0(x,0)$, which is of degree $t \geq 1$. Hence, $I_x$ is a proper ideal of $k[x]$.

Let now $i \geq 2$. If for $k \geq i$, we delete the $(k+1)$th row of $M_\lambda^y$, then we obtain a lower block-triangular matrix of the form

$$\begin{pmatrix} M_1 & 0 \\ * & M_2^{(k)} \end{pmatrix},$$

where $M_1$ is the square submatrix of $M_\lambda^y$ corresponding to the rows $1, \ldots, i$ and to the columns $1, \ldots, i$; this is a lower triangular matrix whose diagonal entries are $a_1, \ldots, a_i$. Consequently, if $a_i = 0$, then $f_k(x,0) = 0$ for all $k \geq i$.

Under the same condition $a_i = 0$, if we delete the $(k+1)$th row of $M_\lambda^y$ for $k < i$, then we obtain a lower block-triangular matrix of the form

$$\begin{pmatrix} M_1^{(k)} & 0 \\ * & M_2 \end{pmatrix},$$

where $M_2$ is the square submatrix of $M_\lambda^y$ corresponding to the rows $i+1, \ldots, t+1$ and to the columns $i, \ldots, t$:

$$M_2 = \begin{pmatrix} a_{i+1,i} - x & a_{i+1} & \cdots & 0 & 0 \\ a_{i+2,i} & a_{i+2,i+1} - x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{t,i} & \cdots & \cdots & a_{t,t-1} - x & a_t \\ a_{t+1,i} & a_{t+1,i+1} & \cdots & a_{t+1,t-1} & a_{t+1,t} - x \end{pmatrix}.$$

Consequently, the polynomials $f_k(x,0)$ for $k < i$ are all divisible by the determinant of $M_2$. Thus, $I_x$ is contained in the ideal generated by $\det(M_2)$, which is a characteristic polynomial of degree $t - i + 1$. Since $t - i + 1 \geq 1$ for all $i = 1, \ldots, t$, we have $I_x \neq k[x]$. □

As an immediate consequence of Section 3.2 and of Proposition 4.1, we obtain the following:

COROLLARY 4.2. *The set of $n$-codimensional ideals of $k[x, y, y^{-1}]$ is the disjoint union*

$$\coprod_{\lambda \vdash n} C_\lambda^y,$$

*where $C^y_\lambda$ is the complement in the affine Gröbner cell $C_\lambda$ of the union of the hyperplanes $a_i = 0$ where $i$ runs over all integers $i = 1, \dots, t$ such that $d_i \geq 1$.*

### 4.2. On the Number of Finite-Codimensional Ideals of $\mathbb{F}_q[x, y, y^{-1}]$

Recall that the positive integer $v(\lambda)$ defined by (3.4) is the number of distinct columns of the partition $\lambda$.

PROPOSITION 4.3. *Let $k = \mathbb{F}_q$. For each partition $\lambda$ of $n$, the set $C^y_\lambda$ is finite, and its cardinality is given by*

$$\operatorname{card} C^y_\lambda = (q-1)^{v(\lambda)} q^{n+\ell(\lambda)-v(\lambda)}.$$

*Proof.* By Corollary 4.2 the set $C^y_\lambda$ is parameterized by $n + \ell(\lambda)$ parameters subject to the sole condition that $v(\lambda)$ of them are not zero. □

COROLLARY 4.4. *For each integer $n \geq 1$, the number $B_n(q)$ of $n$-codimensional ideals of $\mathbb{F}_q[x, y, y^{-1}]$ is equal to $(q-1)q^n B^\circ_n(q)$, where*

$$B^\circ_n(q) = \sum_{\lambda \vdash n} (q-1)^{v(\lambda)-1} q^{\ell(\lambda)-v(\lambda)}.$$

Note that $B^\circ_n(q)$ is a polynomial in $q$ since $v(\lambda) \geq 1$ and $\ell(\lambda) \geq v(\lambda)$ for all partitions. It is of degree $n-1$ and has integer coefficients. The coefficients of $B^\circ_n(q)$ may be negative, as we can see in Table 3.

REMARK 4.5. Let $v_n$ be the valuation of the polynomial $B^\circ_n(q)$, that is, the maximal integer $r$ such that $q^r$ divides $B^\circ_n(q)$. We conjecture that $v_n = 0$, 1, or 2 and

**Table 3** The polynomials $B^\circ_n(q)$

| $n$ | $B^\circ_n(q)$ | $B^\circ_n(1)$ | $B^\circ_n(-1)$ |
|---|---|---|---|
| 1 | $1$ | 1 | 1 |
| 2 | $q+1$ | 2 | 0 |
| 3 | $q^2+q$ | 2 | 0 |
| 4 | $q^3+q^2+q$ | 3 | $-1$ |
| 5 | $q^4+q^3+q^2-1$ | 2 | 0 |
| 6 | $q^5+q^4+q^3+q^2$ | 4 | 0 |
| 7 | $q^6+q^5+q^4+q^3-q-1$ | 2 | 0 |
| 8 | $q^7+q^6+q^5+q^4+q^3-q$ | 4 | 0 |
| 9 | $q^8+q^7+q^6+q^5+q^4-q^2-q$ | 3 | 1 |
| 10 | $q^9+q^8+q^7+q^6+q^5+q^4-q^2-q$ | 4 | 0 |
| 11 | $q^{10}+q^9+q^8+q^7+q^6+q^5-q^3-2q^2-q$ | 2 | 0 |
| 12 | $q^{11}+q^{10}+q^9+q^8+q^7+q^6+q^5-q^3-q^2+1$ | 6 | 0 |

that the infinite word $v_1v_2v_3\dots$ is equal to $0\prod_{n=1}^{\infty} 01^{2n}02^n$. This conjecture is supported by computer calculations.

