Michigan Math. J. 67 (2018), 227–251

Splitting Criteria for Vector Bundles Induced by Restrictions to Divisors

Mihai Halic

Abstract. In this article we obtain criteria for the splitting and triviality of vector bundles by restricting them to partially ample divisors. This allows us to study the problem of splitting on the total space of fibre bundles. The statements are illustrated with examples.

For products of minuscule homogeneous varieties, we show that the splitting of vector bundles can be tested by restricting them to subproducts of Schubert 2-planes. By using known cohomological criteria for multiprojective spaces, we deduce necessary and sufficient conditions for the splitting of vector bundles on products of minuscule varieties.

The triviality criteria are particularly suited to Frobenius split varieties. We prove that a vector bundle on a smooth toric variety, whose anticanonical bundle has stable base locus of codimension at least three, is trivial precisely when its restrictions to the invariant divisors are trivial, with trivializations compatible along the various intersections.

Introduction

Although the problem of deciding the splitting of vector bundles is classical, only relatively few cases have been settled despite numerous efforts: there are cohomological criteria for products of projective spaces and quadrics [9; 4], Grassmannians [25; 22], hypersurfaces in projective spaces [27; 5], and uniformity criteria for vector bundles on minuscule homogeneous varieties [24].

Our guiding principle is that, for investigating the splitting of vector bundles, we should consider restrictions in order to *simplify* the problem. Usually, this process leads to (much) lower-dimensional subvarieties, where we can use further techniques. Indeed, when the geometry of the base variety is involved, the cohomological characterization of the split vector bundles becomes very intricate; this is apparent in the references cited above. For this reason, it is both computationally and theoretically convenient to first reduce the dimension of the base and only afterward to apply cohomological methods.

The statement is strongly supported by applications. In [14], we obtained an *algorithmic* necessary and sufficient condition for the splitting of vector bundles of arbitrary rank on minuscule homogeneous varieties, valid in any characteristic. This goes beyond [24], where vector bundles of sufficiently low rank were considered. The techniques developed inhere show that the splitting of a vector bundle

Received July 19, 2016. Revision received January 18, 2018.

on a product of minuscule homogeneous varieties can be verified by restricting it to the product of the two-dimensional Schubert subvarieties of the factors. By combining this result with [9; 22], we deduce necessary and sufficient conditions for the splitting. To our knowledge, currently there are *no results* in this direction; even a product of Grassmannians is uncovered.

A vector bundle on $\mathbb{P}^n$, $n \geq 3$, splits precisely when its restriction to some hyperplane $\mathbb{P}^{n-1}$ does (see [18]). This was generalized in [3] for restrictions of vector bundles on "*Horrocks varieties*"—a restrictive cohomological condition— to *ample divisors*. The ampleness assumption excludes several natural situations, for example, the case of morphisms, where we wish to restrict vector bundles either to preimages of ample divisors or to relatively ample ones.

Here we generalize the works cited to include q-ample divisors; this covers the case of morphisms mentioned before. We obtain two types of results, *splitting* and *triviality* criteria.

THEOREM (splitting criteria). *Let* $(X, \mathcal{O}_X(1))$ *be a smooth complex projective variety with* $\dim X \geq 3$. *Let* $\mathcal{V}$ *be a vector bundle on* X, $\mathcal{E} := \mathcal{E}nd(\mathcal{V})$ *the bundle of its endomorphisms*, $\mathcal{L} \in \operatorname{Pic}(X)$, *and* $D \in |d\mathcal{L}|$. *The equivalence*

$$[\mathcal{V} \textit{ splits} \Leftrightarrow \mathcal{V} \otimes \mathcal{O}_D \textit{ splits}]$$

holds in any of the following cases:

(a) (*see Proposition 1.8, Remark 1.9*) *If* $\mathcal{L}$ *is* $(\dim X - 3)$-*ample and* $H^1(\mathcal{E}_D \otimes \mathcal{L}_D^{-a}) = 0$ *for all* $a \geq d$. *The parameter* d *is bounded from below by a linear function in the regularity of* $\mathcal{E}$ *with respect to* $\mathcal{O}_X(1)$.

(b) (*see Corollary 2.5*) *If* X *is* 2-*split*, $\mathcal{L}$ *is* $(\dim X - 4)$-*positive*, *and* D *is smooth*.

(c) (*see Theorem 3.4*) *If* X *is* 1-*split*, $\mathcal{L}$ *is globally generated and* $(\dim X - 4)$-*ample*, *and* D *is very general*.

The conditions 1- and 2-split are respectively the notions of "splitting" and "Horrocks variety" in [3]. The statements become more effective in the case where $\mathcal{L}$ is relatively ample with respect to a morphism: in (a), it suffices D to be weakly normal and $\mathcal{E} \otimes \mathcal{L}$ be relatively ample (see Theorem 1.10); in (b), D can be arbitrary (see Corollary 2.5(b)); hence we strengthen Bakhtary's result.

We illustrate the advantage of allowing q-ample line bundles by discussing several explicit examples. The cohomological splitting criteria for products of projective spaces and quadrics [9; 4; 22] involve numerous conditions. The effect of restricting to "subproducts" is a massive reduction of the number of tests. As we mentioned earlier, we obtain a splitting criterion for vector bundles on products of minuscule homogeneous varieties; these include, for example, projective planes, Grassmannian, quadrics, and spinor varieties.

THEOREM (see Theorem 2.9). *Let* $M^{(j)}$, $j = 1, \ldots, t$, *be minuscule homogeneous varieties*, *with* $\dim M^{(j)} \geq 2$. *A vector bundle on* $X := M^{(1)} \times \cdots \times M^{(t)}$ *splits if and only if it does on* $X_{2t} := M_2^{(1)} \times \cdots \times M_2^{(t)}$, *where* $M_2^{(j)} \subset M^{(j)}$ *stands for the union of the two-dimensional Schubert subvarieties.*

The result completely addresses the splitting problem for X as before. Indeed, each factor $M_2^{(j)}$ is either a projective plane or a union of two planes (see [14]), and hence X_{2t} is a union of $(\mathbb{P}^2)^t$. Together with [9; 22], we deduce necessary and sufficient cohomological conditions to probe the splitting of vector bundles.

The trivializable vector bundles are particular cases of the split ones, so the triviality criteria in the next theorem hold in greater generality. Notably, we can eliminate the conditions 1- and 2-split.

THEOREM (triviality criteria). *Let X, $\mathcal{L}$, $\mathcal{V}$ be as before, and let $D \in |\mathcal{L}|$. The equivalence [$\mathcal{V}$ is trivial $\Leftrightarrow$ $\mathcal{V}_D$ is trivial] holds in any of the following situations*:

(a) (*see Theorem 4.2*)
   - *If $\mathcal{L} \in \mathrm{Pic}(X)$ is semiample and* $(\dim X - 3)$*-ample.*
   - *If $\mathcal{L}$ is relatively ample for a morphism $X \xrightarrow{f} Y$ of relative dimension at least three.*

(b) (*see Corollary 4.4*) *If the anticanonical bundle ω_X^{-1} is* $(\dim X - 3)$*-ample, X is Frobenius split by a power of a section σ in ω_X^{-1}, and $D = \mathrm{divisor}(\sigma)$.*

Condition (b) is particularly suited for spherical varieties (e.g. toric varieties), because they satisfy the assumption about the Frobenius splitting. We elaborate the case of toric varieties.

THEOREM (see Theorem 4.6). *Let X be a smooth toric variety, and let Δ be its boundary divisor. We assume that*

$$\operatorname{codim}(\text{stable base locus}(\omega_X^{-1})) \geq 3.$$

(*By* $\operatorname{codim}(\cdot)$ *we mean the maximal codimension of the components.*)

Then, for a vector bundle $\mathcal{V}$ on X, we have the equivalences:

(a) [$\mathcal{V}$ *splits* $\Leftrightarrow$ $\mathcal{V}_{\Delta_m}$ *splits*] *for* $m \gg 0$.
(b) [$\mathcal{V}$ *is trivial* $\Leftrightarrow$ $\mathcal{V}_\Delta$ *is trivial*].

The splitting criteria obtained in this article are based on two technical ingredients: the "universal" criterion Proposition 1.8, on one hand, and various Kodaira-type vanishing theorems for q-ample line bundles, on the other hand. The article discusses the splitting problem from the start. The necessary background about partially ample line bundles is presented in the appendix. For this reason, the role of the latter is twofold:

(a) to recall the definitions and properties of the q-ampleness (see [30; 26]) and Frobenius splitting (see [8]), which are used in the body of the article;
(b) to present a few, possibly new, results:
   - a Kodaira vanishing theorem for relatively ample line bundles on weakly normal varieties (see Theorem A.5(iii)) and for q-ample line bundles on Frobenius-split varieties (see Theorem B.3);
   - a Picard–Lefschetz property in the relative setting (see Theorem A.7);

– a q-ampleness criterion for line bundles, which are not necessarily globally generated (see Theorem A.8).

In this article, X stands for a smooth projective variety over $\mathbb{C}$ of dimension at least three.

1. The General Splitting Principle

DEFINITION 1.1. Let T be a scheme defined over $\mathbb{C}$, and let S be its closed subscheme defined by the sheaf of ideals $\mathcal{I}_S \subset \mathcal{O}_T$; we assume that $H^0(\mathcal{O}_T) = \mathbb{C}$. For a locally free sheaf (a vector bundle) $\mathscr{V}_T$ of rank r on T, we denote $\mathscr{E}_T := \mathcal{E}nd(\mathscr{V}_T)$ its sheaf of endomorphisms; let $\mathscr{V}_S := \mathscr{V}_T \otimes_{\mathcal{O}_T} \mathcal{O}_S$, $\mathscr{E}_S := \mathscr{E}_T \otimes_{\mathcal{O}_T} \mathcal{O}_S$, and so on.

An *eigenvalue* of $h_T \in H^0(\mathscr{E}_T)$ is a complex root of its characteristic polynomial

$$p_{h_T} := \det(t\mathbb{1} - h_T) \in H^0(\mathcal{E}nd(\det \mathscr{V}_T))[t] = H^0(\mathcal{O}_T)[t] = \mathbb{C}[t]. \tag{1.1}$$

We say that $\mathscr{V}_T$ *splits* if it is isomorphic to a direct sum of r invertible sheaves (line bundles) on T.

REMARK 1.2. Let T and h_T be as before. Note that if $\varepsilon \in \mathbb{C}$ is an eigenvalue of p_{h_T}, then $\operatorname{Ker}(\varepsilon\mathbb{1} - h_T) \subset \mathscr{V}_T$ is a nonzero $\mathcal{O}_T$-module. Indeed, for a closed point $x \in T$ with residue field $\mathbb{C}(x) \cong \mathbb{C}$, ε is a (usual) eigenvalue of $h_T \otimes \mathbb{C}(x) \in \operatorname{End}(\mathscr{V}_T \otimes \mathbb{C}(x))$.

LEMMA 1.3 (see [14, Lemma 2.2]). *The following statements hold*:

(i) *$\mathscr{V}_T$ splits if and only if there is $h_T \in H^0(\mathscr{E}_T)$ with r pairwise distinct eigenvalues.*

(ii) *If $H^0(\mathscr{E}_T) \to H^0(\mathscr{E}_S)$ is surjective—in particular, if $H^1(\mathcal{I}_S \otimes \mathscr{E}_T) = 0$—then $\mathscr{V}_T$ splits if and only if $\mathscr{V}_S$ splits.*

DEFINITION 1.4. Let $\mathcal{L}$ be an invertible sheaf (a line bundle) on T, and let $D \in |d\mathcal{L}|$ be an effective divisor. For $m \geq 0$, the mth-order thickening D_m of D is the subscheme of T defined by the ideal $\mathcal{I}_D^{m+1}$, where $\mathcal{I}_D = \mathcal{O}_T(-D) \cong \mathcal{L}^{-d}$.

