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Characterization of Aleksandrov Spaces of Curvature Bounded Above by Means of the Metric Cauchy–Schwarz Inequality

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Abstract. We consider the previously introduced notion of the K-quadrilateral cosine, which is the cosine under parallel transport in model K-space, and which is denoted by cosq_K. In K-space, $|\operatorname{cosq}_K| \le 1$ is equivalent to the Cauchy–Schwarz inequality for tangent vectors under parallel transport. Our principal result states that a geodesically connected metric space (of diameter not greater than $\pi/(2\sqrt{K})$ if $K > 0$) is an $\Re_K$ domain (otherwise known as a CAT(K) space) if and only if always $\operatorname{cosq}_K \le 1$ or always $\operatorname{cosq}_K \ge -1$. (We prove that in such spaces always $\operatorname{cosq}_K \le 1$ is equivalent to always $\operatorname{cosq}_K \ge -1$.) The case of $K = 0$ was treated in our previous paper on quasilinearization. We show that in our theorem the diameter hypothesis for positive K is sharp, and we prove an extremal theorem—isometry with a section of K-plane—when $|\operatorname{cosq}_K|$ attains an upper bound of 1, the case of equality in the metric Cauchy–Schwarz inequality. We derive from our main theorem and our previous result for $K = 0$ a complete solution of Gromov's curvature problem in the context of Aleksandrov spaces of curvature bounded above.

1. Introduction

Classes of Riemannian metrics that satisfy uniform sectional curvature bounds often arise in geometry. In his fundamental papers [1] and [3], Aleksandrov presented the upper and lower curvature conditions for a geodesically connected metric space, that is, a metric space in which any two points can be joined by a shortest. In particular, Aleksandrov introduced the notion of an $\Re_K$ domain, also known as a CAT(K) space, a geodesically connected metric space of curvature $\le K$ in the sense of Aleksandrov, in which shortests depend continuously on their end points, and in which the perimeter of every geodesic triangle is less than $2\pi/\sqrt{K}$ if $K > 0$.

Recall that the K-plane $\mathbb{S}_K$ is the Euclidean plane if $K = 0$, the open hemisphere of radius $1/\sqrt{K}$ if $K > 0$, and the hyperbolic plane of curvature K if $K < 0$. The definition of K-space $\mathbb{S}^3_K$ is similar.

Let $A, B, P, Q \in \mathbb{S}^3_K$, $\overrightarrow{u} = \exp_A^{-1}(P)$, $\overrightarrow{v} = \exp_B^{-1}(Q)$, and let $(\overrightarrow{v})^A_B$ be the tangent vector to $\mathbb{S}^3_K$ at the point A which is (Levi-Civita) parallel to the vector

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$\overrightarrow{v}$ along the unique shortest from the point B to the point A. Then, for the inner product $\langle \overrightarrow{u}, \overrightarrow{v} \rangle := \langle \overrightarrow{u}, (\overrightarrow{v})^A_B \rangle$, the *Cauchy–Schwarz inequality* $|\langle \overrightarrow{u}, \overrightarrow{v} \rangle| \leq \| \overrightarrow{u} \| \| \overrightarrow{v} \|$ holds. For nonzero $\overrightarrow{u}$ and $\overrightarrow{v}$, $\cos \measuredangle(\overrightarrow{u}, (\overrightarrow{v})^A_B)$ can be calculated in terms of the six distances between the points of the quadruple $\{A, B, P, Q\}$. Figure 2 explains the geometric construction of the parallel transport in $\mathbb{S}^3_K$ and suggests the method of calculation of $\cos \measuredangle(\overrightarrow{u}, (\overrightarrow{v})^A_B) = -\cos \measuredangle P'BQ$ in K-space. The resulting function of these six distances is referred to as the *K-quadrilateral cosine* $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})$ where $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ denote the ordered pairs (A, P) and (B, Q), respectively (see (1.1)–(1.3)). Hence, for nonzero $\overrightarrow{u}$ and $\overrightarrow{v}$, the Cauchy–Schwarz inequality is equivalent to the inequality $|\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})| \leq 1$. In a metric space $(\mathcal{M}, \rho)$, the K-quadrilateral cosine $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})$ is defined by (1.1)–(1.3), provided that $\rho(A, P)$, $\rho(B, Q)$, and $\rho(A, B) < \pi/\sqrt{K}$ for positive K. In a general metric space, $|\operatorname{cosq}_K|$ can exceed 1. In this note, we present a deeper metric analysis of Aleksandrov's upper boundedness curvature condition by carrying over the Cauchy–Schwarz inequality condition to general metric spaces by requiring that for every two pairs of two distinct points in a metric space, always $\operatorname{cosq}_K \leq 1$ or always $\operatorname{cosq}_K \geq -1$ (we prove that in a geodesically connected metric space, always $\operatorname{cosq}_K \leq 1$ is equivalent to always $\operatorname{cosq}_K \geq -1$).

In this paper, we present the following three substantial applications of the K-quadrilateral cosine: we characterize Aleksandrov $\Re_K$ domains via boundedness by 1 of all K-quadrilateral cosines in a geodesically connected metric space (Theorem 1.1), K-quadrilateral cosine enables us to obtain an interesting new rigidity result (Theorem 1.2) establishing an isometry to a trapezoid in $\mathbb{S}_K$, and the K-quadrilateral cosine is the main tool in solving Gromov's curvature problem in the context of Aleksandrov spaces of curvature bounded above by K (Theorem 1.3). Loosely speaking, we show that Aleksandrov's upper curvature condition of a geodesically connected metric space can be characterized in terms of a (trigonometric or hyperbolic trigonometric) polynomial inequality involving six distances of independent quadruples of the metric space. We remark that our condition is not local in the sense that when it is applied to a single triangular quadruple, it need not be equivalent to any of the familiar equivalent conditions of Aleksandrov's curvature bound from above.

The quadrilateral cosine cosq_0 was introduced in [19] under the name of function h and was used to construct the generalized Sasaki metric on the set of tangent elements of a metric space and to obtain a pure metric characterization of Riemannian spaces [19; 20].

The generalization of the quadrilateral cosine to nonzero K is not straightforward. Let $K \neq 0$ and $\kappa = \sqrt{|K|}$. In what follows, $\widehat{\kappa} = \kappa = \sqrt{K}$ if $K > 0$ and $\widehat{\kappa} = i\kappa = i\sqrt{-K}$ if $K < 0$. The following definition is equivalent to Definition 3.2 in [7]. We will use the following terminology. An ordered pair (A, P) of points in a metric space is called a *bound vector* $\overrightarrow{AP}$, and the bound vector $\overrightarrow{AP}$ is called nonzero if $A \neq P$.

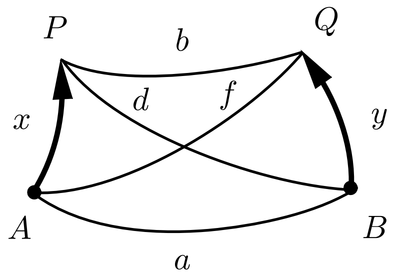

Figure 1 Definition of cosq_K

DEFINITION 1.1. Let $(\mathcal{M}, \rho)$ be a metric space, and let $A, P, B, Q \in \mathcal{M}$ be such that $A \neq P$ and $B \neq Q$. If $K > 0$, then we assume that $\rho(A, P)$, $\rho(B, Q)$, and $\rho(A, B) < \pi/\sqrt{K}$. Set

$$\rho(A, P) = x, \qquad \rho(B, Q) = y, \qquad \rho(A, B) = a,$$
$$\rho(P, Q) = b, \qquad \rho(P, B) = d, \quad \text{and} \quad \rho(A, Q) = f,$$

as shown in Figure 1. Then if $K \neq 0$, the *K-quadrilateral cosine* $\mathrm{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})$ is defined by

$$\mathrm{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = \frac{\cos\widehat{\kappa}b + \cos\widehat{\kappa}x\cos\widehat{\kappa}y}{\sin\widehat{\kappa}x\sin\widehat{\kappa}y} - \frac{(\cos\widehat{\kappa}x + \cos\widehat{\kappa}d)(\cos\widehat{\kappa}y + \cos\widehat{\kappa}f)}{(1+\cos\widehat{\kappa}a)\sin\widehat{\kappa}x\sin\widehat{\kappa}y}.$$

In particular, if $K > 0$, then

$$\mathrm{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = \frac{\cos\kappa b + \cos\kappa x\cos\kappa y}{\sin\kappa x\sin\kappa y} - \frac{(\cos\kappa x + \cos\kappa d)(\cos\kappa y + \cos\kappa f)}{(1+\cos\kappa a)\sin\kappa x\sin\kappa y}, \tag{1.1}$$

and if $K < 0$, then

$$\mathrm{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = \frac{(\cosh\kappa x + \cosh\kappa d)(\cosh\kappa y + \cosh\kappa f)}{(1+\cosh\kappa a)\sinh\kappa x\sinh\kappa y} - \frac{\cosh\kappa b + \cosh\kappa x\cosh\kappa y}{\sinh\kappa x\sinh\kappa y}. \tag{1.2}$$

We recall that the (0-)quadrilateral cosine is defined by

$$\mathrm{cosq}_0(\overrightarrow{AP}, \overrightarrow{BQ}) = \frac{f^2 + d^2 - a^2 - b^2}{2xy}. \tag{1.3}$$

REMARK 1.1. The formula for cosq_K can be stated implicitly in a more symmetric form motivated by Figure 2:

$$(1+\cos\widehat{\kappa}a)(\cos\widehat{\kappa}b + \cos\widehat{\kappa}g) = (\cos\widehat{\kappa}x + \cos\widehat{\kappa}d)(\cos\widehat{\kappa}y + \cos\widehat{\kappa}f),$$

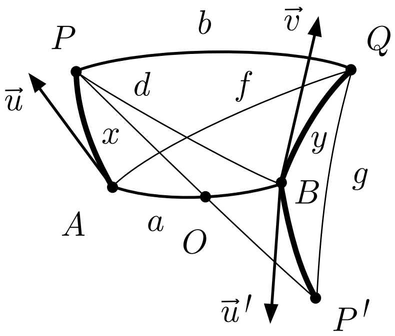

Figure 2 cosq_K in $\mathbb{S}^3_K$

where $\cos\widehat{\kappa}g = \cos\widehat{\kappa}x\cos\widehat{\kappa}y - \mathrm{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})\sin\widehat{\kappa}x\sin\widehat{\kappa}y$. This provides an efficient conceptual setting for cosq_K.

We introduce the following conditions for a metric space $(\mathcal{M}, \rho)$:

(i) The *upper four-point* cosq_K *condition:* $\mathrm{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) \leq 1$ for every pair of nonzero bound vectors $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ in $\mathcal{M}$ such that $\rho(A, P)$, $\rho(B, Q)$, and $\rho(A, B) < \pi/\sqrt{K}$ when $K > 0$.

(ii) The *lower four-point* cosq_K *condition:* $\mathrm{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) \geq -1$ for every pair of nonzero bound vectors $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ in $\mathcal{M}$ such that $\rho(A, P)$, $\rho(B, Q)$, and $\rho(A, B) < \pi/\sqrt{K}$ when $K > 0$.

We say that $(\mathcal{M}, \rho)$ satisfies the *one-sided four-point* cosq_K *condition* if it satisfies either the upper four-point cosq_K condition or the lower four-point cosq_K condition.

Our present main result is given by the following:

THEOREM 1.1. *Let* $K \neq 0$, *and let* $(\mathcal{M}, \rho)$ *be a geodesically connected metric space such that* $\mathrm{diam}(\mathcal{M}) \leq \pi/(2\sqrt{K})$ *when* $K > 0$. *Then* $(\mathcal{M}, \rho)$ *is an* $\Re_K$ *domain with the same diameter restriction if and only if* $(\mathcal{M}, \rho)$ *satisfies the one-sided* cosq_K *condition.*

Theorem 1.1 for $K = 0$ is proved in [8, Thm. 1]. There are striking differences in our approach to nonzero K that require different methods. The lack of linearity for nonzero K in the model space presents substantial conceptual and technical problems.