Let us now give a product formula for the generating function of the sequence of polynomials $B_n(q)$ and an arithmetical interpretation for two values of $B_n^\circ(q)$.

THEOREM 4.6. (a) *Let $B_n(q)$ be the number of ideals of $\mathbb{F}_q[x,y,y^{-1}]$ of codimension $n$. We have*

$$1+\sum_{n\geq 1}\frac{B_n(q)}{q^n}t^n=\prod_{i\geq 1}\frac{1-t^i}{1-qt^i}.$$

(b) *Let $B_n^\circ(q)$ be the polynomial $B_n^\circ(q)=(q-1)^{-1}q^{-n}B_n(q)$. It has integer coefficients and satisfies*

$$B_n^\circ(1)=\sigma_0(n),$$

*where $\sigma_0(n)$ is the number of divisors of $n$, and*

$$B_n^\circ(-1)=\begin{cases}(-1)^{k-1} & \text{if } n=k^2 \text{ for some integer } k,\\ 0 & \text{otherwise.}\end{cases}$$

*Proof.* (a) Since an analogous proof will be used in Remark 4.7 and Section 6.2, we give here a detailed proof. Let $X$ be a set, and let $M$ be the free Abelian monoid on $X$ ($X$ is called a basis of $M$). We say that a function $\varphi : M \to R$ from $M$ to a ring $R$ is *multiplicative* if $\varphi(uv)=\varphi(u)\varphi(v)$ for all couples $(u,v)\in M^2$ of words having no common basis element. Under this condition, it is easy to check the following identity:

$$\sum_{w\in M}\varphi(w)=\prod_{x\in X}\Big(1+\sum_{e\geq 1}\varphi(x^e)\Big). \tag{4.1}$$

Now, identifying each partition with its planar diagram, we consider a partition $\lambda$ as a union of rectangular partitions $i^{e_i}$, with $e_i$ parts of length $i$, for $e_i\geq 1$ and distinct $i\geq 1$, which we denote by the formal product $\lambda=\prod_{i\geq 1} i^{e_i}$. Thus the set of partitions is equal to the free Abelian monoid on $X=\mathbb{N}\setminus\{0\}$ (viewed as a set). Before we apply (4.1), let us remark that $|\lambda|=\sum_i ie_i$ and $\ell(\lambda)=\sum_i e_i$. Moreover, the multisets $\{e_i \mid i\geq 1\}$ and $\{d_i \mid i\geq 1\}$ are equal (recall that the integers $d_i$ are those associated with $\lambda$ in (3.3)); therefore, $v(\lambda)=\sum_{i,d_i\geq 1}1=\operatorname{card}\{i\mid e_i\geq 1\}$.

Let $s$ be a new variable. The function $\lambda\mapsto \operatorname{card} C_\lambda^y s^{|\lambda|}$ computed in Proposition 4.3 is clearly multiplicative. Applying (4.1), we obtain

$$\begin{aligned}1+\sum_{n\geq 1}B_n(q)s^n &= 1+\sum_{|\lambda|\geq 1}\operatorname{card} C_\lambda^y s^{|\lambda|}\\ &=\prod_{i\geq 1}\Big(1+\sum_{e\geq 1}\operatorname{card} C_{i^e}^y s^{ie}\Big)\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i\geq 1}\Bigl(1 + \sum_{e\geq 1}(q-1)q^{ie+e-1}s^{ie}\Bigr) \\
&= \prod_{i\geq 1}\Bigl(1 + (q-1)q^{-1}\sum_{e\geq 1}(q^{i+1}s^{i})^{e}\Bigr) \\
&= \prod_{i\geq 1}\Bigl(1 + (q-1)q^{-1}\frac{q^{i+1}s^{i}}{1-q^{i+1}s^{i}}\Bigr) \\
&= \prod_{i\geq 1}\frac{(1-q^{i+1}s^{i}) + (q-1)q^{i}s^{i}}{1-q^{i+1}s^{i}} \\
&= \prod_{i\geq 1}\frac{1-q^{i}s^{i}}{1-q^{i+1}s^{i}}.
\end{aligned}
$$

Finally, replace $s$ by $q^{-1}t$.

(b) To compute $B_n^{\circ}(1)$, we use the formula of Corollary 4.4. Since the value at $q = 1$ of $(q-1)^{v(\lambda)-1}$ is 1 if $v(\lambda) = 1$ and 0 otherwise and since $v(\lambda) = 1$ if and only if $m_1 = \cdots = m_t = d$, in which case $dt = n$, we have

$$
B_n^{\circ}(1) = \sum_{dt=n} 1 = \sum_{d\mid n, d\geq 1} 1 = \sigma_0(n).
$$

For $B_n^{\circ}(-1)$, we use the infinite product expansion of Item (a): replacing $B_n(q)$ by $(q-1)q^n B_n^{\circ}(q)$, we obtain

$$
1 + \sum_{n\geq 1}(q-1)B_n^{\circ}(q)t^n = \prod_{i\geq 1}\frac{1-t^i}{1-qt^i}.
$$

Setting $q = -1$ yields

$$
1 - 2\sum_{n\geq 1} B_n^{\circ}(-1)t^n = \prod_{i\geq 1}\frac{1-t^i}{1+t^i}.
$$

Now recall the following identity of Gauss (see [12, (7.324)] or [17, 19.9 (i)]):

$$
\prod_{i\geq 1}\frac{1-t^i}{1+t^i} = \sum_{k\in\mathbb{Z}}(-1)^k t^{k^2}. \tag{4.2}
$$