The structure sheaves of the successive thickenings fit into the exact sequences:

$$0 \to \mathcal{L}_D^{-dm} \to \mathcal{O}_{D_m} \to \mathcal{O}_{D_{m-1}} \to 0, \quad m \geq 1. \tag{1.2}$$

We will apply Lemma 1.3 mostly in the case $T = X$ and $S = D_m$ for suitable m. (Recall that X is a smooth, projective variety.) In the framework of formal schemes, we have the following very general statement.

PROPOSITION 1.5. *Let $D \subset X$ be an effective divisor*, $\dim X \geq 2$, *and let $\hat{X} := \varinjlim_m D_m$ denote the formal completion of X along D. If the cohomological dimension* $\operatorname{cd}(X \setminus D) \leq \dim X - 2$, *then $\mathscr{V}$ splits if and only if $\mathscr{V} \otimes \mathcal{O}_{\hat{X}}$ does. The assumption is satisfied if D is* $(\dim X - 2)$*-ample.*

The proposition generalizes [3, Prop. 3.1], which corresponds to the case where D is ample, so $\mathrm{cd}(X \setminus D) = 0$.

Proof. By [16, Thm. 3.4], $H^0(\mathscr{E}) \to H^0(\mathscr{E}_{\hat{X}})$ is an isomorphism. The splitting of $\mathscr{V} \otimes \mathcal{O}_{\hat{X}}$ yields $\hat{h} \in H^0(\mathscr{E}_{\hat{X}})$ with r pairwise distinct eigenvalues. This is in fact induced from $H^0(\mathscr{E}_{D_m})$, $m \gg 0$, and we conclude by Lemma 1.3(ii). The last claim is [26, Prop. 5.1]. □

LEMMA 1.6. *Let T be a projective equidimensional Cohen–Macaulay scheme with $H^0(\mathcal{O}_T) = \mathbb{C}$. Suppose that $\mathcal{L} \in \mathrm{Pic}(T)$ is q-ample, $q \le \dim T - 2$, and consider $D \in |\mathcal{L}^d|$. Then $\mathscr{V}_T$ splits if and only if its restriction $\mathscr{V}_{D_m}$ splits for appropriate $m \gg 0$.*

Proof. We apply Lemma 1.3: it suffices to have $H^1(\mathscr{E}_T \otimes \mathcal{L}^{-d(m+1)}) = 0$. The Serre duality holds for T (see [17, Thm. III.7.6]), so the $(\dim T - 2)$-ampleness of $\mathcal{L}$ implies that this is indeed the case for m large enough. □

REMARK 1.7. Since the surjectivity of $H^0(\mathscr{E}_X) \to H^0(\mathscr{E}_D)$ is implied by the vanishing of $H^1(\mathcal{O}_X(-D) \otimes \mathscr{E}_X)$, the $(\dim X - 2)$-amplitude of $\mathcal{O}_X(D)$ is, except special cases, the weakest possible assumption that allows us to deduce the splitting of vector bundles by restricting them to D.

The following general splitting principle, corresponding to restrictions to partially ample divisors, is the root of the results obtained in this article.

PROPOSITION 1.8. *Let T be a projective equidimensional Cohen–Macaulay scheme such that $H^0(\mathcal{O}_T) = \mathbb{C}$. Suppose that $\mathcal{L} \in \mathrm{Pic}(T)$ is q-ample with $q \le \dim T - 2$ and that $D \in |d\mathcal{L}|$ is an effective divisor. We assume moreover that*

$$H^1(D, \mathscr{E}_D \otimes \mathcal{L}_D^{-a}) = 0 \quad \text{for all } a \ge c. \tag{1.3}$$

Then the following properties hold:

(i) $H^1(T, \mathscr{E}_T \otimes \mathcal{L}^{-a}) = 0$ *for all* $a \ge c$.
(ii) *If $d \ge c$ and $\mathscr{V}_D$ splits, then $\mathscr{V}$ splits too.*

Proof. (i) Using the Serre duality, the $(\dim T - 2)$-ampleness of $\mathcal{L}$ implies

$$a_0 := \max\{a \mid H^1(T, \mathscr{E}_T \otimes \mathcal{L}^{-a}) \ne 0\} < +\infty.$$

The exact sequence $0 \to \mathcal{L}^{-d} \to \mathcal{O}_T \to \mathcal{O}_D \to 0$ yields

$$\cdots \to H^1(\mathscr{E}_T \otimes \mathcal{L}^{-d-a_0}) \to H^1(\mathscr{E}_T \otimes \mathcal{L}^{-a_0}) \to H^1(\mathscr{E}_D \otimes \mathcal{L}_D^{-a_0}) \to \cdots$$

with $-d - a_0 \le -(a_0 + 1)$, so the leftmost term vanishes. If $a_0 \ge c$, then the rightmost and the middle terms also vanish. This contradicts the definition of a_0, and hence $a_0 < c$.

(ii) Since $d \ge c$, $H^0(\mathscr{E}_T) \to H^0(\mathscr{E}_D)$ is surjective. □

REMARK 1.9. (i) The uniform q-ampleness property [30, Thm. 6.4] implies that there is a linear function $l(r) = \lambda r + \mu$, with λ, μ depending only on $\mathcal{L}$, such that Proposition 1.8(i) holds for all $a \geq l(\operatorname{reg}(\mathscr{E}_T))$, where $\operatorname{reg}(\mathscr{E}_T)$ stands for the regularity of $\mathscr{E}_T$ with respect to a (fixed) ample line bundle $\mathcal{O}_T(1)$.

(ii) Condition (1.3) involves *only* $\mathscr{E}_D$, which splits by assumption. This feature is helpful because it is easier to decide the vanishing of the cohomology of line bundles, rather than of vector bundles (i.e., Proposition 1.8(i)). Also, for $q \leq \dim T - 3$, condition (1.3) is indeed fulfilled for $c \gg 0$.

(iii) There are two important classes of q-ample line bundles, the relatively ample and the pull-back of ample line bundles with respect to a morphism. We shall constantly elaborate these two situations; the case of a pull-back typically requires stronger hypotheses.

THEOREM 1.10. *Let $\mathscr{V}$ be a vector bundle on X, $\mathscr{E} := \mathcal{E}nd(\mathscr{V})$, and let $f : X \to Y$ be a surjective morphism with Y projective,*

$$\dim X - \dim Y \geq 3, \quad \mathcal{L} \in \operatorname{Pic}(X)\ f\textit{-relatively ample}.$$

Let $D \in |\mathcal{L}|$ be a reduced weakly normal *divisor, and assume that $\mathscr{E}_D \otimes \mathcal{L}_D$ is relatively ample with respect to $D \to Y$. (In particular, it suffices $\mathscr{E} \otimes \mathcal{L}$ to be relatively ample.) Then we have the equivalence [$\mathscr{V}$ splits $\Leftrightarrow$ $\mathscr{V}_D$ splits].*

The weak normality condition for a divisor (see [21, Prop. 4.1]) is explicit, but it is somewhat technical. However, we can see that the condition is satisfied in the following fairly general situation (a particular case of the $WN1$-property [10, Def. 3.2]):

- $D = D_1 + \cdots + D_t$ is reduced, and D is normal away from its self-intersections of a single irreducible component or of two different components;
- For any point $p \in D$, the local equations (in the analytic topology) of the components of D that are passing through p form a regular sequence in $\hat{\mathcal{O}}_{X,p} \cong \mathbb{C}\{\xi_1, \ldots, \xi_{\dim X}\}$. Hence the locus of the points that belong to at least three branches of D have codimension at least two in D.
- D has *generically* normal crossings: at the generic intersection point of two local (analytic) branches of D, there are local (analytic) coordinates $\{\xi_1, \ldots, \xi_{\dim X}\}$ in $\hat{\mathcal{O}}_{X,p}$ such that the germ of D at p is given by $\{\xi_1\xi_2 = 0\}$.

Proof. Let $\ell \in \operatorname{Pic}(D)$ be a direct summand of $\mathscr{E}_D$; by hypothesis, $\ell_a := \ell \otimes \mathcal{L}^a$ is relatively ample for all $a \geq 1$. Note that D is Gorenstein, so it satisfies Serre's condition S_2. Then Theorem A.5(iii) implies that $H^1(D, \ell_a^{-1}) = 0$, that is, condition (1.3) is fulfilled, and we may apply Proposition 1.8. □

2. Splitting Along Divisors: A "Deterministic" Approach

DEFINITION 2.1. For $s \geq 1$, we say that a scheme T is s-split if

$$H^j(T, \ell) = 0, \quad \text{for } j = 1, \ldots, s, \forall \ell \in \operatorname{Pic}(T). \qquad (s\text{-split})$$

REMARK 2.2. For $s = 1, 2$, we get respectively the "splitting" and "Horrocks scheme" notions introduced in [3]. Examples of projective varieties that satisfy (s-split) are as follows:

(i) arithmetically Cohen–Macaulay varieties X—for example, homogeneous spaces, complete intersections in them—with cyclic Picard group (where $s = \dim X - 1$), and their products;

(ii) projective bundles: if Y is s-split and $\mathcal{M}_1, \dots, \mathcal{M}_r$, $r \geq s+2$, are line bundles on Y, then $X := \mathbb{P}(\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_r)$ is also s-split (see [3, Example 4.9]).

PROPOSITION 2.3. (i) *Suppose T is a projective equidimensional Cohen–Macaulay scheme and D is an effective q-ample divisor on T. The following statements hold*:

(a) *If D is s-split with $s \leq \dim T - (q+1)$, then T is s-split.*

(b) *If D is s-split with $q \leq s \leq \dim T - (q+1)$, then T is $(s+1)$-split.*

(ii) *Suppose X is a smooth projective variety. If X is $(s+1)$-split, then D is s-split in the following cases*:

(a) *D is smooth, and $\mathcal{O}_X(D)$ is $(\dim X - 4)$-positive*;

(b) *D is arbitrary relatively ample for a morphism $X \to Y$ with projective Y and* $\dim X - \dim Y \geq 4$.

Proof. (i)(a) We consider the exact sequences

$$0 \to \mathcal{O}_T((k-1)D) \to \mathcal{O}_T(kD) \to \mathcal{O}_D(kD) \to 0, \quad k \in \mathbb{Z},$$

and tensor them by $\mathcal{L} \in \operatorname{Pic}(X)$. We obtain the surjective homomorphisms

$$H^i(T, \mathcal{L}((k-1)D)) \twoheadrightarrow H^i(T, \mathcal{L}(kD)) \quad \text{for } i \leq s.$$

The q-ampleness of D and the Serre duality on T imply that $H^i(T, \mathcal{L}(kD)) = 0$ for $k \ll 0$, which yields $H^i(T, \mathcal{L}) = 0$.

(b) We have to verify only that $H^{s+1}(T, \mathcal{L}) = 0$. The previous exact sequence yields inclusions $H^{s+1}(T, \mathcal{L}((k-1)D)) \subset H^{s+1}(T, \mathcal{L}(kD))$ for all $k \in \mathbb{Z}$. Again, the q-ampleness of D implies $H^{s+1}(T, \mathcal{L}(kD)) = 0$ for $k \gg 0$, so $H^{s+1}(T, \mathcal{L}) = 0$.

(ii) Since X is smooth, Theorem A.7 implies in both cases that the restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(D)$ is an isomorphism. Consequently, for any $\ell \in \operatorname{Pic}(D)$, there is $\tilde{\ell} \in \operatorname{Pic}(X)$ such that $\tilde{\ell}_D = \ell$. It remains to take the cohomology of the sequence $0 \to \tilde{\ell}(-D) \to \tilde{\ell} \to \ell \to 0$. □

THEOREM 2.4. *Let T be a projective equidimensional Cohen–Macaulay scheme with $H^0(\mathcal{O}_T) = \mathbb{C}$. Suppose D is an effective divisor on T such that D is* 1*-split and $\mathcal{O}_X(D)$ is* $(\dim T - 2)$*-ample. Then we have the equivalence*

$$[\mathscr{V} \text{ splits} \Leftrightarrow \mathscr{V}_D \text{ splits}].$$

Proof. Since $\mathscr{V}_D$ splits and D is 1-split, we have $H^1(\mathscr{E}_D \otimes \mathcal{L}_D^{-a}) = 0$ for all $a \geq 1$. The conclusion follows from Proposition 1.8. □

The interest in allowing partial ampleness for line bundles, which is considerably weaker than the ampleness, is to apply the result for morphisms (e.g., fiber bundles).