REMARK 1.2. In contrast to the well-known conceptually similar Aleksandrov's upper curvature condition, Bruhat–Tits condition, more recent $(2+2)$-point K-comparison condition and others, the metric Cauchy–Schwarz inequality condition in the form of the one-sided cosq_K condition is not designed to provide sufficiency when applied to an individual quadruple. The one-sided cosq_K condition implies Aleksandrov's upper curvature condition for a prescribed quadruple $\mathfrak{Q}$

only when it is applied to a large number of quadruples from the geodesic convex hull of the quadruple $\mathfrak{Q}$. Example 8.1 offers a clear demonstration that our criterion cannot be reduced to a repackaging of Bruhat–Tits condition or any of the similar conditions.

REMARK 1.3. If $K > 0$, then it is possible that $|\operatorname{cosq}_K| > 1$ in an $\Re_K$ domain unless $\operatorname{diam}(\Re_K)$ is not greater than $\pi/(2\sqrt{K})$ (however, we prove that always $|\operatorname{cosq}_K| \leq 1$ in every K-space). Example 4.1 shows that the restriction on the diameter of $(\mathcal{M}, \rho)$ for positive K cannot be dropped and the surprising diameter bound in the hypothesis of Theorem 1.1 is sharp.

REMARK 1.4. A normed vector space of curvature $\leq K$ in the sense of Aleksandrov is an inner product space [2, p. 7]. Hence, we can complement the results of the paper by Schoenberg [23] by deriving from Theorem 1.1 that a normed vector space is an inner product space if and only if it satisfies the one-sided cosq_K condition for some positive K.

If $K = 0$, then the upper four-point cosq_K condition is immediately equivalent to the lower four-point cosq_K condition [8, Introduction]. According to Examples 5.1 and 5.2, in a general metric space, this is not true anymore for nonzero K. However, we derive from Theorem 1.1 and Theorem 4.1 of Section 4 the following:

COROLLARY 1.1. *Let* $K \neq 0$, *and let* $(\mathcal{M}, \rho)$ *be a geodesically connected metric space such that* $\operatorname{diam}(\mathcal{M}) \leq \pi/(2\sqrt{K})$ *when* $K > 0$. *Then* $(\mathcal{M}, \rho)$ *satisfies the upper four-point* cosq_K *condition if and only if* $(\mathcal{M}, \rho)$ *satisfies the lower four-point* cosq_K *condition.*

REMARK 1.5. After establishing that we are in an $\Re_K$ domain, typical use of cosq_K involves Reshetnyak's majorization theorem (Section 2); typically we reason on comparison with our cosq_K construction rather than direct computation.

Recall that a polygonal curve $\mathcal{APQBA}$ in a Riemannian space is called a Levi-Civita parallelogramoid [13] if the distances between A and P and between B and Q are equal, and the vectors $\exp_A^{-1}(P)$ and $\exp_B^{-1}(Q)$ are parallel along a shortest joining A to B. We say that a polygonal curve $\mathcal{APQBA}$ is a Levi-Civita trapezoid if either the vectors $\exp_A^{-1}(P)$ and $\exp_B^{-1}(Q)$ are parallel along the shortest $\mathcal{AB}$ or the vectors $\exp_A^{-1}(P)$ and $-\exp_B^{-1}(Q)$ are parallel along the shortest $\mathcal{AB}$. A convex hull in $\mathbb{S}_K$ of a Levi-Civita trapezoid is called a Levi-Civita trapezoidal domain. In particular, the set of points of a shortest in $\mathbb{S}_K$ is a degenerate Levi-Civita trapezoidal domain. The following theorem generalizing [6, Thm. 15] and [7, Thm. 6.2] describes the extremal cases where cosq_K takes values 1 or -1.

THEOREM 1.2. *Let* $K \neq 0$, *and let* $(\mathcal{M}, \rho)$ *be a geodesically connected metric space such that* $\operatorname{diam}(\mathcal{M}) < \pi/(2\sqrt{K})$ *when* $K > 0$. *If* $(\mathcal{M}, \rho)$ *satisfies the one-sided four-point* cosq_K *condition, and for a pair of nonzero bound vectors* $\overrightarrow{AP}$

and $\overrightarrow{BQ}$ *in* $\mathcal{M}$, $|\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})| = 1$, *then the convex hull of the quadruple* $\{A, P, Q, B\}$ *is isometric to a Levi-Civita trapezoidal domain in* $\mathbb{S}_K$.

REMARK 1.6. By Example 7.1, Theorem 1.2 need not be true if $\operatorname{diam}(\mathcal{M}) = \pi/(2\sqrt{K})$ when $K > 0$.

REMARK 1.7. By Theorem 1.2, if the equality is actually attained in the metric Cauchy–Schwarz inequality for a pair of bound vectors $\overrightarrow{AP}$ and $\overrightarrow{BQ}$, then the bound vector $\overrightarrow{AP}$ is parallel to the bound vector $\overrightarrow{BQ}$ in the sense that $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ span a space isometric to a Levi-Civita trapezoid in the K-plane with the parallel edges AP and BQ. That is, in the trapezoidal domain, the tangent vectors $\overrightarrow{u} = \exp_A^{-1}(P)$ and $\overrightarrow{v} = \exp_B^{-1}(Q)$ are (Levi-Civita) collinear as the case of equality in the Cauchy–Schwarz inequality for Hilbert space suggests. We reiterate that our Theorem 1.2 is not a repackaging of the rigidity implied by equality in Aleksandrov's criterion or equivalent family of rigidity results. This result is qualitatively different from Aleksandrov-type rigidity in that $|\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})| = 1$, in itself, does not guarantee that the four-point configuration can be imbedded in K-space. Example 8.1 can be easily modified to provide a counterexample.

Recall that a semimetric space is a distance space with a positive definite and symmetric distance. A semimetric space $(\mathcal{M}, \rho)$ is said to be *weakly convex* if, for every $A, B \in \mathcal{M}$, there is $\lambda \in (0, 1)$ such that, for every $\varepsilon > 0$, there is $C_\varepsilon \in \mathcal{M}$ satisfying the inequalities $|\rho(A, C_\varepsilon) - \lambda\rho(A, B)| < \varepsilon$ and $|\rho(B, C_\varepsilon) - (1 - \lambda)\rho(A, B)| < \varepsilon$. Cauchy sequences in a semimetric space and the diameter of a semimetric space are defined in the same way as in a metric space. Finally, notice that the upper and the lower four-point cosq_K conditions can also be stated for semimetric spaces. Similar to the case $K = 0$, the following extension to nonzero K of [8, Thm. 5] is derived from Theorem 1.1 and Menger's theorem [10, Thm. 14.1]:

COROLLARY 1.2. *Let* $K \neq 0$, *and let* $(\mathcal{M}, \rho)$ *be a semimetric space such that* $\operatorname{diam}(\mathcal{M}) \leq \pi/(2\sqrt{K})$ *when* $K > 0$. *Then* $(\mathcal{M}, \rho)$ *is a complete* $\Re_K$ *domain with the same diameter restriction if and only if the following conditions are satisfied*: (*a*) $(\mathcal{M}, \rho)$ *is weakly convex.* (*b*) *Each Cauchy sequence in* $(\mathcal{M}, \rho)$ *has a limit.* (*c*) $(\mathcal{M}, \rho)$ *satisfies the one-sided four-point* cosq_K *condition.*

In his foundational paper [24], A. Wald characterized the curvature of a two-dimensional Riemannian space in terms of six distances between the points of independent quadruples. Wald's approach inspires the following important question: is it possible to characterize Aleksandrov's upper curvature condition of a geodesically connected metric space in terms of a (trigonometric or hyperbolic trigonometric) polynomial inequality involving six distances of independent quadruples of the metric space? This question is a part of the general Gromov's curvature problem.

In his book [17], Gromov offered a method to define classes of metric spaces corresponding to Riemannian manifolds with prescribed curvature restrictions by introducing global and local $\mathcal{K}$-curvature classes. Let $r \in \mathbb{N}$, and let M_r denote the set of all symmetric $r \times r$ matrices with zero diagonal entries and nonnegative entries otherwise. Let $\mathcal{X}$ be a set, and let $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a nonnegative function such that, for $P, Q \in \mathcal{X}$, $d(P, Q) = d(Q, P)$ and $d(P, Q) = 0$ if and only if $P = Q$. Then $K_r(\mathcal{X})$ consists of all matrices $A = (a_{ij})$ in M_r such that, for every $A \in K_r(\mathcal{X})$, there is an r-tuple $\{P_1, P_2, \ldots, P_r\} \subseteq \mathcal{X}$ satisfying $a_{ij} = d(P_i, P_j)$, $i, j = 1, 2, \ldots, r$. A subset $\mathcal{K} \subseteq M_r$ defines the (global) $\mathcal{K}$-curvature class as follows. The $\mathcal{K}$-curvature class consists of all $(\mathcal{X}, d)$ such that $K_r(\mathcal{X}) \subseteq \mathcal{K}$. Gromov's curvature problem is the problem of a meaningful geometric description of $\mathcal{K}$-curvature classes ([17], Section 1.19_+, Curvature Problem).

In [8, Thm. 8] we gave a solution of Gromov's curvature problem in the context of $\Re_0$ domains and therefore for Aleksandrov spaces of nonpositive curvature. In this note, we obtain a complete solution of Gromov's curvature problem in the context of $\Re_K$ domains and Aleksandrov spaces of curvature $\leq K$ by solving Gromov's curvature problem for nonzero K as a corollary of Theorem 1.1 and Corollary 1.2.

Let $\mathfrak{M}_G$ be the set of all geodesically connected metric spaces, and let $\mathfrak{M}_S$ denote the set of all semimetric spaces satisfying conditions (a) and (b) of Corollary 1.2. For $\kappa > 0$, let $\mathcal{K}^+(\kappa^2)$ denote the set of matrices $A = (a_{ij}) \in M_4$ such that

$$\begin{aligned}(\cos\kappa a_{23} + \cos\kappa a_{12}\cos\kappa a_{34})(1 + \cos\kappa a_{14})\\ - (\cos\kappa a_{12} + \cos\kappa a_{24})(\cos\kappa a_{34} + \cos\kappa a_{13})\\ \leq \sin\kappa a_{12}\sin\kappa a_{34}(1 + \cos\kappa a_{14})\end{aligned}$$

and $a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34} \leq \pi/(2\kappa)$. For $\mathcal{K}^-(\kappa^2)$, multiply the left-hand side of the above inequality by (-1). In a similar way, we define $\mathcal{K}^+(-\kappa^2)$ as the set of all matrices $A = (a_{ij}) \in M_4$ such that

$$\begin{aligned}(\cosh\kappa a_{12} + \cosh\kappa a_{24})(\cosh\kappa a_{34} + \cosh\kappa a_{13})\\ - (\cosh\kappa a_{23} + \cosh\kappa a_{12}\cosh\kappa a_{34})(1 + \cosh\kappa a_{14})\\ \leq \sinh\kappa a_{12}\sinh\kappa a_{34}(1 + \cosh\kappa a_{14})\end{aligned}$$

and for $\mathcal{K}^-(-\kappa^2)$, multiply the left-hand side of the above inequality by (-1).

THEOREM 1.3. *Let* $\kappa > 0$ *and* $K = \kappa^2$ *if* $K > 0$ *and* $K = -\kappa^2$ *if* $K < 0$. *Then* (*i*) $(\mathcal{X}, \rho) \in \mathfrak{M}_G$ (*respectively* $(\mathcal{X}, \rho) \in \mathfrak{M}_S$) *is in the global* $\mathcal{K}^{\pm}(\kappa^2)$*-curvature class if and only if* $(\mathcal{X}, \rho)$ *is an* $\Re_K$ *domain* (*respectively complete* $\Re_K$ *domain*) *of diameter not greater than* $\pi/(2\kappa)$. (*ii*) $(\mathcal{X}, \rho) \in \mathfrak{M}_G$ (*respectively* $(\mathcal{X}, \rho) \in \mathfrak{M}_S$) *is in the global* $\mathcal{K}^{\pm}(-\kappa^2)$*-curvature class if and only if* $(\mathcal{X}, \rho)$ *is an* $\Re_K$ *domain* (*respectively complete* $\Re_K$ *domain*).

REMARK 1.8. In particular, $(\mathcal{X}, \rho) \in \mathfrak{M}_G$ is in the local $\mathcal{K}^{\pm}(\pm\kappa^2)$-curvature class if and only if $(\mathcal{X}, \rho)$ is an Aleksandrov space of curvature $\leq K$ where $K = \pm\kappa^2$.