It follows that

$$
1 - 2\sum_{n\geq 1} B_n^{\circ}(-1)t^n = 1 + 2\sum_{k\geq 1}(-1)^k t^{k^2},
$$

which allows us to conclude. □

REMARK 4.7. The results of Theorem 4.6 should be compared to the following ones concerning the number $A_n(q)$ of ideals of $\mathbb{F}_q[x, y]$ of codimension $n$. Proceeding as in the proof of Theorem 4.6, we deduce from (3.1) that

$$
1 + \sum_{n\geq 1} A_n(q)s^n = \prod_{i\geq 1}\frac{1}{1-q^{i+1}s^{i}}.
$$

Setting $q = -1$, we have

$$1 + \sum_{n \geq 1} A_n(-1) s^n = \prod_{i \geq 1} \frac{1}{1 - (-1)^{i+1} s^i} = \prod_{m \geq 1} \frac{1}{(1 - s^{2m-1})(1 + s^{2m})}. \tag{4.3}$$

Multiplying by $\prod_{m \geq 1} (1 + s^{2m})^{-1}$ both sides of the Euler identity

$$\prod_{m \geq 1} \frac{1}{1 - s^{2m-1}} = \prod_{i \geq 1} (1 + s^i)$$

(see [17, (19.4.7)]), we deduce that the right-hand side of (4.3) is equal to the infinite product

$$\prod_{m \geq 1} (1 + s^{2m-1}).$$

Thus by [2, Table 14.1, p. 310] or [17, (19.4.4)], the value $A_n(-1)$ is equal to the number[2] of partitions of $n$ with unequal odd parts. Note that $A_n(1)$ is equal to the number[3] of partitions of $n$. See Table 4 for a list of the polynomials $A_n(q)$ $(1 \leq n \leq 12)$.

**Table 4** The polynomials $A_n(q)$

| $n$ | $A_n(q)$ | $A_n(1)$ | $A_n(-1)$ |
|---|---|---|---|
| 1 | $q^2$ | 1 | 1 |
| 2 | $q^4 + q^3$ | 2 | 0 |
| 3 | $q^6 + q^5 + q^4$ | 3 | 1 |
| 4 | $q^8 + q^7 + 2q^6 + q^5$ | 5 | 1 |
| 5 | $q^{10} + q^9 + 2q^8 + 2q^7 + q^6$ | 7 | 1 |
| 6 | $q^{12} + q^{11} + 2q^{10} + 3q^9 + 3q^8 + q^7$ | 11 | 1 |
| 7 | $q^{14} + q^{13} + 2q^{12} + 3q^{11} + 4q^{10} + 3q^9 + q^8$ | 15 | 1 |
| 8 | $q^{16} + q^{15} + 2q^{14} + 3q^{13} + 5q^{12} + 5q^{11} + 4q^{10} + q^9$ | 22 | 2 |
| 9 | $q^{18} + q^{17} + 2q^{16} + 3q^{15}$ $+ 5q^{14} + 6q^{13} + 7q^{12} + 4q^{11} + q^{10}$ | 30 | 2 |
| 10 | $q^{20} + q^{19} + 2q^{18} + 3q^{17} + 5q^{16}$ $+ 7q^{15} + 9q^{14} + 8q^{13} + 5q^{12} + q^{11}$ | 42 | 2 |
| 11 | $q^{22} + q^{21} + 2q^{20} + 3q^{19} + 5q^{18} +$ $+ 7q^{17} + 10q^{16} + 11q^{15} + 10q^{14} + 5q^{13} + q^{12}$ | 56 | 2 |
| 12 | $q^{24} + q^{23} + 2q^{22} + 3q^{21} + 5q^{20} + 7q^{19}$ $+ 11q^{18} + 13q^{17} + 15q^{16} + 12q^{15} + 6q^{14} + q^{13}$ | 77 | 3 |

[2] See Sequence A000700 in [25].

[3] See Sequence A000041 in [25].

## 5. Invertible Gröbner Cells

Let $\mathrm{Hilb}^n((\mathbb{A}^1_k \setminus \{0\})^2)$ be the Hilbert scheme parameterizing finite subschemes of colength $n$ of the two-dimensional torus, that is, of the complement of two distinct intersecting lines in the affine plane. Its $k$-points are in bijection with the set of ideals of $k[x, y, x^{-1}, y^{-1}]$ of codimension $n$. By Section 3.2 this set of ideals is the disjoint union over the partitions $\lambda$ of $n$ of the sets $C_\lambda^{x,y}$, where $C_\lambda^{x,y}$ consists of the ideals $I \in C_\lambda$ such that both $x$ and $y$ are invertible in $k[x, y]/I$. We call $C_\lambda^{x,y}$ the *invertible Gröbner cell* associated with the partition $\lambda$.

When the ground field is finite, so is $C_\lambda^{x,y}$. The aim of this section is to compute the cardinality of $C_\lambda^{x,y}$ when $k = \mathbb{F}_q$.

### *5.1. The Cardinality of an Invertible Gröbner Cell*

Recall the nonnegative integers $d_1, \dots, d_t$ defined by (3.3) and the positive integer $v(\lambda)$ defined by (3.4). We now give a formula for $\operatorname{card} C_\lambda^{x,y}$.

THEOREM 5.1. *Let $k = \mathbb{F}_q$, let $n$ be an integer $\geq 1$, and let $\lambda$ be a partition of $n$. Then*

$$\operatorname{card} C_\lambda^{x,y} = (q-1)^{2v(\lambda)} q^{n-\ell(\lambda)} \prod_{\substack{i=1,\dots,t \\ d_i \geq 1}} \frac{q^{2d_i} - 1}{q^2 - 1}.$$

The theorem will be proved in Section 5.3. It has the following straightforward consequences.