COROLLARY 2.5. *Let X be a smooth 2-split projective variety, and let D be an effective divisor on it; let $X \xrightarrow{f} Y$ be a morphism with projective Y. Then the splitting of $\mathscr{V}_D$ implies the splitting of $\mathscr{V}$ in any of the following cases:*

(a) *f is smooth, $D = f^{-1}(D_Y)$ with $D_Y \subset Y$ a smooth $(\dim Y - 4)$-positive divisor;*

(b) *D is arbitrary, f-relatively ample, and $\dim X - \dim Y \geq 4$.*
(*This generalizes* [3, *Cor. 4.14*] *to the relative case.*)

Proof. In both situations, Proposition 2.3(ii) implies that D is 1-split. □

EXAMPLE 2.6. Let $X = X^{(1)} \times \cdots \times X^{(t)}$ be a product of Fano varieties of dimension at least four, with $\mathrm{Pic}(X^{(j)}) \cong \mathbb{Z}$ for all j. By applying Theorem 2.4 we reduce the problem of splitting of a vector bundle on X to $S_3^{(1)} \times \cdots \times S_3^{(t)}$, where each $S_3^{(j)} \subset X^{(j)}$ is an irreducible, three-dimensional complete intersection.

EXAMPLE 2.7 (Projective bundles). Let Y be a smooth, projective, 1-split variety. Consider $\mathcal{M}_1, \ldots, \mathcal{M}_t \in \mathrm{Pic}(Y)$, $t \geq 3$, and define

$$X := \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_t) \xrightarrow{f} Y, \qquad D := \mathbb{P}(\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_t) \in |\mathcal{O}_f(1)|.$$

(D is 1-split by Remark 2.2(ii).) For any vector bundle $\mathscr{V}$ on X, we have

$$[\mathscr{V} \text{ splits} \Leftrightarrow \mathscr{V}_D \text{ splits}].$$

By repeatedly applying this method we reduce the verification of the splitting of $\mathscr{V}$ to a $\mathbb{P}^2$-subbundle of X over Y.

2.1. *Vector Bundles on Products of Projective Spaces and Quadrics*

A splitting criterion for vector bundles on $X_1 := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ is obtained in [9, corrigendum, Thm. 1.4], [22, Prop. 3]. It generalizes Horrocks' criterion and involves the vanishing of $(n_1 + 1) \cdot \cdots \cdot (n_t + 1)$ cohomology groups.

The result has been extended in [4, Thms. 2.14 and 2.15] to products $X_1 \times X_2$, where X_1 is as before, and X_2 is a product of hyperquadrics $Q_n \subset \mathbb{P}^{n+1}$. The splitting criterion involves a large number of cohomological conditions. By applying Corollary 2.5 we obtain the following:

COROLLARY 2.8. *Let X_1, X_2 be as before.*

(i) *A vector bundle on X_1 splits if and only if it splits along a $\mathbb{P}^2 \times \cdots \times \mathbb{P}^2 \subset X_1$.*

(ii) *A vector bundle on $X_1 \times X_2$ splits if and only if it splits along some $X_1' \times X_2' \subset X_1 \times X_2$, where X_1' is a product of projective planes $\mathbb{P}^2$, and X_2' is a product of copies of Q_3.*

Let us remark that in both cases, by taking restrictions the number of cohomological tests necessary to decide the splitting decreases dramatically.

2.2. *Vector Bundles on Products of Minuscule Varieties*

The minuscule homogeneous varieties are the following: the projective spaces, Grassmannians, spinor varieties, quadrics, the Cayley plane, and Freudenthal's variety. In [14] it is proved that a vector bundle on a minuscule homogeneous variety M, $\dim M \geq 2$, splits if and only if its restriction to the union $M_2 \subset M$ of the two-dimensional Schubert subvarieties splits. (It turns out that M_2 is either $\mathbb{P}^2$ or a union of two copies of $\mathbb{P}^2$ glued along a $\mathbb{P}^1$.) The proof, which does not fit within the frame of this article, exploits the compatible F-splitting of the Schubert varieties and the properties of the minuscule weights.

Rather *surprisingly,* our approach reduces the splitting problem for vector bundles on products of minuscule varieties to (only) products of 2-planes. To our knowledge, *there are no results* even for products of Grassmannians.

THEOREM 2.9. *Let $M^{(j)}$, $j = 1, \ldots, t$, be minuscule homogeneous varieties with* $\dim M^{(j)} \geq 2$, *and let $X := M^{(1)} \times \cdots \times M^{(t)}$.*

A vector bundle $\mathscr{V}$ on X splits if and only if its restriction to $M_2^{(1)} \times \cdots \times M_2^{(t)}$ splits, where each $M_2^{(j)} \subset M^{(j)}$ stands for the union of the two-dimensional Schubert subvarieties.

Proof. The proof is by induction on $(t, \mathrm{rk}\mathscr{V})$. For $t = 1$, see [14]. Let us prove the statement for $\tilde{X} = X \times M$ with X as before and M minuscule: we assume that $\mathscr{V}_{X_2 \times M_2}$ splits, where $X_2 := M_2^{(1)} \times \cdots \times M_2^{(t)} \subset X$. The proof consists of two steps.

CLAIM 1. *$\mathscr{V}_{X \times M_2}$ splits.*

We prove by induction on $\mathrm{rk}(\mathscr{V})$ for vector bundles on $X \times M_2$.

Recall that $\mathbb{Z}^{t+1} \cong \mathrm{Pic}(\tilde{X}) \overset{\cong}{\to} \mathrm{Pic}(X_2 \times M_2)$, so the line bundles on $\tilde{X}$ are of the form $\mathcal{O}_{\tilde{X}}(\underline{\tilde{\alpha}}) = \mathcal{O}_{M^{(1)}}(\alpha_1) \boxtimes \cdots \boxtimes \mathcal{O}_{M^{(t)}}(\alpha_t) \boxtimes \mathcal{O}_M(k) = \mathcal{O}_X(\underline{\alpha}) \boxtimes \mathcal{O}_M(k)$. We deduce that

$$\mathscr{V}_{X_2 \times M_2} \cong \bigoplus_{(\underline{\alpha}, k) \in \mathbb{Z}^{t+1}} (\mathcal{O}_X(\underline{\alpha}) \boxtimes \mathcal{O}_{M_2}(k)^{d_{\underline{\alpha},k}}). \tag{2.1}$$

Let us consider the diagram

$$\begin{tikzcd} X_2 \times M_2 \arrow[r, hook] \arrow[d, "f"] & X \times M_2 \arrow[d, "f"] \\ M_2 \arrow[r, equal] & M_2. \end{tikzcd}$$

The induction hypothesis implies that $\mathscr{V}$ splits on the fibers of f, so

$$\mathscr{V}_{X \times \{z\}} \cong \bigoplus_{\underline{\alpha} \in \mathbb{Z}^t} \mathcal{O}_X(\underline{\alpha})^{d_{\underline{\alpha}}(z)}, \quad z \in M_2.$$

In fact, the multiplicities $d_{\underline{\alpha}}(z)$ are independent of $z \in M_2$. By restricting to $X_2 \times \{z\}$ and using (2.1) we deduce:

$$d_{\underline{\alpha}} := \sum_{k\in\mathbb{Z}} d_{\underline{\alpha},k} = d_{\underline{\alpha}}(z), \quad z \in M_2.$$

Let $\underline{a} \in \mathbb{Z}^t$ be a maximal element of $\{\underline{\alpha} \in \mathbb{Z}^t \mid d_{\underline{\alpha}} \neq 0\}$ for the lexicographic order. Then $f_*(\mathcal{O}_X(-\underline{a}) \otimes \mathscr{V})$ is locally free on M_2 of rank $d_{\underline{a}}$, and (2.1) yields:

$$\mathscr{R} := f_*(\mathcal{O}_X(-\underline{a}) \otimes \mathscr{V}) \cong \bigoplus_{k\in\mathbb{Z}} \mathcal{O}_{M_2}(k)^{d_{\underline{a},k}}.$$

(For this, observe that $H^0(X, \mathcal{O}_X(\underline{\alpha} - \underline{a}))$, $H^0(X_2, \mathcal{O}_X(\underline{\alpha} - \underline{a})) \neq 0$ if and only if all the components of $\underline{\alpha} - \underline{a}$ are nonnegative; the only such $\underline{\alpha}$ is $\underline{a}$ itself.) It follows that we have the exact sequence $0 \to \mathcal{O}_X(\underline{a}) \boxtimes \mathscr{R} \to \mathscr{V} \to \mathscr{V}' \to 0$ on $X \times M_2$.

The first arrow is pointwise injective, so the quotient $\mathscr{V}'$ is locally free on $X \times M_2$, its restriction to $X_2 \times M_2$ splits, and $\operatorname{rk}(\mathscr{V}') < \operatorname{rk}(\mathscr{V})$. Hence, by the induction hypothesis, $\mathscr{V}'$ splits. Finally, we deduce that

$$\mathscr{V} \in \operatorname{Ext}^1(\mathscr{V}', \mathcal{O}_X(\underline{a}) \boxtimes \mathscr{R}) = H^1(X \times M_2, \mathcal{H}om(\mathscr{V}', \mathcal{O}_X(\underline{a}) \boxtimes \mathscr{R})) = 0$$

because $X \times M_2$ is 1-split and $\mathcal{H}om(\mathscr{V}', \mathcal{O}_X(\underline{a}) \boxtimes \mathscr{R})$ is a direct sum of line bundles. We conclude by recurrence that $\mathscr{V}_{X \times M_2}$ splits.

CLAIM 2. *$\mathscr{V}_{X\times M}$ splits.*

We denote by M_d the union of all the d-dimensional Schubert subvarieties $S_d \subset M$. For any $(d + 1)$-dimensional Schubert variety $S_{d+1} \subset M$, the intersection $\partial S_{d+1} := M_d \cap S_{d+1}$ is reduced, and it is the union of the d-dimensional Schubert subvarieties of S_{d+1}; usually, it is called the boundary of S_{d+1}. With this notation, for $d \geq 2$, the following properties hold (see [14] and the references therein):

- $\mathbb{Z} \cdot \mathcal{O}_M(1) = \operatorname{Pic}(M) \to \operatorname{Pic}(S_d)$ is an isomorphism;
- $\mathcal{O}_M(1) \otimes \mathcal{O}_{S_{d+1}} = \mathcal{O}_{S_{d+1}}(\partial S_{d+1})$;
- S_d and ∂S_{d+1} are 1-split.

Thus, for any Schubert variety $S_{d+1} \subset M$, the divisor $X \times \partial S_{d+1} \subset X \times S_{d+1}$ is 1-split and $\dim X$-positive. We deduce the implications:

$$\mathscr{V}_{X\times M_d} \text{ splits} \overset{\partial S_{d+1} \subset M_d}{\Longrightarrow} \mathscr{V}_{X\times\partial S_{d+1}} \text{ splits} \overset{\text{Thm. 2.4}}{\Longrightarrow} \mathscr{V}_{X\times S_{d+1}} \text{ splits.}$$

Clearly, for $S'_{d+1}, S''_{d+1} \subset M$, the splittings of $\mathscr{V}_{X\times S'_{d+1}}$ and $\mathscr{V}_{X\times S''_{d+1}}$ coincide along the (reduced) intersection $X \times (S'_{d+1} \cap S''_{d+1}) \subset X \times M_d$, so we get a splitting of $\mathscr{V}_{X\times M_{d+1}}$. By repeating the argument we deduce that $\mathscr{V}_{X\times M}$ splits. □

REMARK 2.10. The previous theorem holds over algebraically closed ground fields of *arbitrary characteristic,* because [14]—used in the proof—does.