Remark 1.9. For an alternative proof of one of our main theorems [8, Thm. 6] solving Gromov's curvature problem in the context of $\Re_0$-domains, see Sato [22]. Sato's proof implicitly uses the averaging principle [8, Cor. 15]. For nonzero K, the averaging principle need not hold. The lack of linearity is a serious obstacle in extending the ideas of Sato's proof to nonzero K.

Remark 1.10. Lafont and Prassidis [18] established Enflo's 2-roudness condition [14] in $\Re_0$ domains. According to Theorem 6 in [8], a geodesically connected metric space is an $\Re_0$ domain if and only if it satisfies Enflo's 2-roudness condition. We can construct a natural extension of Enflo's 2-roundness condition to nonzero K that holds in $\Re_K$ domains [9]. We do not know if the converse is true.

Remark 1.11. Foertsch, Lytchak, and Schroeder [15] (also, see the correction in [16]) considered a weaker Ptolemaic condition and showed that while each $\Re_0$ domain is Ptolemaic, the converse may not be true.

Section 2 is a short review of Aleksandrov spaces of curvature bounded above. In Section 3, we prove that $|\operatorname{cosq}_K| \le 1$ in K-space. Section 4 presents the proof of $|\operatorname{cosq}_K| \le 1$ in an $\Re_K$ domain of diameter not greater than $\pi/(2\sqrt{K})$ if $K > 0$. We show that, in contrast to $\mathbb{S}^3_K$, the diameter restriction cannot be dropped for an $\Re_K$ domain. In Section 5, we present counterexamples showing that in a non-geodesically connected metric space the upper four-point cosq_K condition need not be equivalent to the lower four-point cosq_K condition. Section 6 contains the proof of our main result–Theorem 1.1. In this section, we assume that $(\mathcal{M}, \rho)$ is a geodesically connected metric space (of diameter not greater than $\pi/(2\sqrt{K})$ if $K > 0$) satisfying the one-sided four-point cosq_K condition. In Section 6.2, we prove that in $(\mathcal{M}, \rho)$ shortests depend continuously on their end points; in particular, any pair of points can be joined by a unique shortest. Hence, by Theorem 9 in [3, Section 3] the global angle comparison in $(\mathcal{M}, \rho)$ will follow from the local angle comparison, that is, locally, each vertex angle of a geodesic triangle $\mathcal{T}$ is not greater than the corresponding angle of the isometric copy of $\mathcal{T}$ in the K-plane. In Section 6.3, we derive the main auxiliary estimate, the cross-diagonal estimate. In Section 6.4, the cross-diagonal estimate lemma is used to derive our major estimate of Section 6, the growth estimate lemma. In Section 6.5, we show that the growth estimate lemma implies that in $(\mathcal{M}, \rho)$, between any pair of shortests starting at a common point A, the proportional angle exists, that is, the limit of $\measuredangle_K X_t A Y_t$ as $t \to 0+$ exists if $\rho(X_t, A)/\rho(Y_t, A) = \text{const}$ (for the notation, see Section 2 and Figure 12). In Section 6.6, following the method of our proof of Proposition 20 in [8], we derive from the existence of proportional angles and growth estimate lemma that in $(\mathcal{M}, \rho)$, between any pair of shortests emanating from a common point, Aleksandrov's angle exists. The existence of Aleksandrov's angle and growth estimate lemma enables us to prove the local angle comparison and thereby the global angle comparison (Section 6.7). In Section 7, we establish an extremal case where $|\operatorname{cosq}_K| = 1$.

2. Aleksandrov's Upper Curvature Condition

In this section, we recall some basic definitions of Aleksandrov geometry.

Let $(\mathcal{M}, \rho)$ be a metric space, and let $\mathcal{L}$ be a curve in $\mathcal{M}$. We denote by $\ell_\rho(\mathcal{L})$ the length of $\mathcal{L}$ in the metric ρ. A rectifiable curve $\mathcal{L}$ joining P to Q is called a *shortest* or *minimal geodesic* (joining P to Q) if $\rho(P, Q) = \ell_\rho(\mathcal{L})$. If $\mathcal{L}$ is a shortest joining P to Q, then often we denote the shortest $\mathcal{L}$ by $\mathcal{PQ}$ if there is no possible ambiguity, and the distance between its end points (or, in general, between a pair of points in $\mathcal{M}$) P and Q by PQ. A subset $\mathcal{U}$ of a metric space is said to be *convex* if every pair of points $P, Q \in \mathcal{U}$ can be joined by a shortest and all shortests joining P to Q are contained in $\mathcal{U}$.

A configuration consisting of three distinct points $A, B, C \in \mathcal{M}$ (*vertices*) and three shortests $\mathcal{AB}$, $\mathcal{BC}$, and $\mathcal{AC}$ (*sides*) is called a *(geodesic) triangle* $\mathcal{T} = ABC$. The *perimeter* $p(\mathcal{T})$ of a triangle $\mathcal{T} = ABC$ (or, in general, of a triple of points $\mathcal{T} = \{A, B, C\}$ in $\mathcal{M}$) is the sum $AB + BC + AC$. The *isometric copy* in the K-plane of the triangle $\mathcal{T}$ is the triangle $\mathcal{T}^K = A^K B^K C^K$ in $\mathbb{S}_K$ having the same side lengths as $\mathcal{T}$: $AB = A^K B^K$, $AC = A^K C^K$, and $BC = B^K C^K$ (if $K > 0$, then we require that $p(\mathcal{T}) < 2\pi/\sqrt{K}$). We let $\measuredangle_K BAC$ denote the angle $\measuredangle B^K A^K C^K$. The area $\sigma(ABC)$ of the triangle ABC is the area of the Euclidean triangle $A^0 B^0 C^0$.

Let $\mathcal{L}$ and $\mathcal{N}$ be two shortest arcs with a common starting point O in a metric space $(\mathcal{M}, \rho)$. Let $X \in \mathcal{L}\backslash\{O\}$ and $Y \in \mathcal{N}\backslash\{O\}$. Set $x = OX$, $y = OY$ and $\measuredangle_K(x, y) = \measuredangle_K XOY$. The *upper* and *lower angles* between the curves $\mathcal{L}$ and $\mathcal{N}$ are defined by

$$\overline{\measuredangle}(\mathcal{L}, \mathcal{N}) = \varlimsup_{x\to 0+, y\to 0+} \measuredangle_K(x, y) \quad \text{and} \quad \underline{\measuredangle}(\mathcal{L}, \mathcal{N}) = \varliminf_{x\to 0+, y\to 0+} \measuredangle_K(x, y).$$

It is known that all these definitions do not depend on K. We say that the angle $\measuredangle(\mathcal{L}, \mathcal{N})$ between $\mathcal{L}$ and $\mathcal{N}$ exists if $\overline{\measuredangle}(\mathcal{L}, \mathcal{N}) = \underline{\measuredangle}(\mathcal{L}, \mathcal{N})$.

The (upper) *K-excess* $\delta_K(\mathcal{T})$ of the triangle $\mathcal{T}$ is defined by

$$\delta_K(\mathcal{T}) = (\overline{\measuredangle} ABC + \overline{\measuredangle} ACB + \overline{\measuredangle} BAC) - (\measuredangle_K ABC + \measuredangle_K ACB + \measuredangle_K BAC).$$

An $\Re_K$ *domain* (otherwise known as a CAT(K) space) is a metric space with the following properties:

(i) $\Re_K$ is convex (that is, $\Re_K$ is geodesically connected).
(ii) If $K > 0$, then the perimeter of every triangle in $\Re_K$ is less than $2\pi/\sqrt{K}$.
(iii) Each triangle $\mathcal{T}$ in $\Re_K$ has nonpositive K-excess $\delta_K(\mathcal{T})$.

We remark that by (ii) all distances in an $\Re_K$ domain are less than $\pi/\sqrt{K}$ when $K > 0$. Another name for an $\Re_K$ domain is a CAT(K) space (when $K > 0$, CAT(K) is slightly more general: if $p(\mathcal{T}) < 2\pi/\sqrt{K}$, then $\delta_K(\mathcal{T}) \leq 0$). We will use Aleksandrov's original notation (see [1] and [3]).

A metric space $(\mathcal{M}, \rho)$ is a *space of curvature* $\leq K$ in the sense of Aleksandrov if each point of $\mathcal{M}$ is contained in some neighborhood that is an $\Re_K$ domain. For more information on Aleksandrov spaces of curvature $\leq K$, see [1; 3; 5], and [11].

We will find useful the following theorem of Reshetnyak [21].

Let $\mathcal{L}$ be a closed rectifiable curve in a metric space $(\mathcal{M}, \rho)$ such that $\ell_\rho(\mathcal{L}) < 2\pi/\sqrt{K}$ if $K > 0$. Let $\mathcal{V}$ be a convex domain in $\mathbb{S}_K$ with the bounding curve $\mathcal{N}$. We say that $\mathcal{V}$ *majorizes* the curve $\mathcal{L}$ if there is a nonexpanding mapping of the domain $\mathcal{V}$ into $\mathcal{M}$ that maps $\mathcal{N}$ onto $\mathcal{L}$ and preserves arc length. The domain $\mathcal{V}$ is called the *majorant* for $\mathcal{L}$.

RESHETNYAK'S MAJORIZATION THEOREM. *In an $\Re_K$ domain, for every rectifiable closed curve $\mathcal{L}$ (whose length is less than $2\pi/\sqrt{K}$ when $K > 0$), there is a convex domain in $\mathbb{S}_K$ that majorizes $\mathcal{L}$.*

Let $(A_1, A_2, \dots, A_n)$ be an n-tuple of distinct points in $(\mathcal{M}, \rho)$. Suppose that for every $j \in \{1, 2, \dots, n-1\}$, the points A_j and A_{j+1} can be joined by a shortest $\mathcal{L}_j = \mathcal{A}_j\mathcal{A}_{j+1}$. Then we call the curve $\mathcal{L} = \mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n$ formed by the consecutive shortests $\mathcal{L}_j$, a *polygonal curve* (with vertices at $A_1, A_2, \dots, A_n$ in $\mathcal{M}$). It is not difficult to see that in Reshetnyak's theorem if $\mathcal{L} = \mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n\mathcal{A}_1$ is a closed polygonal curve, then $\mathcal{N}$ is also a closed polygonal curve $\mathcal{A}_1'\mathcal{A}_2'\dots\mathcal{A}_n'\mathcal{A}_1'$ in $\mathbb{S}_K$. In our notation, we always assume that the vertices of $\mathcal{N}$ are labeled so that $A_jA_{j+1} = A_j'A_{j+1}'$ for every $j = 1, 2, \dots, n$, where $A_{n+1} = A_1$ and $A_{n+1}' = A_1'$.

If $\mathcal{L}$ is a polygonal curve $\mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n$ of length l in a metric space, then the *arc length parameterization of $\mathcal{L}$ relative to A_1* is an arc length parameterization of $\mathcal{L}$, $\mathfrak{g}_{al} = \mathfrak{g}_{al,\mathcal{L}} : [0, l] \to \mathcal{M}$, such that the length of the arc of $\mathcal{L}$ with the end points at A_1 and $\mathfrak{g}_{al}(s)$ is equal to $s \in [0, l]$. The *reduced parameterization of $\mathcal{L}$ relative to A* is the mapping $\mathfrak{g}_r = \mathfrak{g}_{r,\mathcal{L}} : [0, 1] \to \mathcal{M}$ given by $\mathfrak{g}_r(t) = \mathfrak{g}_{al}(tl)$ for every $t \in [0, 1]$. If $l_0 > 0$, then the l_0-arc length proportional parameterization of $\mathcal{L}$ is the mapping $\mathfrak{g}_{l_0,pr} = \mathfrak{g}_{l_0,pr,\mathcal{L}} : [0, l_0] \to \mathcal{M}$ given by $\mathfrak{g}_{l_0,pr}(u) = \mathfrak{g}_{al}(ul/l_0)$.

Let $(\mathcal{M}, \rho)$ be a geodesically connected metric space, and let $\mathcal{F} \subseteq \mathcal{M}$ be a nonempty set. For a pair of points $P, Q \in (\mathcal{M}, \rho)$, we let $\mathcal{G}[P, Q]$ denote the set of points each of which belongs to a shortest joining the points P and Q. We define $\mathcal{G}[\mathcal{F}]$ by $\mathcal{G}[\mathcal{F}] = \bigcup_{P,Q\in\mathcal{F}} \mathcal{G}[P, Q]$. Next, denote $\mathcal{F}$ by $\mathcal{G}^0[\mathcal{F}]$ and $\underbrace{\mathcal{G}[\mathcal{G}[\dots\mathcal{G}[\mathcal{F}]]]}_{n \text{ times}}$ by $\mathcal{G}^n[\mathcal{F}]$. Then the *geodesic convex hull* of $\mathcal{F}$ is defined as $\mathcal{GC}[\mathcal{F}] = \bigcup_{n=0}^{\infty} \mathcal{G}^n[\mathcal{F}]$.