COROLLARY 5.2. *Let $k = \mathbb{F}_q$, and let $\lambda$ be a partition of $n$.*

(a) $\operatorname{card} C_\lambda^{x,y}$ *is a monic polynomial in $q$ with integer coefficients*; *it is of degree $n + \ell(\lambda)$.*

(b) *The polynomial* $\operatorname{card} C_\lambda^{x,y}$ *is divisible by $(q-1)^2$. The quotient*

$$P_\lambda(q) = \frac{\operatorname{card} C_\lambda^{x,y}}{(q-1)^2}$$

*is a monic polynomial in $q$ with integer coefficients and of degree $n+\ell(\lambda)-2$.*

(c) *If the partition $\lambda$ is rectangular, that is, if $v(\lambda) = 1$, in which case $d_2 = \cdots = d_t = 0$ and $d = d_1$ is a divisor of $n$, then*

$$P_\lambda(q) = q^{n-d} \frac{q^{2d} - 1}{q^2 - 1} = q^{n-d}(1 + q^2 + \cdots + q^{2d-2}).$$

*In this case, $P_\lambda(1) = d$.*

(d) *If $v(\lambda) \geq 2$, then $P_\lambda(q)$ is divisible by $(q-1)^2$, and $P_\lambda(1) = 0$.*

REMARK 5.3. The polynomials $P_\lambda(q)$ may have negative coefficients. For instance, if $\lambda$ is the partition of 4 corresponding to $t = 2$, $d_1 = 1$, $d_2 = 2$, then

$$P_\lambda(q) = q^5 - 2q^4 + 2q^3 - 2q^2 + q.$$

The rest of the section is devoted to the proof of Theorem 5.1.

### 5.2. A Criterion for the Invertibility of $x$

In Section 4 we introduced the algebra map $p_y : k[x, y] \to k[x]$ sending $x$ to itself and $y$ to 0. Similarly, let $p_x : k[x, y] \to k[x]$ be the algebra map sending $x$ to 0 and $y$ to itself. Then by Lemma 2.2, the set $C_\lambda^{x,y}$ consists of the ideals $I \in C_\lambda$ such that $p_x(I) = k[y]$ and $p_y(I) = k[x]$. We already have a criterion for $p_y(I) = k[x]$ (see Proposition 4.1). We shall now give a necessary and sufficient condition for $p_x(I)$ to be equal to $k[y]$.

Resuming the notation of Section 4, we see that $p_x(I)$ can be identified with the ideal of $k[y]$ generated by the polynomials $f_0(0, y), \dots, f_t(0, y) \in k[y]$ obtained from the polynomials $f_0(x, y), \dots, f_t(x, y)$ by setting $x = 0$. The polynomials $f_0(0, y), \dots, f_t(0, y)$ are the maximal minors of the matrix $M_\lambda^x$ obtained from the matrix $M_\lambda$ of (3.5) by setting $x = 0$.

Let $\mu_i$ be the determinant of the submatrix $M_i$ of $M_\lambda^x$ corresponding to the rows $(i+1), \dots, (t+1)$ and to the columns $i, \dots, t$. We have $\mu_t = p_{t+1,t}$ and

$$\mu_i = \begin{vmatrix} p_{i+1,i} & y^{d_{i+1}} + p_{i+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{t,i} & p_{t,i+1} & \cdots & y^{d_t} + p_t \\ p_{t+1,i} & p_{t+1,i+1} & \cdots & p_{t+1,t} \end{vmatrix}$$

if $1 \le i < t$. Expanding $\mu_i$ along its first column, we obtain

$$\mu_i = \sum_{j=1}^{t-i+1} p_{i+j,i} q_{i+j,i}, \tag{5.1}$$

where

$$q_{i+j,i} = \begin{cases} \mu_{i+1} & \text{if } j = 1, \\ (-1)^{j-1}(y^{d_{i+1}} + p_{i+1}) & \\ \qquad \cdots (y^{d_{i+j-1}} + p_{i+j-1})\mu_{i+j} & \text{if } 1 < j < t - i + 1, \\ (-1)^{t-i}(y^{d_{i+1}} + p_{i+1}) & \\ \qquad \cdots (y^{d_{t-1}} + p_{t-1})(y^{d_t} + p_t) & \text{if } j = t - i + 1. \end{cases} \tag{5.2}$$

Observe also that

$$f_i(0, y) = \begin{cases} \mu_1 & \text{if } i = 0, \\ (y^{d_1} + p_1) \cdots (y^{d_i} + p_i)\mu_{i+1} & \text{if } 1 \le i < t, \\ (y^{d_1} + p_1) \cdots (y^{d_t} + p_t) & \text{if } i = t. \end{cases} \tag{5.3}$$

LEMMA 5.4. *If $1 \le i \le j \le t$, then $\mu_i$ belongs to the ideal $(\mu_j, y^{d_j} + p_j)$ generated by $\mu_j$ and $(y^{d_j} + p_j)$.*

*Proof.* The case $i = j$ is obvious. Otherwise, consider the matrix $M_i$ whose determinant is $\mu_i$; the column of $M_i$ containing the entry $y^{d_j} + p_j$ can be written as the sum of a column containing only the entry $y^{d_j} + p_j$, the other entries being zero, and of a column whose top entry is zero and the bottom ones form the first column of the matrix $M_j$ whose determinant is $\mu_j$. Therefore by the multilinearity property of determinants, $\mu_i$ is the sum of a determinant that is a multiple of $y^{d_j}+p_j$ and of another determinant that is a multiple of $\mu_j$; indeed, this second determinant is block-triangular with one diagonal block equal to $\mu_j$. $\square$

Here is our criterion for the invertibility of $x$.

PROPOSITION 5.5. *We have $p_x(I_\lambda)=k[y]$ if and only if $y^{d_i}+p_i$ and $\mu_i$ are coprime for all $i=1,\dots,t$.*

*Proof.* (a) Let us first check the sufficiency. The fact that $y^{d_t}+p_t$ and $\mu_t$ are coprime implies that by (5.3) the gcd of $f_t(0,y)$ and of $f_{t-1}(0,y)$ is $(y^{d_1}+p_1)\cdots(y^{d_{t-1}}+p_{t-1})$. Now the gcd of the latter and of $f_{t-2}(0,y)$ is $(y^{d_1}+p_1)\cdots(y^{d_{t-2}}+p_{t-2})$ in view of the fact that $y^{d_{t-1}}+p_{t-1}$ and $\mu_{t-1}$ are coprime. Repeating this argument, we find that the gcd of $f_0(0,y),\dots,f_t(0,y)$ is 1, which implies that $p_x(I_\lambda)=k[y]$.