Table 1 further lists the minuscule homogeneous varieties and their two-dimensional Schubert subvarieties (see [14]). Although the odd-dimensional quadrics are not minuscule, we include them in the table because Corollary 2.5

Table 1 Minuscule homogeneous varieties

Variety M	Subvariety M_2 to test splitting
$\mathbb{P}^{n-1}$	$\mathbb{P}^2$
$\mathrm{Gr}(k; n)$, $1 < k < n-1$	$\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$
spinor variety $\mathcal{S}_n$	$\mathbb{P}^2$
quadrics $\mathcal{Q}_4$, $\mathcal{Q}_5$	$\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$
quadric $\mathcal{Q}_n$, $n \geq 6$	$\mathbb{P}^2$
Cayley plane $\mathbb{OP}^2$	$\mathbb{P}^2$
Freudenthal variety	$\mathbb{P}^2$

allows us to reduce the splitting problem to the case of the even-dimensional quadrics by restricting to a hyperplane section.

Hence Theorem 2.9 reduces the splitting problem for a product of minuscule varieties to a union of products $(\mathbb{P}^2)^t$. Furthermore, the cohomological characterization of split vector bundles on multiprojective spaces is known.

THEOREM ([9, corrigendum, Thm. 1.4], [22, Prop. 3]). *Let $\mathcal{W}$ be a normalized vector bundle on $P := (\mathbb{P}^2)^t$ such that*:

$$H^k(P, \mathcal{W} \otimes \mathcal{O}_P(\delta_1, \ldots, \delta_t)) = 0, \quad -2 \leq \delta_j \leq 0, j = 1, \ldots, t, k = -1 - \sum_{j=1}^{t} \delta_j.$$

(*Here $\mathcal{O}_P(\delta_1, \ldots, \delta_t)$ stands for the tensor product of the line bundles $\mathcal{O}_{\mathbb{P}^2}(\delta_j)$ on the factors.*) *Then $\mathcal{W}$ contains $\mathcal{O}_P$ as a direct summand.*

By combining the two results (apply to $\mathcal{W} := \mathcal{V}_P$ and normalize) we obtain *algorithmically computable conditions* to probe the splitting of vector bundles of *arbitrary rank* on products of minuscule homogeneous varieties.

3. Splitting Along Divisors: A "Probabilistic" Approach

In this section we obtain splitting criteria for vector bundles by restricting them to zero loci of *generic* sections of *globally generated* partially positive line bundles. The global generation allows us to replace the (2-split) by the weaker (1-split) condition, but we have to consider very general test divisors instead of arbitrary ones. This explains the "probabilistic" attribute used in the title.

Note that if $\mathcal{L}$ is a q-ample line bundle on X such that $\mathcal{L}^d$ is globally generated for some $d \geq 1$, then $\mathcal{L}$ is q-positive, and the fibers of the morphism $f : X \to |\mathcal{L}^d|$ are at most q-dimensional (see [23, Thm. 1.4]), qnd thus

$$\kappa(\mathcal{L}) \geq \dim(\mathrm{Image}(f)) \geq \dim X - q.$$

Henceforth we replace $\mathcal{L}^d$ by $\mathcal{L}$.

Let the situation be as before. We start with general considerations: the equations defining X, $\mathcal{L}$, $\mathcal{V}$ involve finitely many coefficients in $\mathbb{C}$. By adjoining them to $\mathbb{Q}$ we obtain a field extension $\mathbb{Q} \hookrightarrow \Bbbk$ of finite type (which depends on $\mathcal{V}$), so we may assume that $\Bbbk \hookrightarrow \mathbb{C}$; its algebraic closure is *countable.* After replacing $\Bbbk$ by $\bar{\Bbbk}$, we may assume that X, $\mathcal{L}$, $\mathcal{V}$ are defined over a countable algebraically closed subfield $\Bbbk$ of $\mathbb{C}$; we denote these objects by $X_{\Bbbk}$, $\mathcal{L}_{\Bbbk}$, $\mathcal{V}_{\Bbbk}$.

The sheaf $\mathcal{G} := \mathrm{Ker}(H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L})$ is locally free, and the incidence variety

$$\mathcal{D} := \{([s], x) \mid s(x) = 0\} \subset |\mathcal{L}| \times X$$

is naturally isomorphic to the projective bundle $\mathbb{P}(\mathcal{G})$ over X. The projections of $\mathcal{D}$ onto $|\mathcal{L}|$, X are denoted by π, ρ:

$$\begin{tikzcd} & \mathcal{D} \arrow[r, hook] \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & {|\mathcal{L}| \times X} \\ {|\mathcal{L}|} & & X. \end{tikzcd}$$

For any open $S \subset |\mathcal{L}|$, let $\mathcal{D}_S := \pi^{-1}(S)$; for $s \in |\mathcal{L}|$, let $D_s := \mathrm{divisor}(s) \subset X$.

All these objects are defined over $\Bbbk$, and they are denoted by $\mathcal{L}_{\Bbbk}$, $\mathcal{D}_{\Bbbk}$, $\pi_{\Bbbk}$, $\rho_{\Bbbk}$. Let $K_{\mathbb{C}} := \mathbb{C}(|\mathcal{L}|)$ and $K_{\Bbbk} := \Bbbk(|\mathcal{L}_{\Bbbk}|)$ be the function fields of the projective spaces $|\mathcal{L}|$ and $|\mathcal{L}_{\Bbbk}|$, respectively.

DEFINITION 3.1. We say that a property holds for a *very general* point of some parameter space if it holds on the complement of countably many proper subvarieties of that parameter space. In our case, we are interested in the splitting of $\mathcal{V}_{D_s}$ for very general $s \in |\mathcal{L}|$.

LEMMA 3.2. (i) *If $\mathcal{V}_{D_s}$ splits for a very general $s \in |\mathcal{L}|$, then $(\rho^*\mathcal{V}) \otimes \bar{K}_{\mathbb{C}}$ splits.*
(ii) *If $(\rho^*\mathcal{V}) \otimes \bar{K}_{\mathbb{C}}$ splits, then there is an* analytic *open ball $\mathbb{B} \subset |\mathcal{L}|$ such that D_s is smooth, for all $s \in \mathbb{B}$, and $(\rho^*\mathcal{V})_{\mathbb{B}}$ splits over $\mathcal{D}_{\mathbb{B}}$.*

Proof. (i) Let $\tau : |\mathcal{L}| \to |\mathcal{L}_{\Bbbk}|$ be the trace morphism. Since $\Bbbk$ is countable and $\mathbb{C}$ is not, $\tau(s)$ is the generic point of $|\mathcal{L}_{\Bbbk}|$ for very general $s \in |\mathcal{L}|$. Hence $\Bbbk(\tau(s)) = \Bbbk(|\mathcal{L}_{\Bbbk}|) = K_{\Bbbk}$, and we obtain the Cartesian diagram

$$\begin{tikzcd} D_s \arrow[r] \arrow[d] & \mathcal{D}_{\bar{K}_{\Bbbk}} \arrow[r] \arrow[d] & \mathcal{D}_{\Bbbk} \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\bar{K}_{\Bbbk}) \arrow[r] & {|\mathcal{L}_{\Bbbk}|.} \end{tikzcd}$$

For varieties defined over algebraically closed fields ($\mathbb{C}$ and $\bar{K}_{\Bbbk}$ in our case), the property of a vector bundle to be split commutes with base change. Then $\mathcal{V}_{\bar{K}_{\Bbbk}}$ splits on $\mathcal{D}_{\bar{K}_{\Bbbk}}$, so the same holds for $\mathcal{V}_{\bar{K}_{\mathbb{C}}}$.

(ii) We note that $\mathcal{V}_{\mathcal{D}_{\bar{K}_{\mathbb{C}}}}$ in fact splits over an intermediate field $K_{\mathbb{C}} \hookrightarrow K' \hookrightarrow \bar{K}_{\mathbb{C}}$ finitely generated (and also algebraic) over $K_{\mathbb{C}}$. Thus there is an open affine $S \subset |\mathcal{L}|$, an affine variety S' over $\mathbb{C}$, and a *finite morphism* $S' \xrightarrow{\sigma} S$ such that the direct summands of $\mathcal{V}_{\mathcal{D}_{\bar{K}_{\mathbb{C}}}}$ are defined over $\mathbb{C}[S']$; thus $(\rho^*\mathcal{V})_{S'}$ splits. For S sufficiently small, Bertini's theorem implies that D_s is smooth for all $s \in S$.

Finally, there are open balls $\mathbb{B}' \subset S'$ and $\mathbb{B} \subset S$ such that $\sigma : \mathbb{B}' \to \mathbb{B}$ is an *analytic* isomorphism. Then the splitting of $(\rho^*\mathscr{V})_{\mathbb{B}'}$ descends to $(\rho^*\mathscr{V})_{\mathbb{B}}$ on $\mathcal{D}_{\mathbb{B}}$. □

REMARK 3.3. The previous lemma makes precise the meaning of a very general point $s \in |\mathcal{L}|$: its coordinates should be algebraically independent over the definition field of X, $\mathcal{L}$, $\mathscr{V}$. In particular, the notion of a very general point depends on $\mathscr{V}$ itself, in fact on its field of definition.

Often we wish to have statements that involve *general* points, rather than very general ones. The splitting of $\mathscr{V}_{D_s}$ for general $s \in |\mathcal{L}|$ means, by definition, that $(\rho^*\mathscr{V}) \otimes K_{\mathbb{C}}$ splits. This condition is *more restrictive* than the splitting of $(\rho^*\mathscr{V}) \otimes \bar{K}_{\mathbb{C}}$.

THEOREM 3.4. *Suppose that* $\mathcal{L} \in \operatorname{Pic}(X)$ *is globally generated and* $(\dim X - 4)$-*positive and that* $D \in |\mathcal{L}|$ *is very general* (*thus smooth*). *If* X *is* 1-*split*, *then*

$$[\mathscr{V} \textit{ splits} \Leftrightarrow \mathscr{V}_D \textit{ splits}].$$

The interest in this result is that it allows us to test the splitting of vector bundles along divisors that are not 1-split; this situation arises especially in low dimensions. Otherwise, of course, we apply the "deterministic" Theorem 2.4.

Proof. Let $\mathbb{B}$ be as in Lemma 3.2. By [26, Prop. 5.1],the cohomological dimension $\operatorname{cd}(X \setminus D_s) \le \dim X - 4$ for all $s \in \mathbb{B}$, which implies that, for all $o, s, t \in \mathbb{B}$, the intersections $D_{st} := D_s \cap D_t$ and $D_{ost} := D_o \cap D_s \cap D_t$ are nonempty and connected (see [16, Ch. III, Cor. 3.9]). Note that the intersections are generically transverse, because $\mathcal{L}$ is globally generated.

CLAIM 1. *Let* $s, t \in \mathbb{B}$ *such that* D_s *and* D_t *intersect transversally. Then we have*:

$$\operatorname{res}^X_{D_s} : \operatorname{Pic}(X) \to \operatorname{Pic}(D_s) \quad \textit{is an isomorphism};$$
$$\operatorname{res}^{D_s}_{D_{st}} : \operatorname{Pic}(D_s) \to \operatorname{Pic}(D_{st}) \quad \textit{is injective}.$$

The first statement is proved in Theorem A.7, and the second in Proposition A.6.