3. K-Quadrilateral Cosine in K-Space

In this section, we prove that $|\operatorname{cosq}_K| \le 1$ in $\mathbb{S}_K^3$.

Let $K \neq 0$. Let $\{A, B, P, Q\}$ be a quadruple of distinct points in $\mathbb{S}_K^3$. Let O be the midpoint of the shortest arc $\mathcal{AB}$. If $PO < \pi/(2\sqrt{K})$ when $K > 0$, then we can use the following constructive interpretation of cosq_K in $\mathbb{S}_K^3$. Indeed, let P' be the point symmetric to the point P relative to O, that is, O is the midpoint of the shortest arc $\mathcal{PP}'$, as illustrated in Figure 2. Then $\vec{u} = \exp_A^{-1}(P)$ is (Levi-Civita) parallel along $\mathcal{AB}$ to the vector $\vec{u}'' = -\vec{u}'$, where $\vec{u}' = \exp_B^{-1}(P')$. Let $\vec{v} = \exp_B^{-1}(Q)$. In [7, Lemma 3.1], we showed that $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = -\cos\angle P'BQ = \cos\angle(\vec{v}, \vec{u}'')$. Hence, for the K-quadrilateral cosine in $\mathbb{S}_K^3$,

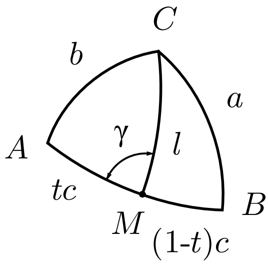

Figure 3 Sketch for Lemma 3.1

we always have

$$|\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})| \le 1$$

as long as $PO < \pi/(2\sqrt{K})$ when $K > 0$.

Next, we show that the restriction $PO < \pi/(2\sqrt{K})$ for positive K can be dropped for $\mathbb{S}^3_K$ itself. We begin with the following simple corollary of the spherical cosine formula.

LEMMA 3.1. *Let $K > 0$, and let $\mathcal{T} = ABC$ be a nondegenerate triangle in $\mathbb{S}_K$. Let $M \in \mathcal{AB}\backslash\{A, B\}$. Set $a = BC$, $b = AC$, $c = AB$, $l = MC$, and $t = AM/c$, as shown in Figure 3. Then*

$$\cos\kappa l = \frac{\cos\kappa a \sin\kappa tc + \cos\kappa b \sin\kappa(1-t)c}{\sin\kappa c}.$$

In particular, if M is the midpoint of the shortest $\mathcal{AB}$, then we obtain a familiar spherical Bruhat–Tits equality:

$$\cos\kappa l = \frac{\cos\kappa a + \cos\kappa b}{2\cos\frac{\kappa c}{2}}$$

(*for $K = 0$, see the Bruhat–Tits inequality in* [12, *Lemma 3.2.1*]).

By K-concavity in $\Re_K$ [3, Section 3, Thm. 2], we also have the following:

COROLLARY 3.1. *Let $K > 0$, let $\mathcal{T} = ABC$ be a nondegenerate triangle in $\Re_K$, and let $M \in \mathcal{AB}\backslash\{A, B\}$. Set $a = BC$, $b = AC$, $c = AB$, $l = MC$, and $t = AM/c$. Then*

$$\cos\kappa l \ge \frac{\cos\kappa a \sin\kappa tc + \cos\kappa b \sin\kappa(1-t)c}{\sin\kappa c}.$$

COROLLARY 3.2. *Let $K > 0$, let $\mathcal{T} = ABC$ be a nondegenerate triangle in $\Re_K$, and let $M \in \mathcal{AB}\backslash\{A, B\}$. Let $AC, BC \le \pi/(2\kappa)$. Then $CM \le \pi/(2\kappa)$. In addition, if either AC or BC is less than $\pi/(2\kappa)$, then $CM < \pi/(2\kappa)$.*

Finally, we show that cosq_K remains the same in the half-sphere after cutting the lengths of bound vectors in half.

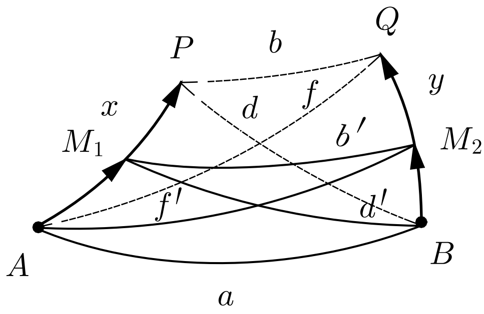

Figure 4 Sketch for Lemma 3.2

LEMMA 3.2. *Let $K > 0$, and let $\overrightarrow{AP}$, $\overrightarrow{BQ}$ be a pair of nonzero bound vectors in $\mathbb{S}^3_K$. Let M_1 and M_2 be the midpoints of the shortests $\mathcal{AP}$ and $\mathcal{BQ}$, respectively. Then*

$$\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = \operatorname{cosq}_K(\overrightarrow{AM_1}, \overrightarrow{BM_2}).$$

Proof. We have:

$$\operatorname{cosq}_K(\overrightarrow{AM_1}, \overrightarrow{BM_2}) = \frac{\cos\kappa b' + \cos\frac{\kappa x}{2}\cos\frac{\kappa y}{2}}{\sin\frac{\kappa x}{2}\sin\frac{\kappa y}{2}} - \frac{(\cos\frac{\kappa x}{2} + \cos\kappa d')(\cos\frac{\kappa y}{2} + \cos\kappa f')}{(1+\cos\kappa a)\sin\frac{\kappa x}{2}\sin\frac{\kappa y}{2}},$$

where the notation is given in Figure 4. By the Bruhat–Tits equality (Lemma 3.1),

$$\cos\kappa d' = \frac{\cos\kappa a + \cos\kappa d}{2\cos\frac{\kappa x}{2}} \text{ (triangle } ABP),$$

$$\cos\kappa f' = \frac{\cos\kappa a + \cos\kappa f}{2\cos\frac{\kappa y}{2}} \text{ (triangle } ABQ),$$

$$\cos\kappa g = \frac{\cos\kappa b + \cos\kappa d}{2\cos\frac{\kappa y}{2}}, \quad g = PM_2 \text{ (triangle } PQB),$$

$$\cos\kappa b' = \frac{\cos\kappa g + \cos\kappa f'}{2\cos\frac{\kappa x}{2}} \text{ (triangle } APM_2),$$

whence $\cos\kappa b' = (\cos\kappa a + \cos\kappa b + \cos\kappa d + \cos\kappa f)/(4\cos\frac{\kappa x}{2}\cos\frac{\kappa y}{2})$. Hence,

$$\begin{aligned}
&\operatorname{cosq}_K(\overrightarrow{AM_1}, \overrightarrow{BM_2}) \\
&\quad = (1+\cos\kappa a)\Big[\cos\kappa a + \cos\kappa b + \cos\kappa d + \cos\kappa f \\
&\qquad + 4\cos^2\frac{\kappa x}{2}\cos^2\frac{\kappa y}{2}\Big] - \left(2\cos^2\frac{\kappa x}{2} + \cos\kappa a + \cos\kappa d\right) \\
&\qquad \times \left(2\cos^2\frac{\kappa y}{2} + \cos\kappa a + \cos\kappa f\right)[(1+\cos\kappa a)\sin\kappa x \sin\kappa y]^{-1}
\end{aligned}$$

$$= [(1 + \cos\kappa a)(\cos\kappa a + \cos\kappa b + \cos\kappa d + \cos\kappa f + 1 + \cos\kappa x + \cos\kappa y + \cos\kappa x \cos\kappa y) - (1 + \cos\kappa x + \cos\kappa a + \cos\kappa d) \times (1 + \cos\kappa y + \cos\kappa a + \cos\kappa f)]/[(1 + \cos\kappa a)\sin\kappa x \sin\kappa y].$$

After elementary but tedious simplifications of the last expression, we get:

$$\begin{aligned}\operatorname{cosq}_K(\overrightarrow{AM_1}, \overrightarrow{BM_2}) &= [(1 + \cos\kappa a)\sin\kappa x \sin\kappa y]^{-1} \\ &\quad \times (\cos\kappa b + \cos\kappa a \cos\kappa b + \cos\kappa a \cos\kappa x \cos\kappa y \\ &\quad - \cos\kappa x \cos\kappa f - \cos\kappa y \cos\kappa d - \cos\kappa d \cos\kappa f) \\ &= \operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}),\end{aligned}$$

as needed. □

Let $K > 0$. Recall that all distances in $\mathbb{S}^3_K$ are less than $\pi/\sqrt{K}$. By Lemma 3.2, there is no restriction in assuming that AP and BQ are as small as we wish. Hence, without loss of generality, we can assume that $PO < \pi/(2\sqrt{K})$ (see Figure 2). So, we get the following:

COROLLARY 3.3. *Let* $K \neq 0$. *Then for every pair of nonzero bound vectors* $\overrightarrow{AP}$ *and* $\overrightarrow{BQ}$ *in* $\mathbb{S}^3_K$, $|\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})| \leq 1$.

4. K-Quadrilateral Cosine in an $\Re_K$ Domain

The main goal of this section is to show that $|\operatorname{cosq}_K| \leq 1$ in an $\Re_K$ domain of diameter not greater than $\pi/2\sqrt{K}$ if $K > 0$. In addition, for $K > 0$, we present examples of $\Re_K$ domains of diameter greater than $\pi/(2\sqrt{K})$ and arbitrarily close to $\pi/(2\sqrt{K})$ for which $|\operatorname{cosq}_K| \leq 1$ does not hold.

The following theorem is a minor generalization of Theorem 4.2 in [7].

THEOREM 4.1. *Let* $K \neq 0$, *and let* $\mathfrak{Q} = \{A, P, B, Q\}$ *be a quadruple of points in an* $\Re_K$ *domain such that* $A \neq P$, $B \neq Q$, *and* $\operatorname{diam}(\mathfrak{Q}) \leq \pi/(2\sqrt{K})$ *if* $K > 0$. *Then*

$$|\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})| \leq 1.$$

Proof. It is sufficient to consider the case of positive K. If $\operatorname{diam}(\mathfrak{Q}) < \pi/(2\sqrt{K})$, then by [7, Thm. 4.2], $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) \geq -1$. For the reader's convenience, we include some omitted details in [7] of the proof of the inequality $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) \leq 1$. Consider the closed polygonal curve $\mathcal{L} = \mathcal{APQBA}$, as shown in Figure 1. We will follow the part of the proof of Reshetnyak's Lemma 2 in [21] corresponding to the case of K-fans consisting of two triangles in $\mathbb{S}_K$ (a particular case of Reshetnyak's majorization theorem). Namely, under the hypothesis of Theorem 4.1, in addition to the existence of a convex domain $\mathcal{V} \subseteq \mathbb{S}_K$ majorizing

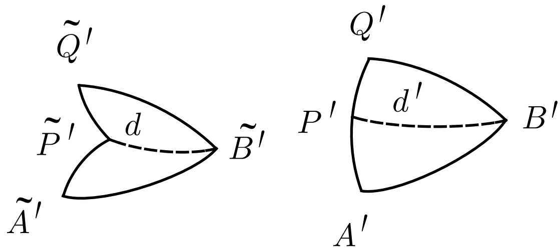

Figure 5 Sketch for Theorem 4.1

the polygonal curve $\mathcal{L}$, Reshetnyak's proof also implies that the domain $\mathcal{V}$ can be selected so that

$$\begin{aligned} d &= PB \le d' = P'B' < \pi/\big(2\sqrt{K}\big), \\ f &= AQ \le f' = A'Q' < \pi/\sqrt{K}, \end{aligned} \tag{4.1}$$

where $\mathcal{L}' = \mathcal{A}'\mathcal{P}'\mathcal{Q}'\mathcal{B}'\mathcal{A}'$ is the bounding curve of $\mathcal{V}$. Indeed, as shown in the proof of Lemma 2 in [21], there is a quadrangular domain $\mathcal{F}$ in $\mathbb{S}_K$ bounded by a quadrangle $\widetilde{\mathcal{L}}' = \widetilde{A}'\widetilde{P}'\widetilde{Q}'\widetilde{B}'$ such that

$$\begin{aligned} &AP = \widetilde{A}'\widetilde{P}', \qquad AB = \widetilde{A}'\widetilde{B}', \qquad PB = \widetilde{P}'\widetilde{B}' \quad \text{and} \\ &PQ = \widetilde{P}'\widetilde{Q}', \qquad BQ = \widetilde{B}'\widetilde{Q}'. \end{aligned}$$