(b) Conversely, suppose that $y^{d_j}+p_j$ and $\mu_j$ are not coprime for some $j$, that is, $(\mu_j, y^{d_j}+p_j)\neq k[y]$. By (5.3) and Lemma 5.4, $f_0(0,y),\dots,f_{j-1}(0,y)$ belong to the ideal $(\mu_j, y^{d_j}+p_j)$. On the other hand, again by (5.3), the remaining polynomials $f_j(0,y),\dots,f_t(0,y)$ are divisible by $y^{d_j}+p_j$ and hence belong to $(\mu_j, y^{d_j}+p_j)$. Therefore, $p_x(I_\lambda)\subseteq(\mu_j, y^{d_j}+p_j)\neq k[y]$. $\square$

For the proof of Theorem 5.1, we also need the following result.

LEMMA 5.6. *If $y^{d_j}+p_j$ and $\mu_j$ are coprime for all $j>i$, then the polynomials $q_{i+1,i},\dots,q_{t+1,i}$ of (5.2) are coprime.*

*Proof.* Note that the polynomials $q_{r,s}$ are defined for $r>s$ in (5.2). Proceeding as in Part (a) of the proof of Proposition 5.5 and using (5.2), we show by descending induction on $j$ that the gcd of $q_{j+1,i},\dots,q_{t+1,i}$ is

$$(y^{d_{i+1}}+p_{i+1})\cdots(y^{d_j}+p_j).$$

In particular, for $j=i+1$, the gcd of $q_{i+2,i},\dots,q_{t+1,i}$ is $(y^{d_{i+1}}+p_{i+1})$. The conclusion follows from this fact together with the coprimality of $(y^{d_{i+1}}+p_{i+1})$ and of $q_{i+1,i}=\mu_{i+1}$. $\square$

### *5.3. Proof of Theorem 5.1*

We have to count the possible matrices $M_\lambda$ such that $M_\lambda^x$ and $M_\lambda^y$ are invertible; equivalently, to count the matrices $M_\lambda^x$ since $x$ in $M_\lambda$ appears with constant coefficients. By Propositions 4.1 and 5.5, it suffices to count the matrices $M_\lambda^x$ over $\mathbb{F}_q[y]$ such that $p_i(0)\neq 0$ and $y^{d_i}+p_i$ and $\mu_i$ are coprime for all $i=1,\dots,t$. We consider these conditions successively for $i=t,t-1,\dots,1$.

Assume first that all integers $d_1,\dots,d_t$ are nonzero. For $i=t$, $y^{d_t}+p_t$ is a monic polynomial of degree $d_t$ with nonzero constant term, $\mu_t=p_{t+1,t}$ is of degree $<d_t$, and both polynomials are coprime. It follows from Lemma 2.5 (or from

Proposition 2.3 with $d = d_t$ and $h = 1$) that we have $(q-1)^2(q^{2d_t}-1)/(q^2-1)$ possible choices for the last column of $M_\lambda$.

For $i = t-1$, it follows from (5.1) that $\mu_{t-1} = P_1Q_1 + P_2Q_2$, where $Q_1 = q_{t,t-1}$ and $Q_2 = -q_{t+1,t-1}$, which are coprime by Lemma 5.6, $P_1 = p_{t,t-1}$ and $P_2 = p_{t+1,t-1}$, which are both polynomials of degree $< d_{t-1}$. The polynomial $P = y^{d_{t-1}} + p_{t-1}$ is monic of degree $d_{t-1}$ with nonzero constant term, and $Q = \mu_{t-1} = P_1Q_1 + P_2Q_2$ is coprime to $P$ by the coprimality condition. It then follows from Proposition 2.3 applied to the case $d = d_{t-1}$ and $h = 2$ that there are

$$(q-1)^2 q^{d_{t-1}} \frac{q^{2d_{t-1}}-1}{q^2-1}$$

possible choices for the $(t-1)$th column of $M_\lambda$.

In general, the polynomial $P = y^{d_i} + p_i$ is monic of degree $d_{t-1}$ with nonzero constant term and is assumed to be coprime to $Q = \mu_i = \sum_{j=1}^{t-i+1} p_{i+j,i}q_{i+j,i}$. By Lemma 5.6 the polynomials $q_{i+1,i}, \ldots, q_{t+1,i}$ are coprime. Applying Proposition 2.3 to the case $d = d_i$ and $h = t+1-i$, we see that there are

$$(q-1)^2 q^{(t-i)d_i} \frac{q^{2d_i}-1}{q^2-1}$$

possible choices for the $i$th column of $M_\lambda^x$.

In the end we obtain a number of possible entries for $M_\lambda^x$ equal to

$$\prod_{i=1}^{t} (q-1)^2 q^{(t-i)d_i} \frac{q^{2d_i}-1}{q^2-1} = q^{n-\ell(\lambda)} \prod_{i=1}^{t} (q-1)^2 \frac{q^{2d_i}-1}{q^2-1}$$

since $\ell(\lambda) = \sum_{i=1}^t d_i$ and $n = |\lambda| = \sum_{i=1}^t (t-i+1)d_i$. We have thus proved the theorem when $d_1, \ldots, d_t$ are all nonzero.