CLAIM 2. $\rho^* : \operatorname{Pic}(X) \to \operatorname{Pic}(\mathcal{D}_{\mathbb{B}})$ *is an isomorphism.*

Indeed, fix $o \in \mathbb{B}$ and consider the diagram

$$\begin{tikzcd}
 & \operatorname{Pic}(D_s) \arrow[dr, "\operatorname{res}^{D_s}_{D_{os}}"] & \\
\operatorname{Pic}(X) \arrow[ur, "\operatorname{res}^X_{D_s}"] \arrow[rr, "\operatorname{res}^X_{D_{os}}"] \arrow[dr, "\operatorname{res}^X_{D_o}"'] & & \operatorname{Pic}(D_{os}). \\
 & \operatorname{Pic}(D_o) \arrow[ur, "\operatorname{res}^{D_o}_{D_{os}}"'] &
\end{tikzcd}$$

The composition $\operatorname{Pic}(X) \xrightarrow{\rho^*} \operatorname{Pic}(\mathcal{D}_{\mathbb{B}}) \xrightarrow{\operatorname{res}_{D_o}} \operatorname{Pic}(D_o)$ is bijective, so ρ^* is injective. For the surjectivity, take $\ell \in \operatorname{Pic}(\mathcal{D}_{\mathbb{B}})$. If $\ell_{D_o} \cong \mathcal{O}_{D_o}$, then $\ell_{D_s} \in \operatorname{Pic}^0(D_s)$,for all $s \in \mathbb{B}$, so

$$\{s \in \mathbb{B} \mid \ell_{D_s} \ncong \mathcal{O}_{D_s}\} = \{s \in \mathbb{B} \mid h^0(\ell_{D_s}) = 0\}$$

is open in $\mathbb{B}$, and hence $S := \{s \in \mathbb{B} \mid \ell_{D_s} \cong \mathcal{O}_{D_s}\}$ is closed.

On the other hand, the diagram implies, by restricting to D_{os}, that S contains all s such that D_s intersects D_o transversally; thus S is dense, so $S = \mathbb{B}$. It follows that $\ell \cong \mathcal{O}_{\mathcal{D}_{\mathbb{B}}}$. For an arbitrary $\ell \in \mathrm{Pic}(\mathcal{D}_{\mathbb{B}})$, let $\mathcal{L} \in \mathrm{Pic}(X)$ be such that $\ell_{D_o} \cong \mathcal{L}_{D_o}$, so $((\rho^*\mathcal{L}^{-1}) \otimes \ell)_{D_o}$ is trivial. This concludes the proof of Claim 2.

Since $\mathscr{V}_{\mathbb{B}}$ splits, we deduce that $(\rho^*\mathscr{V})_{\mathbb{B}} \cong \rho^*(\bigoplus_{j\in J} \mathcal{L}_j^{\oplus d_j})$ with $\mathcal{L}_j \in \mathrm{Pic}(X)$ pairwise nonisomorphic (J is some index set). We consider the following partial order on line bundles:

$$\mathcal{L} \prec \mathcal{M} \quad \Leftrightarrow \quad \mathcal{L} \not\cong \mathcal{M} \quad \text{and} \quad H^0(\mathcal{L}^{-1} \otimes \mathcal{M}) \neq 0.$$

Note that for $\mathcal{L} \not\cong \mathcal{M}$, only the following mutually exclusive possibilities can occur: $\mathcal{L}, \mathcal{M}$ are not comparable, or $\mathcal{L} \prec \mathcal{M}$, or $\mathcal{M} \prec \mathcal{L}$. It is not possible that $\mathcal{L} \prec \mathcal{M}$ and $\mathcal{M} \prec \mathcal{L}$, because then necessarily holds $\mathcal{L} \cong \mathcal{M}$.

For $s \in \mathbb{B}$, let $J_{s,\max} \subset J$ be the set of maximal elements of $\{\mathcal{L}_j \otimes \mathcal{O}_{D_s}\}_{j\in J} \subset \mathrm{Pic}(D_s)$ with respect to $\prec$. By semicontinuity the set $\{t \in \mathbb{B} \mid J_{s,\max} \subset J_{t,\max}\}$ is open. Thus, after shrinking $\mathbb{B}$, we may assume that $J_{s,\max} \subset J$ is independent of s; we denote it by $J_{\max}$.

The maximality property implies that there is a natural pointwise injective homomorphism:

$$h : \bigoplus_{\mu\in J_{\max}} \rho^*\mathcal{L}_\mu \otimes \underbrace{\pi^*\pi_*(\rho^*(\mathcal{L}_\mu^{-1} \otimes \mathscr{V}))}_{\overset{\circledast}{\cong} \mathcal{O}_{\mathcal{D}_{\mathbb{B}}}^{\oplus d_\mu}} \to (\rho^*\mathscr{V})_{\mathcal{D}_{\mathbb{B}}}. \tag{3.1}$$

CLAIM 3. *h descends to X after a suitable base change in $\mathcal{O}_{\mathcal{D}_{\mathbb{B}}}^{\oplus d_\mu}$ by an analytic map $\beta : \mathbb{B} \to \prod_{\mu\in J_{\max}} \mathrm{GL}(d_\mu)$.*

Indeed, we fix $o \in \mathbb{B}$ and a basis in $H^0(D_o, \mathcal{L}_\mu^{-1} \otimes \mathscr{V}) \cong \mathbb{C}^{d_\mu}$ for all $\mu \in J_{\max}$. (Bases are represented as square matrices whose columns are the vectors of the basis.) For any $s \in \mathbb{B}$, D_{os} is nonempty and connected, so there is a *unique* basis in $H^0(D_s, \mathcal{L}_\mu^{-1} \otimes \mathscr{V}) \cong \mathbb{C}^{d_\mu}$ whose restriction to D_{os} coincides with the restriction of the basis along D_o. (As $s \in \mathbb{B}$ varies, the transition matrices from the trivialization $\circledast$ in (3.1) to these new bases yield the map β.)

We observe that, after this reparameterization, h descends to the open set $\mathcal{U} := \rho(\mathcal{D}_{\mathbb{B}}) \subset X$. Indeed, define

$$\bar{h} : \Big(\bigoplus_{\mu\in J_{\max}} \mathcal{L}_\mu^{\oplus d_\mu}\Big) \otimes \mathcal{O}_{\mathcal{U}} \to \mathscr{V}_{\mathcal{U}}, \bar{h}(x) := h_s(x) \quad \text{for some } s \in \mathbb{B}, x \in D_s. \tag{3.2}$$

To prove that $\bar{h}(x)$ is independent of $s \in \mathbb{B}$, we must show that the restrictions to D_{st} of the new bases in $H^0(D_s, \mathcal{L}_\mu^{-1} \otimes \mathscr{V})$, $H^0(D_t, \mathcal{L}_\mu^{-1} \otimes \mathscr{V})$ coincide, for all $s, t \in \mathbb{B}$; it is enough to check this on the triple intersection $D_{ost} = D_o \cap D_{st}$ (which is nonempty and connected), where both bases are induced from D_o.

The homomorphism (3.2) yields the extension of locally free sheaves on $\mathcal{U}$:

$$0 \to \Big(\bigoplus_{\mu\in J_{\max}} \mathcal{L}_\mu^{\oplus d_\mu}\Big) \otimes \mathcal{O}_{\mathcal{U}} \to \mathscr{V}_{\mathcal{U}} \to \mathscr{W}_{\mathcal{U}} \to 0,$$
$$\rho^*(\mathscr{W}_{\mathcal{U}}) \cong \rho^*\Big(\bigoplus_{j\in J\setminus J_{\max}} \mathcal{L}_j^{\oplus d_j} \otimes \mathcal{O}_{\mathcal{U}}\Big).$$

The homomorphism on the left is pointwise injective. Recursively, we deduce that $\mathscr{V}_{\mathcal{U}}$ is obtained as a successive extension of the line bundles $\mathcal{L}_j \otimes \mathcal{O}_{\mathcal{U}}$, $j \in J$. (Note that, in the gluing process, we did not use that $\mathscr{V}$ is defined on all X; we used only its restriction to $\mathcal{U}$.)

Since $\mathcal{U}$ is an analytic neighborhood of D_o, we get induced extensions on the thickenings $(D_o)_m$, $m \geq 0$ (see Definition 1.4). However, X is 1-split, and D_o is $(\dim X - 4)$-ample, so

$$0 = \mathrm{Ext}^1(\mathcal{L}, \mathcal{M}) \to \mathrm{Ext}^1(\mathcal{L}_{(D_o)_m}, \mathcal{M}_{(D_o)_m}) \tag{3.3}$$

is an isomorphism for all $\mathcal{L}, \mathcal{M} \in \mathrm{Pic}(X)$, $m \gg 0$. It follows that $\mathscr{V}_{(D_o)_m}$ splits for $m \gg 0$. By applying Lemma 1.6 we deduce that $\mathscr{V}$ splits on X. (Here it is necessary to have $\mathscr{V}$ defined on the whole X.) □

Example 3.5. We have seen in Example 2.7 that the splitting of a vector bundle on a projective bundle over a 1-split variety can be verified along an arbitrary $\mathbb{P}^2$-subbundle. At this stage, the reduction process in Theorem 2.4 stops. However, in this example we show that Theorem 3.4 allows us to decrease further the dimension of the test subvarieties.

Let $(S, \mathcal{O}_S(1))$ be an arithmetically Cohen–Macaulay surface with $\mathrm{Pic}(S) = \mathbb{Z}\cdot\mathcal{O}_S(1)$. (A necessary and sufficient cohomological condition for the splitting of vector bundles on such surfaces has been obtained in [15].) Let $\mathcal{A}$ be a very ample multiple of $\mathcal{O}_S(1)$. We observe that the four-fold

$$X := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{A}^{-m} \oplus \mathcal{A}^{-m-n}) \xrightarrow{f} S, \quad m, n \geq 0,$$

is 1-split. The line bundle $\mathcal{L} := \mathcal{O}_f(1) \otimes f^*\mathcal{A}$ is very ample on X, and the general $D \in |\mathcal{L}|$ is a smooth $\mathbb{P}^1$-fibre bundle over S. In particular, D is not 1-split, so Theorem 2.4 does not apply. However, the "probabilistic" Theorem 3.4 still applies: a vector bundle $\mathscr{V}$ on X splits if and only if it does on a very general D.

4. Triviality Criteria

Finally, in this section we restrict our discussion to the case of the trivializable vector bundles. The motivation is, first, that the general "effective splitting criterion", Proposition 1.8, is not explicit enough, especially for partially ample line bundles that are pulls-back (see Remark 1.9, Theorem 1.10). Second, it is desirable to remove the 1- and 2-split conditions that appear throughout Sections 2 and 3 and that are imposed precisely to ensure the vanishing (1.3).

Unfortunately, the Kodaira vanishing does not hold for q-ample line bundles. Thus, to obtain effective results in this situation, we must find appropriate conditions that imply the Kodaira vanishing. These lines of thought lead to the triviality criteria.

LEMMA 4.1. *Assume that* $\mathcal{L} \in \mathrm{Pic}(X)$ *is* $(\dim X - 2)$*-ample and satisfies*

$$H^i(X, \mathcal{L}^{-a}) = 0 \quad \textit{for all } a \geq 1 \textit{ and } i = 0, 1, 2.$$

For any vector bundle $\mathscr{V}$ *on* X *and* $D \in |\mathcal{L}|$, *we have* $[\mathscr{V} \cong \mathcal{O}_X^{\oplus r} \Leftrightarrow \mathscr{V}_D \cong \mathcal{O}_D^{\oplus r}]$.

Proof. The hypothesis implies that $H^i(\mathcal{L}_D^{-a}) = 0$ for all $a \geq 1$, $i = 0, 1$, and that $H^0(\mathcal{O}_{D_a}) = \mathbb{C}$ for $a \geq 0$, so $H^0(\mathscr{E}) \to H^0(\mathscr{E}_D) = \mathrm{End}(\mathbb{C}^r)$ is an isomorphism, by Proposition 1.8. Hence $\mathscr{V}$ splits; let us write $\mathscr{V} = \bigoplus_{j \in J} \mathcal{M}_j^{d_j}$ with $\mathcal{M}_j$ pairwise nonisomorphic and $\sum_{j \in J} d_j = r$.

The previous discussion shows that the finite-dimensional $\mathbb{C}$-algebra $H^0(\mathscr{E})$ is isomorphic to $\mathrm{End}(\mathbb{C}^r)$. The uniqueness of the Wedderburn–Malcev decomposition implies that J consists of a single element, so $\mathscr{V} \cong \mathcal{M}^{\oplus r}$ for some $\mathcal{M} \in \mathrm{Pic}(X)$.