If $\mathcal{F}$ is convex, then we put $\mathcal{F} = \mathcal{V}$, and we have $d = \widetilde{d}' = \widetilde{P}'\widetilde{B}' < \pi/(2\sqrt{K})$ and (as shown in Reshetnyak's proof) $f = AQ \le \widetilde{f}' = \widetilde{A}'\widetilde{Q}' < \pi/\sqrt{K}$. Now suppose that the quadrangular domain $\mathcal{F}$ is not convex. Then either the angle of the quadrangle $\widetilde{\mathcal{L}}'$ at its vertex $\widetilde{P}'$ is greater than π, or the angle at its vertex $\widetilde{B}'$ is greater than π. For definiteness, suppose that the angle of $\widetilde{\mathcal{L}}'$ at $\widetilde{P}'$ is greater than π, as shown in Figure 5. Let $\mathcal{V} \subseteq \mathbb{S}_K$ be the domain bounded by the triangle $A'Q'B'$ obtained from the polygonal curve $\widetilde{\mathcal{L}}'$ by rectifying the arc $\widetilde{A}'\widetilde{P}'\widetilde{Q}'$. Then by [3, Section 3, Lemma 2], $d = \widetilde{d}' < d' = P'B'$ and $f \le A'Q' < \pi/\sqrt{K}$ (as shown in Reshetnyak's proof). By Corollary 3.2, $d' < \pi/(2\sqrt{K})$; so, inequalities (4.1) hold.

By (4.1) and because $\operatorname{diam}(\Re_K) < \pi/(2\sqrt{K})$, we see that the difference of the products

$$\begin{aligned} &(\cos\kappa x + \cos\kappa d)(\cos\kappa y + \cos\kappa f) - (\cos\kappa x + \cos\kappa d')(\cos\kappa y + \cos\kappa f') \\ &\quad = \cos\kappa x(\cos\kappa f - \cos\kappa f') + \cos\kappa y(\cos\kappa d - \cos\kappa d') \\ &\qquad + \cos\kappa f(\cos\kappa d - \cos\kappa d') + \cos\kappa d'(\cos\kappa f - \cos\kappa f') \end{aligned}$$

is nonnegative. So, the inequality $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) \le \operatorname{cosq}_K(\overrightarrow{A'P'}, \overrightarrow{B'Q'})$ follows. By Corollary 3.3,

$$\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) \le 1,$$

as needed.

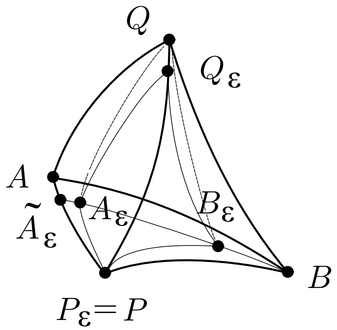

Figure 6 diam$(\mathfrak{Q}) = \pi/(2\sqrt{K})$

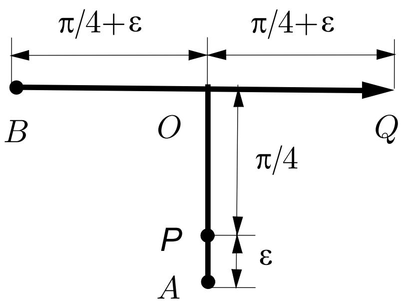

Figure 7 Sketch for Example 4.1

Now, we consider the case where diam$(\mathfrak{Q}) = \pi/(2\sqrt{K})$. If $\varepsilon > 0$ is sufficiently small, then by invoking Corollary 3.2, it is not difficult to select points A_ε, P_ε, B_ε, and Q_ε in $\Re_K$ such that the distances AA_ε, PP_ε, BB_ε, and QQ_ε do not exceed ε and such that diam$\{A_\varepsilon, P_\varepsilon, B_\varepsilon, Q_\varepsilon\} < \pi/(2\kappa)$. One of such configurations is shown in Figure 6. From the first part of the proof, we see that $|\operatorname{cosq}_K(\overrightarrow{A_\varepsilon P}, \overrightarrow{B_\varepsilon Q_\varepsilon})| \le 1$ for every small positive ε. Hence, by passing to the limit as $\varepsilon \to 0+$, we get $|\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})| \le 1$, as claimed. □

The following example shows that for positive K, the restriction on the diameter of $\Re_K$ cannot be dropped and the diameter bound in Theorem 4.1 is sharp. For simplicity, we consider $K = 1$.

EXAMPLE 4.1. Let $\varepsilon > 0$. Consider the T-shaped graph $(\mathcal{M}_\varepsilon, \rho_\varepsilon)$ obtained by gluing a segment of straight line AO of length $\pi/4 + \varepsilon$ to the middle O of another segment of straight line BQ of length $\pi/2 + 2\varepsilon$, as shown in Figure 7. It is readily seen that $(\mathcal{M}_\varepsilon, \rho_\varepsilon)$ is an $\Re_0$ domain and that the perimeter of every triangle in $(\mathcal{M}_\varepsilon, \rho_\varepsilon)$ is less than 2π for small positive ε. Hence, $(\mathcal{M}_\varepsilon, \rho_\varepsilon)$ is also an $\Re_1$ domain. Notice that diam$(\mathcal{M}_\varepsilon) = \pi/2 + 2\varepsilon$. Let $P \in \mathcal{AO}\backslash\{A, O\}$ be such that $AP = \varepsilon$.

(a)

$$\begin{aligned}\operatorname{cosq}_1(\overrightarrow{BQ}, \overrightarrow{AP}) &= \frac{\cos(\frac{\pi}{2}+\varepsilon)+\cos(\frac{\pi}{2}+2\varepsilon)\cos\varepsilon}{\sin(\frac{\pi}{2}+2\varepsilon)\sin\varepsilon} \\ &\quad - \frac{2\cos(\frac{\pi}{2}+2\varepsilon)[\cos\varepsilon+\cos(\frac{\pi}{2}+\varepsilon)]}{[1+\cos(\frac{\pi}{2}+2\varepsilon)]\sin(\frac{\pi}{2}+2\varepsilon)\sin\varepsilon} \\ &= \frac{1+\sin 2\varepsilon}{1-\sin 2\varepsilon} > 1\end{aligned}$$

for every $\varepsilon \in (0, \pi/4)$ and therefore for small positive ε.

(b) In a similar way,

$$\begin{aligned}\operatorname{cosq}_1(\overrightarrow{BQ}, \overrightarrow{PA}) &= \frac{\cos(\frac{\pi}{2}+2\varepsilon)+\cos(\frac{\pi}{2}+2\varepsilon)\cos\varepsilon}{\sin(\frac{\pi}{2}+2\varepsilon)\sin\varepsilon} \\ &\quad - \frac{[\cos(\frac{\pi}{2}+2\varepsilon)+\cos(\frac{\pi}{2}+\varepsilon)][\cos\varepsilon+\cos(\frac{\pi}{2}+2\varepsilon)]}{[1+\cos(\frac{\pi}{2}+\varepsilon)]\sin(\frac{\pi}{2}+2\varepsilon)\sin\varepsilon} \\ &= -\frac{(1+\sin 2\varepsilon)\cos\varepsilon}{(1-\sin\varepsilon)\cos 2\varepsilon} < -1\end{aligned}$$

for every $\varepsilon \in (0, \pi/4)$ and therefore for small positive ε.

So, for small positive ε, the metric space $(\mathcal{M}_\varepsilon, \rho_\varepsilon)$ is an $\Re_1$ domain, the diameter of $(\mathcal{M}_\varepsilon, \rho_\varepsilon)$ is greater than $\pi/2$, and

$$\lim_{\varepsilon\to 0+} \operatorname{diam}(\mathcal{M}_\varepsilon) = \pi/2,$$

whereas cosq_1 takes values greater than 1 and less than -1.

We note that Example 4.1 can be modified so that for any $K < 1$ and $\varepsilon > 0$, we can find a construction on a space of curvature between K and 1 with diameter less than $\frac{\pi}{2} + \varepsilon$ yielding $|\operatorname{cosq}_1| > 1$.

5. Testing cosq_K. Counterexamples

We begin with the discussion of testing a metric space for the one-sided four-point cosq_K condition. We present counterexamples showing that in general the upper four-point cosq_K condition is different from the lower four-point cosq_K condition.

Let $K \in \mathbb{R}$, and let $\mathfrak{Q} = \{A, P, B, Q\}$ be a quadruple of distinct points in a metric space $(\mathcal{M}, \rho)$ such that the perimeter of every triple $\{A, B, C\}$ in $\mathfrak{Q}$ is less than $2\pi/\sqrt{K}$ when $K > 0$. For every triple $X, Y, Z \in \mathfrak{Q}$, the absolute value of the K-quadrilateral cosine between any pair of nonzero bound vectors with heads and tails in the triple $\{X, Y, Z\}$ always does not exceed one. Indeed, each such triple can be embedded isometrically into $\mathbb{S}_K$; hence, by Corollary 3.3, $|\operatorname{cosq}_K|$ does not exceed 1 for every pair of such bound vectors. So, by recalling that cosq_K is symmetric, we need consider only the following 12 main cases given in Table 1, where the two nonzero bound vectors have no point in common.

Table 1 Twelve main cases

Case	I	II	III	IV	V	VI
cosq_1	$\overrightarrow{AP}, \overrightarrow{BQ}$	$\overrightarrow{AP}, \overrightarrow{QB}$	$\overrightarrow{AB}, \overrightarrow{PQ}$	$\overrightarrow{AB}, \overrightarrow{QP}$	$\overrightarrow{AQ}, \overrightarrow{PB}$	$\overrightarrow{AQ}, \overrightarrow{BP}$

Case	VII	VIII	IX	X	XI	XII
cosq_1	$\overrightarrow{PA}, \overrightarrow{BQ}$	$\overrightarrow{PA}, \overrightarrow{QB}$	$\overrightarrow{PB}, \overrightarrow{QA}$	$\overrightarrow{PQ}, \overrightarrow{BA}$	$\overrightarrow{BA}, \overrightarrow{QP}$	$\overrightarrow{BP}, \overrightarrow{QA}$

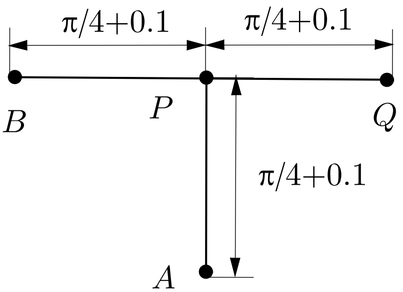

Figure 8 Sketch for Example 5.1, part (a)

The following examples show that the upper and lower four-point cosq_K conditions are not equivalent for nonzero K. For simplicity, we consider $K = \pm 1$. Adjustment for arbitrary nonzero K is straightforward.

EXAMPLE 5.1 ($K = 1$). (a) **The lower four-point** cosq_1 **condition holds, whereas the upper four-point** cosq_1 **condition fails.** Consider the T-shaped graph obtained by gluing a segment of straight line AP of length $\pi/4 + 0.1$ to the middle P of another segment of straight line BQ of length $\pi/2 + 0.2$, as shown in Figure 8. Let $\mathcal{M} = \{A, P, B, Q\}$ with the induced metric ρ. All 12 main (approximate) values of cosq_1 for the four-point metric space $(\mathcal{M}, \rho)$ are given in Table 2.