Let $E$ be the subset of $\{1, \ldots, t\}$ consisting of those subscripts $i$ for which $d_i = 0$. (Note that 1 does not belong to $E$ since $d_1 > 0$.) Assume now that $E$ is nonempty and set $t' = t - \operatorname{card} E$. By assumption $t' < t$. For any positive integer $i \le t'$, let $d'_i$ be equal to the $i$th nonzero $d_i$. The integers $d'_1 = d_1, d'_2, \ldots, d'_{t'}$ are positive.

Recall that if $i \in E$, then the $i$th column of the matrix $M_\lambda^x$ is zero except for the $(i,i)$-entry, which is 1. Permuting rows and columns, we may rearrange $M_\lambda^x$ into a matrix $M'_\lambda$ of the form

$$M'_\lambda = \begin{pmatrix} M_\nu^x & 0 \\ N & I_{t-t'} \end{pmatrix},$$

where $I_{t-t'}$ is the identity matrix of size $(t-t')$. The $(t'+1) \times t'$-matrix $M_\nu^x$ has the same form as $M_\lambda^x$ with $t$ replaced by $t'$, the sequence $d_1, \ldots, d_t$ by the shorter sequence $d'_1, \ldots, d'_{t'}$, and the partition $\lambda$ by the partition $\nu$ associated with the sequence $d'_1, \ldots, d'_{t'}$.

Let $f'_i$ be the determinant of the square matrix obtained from $M'_\lambda$ by deleting its $(i+1)$th row. It is clear that up to sign and to reordering the maximal minors

$f'_0, \dots, f'_t$ of $M'_\lambda$ are the same as those of $M_\lambda$. In view of the special form of $M'_\lambda$, observe that

$$f'_i = \begin{cases} f_i^{(\nu)} & \text{if } 0 \le i \le t', \\ 0 & \text{if } t' < i \le t, \end{cases}$$

where $f_i^{(\nu)}$ is the determinant of the $t' \times t'$-matrix obtained from $M_\nu$ by deleting its $(i+1)$th row.

The number of possible entries of $M_\lambda$, which is the same as the number of possible entries of $M'_\lambda$, is then the product of the number of possible entries of $N$, which is a power of $q$, and of the number of possible entries of $M_\nu$. Since $d'_1, \dots, d'_{t'}$ are positive, by the first part of the proof, we know that the number of possible entries of $M_\nu$ is the product of a power of $q$ by

$$\prod_{i=1}^{t'} (q-1)^2 \frac{q^{2d'_i} - 1}{q^2 - 1}.$$

In other words, the number of possible entries of $M_\lambda$ is

$$q^c \prod_{\substack{i=1,\dots,t \\ d_i \ge 1}} (q-1)^2 \frac{q^{2d_i} - 1}{q^2 - 1}$$

for some nonnegative integer $c$. Now since the invertible Gröbner cell $C_\lambda^{x,y}$ is a Zarisky open subset of the affine Gröbner cell $C_\lambda$, the degree of the previous polynomial in $q$ must be the same as the degree of the cardinal of $C_\lambda$, which is $q^{n+\ell(\lambda)}$ by Section 3.1. This suffices to establish that $c = n - \ell(\lambda)$ and to complete the proof of the theorem.

### 5.4. *Proof of Theorem 1.1*

By our remark at the beginning of Section 5, the number $C_n(q)$ of ideals of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of codimension $n$ is given by

$$C_n(q) = \sum_{\lambda \vdash n} \operatorname{card} C_\lambda^{x,y}, \tag{5.4}$$

where $C_\lambda^{x,y}$ is the *invertible Gröbner cell* associated with the partition $\lambda$. The equality in Theorem 1.1 follows then from the formula for $\operatorname{card} C_\lambda^{x,y}$ given in Theorem 5.1.

By Corollary 5.2 (a) $\operatorname{card} C_\lambda^{x,y}$ is a monic polynomial of degree $n + \ell(\lambda)$ with integer coefficients. Therefore, $C_n(q)$ has integer coefficients, and its degree is $\max\{n + \ell(\lambda) \mid \lambda \vdash n\}$. Now $\ell(\lambda)$ is maximal if and only if $\lambda = 1^n$, in which case $\ell(\lambda) = n$. Therefore $C_n(q)$ is monic, and its degree is $2n$.

Since $\nu(\lambda) \ge 1$, it follows from the formula in Theorem 5.1 that $\operatorname{card} C_\lambda^{x,y}$ is divisible by $(q-1)^2$ for each invertible Gröbner cell. Therefore, the polynomial $C_n(q)$ is divisible by $(q-1)^2$.

## 6. Proofs of the Corollaries

We now start the proofs of Corollaries 1.2 and 1.4.

### 6.1. *Proof of Corollary 1.2*

Since $C_n(q)$ and $(q-1)^2$ are both monic with integer coefficients, so is $P_n(q)$. The latter is the sum over all partitions of $n$ of the polynomials $P_\lambda(q)$ (introduced in Corollary 5.2(b)). By Corollary 5.2(c)–(d), we have $P_\lambda(1)=0$ if $v(\lambda)\geq 2$ and if $v(\lambda)=1$, then $\lambda$ is of the form $t^d$, where $dt=n$, in which case $P_\lambda(1)=d$. The desired formula for $P_n(1)$ follows.

### 6.2. *Proof of Corollary 1.4*

As in the proof of Theorem 4.6, we consider each partition $\lambda$ as a union of rectangular partitions $i^{e_i}$, with $e_i$ parts of length $i$, for $e_i\geq 1$ and distinct $i\geq 1$. Recall that $|\lambda|=\sum_i ie_i$, $\ell(\lambda)=\sum_i e_i$, and $v(\lambda)=\sum_i 1$. To indicate the dependance of $e_i$ on $\lambda$, we write $e_i=e_i(\lambda)$. We then obtain the following statement.