Since $\mathscr{V}_D = \mathcal{O}_D^{\oplus r}$, both $\mathcal{M}_D$ and $\mathcal{M}_D^{-1}$ admit nontrivial sections, so $\mathcal{M}_D \cong \mathcal{O}_D$. According to Proposition A.6, $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is injective, so $\mathcal{M}$ is trivial. □

THEOREM 4.2. *Consider a vector bundle* $\mathscr{V}$ *on* X, $\mathcal{L} \in \mathrm{Pic}(X)$, *and* $D \in |\mathcal{L}|$. *In any of the following cases, we have* $[\mathscr{V} \cong \mathcal{O}_X^{\oplus r} \Leftrightarrow \mathscr{V}_D \cong \mathcal{O}_D^{\oplus r}]$:

(a) $\mathcal{L} \in \mathrm{Pic}(X)$ *is semiample and* $(\dim X - 3)$*-ample*;

(b) $\mathcal{L}$ *is relatively ample for a morphism* $X \xrightarrow{f} Y$ *with* $\dim(X) - \dim(Y) \geq 3$.

Proof. In both cases, Theorem A.5 implies $H^i(X, \mathcal{L}^{-a}) = 0$ for $a \geq 1$, $i = 0, 1, 2$, so we can apply Lemma 4.1. □

4.1. The Case of Frobenius Split (F-Split) Varieties

These objects are ubiquitous, especially in representation theory. Examples of F-split varieties (defined in characteristic zero) include Fano varieties (see [8, Exercise 1.6E(5)]), spherical varieties, in particular projective homogeneous varieties and toric varieties (see [7], [29, Section 31]). The notions and properties that are relevant for us are summarized in Appendix B.

THEOREM 4.3. *Let* D *be a* $(\dim X - 3)$*-ample effective divisor, which is F-split. Then*

$$\mathscr{V} \cong \mathcal{O}_X^{\oplus r} \quad \Leftrightarrow \quad \mathscr{V}_D \cong \mathcal{O}_D^{\oplus r}.$$

The F-splitting allows us to handle more "exotic" situations. Many examples arise from varieties X that are compatibly split with respect to a divisor D.

Proof. Theorem B.3 implies $H^1(D, \mathcal{O}_D(-aD)) = 0$ for all $a \geq 1$, since $\mathcal{O}_D(D)$ is $(\dim D - 2)$-ample. The conclusion follows from Propositions 1.8 and A.6. □

COROLLARY 4.4. *Let X be a smooth projective variety whose anticanonical line bundle ω_X^{-1} is* $(\dim X - 3)$*-ample. Assume that X is F-split by $\sigma \in H^0(\omega_X^{-1})$, and denote $D :=$* divisor(σ). *Then* $[\mathcal{V} \cong \mathcal{O}_X^{\oplus r} \Leftrightarrow \mathcal{V}_D \cong \mathcal{O}_D^{\oplus r}]$.

The criterion applies, in particular, in the following cases:

(a) *X is a Fano variety of dimension at least three*;
(b) *X is a spherical variety whose anticanonical bundle is* $(\dim X - 3)$*-ample*.

Proof. The hypothesis implies that D is F-split, compatibly with the splitting defined by σ.

Fano and spherical varieties are Frobenius split, compatibly with suitable anticanonical divisors (see [8, 1.6.E(5), p. 56] and [7, Thm. 1], respectively). □

4.2. The Case of Toric Varieties

A nontrivial application of the ideas developed inhere arises when $X := X_\Sigma$ is the smooth projective toric variety defined by the regular fan Σ.

REMARK 4.5. (i) X is F-split, compatibly with the invariant divisors D_ρ, $\rho \in \Sigma(1)$, and their intersections (see [8, Exercise 1.3E(6)]).
(ii) Consider the torus-invariant anticanonical divisor

$$\Delta := \sum_{\rho \in \Sigma(1)} D_\rho. \tag{4.1}$$

Then $X \setminus \Delta \cong (\mathbb{C}^*)^{\dim X}$, so its cohomological dimension equals $\mathrm{cd}(X \setminus \Delta) = 0$.

THEOREM 4.6. *Let $\mathcal{V}$ be an arbitrary vector bundle on the smooth toric variety X. The following statements hold*:

(i) *Let $\hat{X} := \varprojlim_m \Delta_m$ be the formal completion of X along Δ. If* $\dim X \geq 2$, *then we have the equivalence* $[\mathcal{V}$ *splits* $\Leftrightarrow \mathcal{V} \otimes \mathcal{O}_{\hat{X}}$ *splits*$]$.
(ii) *Assume that* $\dim X$, $\mathrm{codim}(\mathrm{base\,locus}(\omega_X^{-1})) \geq 3$. *Then we have*:
   (a) $[\mathcal{V}$ *splits* $\Leftrightarrow \mathcal{V}_{\Delta_m}$ *splits*$]$ *for* $m \gg 0$. (*See Remark* 1.9(*i*) *for a lower bound on* m.)
   (b) $[\mathcal{V}$ *is trivial* $\Leftrightarrow \mathcal{V}_\Delta$ *is trivial*$]$.

Proof. (i) See Proposition 1.5.

(ii) Theorem A.8 implies that ω_X^{-1} is $(\dim X - 3)$-ample. Now the conclusion follows from Proposition 1.8 and Corollary 4.4, respectively. □

REMARK 4.7. (i) We may wonder if it is possible to have a splitting criterion for toric varieties that involves an *irreducible* torus-invariant $(\dim X - 2)$-ample divisor. In general, the answer is "no"; *reducible* divisors are necessary for the following reason. If D is such an irreducible divisor, then $\mathrm{cd}(X \setminus D) \leq \dim X - 2$, so D must intersect all the other torus-invariant divisors. Hence

Σ has the following property: if $\xi_D \in \Sigma(1)$ defines D, then ξ_D, ξ span a cone of Σ for all $\xi \in \Sigma(1) \setminus \{\xi_D\}$. This condition is clearly not satisfied in general.

(ii) It is rather surprising that the issue concerning the bare existence of nontrivial vector bundles on toric varieties is not settled yet in general (see [13]).

Appendix A: About q-Ample and q-Positive Line Bundles

In this section we summarize the notions and the results about partial positivity for line bundles that are used in this note. Henceforth X stands for a smooth projective variety of arbitrary dimension defined over $\mathbb{C}$.

DEFINITION A.1. Consider a line bundle $\mathcal{L}$ on X.

(i) (see [30]) $\mathcal{L}$ is *q-ample* if for any coherent sheaf $\mathcal{F}$ on X, there is $m_{\mathcal{F}} > 0$ such that $H^i(X, \mathcal{F} \otimes \mathcal{L}^m) = 0$ for all $m \geq m_{\mathcal{F}}$ and $i > q$.

(ii) (see [1; 12]) $\mathcal{L}$ is *q-positive* if it admits a Hermitian metric whose curvature is positive definite on a subspace of $T_{X,x}$ of dimension at least $\dim X - q$ for all $x \in X$; equivalently, the curvature has, at each point $x \in X$, at most q negative or zero eigenvalues.

(iii) $\mathcal{L}$ is *semiample* if a tensor power of it is globally generated.

(iv) Assume that there exists $a \geq 1$ such that $H^0(X, \mathcal{L}^a) \neq 0$. The *Kodaira–Iitaka dimension* and the *(stable) base locus* of $\mathcal{L}$ are defined as follows (see [20, Section 2.1]):

$$\begin{aligned}\kappa(\mathcal{L}) &:= \text{transcend. deg.}_{\mathbb{C}}\Big(\bigoplus_{a\geq 0} H^0(X, \mathcal{L}^a)\Big) - 1 \\ &= \max_{a\geq 1} \dim(\mathrm{Image}(X \dashrightarrow |\mathcal{L}^a|)); \\ \text{base locus}(\mathcal{L}) &:= \bigcap_{s\in H^0(X,\mathcal{L})} \text{divisor}(s), \\ \text{stable base locus}(\mathcal{L}) &:= \bigcap_{a\geq 1} \text{base locus}(\mathcal{L}^a)_{\text{red}}.\end{aligned}$$

REMARK A.2. (i) Any q-positive line bundle is q-ample (see [1, Prop. 28], [12, Prop. 2.1]), but the converse is false (see [26, Thm. 10.3]).

(ii) If $\ell, \mathcal{L} \in \mathrm{Pic}(X)$ and $\mathcal{L}$ is q-ample, then $\ell \otimes \mathcal{L}^m$ is q-ample for $m \gg 0$. This is a direct consequence of the uniform q-ampleness property [30, Thm. 6.4].

(iii) Definition A.1(i) makes sense for any projective scheme, not necessarily smooth. This more general situation occurs in Theorem B.3.

(iv) If $\mathcal{L}$ is q-ample (positive), then it is also q'-ample (positive) for any $q' \geq q$; the larger the value of q, the weaker the restriction on $\mathcal{L}$. For example, the $\dim X$-ampleness (positivity) is an empty condition.

THEOREM A.3. *Suppose $\mathcal{L}$ is semiample and q-ample. Then, $\mathcal{L}$ is q-positive, and*

$$\dim X \leq q + \kappa(\mathcal{L}).$$

Proof. See [23, Thm. 1.4]. For suitable a, the image of the morphism $X \to |\mathcal{L}^a|$ is $\kappa(\mathcal{L})$-dimensional; we denote the image by Y. Hence $\dim X - \kappa(\mathcal{L})$ equals the dimension of the generic fibre of $X \to Y$; by the same theorem the dimension of all the fibers is bounded above by q. □

LEMMA A.4. *Let X, Y be smooth projective varieties, and let $f : X \to Y$ be a smooth surjective morphism of relative dimension δ. Then the following implications hold*:

(i) *If $\mathcal{M} \in \mathrm{Pic}(Y)$ is q-ample, then $\mathcal{L} := f^*\mathcal{M}$ is $(\delta + q)$-ample*;
(ii) *If $\mathcal{M} \in \mathrm{Pic}(Y)$ is q-positive, then $\mathcal{L} := f^*\mathcal{M}$ is $(\delta + q)$-positive*;
(iii) *If $\mathcal{L} \in \mathrm{Pic}(X)$ is f-relatively ample, then $\mathcal{L}$ is $\dim Y$-positive.*

Proof. Leray's spectral sequence implies (i); for (ii), the pull-back metric on $\mathcal{L}$ satisfies (A.1).

(iii) Note that $\mathcal{L}' := \mathcal{L} \otimes f^*\mathcal{A}$ is ample for $\mathcal{A} \in \mathrm{Pic}(Y)$ sufficiently ample. Then $m\mathcal{L}'$, $m \gg 0$, defines an embedding $\iota : X \to \mathbb{P}$ into some projective space; the morphism $(f, \iota) : X \to Y \times \mathbb{P}$ is an embedding too, and $\mathcal{L}^m = (f, \iota)^*(\mathcal{A}^{-m} \boxtimes \mathcal{O}_{\mathbb{P}}(1))$. The restriction to X of the product metric on $\mathcal{A}^{-m} \boxtimes \mathcal{O}_{\mathbb{P}}(1)$ is positive definite on $\mathrm{Ker}(\mathrm{d}f)$. □

THEOREM A.5. (i) *Assume that $\mathcal{L} \in \mathrm{Pic}(X)$ is* semiample *and* q-ample, $q \leq \dim X - 1$. *Then*

$$H^i(X, \mathcal{L}^{-a}) = 0, \quad \forall i \leq \dim X - q - 1, \forall a \geq 1.$$

(ii) (relative Kodaira vanishing) *Consider a morphism $X \xrightarrow{f} Y$ with Y projective, and let $\mathcal{L} \in \mathrm{Pic}(X)$ be f*-relatively ample. *Then*

$$H^i(X, \mathcal{L}^{-1}) = 0, \quad \forall i < \dim X - \dim Y.$$

(iii) *Let Z be a projective equidimensional reduced weakly normal variety that satisfies the condition S_2 of Serre. Let $\mathcal{L} \in \mathrm{Pic}(Z)$ be relatively ample for $Z \xrightarrow{f} Y$ with Y projective and* $\dim Z - \dim Y \geq 2$. *Then* $H^1(Z, \mathcal{L}^{-1}) = 0$.