(b) **The upper four-point** cosq_1 **condition holds, whereas the lower four-point** cosq_1 **condition fails.** Consider the quadruple $\mathfrak{Q} = \{A, P, B, Q\}$ in $\mathbb{S}_1$ with the metric $\rho_{\mathbb{S}_1}$ such that the point P is symmetric to the point Q w.r.t. the midpoint of the shortest $\mathcal{AB}$. All six distances between the pairs of points of $\mathfrak{Q}$ are shown in Figure 9 with $\varepsilon = 0$. Then $\text{cosq}_1(\overrightarrow{AP}, \overrightarrow{BQ}) = -1$. Now we change the metric $\rho_{\mathbb{S}_1}$ by increasing the distance between P and Q by a positive ε and leaving all other distances the same. If ε is sufficiently small, then the new distance ρ_ε is a metric. For $\varepsilon = 0.1$, all 12 main (approximate) values of cosq_1 for the four-point metric space $(\mathfrak{Q}, \rho_{0.1})$ are given in Table 3.

Table 2 Example 5.1, part (a)

Case	I	II	III	IV	V	VI
cosq_1	1.496	1.496	−0.58	1.496	−0.58	1.496

Case	VII	VIII	IX	X	XI	XII
cosq_1	−0.58	−0.58	−0.58	−0.58	1.496	1.496

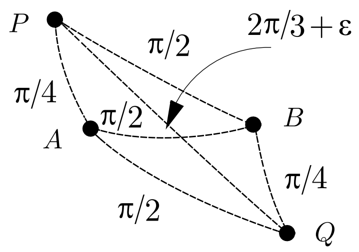

Figure 9 Sketch for Example 5.1, part (b)

Table 3 Example 5.1, part (b)

Case	I	II	III	IV	V	VI
cosq_1	−1.168	0.826	0.871	−0.107	0.707	−1.084

Case	VII	VIII	IX	X	XI	XII
cosq_1	0.826	−1.404	−1.202	−0.107	0.871	0.707

EXAMPLE 5.2 ($K = -1$). We use the same approach to construction of counterexamples for $K = -1$ as in part (b) of Example 5.1. Let $\mathfrak{Q} = \{A, P, B, Q\}$ be a four-element set.

(a) **The lower four-point** cosq_{-1} **condition holds, whereas the upper four-point** cosq_{-1} **condition fails.** The six (symmetric) distances between the pairs of points in $\mathfrak{Q}$ are given by

$$\rho(A, P) = \rho(B, Q) = 1, \qquad \rho(A, B) = 2,$$
$$\rho(P, Q) = 2.697, \quad \text{and} \quad \rho(A, Q) = \rho(B, P) = 2.44.$$

All 12 main (approximate) values of cosq_{-1} for the four-point metric space $(\mathfrak{Q}, \rho)$ are given in Table 4.

Table 4 Example 5.2, part (a)

Case	I	II	III	IV	V	VI
$\cos q_{-1}$	1.0347	−0.8133	0.7495	−0.9998	0.4534	−0.9133

Case	VII	VIII	IX	X	XI	XII
$\cos q_{-1}$	−0.8133	0.1465	−0.9511	−0.9998	0.7495	0.4534

Table 5 Example 5.2, part (b)

Case	I	II	III	IV	V	VI
$\cos q_{-1}$	−1.184	0.922	0.522	−0.944	0.807	−1.008

Case	VII	VIII	IX	X	XI	XII
$\cos q_{-1}$	0.922	−1.077	−1.003	−0.944	0.522	0.807

(b) **The upper four-point** cosq_{-1} **condition holds, whereas the lower four-point** cosq_{-1} **condition fails.** The six distances between the pairs of points in $\mathfrak{Q}$ are given by

$$\rho(A, P) = \rho(B, Q) = 1, \qquad \rho(A, B) = 2,$$
$$\rho(P, Q) = 3.027, \quad \text{and} \quad \rho(A, Q) = \rho(B, P) = 2.43.$$

All 12 main (approximate) values of cosq_{-1} for the four-point metric space $(\mathfrak{Q}, \rho)$ are given in Table 5.

6. Proof of Theorem 1.1

6.1. Sketch of the Proof

Let $(\mathcal{M}, \rho)$ be a geodesically connected metric space (of diameter not greater than $\pi/(2\sqrt{K})$ for positive K) satisfying the one-sided four-point cosq_K condition for nonzero K. Theorem 1.1 is proved once we establish the angle comparison: for every geodesic triangle $\mathcal{T} = ABC$ in $(\mathcal{M}, \rho)$, $\measuredangle ABC \leq \measuredangle_K ABC$, $\measuredangle BAC \leq \measuredangle_K BAC$, and $\measuredangle ACB \leq \measuredangle_K ACB$. We begin by proving Lemma 6.1 stating that shortests in $(\mathcal{M}, \rho)$ depend continuously on their end points. One of Aleksandrov's theorems and Lemma 6.1 enable us to reduce the derivation of the global angle comparison estimate to the proof of the local angle comparison. The cross-diagonal estimate lemma (Lemma 6.2) is one of the main steps in the proof of the major growth estimate lemma (Lemma 6.3). Both of these estimates are

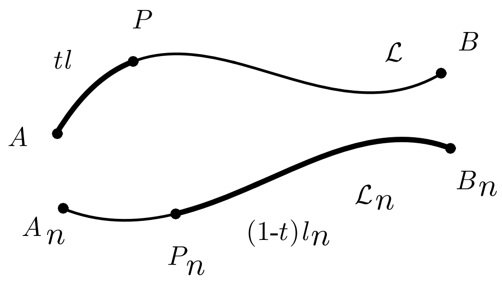

Figure 10 Sketch for Lemma 6.1

derived from the one-sided four-point cosq_K condition. We employ the growth estimate to prove the "almost monotonicity" of the angles $\alpha_0(t)$ (Corollary 6.3) and existence of proportional angles (Corollary 6.4), an important auxiliary step in proving the existence of Aleksandrov angles (Proposition 1). Now we have all necessary means needed for derivation of the local angle comparison inequality. We begin with the identity corresponding to the growth estimate in $\mathbb{S}_K$ (Proposition 2). We consider a sufficiently small geodesic triangle $\mathcal{T} = ABC$ in $(\mathcal{M}, \rho)$. The existence of Aleksandrov angles gives us the freedom of selecting the points in shortests $\mathcal{AB}$ and $\mathcal{AC}$ respectively approaching to the vertex A in a special way. For every small positive t, we select $\widehat{X}_t \in \mathcal{AB}$ and $\widehat{Y}_t \in \mathcal{AC}$, $\widehat{X}_t, \widehat{Y}_t \to A$ as $t \to 0+$ (see Section 6.7), so that $\measuredangle A^K \widehat{X}_t^K \widehat{Y}_t^K$ and $\measuredangle \widehat{X}_t^K A^K \widehat{Y}_t^K$ converge as $t \to 0+$ (Lemma 6.5). Hence, it is possible to pass to the limit in the growth estimate as $t \to 0+$. The limit form of the growth estimate and the identity of Proposition 2 enable us to derive the local angle comparison estimate (Proposition 3).

6.2. Continuity and Uniqueness of Shortests

The main result of this section is the following:

LEMMA 6.1. *Let* $K \neq 0$, *and let* $(\mathcal{M}, \rho)$ *be a metric space such that* $\operatorname{diam}(\mathcal{M}) \leq \pi/(2\sqrt{K})$ *when* $K > 0$. *Let* $\mathcal{L} = \mathcal{AB}$ *be a shortest, and let* $(\mathcal{L}_n = \mathcal{A}_n\mathcal{B}_n)_{n=1}^{\infty}$ *be a sequence of shortests in* $(\mathcal{M}, \rho)$ *such that* $\lim_{n\to\infty} A_n = A$ *and* $\lim_{n\to\infty} B_n = B$. *Let* $\mathfrak{g}_r$ *be the reduced parameterization of* $\mathcal{L}$ *relative to* A, *and let* $\mathfrak{g}_{r,n}$ *be the reduced parameterization of* $\mathcal{L}_n$ *relative to* A_n, $n = 1, 2, \ldots$ (*see Section* 2). *If* $(\mathcal{M}, \rho)$ *satisfies the one-sided four-point* cosq_K *condition, then the sequence* $(\mathfrak{g}_{r,n})_{n=1}^{\infty}$ *converges uniformly to* $\mathfrak{g}_r$ *on the closed interval* $[0, 1]$.

Proof. Let $\mathcal{L} = \mathcal{AB}$ and $\mathcal{L}_n = \mathcal{A}_n\mathcal{B}_n$, $n = 1, 2, \ldots$. We can assume that $l = \ell_\rho(\mathcal{L}) > 0$ and $l_n = \ell_\rho(\mathcal{L}_n) > 0$ for every n. For $t \in (0, 1)$, set $P = \mathfrak{g}_r(t)$, $P_n = \mathfrak{g}_{r,n}(t)$, and $\overline{\delta} = \overline{\lim}_{n\to\infty} PP_n$, see Figure 10.

I. Let $(\mathcal{M}, \rho)$ **satisfy the upper four point** cosq_K **condition.** Consider the nonzero bound vectors $\overrightarrow{AP}$ and $\overrightarrow{P_nB_n}$. By the upper four-point cosq_K condition,

$$\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{P_nB_n}) = \frac{\cos\widehat{\kappa}PB_n + \cos\widehat{\kappa}AP\cos\widehat{\kappa}P_nB_n}{\sin\widehat{\kappa}AP\sin\widehat{\kappa}P_nB_n} - \frac{(\cos\widehat{\kappa}AP + \cos\widehat{\kappa}PP_n)(\cos\widehat{\kappa}P_nB_n + \cos\widehat{\kappa}B_nA)}{(1+\cos\widehat{\kappa}AP_n)\sin\widehat{\kappa}AP\sin\widehat{\kappa}P_nB_n}$$

does not exceed 1. By letting $n \to \infty$, we get

$$\begin{aligned}
&\overline{\lim}_{n\to\infty}\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{P_nB_n}) \\
&\quad= \frac{\cos\widehat{\kappa}(1-t)l + \cos\widehat{\kappa}tl\cos\widehat{\kappa}(1-t)l}{\sin\widehat{\kappa}tl\sin\widehat{\kappa}(1-t)l} \\
&\qquad - \frac{(\cos\widehat{\kappa}tl + \cos\widehat{\kappa}\overline{\delta})[\cos\widehat{\kappa}(1-t)l + \cos\widehat{\kappa}l]}{(1+\cos\widehat{\kappa}tl)\sin\widehat{\kappa}tl\sin\widehat{\kappa}(1-t)l} \\
&\quad= 1 + \frac{\cos\widehat{\kappa}(1-t)l + \cos\widehat{\kappa}l}{\sin\widehat{\kappa}tl\sin\widehat{\kappa}(1-t)l} - \frac{(\cos\widehat{\kappa}tl + \cos\widehat{\kappa}\overline{\delta})[\cos\widehat{\kappa}(1-t)l + \cos\widehat{\kappa}l]}{(1+\cos\widehat{\kappa}tl)\sin\widehat{\kappa}tl\sin\widehat{\kappa}(1-t)l} \\
&\quad= 1 + \frac{[1-\cos(\widehat{\kappa}\overline{\delta})][\cos(\widehat{\kappa}(1-t)l) + \cos(\widehat{\kappa}l)]}{[1+\cos(\widehat{\kappa}tl)]\sin(\widehat{\kappa}tl)\sin(\widehat{\kappa}(1-t)l)} \leq 1. \qquad (6.1)
\end{aligned}$$

If $K > 0$, then

$$\frac{(1-\cos\kappa\overline{\delta})[\cos\kappa(1-t)l + \cos\kappa l]}{(1+\cos\kappa tl)\sin\kappa tl\sin\kappa(1-t)l} \leq 0.$$

Because $\operatorname{diam}(\mathcal{M}) \leq \pi/(2\kappa)$, $\overline{\delta} = 0$ follows.

If $K < 0$, then

$$\frac{(\cosh\kappa\overline{\delta} - 1)[\cosh\kappa(1-t)l + \cosh\kappa l]}{(1+\cosh\kappa tl)\sinh\kappa tl\sinh\kappa(1-t)l} \leq 0,$$

whence $\overline{\delta} = 0$ follows.