PROPOSITION 6.1. *Let $s_1,s_2,\dots$ be new variables. We have the infinite product expansion*

$$1+\sum_\lambda \operatorname{card} C_\lambda^{x,y} s_1^{e_1(\lambda)} s_2^{e_2(\lambda)}\cdots=\prod_{i\geq 1}\frac{(1-q^i s_i)^2}{(1-q^{i+1}s_i)(1-q^{i-1}s_i)}.$$

*Proof.* Proceeding as in the proof of Theorem 4.6 and using Theorem 5.1, we deduce that the left-hand side is equal to

$$1+\sum_\lambda\prod_{i\geq 1}(q-1)^2\frac{q^{2e_i(\lambda)}-1}{q^2-1}q^{ie_i(\lambda)-e_i(\lambda)}s_i^{e_i(\lambda)},$$

which in turn is equal to

$$\begin{aligned}
&\prod_{i\geq 1}\left(1+\frac{(q-1)^2}{q^2-1}\sum_{e_i\geq 1}((q^{i+1}s_i)^{e_i(\lambda)}-(q^{i-1}s_i)^{e_i(\lambda)})\right)\\
&\quad=\prod_{i\geq 1}\left(1+\frac{(q-1)^2}{q^2-1}\left(\frac{q^{i+1}s_i}{1-q^{i+1}s_i}-\frac{q^{i-1}s_i}{1-q^{i-1}s_i}\right)\right)\\
&\quad=\prod_{i\geq 1}\left(1+\frac{(q-1)^2}{q^2-1}\frac{(q^2-1)q^{i-1}s_i}{(1-q^{i+1}s_i)(1-q^{i-1}s_i)}\right)\\
&\quad=\prod_{i\geq 1}\left(1+\frac{(q-1)^2q^{i-1}s_i}{(1-q^{i+1}s_i)(1-q^{i-1}s_i)}\right)\\
&\quad=\prod_{i\geq 1}\frac{(1-q^i s_i)^2}{(1-q^{i+1}s_i)(1-q^{i-1}s_i)}.
\end{aligned}$$

□

*Proof of Corollary 1.4.* (a) Replace $s_i$ by $(t/q)^i$ in Proposition 6.1, use (5.4) and Theorem 1.1, and observe that $(1-qt^i)(1-q^{-1}t^i) = 1-(q+q^{-1})t^i + t^{2i}$.

(b) The infinite product is clearly invariant under the transformation $q \leftrightarrow q^{-1}$; thus, $C_n(q^{-1}) = q^{-2n}C_n(q)$. Together with $\deg C_n(q) = 2n$, this implies that $C_n(q)$ is palindromic. The polynomial $P_n(q)$ is palindromic as a quotient of two palindromic polynomials. $\square$

### 6.3. *An Alternative Proof of Corollary 1.4(a)*

After we made public a first version of this article, we learnt of an alternative geometric approach to the polynomials $C_n(q)$. Indeed, Göttsche and Soergel determined the mixed Hodge structure of the punctual Hilbert schemes of any smooth complex algebraic surface (see [15, Thm. 2]). Applying their result to the Hilbert scheme $H^n_{\mathbb{C}} = \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times)$ of $n$ points of the complex two-dimensional torus, Hausel, Letellier, and Rodriguez-Villegas observed in [20, Thm. 4.1.3] that the compactly supported mixed Hodge polynomial $H_c(H^n_{\mathbb{C}}; q, u)$ of $H^n_{\mathbb{C}}$ fits into the equality of formal power series

$$1 + \sum_{n\geq 1} H_c(H^n_{\mathbb{C}}; q, u)\frac{t^n}{q^n} = \prod_{i\geq 1} \frac{(1+u^{2i+1}t^i)^2}{(1-u^{2i+2}qt^i)(1-u^{2i}q^{-1}t^i)}. \tag{6.1}$$

Setting $u=-1$ in (6.1), we obtain an infinite product expansion for the generating function of the $E$-polynomial $E(H^n_{\mathbb{C}}; q) = H_c(H^n_{\mathbb{C}}; q, -1)$ of $H^n_{\mathbb{C}}$, namely

$$1 + \sum_{n\geq 1} E(H^n_{\mathbb{C}}; q)\frac{t^n}{q^n} = \prod_{i\geq 1} \frac{(1-t^i)^2}{1-(q+q^{-1})t^i + t^{2i}}. \tag{6.2}$$

Now, $H^n_{\mathbb{C}}$ is strongly polynomial-count in the sense of Nick Katz (see [21, Appendix]), probably a well-known fact (which also follows from the computations in the present paper). Therefore, by [21, Thm. 6.1.2] the $E$-polynomial counts the number of elements of $H^n$ over the finite field $\mathbb{F}_q$, which is also the number $C_n(q)$ of ideals of codimension $n$ of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$. Thus (6.2) implies the equality of Corollary 1.4(a).

REMARK 6.2. In the same vein as before, there is a geometric explanation of the palindromicity of the polynomials $C_n(q)$. De Cataldo, Hausel, and Migliorini [9] observed that any diffeomorphism between $\mathbb{C}^\times \times \mathbb{C}^\times$ and the cotangent bundle $E \times \mathbb{C}$ of the elliptic curve $E = \mathbb{C}/\mathbb{Z}[i]$ induces a linear isomorphism of graded vector spaces between the cohomology groups of the corresponding Hilbert schemes: $H^*(H^n_{\mathbb{C}}, \mathbb{Q}) \cong H^*(\mathrm{Hilb}^n(E\times\mathbb{C}), \mathbb{Q})$. This isomorphism does not preserve the mixed Hodge structures, as that on the right-hand side is pure, whereas that on the left-hand side is not. Nevertheless, such an isomorphism identifies the weight filtration on $H^*(H^n_{\mathbb{C}}, \mathbb{Q})$ with the perverse Leray filtration on $H^*(\mathrm{Hilb}^n(E\times\mathbb{C}), \mathbb{Q})$ associated with the natural projective map from $\mathrm{Hilb}^n(E\times\mathbb{C})$ to the $n$th symmetric product of $\mathbb{C}$ induced by the projection of $E\times\mathbb{C}$ on the second factor. The perverse Leray filtration is "palindromic" as a consequence of the relative hard Lefschetz theorem for that map (see [9, Thms. 4.1.1 and 4.3.2]).