Proof. (i) The Grauert–Riemenschneider theorem (see [28, Thm. 7.73]) yields that, for all $a \geq 1$ and $i \leq \kappa(\mathcal{L}) - 1$, we have $H^i(X, \mathcal{L}^{-a}) = 0$. Now, we use the inequality in Theorem A.3.

(ii) The claim follows from [19, Thms. 1–3]: $R^i f_*(\omega_X \otimes \mathcal{L}) = 0$ for all $i > 0$. Indeed, for $i < \dim X - \dim Y$, the Leray-spectral sequence implies that

$$H^i(X, \mathcal{L}^{-1}) \cong H^{\dim X - i}(X, \omega_X \otimes \mathcal{L}) = H^{\dim X - i}(Y, f_*(\omega_X \otimes \mathcal{L})) = 0.$$

An independent proof can be obtained by following the lines of part (iii).

(iii) The previous argument does not apply because Z is not necessarily smooth. Without loss of generality, we may assume that f is surjective. For $\dim Y = 0$, the vanishing is proved in [2, Thm. 3.1]. Now take a very ample line bundle $\mathcal{A}$ on Y such that $\mathcal{L} \otimes f^*\mathcal{A}$ is ample on Z. Bertini's theorem implies that

$Z' := f^{-1}(Y')$ satisfies the same assumptions as Z for general $Y' \in |\mathcal{A}|$ (see [11, Thm. 1] and also [10, Cor. 1.9]). Finally, observe that in the exact sequence

$$\cdots \to H^1(Z, (\mathcal{L} \otimes \mathcal{A})^{-1}) \to H^1(Z, \mathcal{L}^{-1}) \to H^1(Z', (\mathcal{L} \otimes \mathcal{O}_{Z'})^{-1}) \to \cdots,$$

both extremities vanish: by [2, Thm. 3.1] for the left-hand side, and by the induction hypothesis for the right-hand side. □

PROPOSITION A.6. *Let* $\mathcal{L} \in \mathrm{Pic}(X)$ *be* $(\dim X - 2)$*-ample, and let* $D \in |\mathcal{L}|$. *Assume that*

$$H^i(D, \mathcal{L}_D^{-a}) = 0, \quad \forall a \geq 1 \text{ and } i = 0, 1.$$

$$(\textit{It suffices to have } H^i(X, \mathcal{L}^{-a}) = 0, \forall a \geq 1, i = 0, 1, 2.)$$

Then the restriction $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ *is injective.*

The vanishing condition is satisfied if $\mathcal{L}$ *is semiample with* $\kappa(\mathcal{L}) \geq 3$, *in particular, for* $\mathcal{L}$ *semiample and* $(\dim X - 3)$*-ample.*

Proof. We use the hypothesis in the exact sequence $0 \to \mathcal{L}_D^{-a} \to \mathcal{O}_{D_a}^{\times} \to \mathcal{O}_{D_{a-1}}^{\times} \to 0$ (see, e.g., [16, p. 179]) and deduce that the homomorphism $\mathrm{Pic}(D_a) \to \mathrm{Pic}(D_{a-1})$ is injective for all $a \geq 1$.

Take $\mathcal{M} \in \mathrm{Ker}(\mathrm{Pic}(X) \to \mathrm{Pic}(D))$, so $\mathcal{M}_D \cong \mathcal{O}_D$; it follows that $\mathcal{M}_{D_a} \cong \mathcal{O}_{D_a}$ for all $a \geq 0$. Note that $\mathbb{C} \cong H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_{D_a})$ is an isomorphism for all $a \geq 0$, so $H^0(\mathcal{M}_{D_a}) \cong \mathbb{C} \cong H^0(\mathcal{M}_{D_a}^{-1})$. But the restrictions $H^0(\mathcal{M}) \to H^0(\mathcal{M}_{D_a})$ and $H^0(\mathcal{M}^{-1}) \to H^0(\mathcal{M}_{D_a}^{-1})$ are isomorphisms for $a \gg 0$, since $\mathcal{L}$ is $(\dim X - 2)$-ample, and thus $\mathcal{M} \cong \mathcal{O}_X$. The last statement follows from the Grauert–Riemenschneider theorem [28, Thm. 7.73] and the inequality in Theorem A.3. □

THEOREM A.7. *Let* $D \subset X$ *be an effective divisor. The restriction* $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ *is an isomorphism in the following cases*:

(a) D *is* smooth *and* q*-positive with* $q \leq \dim X - 4$;
(b) (relative Picard–Lefschetz) D *is* arbitrary *relatively ample for a morphism* $X \xrightarrow{f} Y$ *with* Y *quasi-projective and* $\dim X - \dim Y \geq 4$.

Proof. (a) The q-positivity of $\mathcal{L}$ implies that $H^i(X; \mathbb{Z}) \to H^i(D; \mathbb{Z})$, $i \leq 2$, are isomorphisms (see [6, Thm. III], [26, Lemma 10.1]), so the same holds with $\mathbb{C}$-coefficients. The Hodge decomposition yields $H^i(X, \mathcal{O}_X) = H^{0,i}(X) \xrightarrow{\cong} H^{0,i}(D) = H^i(D, \mathcal{O}_D)$ for $i \leq 2$. By comparing the exponential sequences for X and D we deduce that $\mathrm{Pic}(X) \xrightarrow{\cong} \mathrm{Pic}(D)$.

(b) We may assume that f is surjective and Y projective. For $\dim Y = 0$, this is the Picard–Lefschetz theorem [16, IV§3, Thm. 3.1]. Now we make the inductive step. Let $\mathcal{A}$ be a very ample line bundle on Y such that $\mathcal{O}_X(D) \otimes f^*\mathcal{A}$ is ample on X. Bertini's theorem implies that the general hyperplane section $Y' \subset Y$ has the following properties:

– $X' := f^{-1}(Y')$ is smooth;

– X' does not contain the support of any irreducible component of D, that is, the schematic intersection $D \cdot X'$ of D, X' is a divisor in X'.

The divisor $D + X'$ is ample on X, so the Picard–Lefschetz theorem (see *idem*) implies that $\mathrm{Pic}(X) \to \mathrm{Pic}(D + X')$ is an isomorphism. Since X' intersects D properly, a line bundle (an invertible sheaf) $\ell \in \mathrm{Pic}(D + X')$ is uniquely determined by

– a pair $(\ell_D, \ell_{X'}) \in \mathrm{Pic}(D) \times \mathrm{Pic}(X')$ and
– an isomorphism $\ell_D \otimes \mathcal{O}_{D \cdot X'} \cong \ell_{X'} \otimes \mathcal{O}_{D \cdot X'}$.[1]

By the induction hypothesis, $\mathrm{Pic}(X') \to \mathrm{Pic}(D \cdot X')$ is an isomorphism, hence $\mathrm{Pic}(D + X') \to \mathrm{Pic}(D)$ is an isomorphism too. The conclusion follows. □

In this article, we often require the $(\dim X - 3)$- or $(\dim X - 2)$-ampleness of an effective divisor, so we need a practical method to verify this condition.

THEOREM A.8. *Let Δ be an effective divisor on X, and let $\mathcal{L} := \mathcal{O}_X(\Delta)$. We assume*:

$$\begin{array}{rl} \text{(i)} & \mathrm{cd}(X \setminus \Delta) \leq \dim X - c, \\ \text{(ii)} & \mathrm{codim}(\text{stable base locus}(\mathcal{L})) \geq c, \end{array} \quad \text{for } c \geq 2.$$

Then $\mathcal{L}$ is $(\dim X - c)$-ample. (Concerning the case $c = 1$, note that $\mathcal{L}$ is automatically $(\dim X - 1)$-ample by [30, *Thm. 9.1*].)

In (i), the notation $\mathrm{cd}(\cdot)$ stands for the cohomological dimension; according to [26, Prop. 5.1], if Δ is q-ample, then $\mathrm{cd}(X \setminus \Delta) \leq q$. In (ii), by $\mathrm{codim}(\cdot)$ we mean the *maximal* codimension of the components.

Proof. Let us analyze assumption (ii). The statement of the theorem is invariant after replacing $\mathcal{L}$ by a positive power $\mathcal{L}^a$, and thus we may assume that

$$\text{stable base locus}(\mathcal{L}) = \text{base locus}(\mathcal{L})_{\mathrm{red}}.$$

The codimension of the former is at least c, so there exist at least c algebraically independent sections in $\mathcal{L}$, that is, $\kappa(\mathcal{L}) \geq c - 1$. Bertini's theorem implies that, for general divisors $D_1, \ldots, D_{c-1} \in |\mathcal{L}|$, the schematic intersection $Z_t := D_1 \cdot \cdots \cdot D_t, t \leq c - 1$, has codimension t in X. Furthermore, since $\mathrm{codim}(\text{base locus}(\mathcal{L})) \geq \mathrm{codim}\, Z_{c-1} + 1$, there is a section in $\mathcal{L}$ that vanishes along a nontrivial divisor $Z_c \subset Z_{c-1}$; otherwise, some component of Z_{c-1} would be contained in base locus$(\mathcal{L})$.

We deduce the following exact sequences of sheaves on X for $t = 1, \ldots, c$:

$$0 \to \mathcal{L}^{m-1} \otimes \mathcal{O}_{Z_{t-1}} \to \mathcal{L}^m \otimes \mathcal{O}_{Z_{t-1}} \to \mathcal{L}^m \otimes \mathcal{O}_{Z_t} \to 0, \quad (Z_0 := X). \quad \text{(A.1)}$$

[1] Let $U \subset \mathrm{Spec}(\mathbb{C}[\xi_1, \ldots, \xi_{\dim X - 1}, y])$ be a local analytic chart on X such that $\{y = 0\}$ and $\{f(\underline{\xi}, y) = 0\}$ are the local equations of X' and D, respectively. The exact sequence $0 \to \frac{\mathbb{C}[\underline{\xi}, y]}{\langle y \cdot f \rangle} \to \frac{\mathbb{C}[\underline{\xi}, y]}{\langle y \rangle} \oplus \frac{\mathbb{C}[\underline{\xi}, y]}{\langle f \rangle} \to \frac{\mathbb{C}[\underline{\xi}, y]}{\langle y, f \rangle} \to 0$ shows that the germs of regular functions on $D + X' = \mathrm{Var}(y \cdot f)$ consist of pairs of regular functions on D, X' that agree on $D \cdot X' = \mathrm{Var}(y, f)$.

Now we use (i): since $\operatorname{cd}(X \setminus \Delta) \le \dim X - c$, any coherent sheaf $\mathcal{G}$ on X satisfies (see [26, Eq. (5.1)])

$$\varinjlim_m H^i(X, \mathcal{G} \otimes \mathcal{L}^m) = H^i(X \setminus \Delta, \mathcal{G}) = 0, \quad \forall i > \dim X - c. \tag{A.2}$$

The proof of the theorem is by induction on c. Recall that it is enough to check the q-ampleness property for locally free sheaves $\mathcal{F}$ on X; we fix one. For $c = 1$, we tensor by $\mathcal{F}$ the sequence (A.1), $t = 1$, and obtain

$$H^{\dim X}(X, \mathcal{F} \otimes \mathcal{L}^{m-1}) \to H^{\dim X}(X, \mathcal{F} \otimes \mathcal{L}^m) \to 0.$$

Thus the sequence of the cohomology groups eventually becomes stationary. By inserting into (A.2) we deduce that $H^{\dim X}(X, \mathcal{F} \otimes \mathcal{L}^m) = 0$ for $m \gg 0$.