II. Let $(\mathcal{M}, \rho)$ **satisfy the lower four-point** cosq_K **condition.** In a manner similar to I, we get

$$\begin{aligned}
&\underline{\lim}_{n\to\infty}\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{B_nP_n}) \\
&\quad= \frac{\cos\widehat{\kappa}\overline{\delta} + \cos\widehat{\kappa}tl\cos\widehat{\kappa}(1-t)l}{\sin\widehat{\kappa}tl\sin\widehat{\kappa}(1-t)l} - \frac{[\cos\widehat{\kappa}tl + \cos\widehat{\kappa}(1-t)l]^2}{(1+\cos\widehat{\kappa}l)\sin\widehat{\kappa}tl\sin\widehat{\kappa}(1-t)l} \geq -1,
\end{aligned}$$

whence

$$\frac{[\cos\widehat{\kappa}\overline{\delta} + \cos\widehat{\kappa}(1-2t)l](1+\cos\widehat{\kappa}l)}{(1+\cos\widehat{\kappa}l)\sin\widehat{\kappa}tl\sin\widehat{\kappa}(1-t)l} - \frac{[\cos\widehat{\kappa}tl + \cos\widehat{\kappa}(1-t)l]^2}{(1+\cos\widehat{\kappa}l)\sin\widehat{\kappa}tl\sin\widehat{\kappa}(1-t)l}$$

is nonnegative. Notice that

$$\begin{aligned}
[\cos\widehat{\kappa}tl + \cos\widehat{\kappa}(1-t)l]^2 &= 4\cos^2\frac{\widehat{\kappa}l}{2}\cos^2\frac{\widehat{\kappa}(1-2t)l}{2} \\
&= (1+\cos\widehat{\kappa}l)(1+\cos\widehat{\kappa}(1-2t)l).
\end{aligned}$$

Hence,

$$[\cos\widehat{\kappa}\overline{\delta}+\cos\widehat{\kappa}(1-2t)l](1+\cos\widehat{\kappa}l)-[\cos\widehat{\kappa}tl+\cos\widehat{\kappa}(1-t)l]^2$$
$$=(\cos\widehat{\kappa}\overline{\delta}-1)(1+\cos\widehat{\kappa}l).$$

So,

$$\varliminf_{n\to\infty}\operatorname{cosq}_K(\overrightarrow{AP},\overrightarrow{B_nP_n})+1=\frac{\cos\widehat{\kappa}\overline{\delta}-1}{\sin\widehat{\kappa}tl\sin\widehat{\kappa}(1-t)l}\geq 0.$$

If $K>0$, then

$$\frac{\cos\kappa\overline{\delta}-1}{\sin\kappa tl\sin\kappa(1-t)l}\geq 0,$$

whence $\overline{\delta}=0$ follows.

If $K<0$, then

$$\frac{\cosh\widehat{\kappa}\overline{\delta}-1}{\sinh\kappa tl\sinh\kappa(1-t)l}\leq 0,$$

whence $\overline{\delta}=0$.

By I and II, $\mathfrak{g}_{r,n}(t)$ converges pointwise to $\mathfrak{g}_r(t)$ for every $t\in[0,1]$ as $n\to\infty$. It is not difficult to see that the sequence $(\mathfrak{g}_{r,n})_{n=1}^{\infty}$ also converges uniformly to $\mathfrak{g}_r$ on the closed interval $[0,1]$.

The proof of Lemma 6.1 is complete. □

COROLLARY 6.1. *Let $K\neq 0$, and let $(\mathcal{M},\rho)$ be a metric space such that* $\operatorname{diam}(\mathcal{M})$ *is not greater than $\pi/(2\sqrt{K})$ when $K>0$. If $(\mathcal{M},\rho)$ satisfies the one-sided four-point* cosq_K *condition, then every pair of points in $\mathcal{M}$ can be joined by at most one shortest.*

6.3. Cross-Diagonal Estimate Lemma

Let $(\mathcal{M},\rho)$ be a metric space. Let A, B, C be three distinct points in $\mathcal{M}$, $0<\underline{m}\leq\overline{m}<+\infty$, and $s,t\in(0,1]$ satisfying the following conditions:

M1. The points A and B can be joined by a shortest $\mathcal{L}$, and the points A and C can be joined by a shortest $\mathcal{N}$.
M2. If $K>0$, then $AB, AC\leq\pi/(2\sqrt{K})$.
M3. $\underline{m}\leq s/t\leq\overline{m}$.

From now on, we will use the following notation:

$$X_s=\mathfrak{g}_{r,\mathcal{L}}(s),\qquad Y_t=\mathfrak{g}_{r,\mathcal{N}}(t),\quad s,t\in(0,1].$$
$$x=AB,\qquad y=AC,\qquad z=BC,\qquad d_{s,t}=BY_t,$$
$$f_{s,t}=CX_s,\qquad z_{s,t}=X_sY_t,$$

as illustrated in Figure 11, and we put $\lambda=\max\{x,y\}$, $\eta=x/y$, and $\xi=\lambda\max\{s,t\}$. Also, for $K\in\mathbb{R}$, set

$$\alpha_K(s,t)=\measuredangle_K X_sAY_t,\qquad \beta_K(s,t)=\measuredangle_K AX_sY_t,\qquad \gamma_K(s,t)=\measuredangle_K AY_tX_s.$$

If $p>0$, then we write $\varphi(s,t)=\mathcal{O}(\xi^p)$ when there is a constant $C>0$ such that $|\varphi(s,t)|\leq C\xi^p$ for sufficiently small s and t. If C is a constant depending

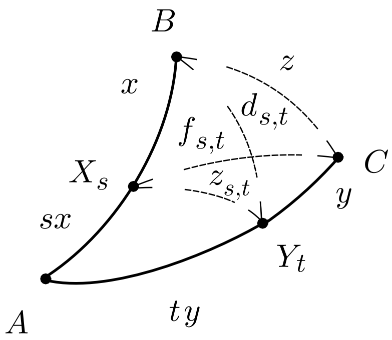

Figure 11 Sketch for the cross-diagonal lemma

on $M_1, M_2, \ldots, M_k$, that is, $C = C(M_1, M_2, \ldots, M_k)$, then we write $\varphi(s,t) = \mathcal{O}_{M_1,M_2,\ldots,M_k}(\xi^p)$.

If $\mathcal{T} = ABC$ is a triangle in $\mathbb{S}_1$, then $f_{s,t}$ is not less than the length of the orthogonal projection of the shortest $\mathcal{X}_s\mathcal{C}$ onto the shortest $\mathcal{AC}$. So, if O_s is the orthogonal projection of the point X_s onto the shortest $\mathcal{AC}$, then $f_{s,t} \geq y - AO_s$. It is not difficult to see that AO_s approximately equals $(sx)\cos\alpha_0(s,t)$. Hence, approximately, $f_{s,t}$ is bounded below by $y - (sx)\cos\alpha_0(s,t)$. The following lemma states a similar estimate for a triangle $\mathcal{T} = ABC$ in a metric space satisfying the one-sided four-point cosq_K condition.

LEMMA 6.2. *Let $K \neq 0$ and $0 < \underline{m} \leq \overline{m} < +\infty$. Let A, B, C be three distinct points in a metric space $(\mathcal{M}, \rho)$ and $s, t \in (0,1]$ satisfying M1–M3. Suppose that $(\mathcal{M}, \rho)$ satisfies the one-sided four-point* cosq_K *condition.*

(i) *If $K > 0$, then*

$$\cos\kappa f_{s,t} \leq \cos\kappa y + \kappa(sx)\sin\kappa y\cos\alpha_0(s,t) + \mathcal{O}(\xi^2),$$
$$\cos\kappa d_{s,t} \leq \cos\kappa x + \kappa(ty)\sin\kappa x\cos\alpha_0(s,t) + \mathcal{O}(\xi^2).$$

(ii) *If $K < 0$, then*

$$\cosh\kappa f_{s,t} \geq \cosh\kappa y - \kappa(sx)\sinh\kappa y\cos\alpha_0(s,t) + \mathcal{O}(\xi^2),$$
$$\cosh\kappa d_{s,t} \geq \cosh\kappa x - \kappa(ty)\sinh\kappa x\cos\alpha_0(s,t) + \mathcal{O}(\xi^2),$$

where $\mathcal{O}(\xi^2) = \mathcal{O}_{\lambda,\eta,\underline{m},\overline{m},K}(\xi^2)$.

Proof. **I. Let $(\mathcal{M}, \rho)$ satisfy the upper four-point** cosq_K **condition**. For brevity, set $h_{s,t} = \mathrm{cosq}_K(\overrightarrow{X_sC}, \overrightarrow{AY_t})$. Then

$$h_{s,t} = \frac{\cos\widehat{\kappa}(1-t)y + \cos\widehat{\kappa} f_{s,t}\cos\widehat{\kappa}ty}{\sin\widehat{\kappa} f_{s,t}\sin\widehat{\kappa}ty} - \frac{(\cos\widehat{\kappa} f_{s,t} + \cos\widehat{\kappa}y)(\cos\widehat{\kappa}ty + \cos\widehat{\kappa}z_{s,t})}{(1+\cos\widehat{\kappa}sx)\sin\widehat{\kappa} f_{s,t}\sin\widehat{\kappa}ty}$$

$$= \frac{\cos\widehat{\kappa}y + \widehat{\kappa}ty\sin\widehat{\kappa}y - \frac{1}{2}\widehat{\kappa}^2(ty)^2\cos\widehat{\kappa}y + \mathcal{O}(\xi^3)}{\sin\widehat{\kappa}f_{s,t}\sin\widehat{\kappa}ty}$$
$$+ \frac{\cos\widehat{\kappa}f_{s,t}[1 - \frac{1}{2}\widehat{\kappa}^2(ty)^2 + \mathcal{O}(\xi^4)]}{\sin\widehat{\kappa}f_{s,t}\sin\widehat{\kappa}ty}$$
$$- \frac{\cos\widehat{\kappa}f_{s,t} + \cos\widehat{\kappa}y}{\sin\widehat{\kappa}f_{s,t}\sin\widehat{\kappa}ty}\left[\frac{1}{2} + \frac{1}{8}\widehat{\kappa}^2(sx)^2 + \mathcal{O}(\xi^4)\right]$$
$$\times\left[2 - \frac{1}{2}\widehat{\kappa}^2(ty)^2 - \frac{1}{2}\widehat{\kappa}^2 z_{s,t}^2 + \mathcal{O}(\xi^4)\right].$$

After lengthy but routine simplifications and using the upper four-point cosq_K condition, we get:

$$h_{s,t} = \frac{\widehat{\kappa}(ty)\sin\widehat{\kappa}y - \widehat{\kappa}^2(sx)(ty)\frac{\cos\widehat{\kappa}y + \cos\widehat{\kappa}f_{s,t}}{2}\cos\alpha_0(s,t) + \mathcal{O}(\xi^3)}{\widehat{\kappa}(ty)[1 + \mathcal{O}(\xi^2)]\sin\widehat{\kappa}f_{s,t}}$$
$$= \frac{\sin\widehat{\kappa}y - \widehat{\kappa}(sx)\frac{\cos\widehat{\kappa}y + \cos\widehat{\kappa}f_{s,t}}{2}\cos\alpha_0(s,t)}{\sin\widehat{\kappa}f_{s,t}} + \mathcal{O}(\xi^2)$$
$$\le 1.$$

Set $\mu = (\cos\widehat{\kappa}y + \cos\widehat{\kappa}f_{s,t})/2$. By the triangle inequality, $|f_{s,t} - y| \le sx$. Hence, $\mu = \cos\widehat{\kappa}y + \mathcal{O}(\xi)$ follows, and we have:

$$h_{s,t} = \frac{\sin\widehat{\kappa}y - \widehat{\kappa}(sx)\cos\widehat{\kappa}y\cos\alpha_0(s,t)}{\sin\widehat{\kappa}f_{s,t}} + \mathcal{O}(\xi^2) \le 1. \tag{6.2}$$

So, if $K > 0$, then we get

$$\sin\kappa y - \kappa(sx)\cos\kappa y\cos\alpha_0(s,t) \le \sin\kappa f_{s,t} + \mathcal{O}(\xi^2),$$

and if $K < 0$, then we get

$$\sinh\kappa y - \kappa(sx)\cosh\kappa y\cos\alpha_0(s,t) \le \sinh\kappa f_{s,t} + \mathcal{O}(\xi^2).$$

Now, by writing $\cos\kappa f_{s,t} = \sqrt{1 - \sin^2\kappa f_{s,t}}$ if $K > 0$ and $\cosh\kappa f_{s,t} = \sqrt{1 + \sinh^2\kappa f_{s,t}}$ if $K < 0$, it is not difficult to derive the inequalities of (i) and (ii) of the lemma for $f_{s,t}$.