Note that Hausel, Letellier, and Rodriguez-Villegas observed a similar palindromicity for the $E$-polynomial of certain character varieties and termed it "curious Poincaré duality" in [19, Cor. 5.2.4] (see also [21, Cor. 3.5.3] and [18, Cor. 1.4]).

REMARK 6.3. The natural action of the group $\mathbb{C}^\times \times \mathbb{C}^\times$ on itself induces an action on the Hilbert scheme $H^n_{\mathbb{C}}$. Consider the GIT quotient $\widetilde{H}^n_{\mathbb{C}} = H^n_{\mathbb{C}}//(\mathbb{C}^\times \times \mathbb{C}^\times)$. Using [21, Thm. 2.2.12] and [19, Section 5.3], we see that the $E$-polynomial of $\widetilde{H}^n_{\mathbb{C}}$ is given by

$$E(\widetilde{H}^n_{\mathbb{C}}; q) = E(H^n_{\mathbb{C}}; q)/(q-1)^2 = C_n(q)/(q-1)^2 = P_n(q).$$

Recall from Introduction (see also Appendix) that the coefficients of $P_n(q)$ are all nonnegative. Therefore, $\widetilde{H}^n_{\mathbb{C}}$ provides an example of a polynomial-count variety with odd cohomology and a counting polynomial with nonnegative coefficients. This implies nontrivial cancellation for the mixed Hodge numbers of $\widetilde{H}^n_{\mathbb{C}}$. No similar positivity phenomenon was observed for the character varieties investigated by Hausel, Letellier, and Rodriguez-Villegas.

### 6.4. *Proof of Corollary 1.5*

It is well known that the generating function of the partition function $p(n)$ is

$$\sum_{n\geq 0} p(n)t^n = \prod_{i\geq 1} \frac{1}{1-t^i}.$$

By [1, Eq. (1.11) and Thm. 1] the generating function for the integers $N_V(m,n)$ is

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} N_V(m,n)q^m t^n = \prod_{i\geq 1} \frac{1-t^i}{1-(q+q^{-1})t^i + t^{2i}}.$$

Thus by Corollary 1.4 we have

$$\begin{aligned}\Bigl(\sum_{n\geq 0} p(n)t^n\Bigr)\Bigl(1+\sum_{n\geq 1}\frac{C_n(q)}{q^n}t^n\Bigr) &= \prod_{i\geq 1}\frac{1}{1-t^i}\prod_{i\geq 1}\frac{(1-t^i)^2}{1-(q+q^{-1})t^i+t^{2i}} \\ &= \prod_{i\geq 1}\frac{1-t^i}{1-(q+q^{-1})t^i+t^{2i}} \\ &= \sum_{m\in\mathbb{Z}}\sum_{n\geq 0} N_V(m,n)q^m t^n.\end{aligned}$$

The desired formula follows immediately.

## Appendix. The Coefficients of the Polynomials $C_n(q)$ and $P_n(q)$

We now state the results of the companion paper [22] on the coefficients of the polynomials $C_n(q)$ and $P_n(q)$.

Since $C_n(q)$ and $P_n(q)$ are palindromic of respective degrees $2n$ and $2n-2$, we may expand $C_n(q)$ and $P_n(q)$ as follows:

$$C_n(q) = c_{n,0}q^n + \sum_{i=1}^{n} c_{n,i}(q^{n+i} + q^{n-i}),$$

where $c_{n,0}, c_{n,1}, c_{n,2} \ldots$ are integers, and

$$P_n(q) = a_{n,0}q^{n-1} + \sum_{i=1}^{n-1} a_{n,i}(q^{n+i-1} + q^{n-i+1}),$$

where $a_{n,0}, a_{n,1}, a_{n,2} \ldots$ are integers.

By Theorem 1.1 of [22] the coefficients $c_{n,i}$ of $C_n(q)$ are given by the following formulas: (a) For the central coefficients $c_{n,0}$, we have

$$c_{n,0} = \begin{cases} 2(-1)^k & \text{if } n = k(k+1)/2 \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) For the noncentral coefficients ($i \geq 1$), we have

$$c_{n,i} = \begin{cases} (-1)^k & \text{if } n = k(k+2i+1)/2 \text{ for some integer } k \geq 1, \\ (-1)^{k-1} & \text{if } n = k(k+2i-1)/2 \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in Item (b) the first two conditions are mutually exclusive.

As for the coefficients of $P_n(q)$, the coefficient $a_{n,i}$ is by [22, Thm. 1.2] equal to the number of divisors $d$ of $n$ such that

$$\frac{i + \sqrt{2n + i^2}}{2} < d \leq i + \sqrt{2n + i^2}.$$

It follows that all coefficients $a_{n,i}$ of $P_n(q)$ are nonnegative integers.

ACKNOWLEDGMENTS. We are grateful to Olivier Benoist, François Bergeron, Mark Haiman, Emmanuel Letellier, and Luca Migliorini for useful discussions, to Pierre Baumann for suggesting the proof of Lemma 2.5, and to the referee for having spotted an inaccuracy in a first version of the proof of Theorem 5.1 We also thank Frank Garvan for having pointed out the relationship with the crank of partitions and José Manuel Rodríguez Caballero for having brought the reference [23] to our attention.

The second-named author is grateful to the Université de Strasbourg for the invited professorship, which allowed him to spend the month of June 2014 at IRMA; he was also supported by NSERC (Canada).

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