Now we proceed with the inductive step: assume that the theorem holds for c and prove it for $c+1$. The induction hypothesis implies that we must only show that $H^{\dim X - c}(X, \mathcal{F} \otimes \mathcal{L}^m) = 0$ for $m \gg 0$.

– The sequence (A.1), $t = c+1$, tensored by $\mathcal{F}$ yields

$$H^{\dim X - c}(X, (\mathcal{F} \otimes \mathcal{O}_{Z_c}) \otimes \mathcal{L}^{m-1}) \to H^{\dim X - c}(X, (\mathcal{F} \otimes \mathcal{O}_{Z_c}) \otimes \mathcal{L}^m) \to 0,$$

because $\dim Z_{c+1} = \dim X - c - 1$. For $\mathcal{G} = \mathcal{F} \otimes \mathcal{O}_{Z_c}$, (A.2) implies, in the same way as before, that

$$H^{\dim X - c}(X, (\mathcal{F} \otimes \mathcal{O}_{Z_c}) \otimes \mathcal{L}^m) = 0 \quad \text{for } m \gg 0.$$

– Insert this into the long sequence in cohomology corresponding to (A.1), $t = c$, and find for $m \gg 0$:

$$H^{\dim X - c}(X, (\mathcal{F} \otimes \mathcal{O}_{Z_{c-1}}) \otimes \mathcal{L}^{m-1}) \to H^{\dim X - c}(X, (\mathcal{F} \otimes \mathcal{O}_{Z_{c-1}}) \otimes \mathcal{L}^m) \to 0.$$

Then (A.2) with $\mathcal{G} = \mathcal{F} \otimes \mathcal{O}_{Z_{c-1}}$ yields $H^{\dim X - c}(X, (\mathcal{F} \otimes \mathcal{O}_{Z_{c-1}}) \otimes \mathcal{L}^m) = 0$ for $m \gg 0$.

– Repeat this procedure—use (A.1) for $t = c-1, \dots, 1$ and (A.2)—until we get

$$H^{\dim X - c}(X, \mathcal{F} \otimes \mathcal{L}^m) = 0 \quad \text{for } m \gg 0.$$

This completes the proof of the theorem. □

Appendix B: About Frobenius Split (F-Split) Varieties

We recall the relevant definitions; the reference for the concept of Frobenius splitting is the book [8], and also [29, Section 31] for applications.

DEFINITION B.1 (see [8, Def. 1.1.3 and Section 1.6]). Let Z_p be a projective variety over $\bar{\mathbb{F}}_p$ (the algebraic closure of the field $\mathbb{Z}/p\mathbb{Z}$). The absolute Frobenius morphism F of Z_p determines the sheaf homomorphism $F^\sharp : \mathcal{O}_{Z_p} \to F_*\mathcal{O}_{Z_p}$. We say that Z_p is *F-split* if there is an $\mathcal{O}_{Z_p}$-linear homomorphism

$$\varphi : F_*\mathcal{O}_{Z_p} \to \mathcal{O}_{Z_p} \quad \text{such that } \varphi \circ F^\sharp = \mathbb{1}_{\mathcal{O}_{Z_p}}.$$

A closed subscheme $Y \subset X$ defined by the sheaf of ideals $\mathcal{I}_Y$ is *compatibly split* if $\varphi(F_*\mathcal{I}_Y) = \mathcal{I}_Y$.

If Z is a smooth projective variety defined over a field of characteristic zero, then there is a finite set $\mathbf{s}$ of primes, a finitely generated $\mathbb{Z}[\mathbf{s}^{-1}]$-algebra R, and a smooth $\mathrm{Spec}(R)$-scheme $\mathcal{Z}$ such that $Z = \mathcal{Z} \times_R \mathbb{C}$. If $\mathcal{L} \in \mathrm{Pic}(Z)$, then we may choose R so that $\mathcal{L}$ also extends over $\mathrm{Spec}(R)$.

Definition B.2. For a maximal ideal $\mathfrak{m} \in \mathrm{Spec}(R)$, the residue field $k(\mathfrak{m})$ is a finite extension of $\mathbb{F}_p$ with $p \notin \mathbf{s}$. The variety $Z_p := \mathcal{Z} \times_R \overline{k(\mathfrak{m})}$ is called *a reduction modulo p* of Z. (Note that $\overline{k(\mathfrak{m})} \cong \overline{\mathbb{F}}_p$.)

We say that Z is *F-split* if Z_p is *F-split* at infinitely many $\mathfrak{m} \in \mathrm{Spec}(R)$. (Such a subset is automatically dense in $\mathrm{Spec}(R)$.)

In our context, the importance of the Frobenius splitting is captured in the following Kodaira vanishing theorem for q-ample line bundles, which (apparently) has not been observed so far.

Theorem B.3. *Let Z be a projective equidimensional Cohen–Macaulay F-split variety over $\mathbb{C}$, and let $\mathcal{L} \in \mathrm{Pic}(Z)$ be q-ample. Then $H^i(Z, \mathcal{L}^{-1}) = 0$ for all $i < \dim Z - q$.*

In this note, we will apply the result in the case where Z is a compatibly split, normal crossing divisor of a smooth variety X.

Proof of Theorem B.3. Consider $\mathcal{Z} \xrightarrow{\pi} \mathrm{Spec}(R)$ as before, that is, such that $\mathcal{L}$ extends to $\mathscr{L} \to \mathcal{Z}$. Then for all primes p large enough, $\mathscr{L}_p \in \mathrm{Pic}(Z_p)$ is still q-ample (see [30, Thm. 8.1]), so $H^i(Z_p, \mathscr{L}_p^{-m}) = 0$ for $i < \dim Z - q$ and $m \gg 0$ by Serre duality (see [17, Ch. III, Cor. 7.7]). The F-splitting property implies that $H^i(Z_p, \mathscr{L}_p^{-1}) = 0$ (see [8, Lemma 1.2.7]). Finally, the generic rank of the coherent sheaf $R^i\pi_*\mathscr{L}^{-1}$ on $\mathrm{Spec}(R)$ is constant, and the conclusion follows from the fact that the vanishing holds for infinitely many primes p. $\square$

Remark B.4. (i) The F-splitting of a nonsingular variety X_p (defined in characteristic p) is given by an element in $H^0(X_p, \omega_{X_p}^{1-p})$ satisfying a certain algebraic equation, where ω_{X_p} stands for the canonical sheaf (see [8, Thm. 1.3.8]).

(ii) In *characteristic zero,* an important source of F-splittings arises from varieties X that have the property that their reduction X_p modulo p is split by the $(p-1)$th power of (the mod p reduction of) a section $\sigma \in H^0(X, \omega_X^{-1})$; in this case, $D := \mathrm{divisor}(\sigma)$ is a compatibly split subvariety of X (see [8, Thm. 1.4.10]). By abuse of language, we say that X is F-split by $\sigma \in H^0(X, \omega_X^{-1})$.

The latter category includes *spherical varieties* (see [7])—in particular projective homogeneous and toric varieties—and also Fano varieties (see [8, Exercise 1.6.E(5)]).

Acknowledgment. I heartily thank the referee for his very quick and careful reading of the manuscript.

References

[1] A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes,* Bull. Soc. Math. France 90 (1962), 193–259.

[2] D. Arapura and D. B. Jaffe, *On Kodaira vanishing for singular varieties,* Proc. Amer. Math. Soc. 105 (1989), 911–916.

[3] P. Bakhtary, *Splitting criteria for vector bundles on higher dimensional varieties,* Pacific J. Math. 252 (2011), 19–29.

[4] E. Ballico and F. Malaspina, *Vector bundles on products of projective spaces and hyperquadrics,* Rend. Istit. Mat. Univ. Trieste 41 (2009), 13–26.

[5] J. Biswas and G. V. Ravindra, *Arithmetically Cohen–Macaulay bundles on complete intersection varieties of sufficiently high multidegree,* Math. Z. 265 (2010), 493–509.

[6] R. Bott, *On a theorem of Lefschetz,* Michigan Math. J. 6 (1959), 211–216.

[7] M. Brion and S. P. Inamdar, *Frobenius splitting of spherical varieties,* Algebraic groups and their generalizations: classical methods (W. Haboush et al. eds.), Proc. Sympos. Pure Math., 56, pp. 207–218, 1994.

[8] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory,* Progr. Math., 231, Birkhäuser, 2005.

[9] L. Costa and R. Miró-Roig, *Cohomological characterization of vector bundles on multiprojective spaces,* J. Algebra 294 (2005), 73–96, corrigendum ibid. 319 (2008), 1336–1338.

[10] C. Cumino, S. Greco, and M. Manaresi, *Bertini theorems for weak normality,* Compos. Math. 48 (1983), 351–362.

[11] ———, *An axiomatic approach to the second theorem of Bertini,* J. Algebra 98 (1986), 171–182.

[12] J.-P. Demailly, T. Peternell, and M. Schneider, *Holomorphic line bundles with partially vanishing cohomology,* Proceedings of the Hirzebruch 65 conference on algebraic geometry (M. Teicher ed.), pp. 165–198, Bar-Ilan Univ, Ramat Gan, 1996.

[13] S. Gharib and K. Karu, *Vector bundles on toric varieties,* C. R. Math. Acad. Sci. Paris 350 (2012), 209–212, Corrigendum ibid. 350 (2012), 965.

[14] M. Halic, *Splitting criteria for vector bundles on minuscule homogeneous varieties,* Transform. Groups 22 (2017), 753–765.

[15] M. Halic and R. Tajarod, *A cohomological splitting criterion for locally free sheaves on arithmetically Cohen–Macaulay surfaces,* Math. Proc. Cambridge Philos. Soc. 155 (2013), 517–527.

[16] R. Hartshorne, *Ample subvarieties of algebraic varieties,* Lecture Notes in Math., 156, Springer-Verlag, 1970.

[17] ———, *Algebraic geometry,* 8th edition, Springer-Verlag, 1997.

[18] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring,* Proc. Lond. Math. Soc. 14 (1964), 689–713.

[19] Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the minimal model problem,* Algebraic geometry, Sendai, 1985 (T. Oda ed.), Adv. Stud. Pure Math., 10, pp. 283–360, North-Holland, 1987.

[20] R. Lazarsfeld, *Positivity in algebraic geometry, I,* Springer-Verlag, 2004.

[21] J. Leahy and M. Vitulli, *Weakly normal varieties: the multicross singularity and some vanishing theorems on local cohomology,* Nagoya Math. J. 83 (1981), 137–152.

[22] F. Malaspina, *A few splitting criteria for vector bundles,* Ric. Mat. 57 (2008), 55–64.

[23] S. Matsumura, *Asymptotic cohomology vanishing and a converse to the Andreotti–Grauert theorem on surfaces,* Ann. Inst. Fourier (Grenoble) 63 (2013), 2199–2221.

[24] R. Muñoz, G. Occhetta, and L. Solá Conde, *Uniform vector bundles on Fano manifolds and applications,* J. Reine Angew. Math. 664 (2012), 141–162.

[25] G. Ottaviani, *Some extensions of Horrocks criterion to vector bundles on Grassmannians and quadrics,* Ann. Mat. Pura Appl. (4) 155 (1989), 317–341.

[26] J. Ottem, *Ample subvarieties and q-ample divisors,* Adv. Math. 229 (2012), 2868–2887.

[27] T. Sawada, *A sufficient condition for splitting of arithmetically Cohen–Macaulay bundles on general hypersurfaces,* Comm. Algebra 38 (2010), 1633–1639.

[28] B. Shiffman and A. J. Sommese, *Vanishing theorems on complex manifolds,* Progr. Math., 56, Birkhäuser, Boston, 1985.

[29] D. Timashev, *Homogeneous spaces and equivariant embeddings,* Encyclopaedia Math. Sci., 138, Springer-Verlag, 2011.

[30] B. Totaro, *Line bundles with partially vanishing cohomology,* J. Eur. Math. Soc. (JEMS) 15 (2013), 731–754.

CRM
UMI 3457
Montréal H3T 1J4
Canada

mihai.halic@gmail.com