II. Let $(\mathcal{M}, \rho)$ satisfy the lower four-point cosq_K condition. Set

$$g_{s,t} = \operatorname{cosq}_K(\overrightarrow{X_sC}, \overrightarrow{Y_tA}).$$

Then

$$g_{s,t} = \frac{(1 + \cos\widehat{\kappa}z_{s,t})(\cos\widehat{\kappa}y + \cos\widehat{\kappa}f_{s,t}\cos\widehat{\kappa}ty)}{(1 + \cos\widehat{\kappa}z_{s,t})\sin\widehat{\kappa}f_{s,t}\sin\widehat{\kappa}ty}$$
$$- \frac{[\cos\widehat{\kappa}f_{s,t} + \cos\widehat{\kappa}(1-t)y](\cos\widehat{\kappa}ty + \cos\widehat{\kappa}sx)}{(1 + \cos\widehat{\kappa}z_{s,t})\sin\widehat{\kappa}f_{s,t}\sin\widehat{\kappa}ty}$$
$$\ge -1.$$

Let I denote the numerator of $g_{s,t}$. We have:

$$I = \left[2 - \frac{1}{2}\widehat{\kappa}^2 z_{s,t}^2 + \mathcal{O}(\xi^4)\right]\left\{\cos\widehat{\kappa}y + \cos\widehat{\kappa}f_{s,t}\left[1 - \frac{1}{2}\widehat{\kappa}^2(ty)^2 + \mathcal{O}(\xi^4)\right]\right\}$$
$$- \left[\cos\widehat{\kappa}f_{s,t} + \cos\widehat{\kappa}y + \widehat{\kappa}(ty)\sin\widehat{\kappa}y - \frac{1}{2}\widehat{\kappa}^2(ty)^2\cos\widehat{\kappa}y + \mathcal{O}(\xi^3)\right]$$
$$\times\left[2 - \frac{1}{2}\widehat{\kappa}^2(ty)^2 - \frac{1}{2}\widehat{\kappa}^2(sx)^2 + \mathcal{O}(\xi^4)\right].$$

After elementary simplifications, we get

$$I = -2\widehat{\kappa}(ty)\sin\widehat{\kappa}y + \widehat{\kappa}^2(\cos\widehat{\kappa}y + \cos\widehat{\kappa}f_{s,t})(sx)(ty)\cos\alpha_0(s,t)$$
$$+ \widehat{\kappa}(ty)^2(\cos\widehat{\kappa}y - \cos\widehat{\kappa}f_{s,t}) + \mathcal{O}(\xi^3).$$

By the triangle inequality, $|y - f_{s,t}| \leq sx$. Hence, $\cos\widehat{\kappa}y - \cos\widehat{\kappa}f_{s,t} = \mathcal{O}(\xi)$. So,

$$I = -2\widehat{\kappa}(ty)\sin\widehat{\kappa}y + \widehat{\kappa}^2(\cos\widehat{\kappa}y + \cos\widehat{\kappa}f_{s,t})(sx)(ty)\cos\alpha_0(s,t) + \mathcal{O}(\xi^3).$$

Hence,

$$g_{s,t} = \frac{-2\widehat{\kappa}(ty)\sin\widehat{\kappa}y + \widehat{\kappa}^2(\cos\widehat{\kappa}y + \cos\widehat{\kappa}f_{s,t})(sx)(ty)\cos\alpha_0(s,t) + \mathcal{O}(\xi^3)}{2[1 + \mathcal{O}(\xi^2)]\widehat{\kappa}(ty)\sin\widehat{\kappa}f_{s,t}}$$
$$= \frac{-\sin\widehat{\kappa}y + \widehat{\kappa}\frac{\cos\widehat{\kappa}y + \cos\widehat{\kappa}f_{s,t}}{2}(sx)\cos\alpha_0(s,t)}{\sin\widehat{\kappa}f_{s,t}} + \mathcal{O}(\xi^2) \geq -1,$$

which implies (6.2). Hence, the inequalities of (i) and (ii) for $f_{s,t}$ follow.

Derivation of the inequalities of parts (i) and (ii) for $d_{s,t}$ is similar.

The proof of the cross-diagonal lemma is complete. □

6.4. *Growth Estimate Lemma*

We keep the notation of Section 6.3. To illustrate the estimates of Lemma 6.3, consider a geodesic triangle $\mathcal{T} = ABC$ in $\mathbb{S}_1$ (for the notation, see Figure 11). Let $z_\perp$ denote the length of the orthogonal projection of the shortest $\mathcal{BC}$ onto the (possibly extended) shortest $\mathcal{X}_s\mathcal{Y}_t$. For small x and y, we can treat the triangle $\mathcal{T}$ as approximately Euclidean triangle. Then it is not difficult to see that $z_\perp$ is approximately equal to $x\cos\beta_0(s,t) + y\cos\gamma_0(s,t)$. So, for small x and y, the length z is approximately bounded below by $x\cos\beta_0(s,t) + y\cos\gamma_0(s,t)$. Lemma 6.3 establishes similar estimates for metric spaces satisfying the one-sided four-point cosq_K condition.

LEMMA 6.3. *Let $K \neq 0$ and $0 < \underline{m} \leq \overline{m} < +\infty$. Let A, B, C be three distinct points in a metric space $(\mathcal{M}, \rho)$ and $s, t \in (0, 1]$ satisfying M1–M3 of Section 6.3. In addition, suppose that $(\mathcal{M}, \rho)$ satisfies the one-sided four-point cosq_K condition. Let $\mathcal{A} \subseteq (0, 1] \times (0, 1]$ be such that $(0, 0)$ is an accumulation point of the set $\mathcal{A}$ and $0 < \underline{m} \leq z_{s,t}/(sx)$ for every $(s, t) \in \mathcal{A}$.*

(i) *If $K > 0$, then for every $(s, t) \in \mathcal{A}$,*

$$\sin\kappa y\cos\gamma_0(s,t) + \frac{\cos\kappa y + \cos\kappa z}{1 + \cos\kappa x}\sin\kappa x\cos\beta_0(s,t) \leq \sin\kappa z + \mathcal{O}(\xi),$$

(ii) *If* $K < 0$, *then for every* $(s,t) \in \mathcal{A}$,

$$\sinh \kappa y \cos \gamma_0(s,t) + \frac{\cosh \kappa y + \cosh \kappa z}{1 + \cosh \kappa x} \sinh \kappa x \cos \beta_0(s,t) \le \sinh \kappa z + \mathcal{O}(\xi),$$

where $\mathcal{O}(\xi) = \mathcal{O}_{\lambda,\eta,\underline{m},\overline{m},K}(\xi)$.

Proof. We consider $(s,t) \in \mathcal{A}$.

I. Let $(\mathcal{M}, \rho)$ **satisfy the upper four-point** cosq_K **condition**. Set

$$p_{s,t} = \mathrm{cosq}_K(\overrightarrow{X_s Y_t}, \overrightarrow{BC}),$$

see Figure 11. Then

$$\begin{aligned}
p_{s,t} &= \frac{\cos \widehat{\kappa}(1-t)y + \cos \widehat{\kappa} z_{s,t} \cos \widehat{\kappa} z}{\sin \widehat{\kappa} z_{s,t} \sin \widehat{\kappa} z} \\
&\quad - \frac{(\cos \widehat{\kappa} z_{s,t} + \cos \widehat{\kappa} d_{s,t})(\cos \widehat{\kappa} z + \cos \widehat{\kappa} f_{s,t})}{[1 + \cos \widehat{\kappa}(1-s)x] \sin \widehat{\kappa} z_{s,t} \sin \widehat{\kappa} z} \\
&= \frac{\cos \widehat{\kappa} y + \widehat{\kappa}(ty) \sin \widehat{\kappa} y + \mathcal{O}(\xi^2) + \cos \widehat{\kappa} z [1 + \mathcal{O}(\xi^2)]}{\sin \widehat{\kappa} z_{s,t} \sin \widehat{\kappa} z} \\
&\quad - \frac{(1 + \cos \widehat{\kappa} d_{s,t})(\cos \widehat{\kappa} z + \cos \widehat{\kappa} f_{s,t}) + \mathcal{O}(\xi^2)}{\kappa z_{s,t} \sin \widehat{\kappa} z} \\
&\quad \times [1 + \mathcal{O}(\xi^2)] \times \left[\frac{1}{1 + \cos \widehat{\kappa} x} - \frac{\widehat{\kappa}(sx) \sin \widehat{\kappa} x}{(1 + \cos \widehat{\kappa} x)^2} + \mathcal{O}(\xi^2) \right].
\end{aligned}$$

For brevity, set $\mu = \cos \widehat{\kappa} z + \cos \widehat{\kappa} y$ and $\nu = 1 + \cos \widehat{\kappa} x$. Let $K > 0$. By part (i) of the cross-diagonal estimate lemma (Lemma 6.2),

$$\begin{aligned}
p_{s,t} \ge{}& \frac{\mu + \kappa(ty) \sin \kappa y + \mathcal{O}(\xi^2)}{\kappa z_{s,t} \sin \kappa z} \\
& - \frac{[\nu + \kappa(ty) \sin \kappa x \cos \alpha_0(s,t)][\mu + \kappa(sx) \sin \kappa y \cos \alpha_0(s,t)]}{\nu \kappa z_{s,t} \sin \kappa z} \\
& \times \left[1 - \frac{\kappa(sx) \sin \kappa x}{\nu} + \mathcal{O}(\xi^2) \right].
\end{aligned}$$

After elementary simplifications and using the upper four-point cosq_K condition, we get:

$$\begin{aligned}
1 &\ge p_{s,t} \\
&\ge \frac{(ty) \sin \kappa y - (sx) \sin \kappa y \cos \alpha_0(s,t)}{z_{s,t} \sin \kappa z} \\
&\quad + \frac{\mu}{\nu} \frac{(sx) \sin \kappa x - (ty) \sin \kappa x \cos \alpha_0(s,t)}{z_{s,t} \sin \kappa z} + \mathcal{O}(\xi). \qquad (6.3)
\end{aligned}$$

By recalling that $\cos \alpha_0(s,t) = [(sx)^2 + (ty)^2 - z_{s,t}^2]/[2(ty)(sx)]$, we readily see that

$$\begin{aligned}
(ty) \sin \kappa y - (sx) \sin \kappa y \cos \alpha_0(s,t) &= z_{s,t} \sin \kappa y \cos \gamma_0(s,t), \\
(sx) \sin \kappa x - (ty) \sin \kappa x \cos \alpha_0(s,t) &= z_{s,t} \sin \kappa x \cos \beta_0(s,t).
\end{aligned}$$

Finally, we get:

$$\frac{\sin\kappa y\cos\gamma_0(s,t)+\frac{\cos\kappa y+\cos\kappa z}{1+\cos\kappa x}\sin\kappa x\cos\beta_0(s,t)}{\sin\kappa z}\le 1+\mathcal{O}(\xi),$$

and the inequality of part (i) follows. The case of negative K is similar, and we leave it to the reader.

II. Let $(\mathcal{M},\rho)$ **satisfy the lower four-point** cosq_K **condition**. Set

$$q_{s,t}=\operatorname{cosq}_K(\overrightarrow{Y_tX_s},\overrightarrow{BC}).$$

Then

$$\begin{aligned}
q_{s,t}&=\frac{\cos\widehat{\kappa}f_{s,t}+\cos\widehat{\kappa}z_{s,t}\cos\widehat{\kappa}z}{\sin\widehat{\kappa}z_{s,t}\sin\widehat{\kappa}z}-\\
&\quad-\frac{[\cos\widehat{\kappa}z_{s,t}+\cos\widehat{\kappa}(1-s)x][\cos\widehat{\kappa}z+\cos\widehat{\kappa}(1-t)y]}{(1+\cos\widehat{\kappa}d_{s,t})\sin\widehat{\kappa}z_{s,t}\sin\widehat{\kappa}z}\\
&=(1+\cos\widehat{\kappa}d_{s,t})\{[\cos\widehat{\kappa}f_{s,t}+\cos\widehat{\kappa}z+\mathcal{O}(\xi^2)]\\
&\quad-[1+\cos\widehat{\kappa}x+\widehat{\kappa}(sx)\sin\widehat{\kappa}x+\mathcal{O}(\xi^2)]\\
&\quad\times[\cos\widehat{\kappa}z+\cos\widehat{\kappa}y+\widehat{\kappa}(ty)\sin\widehat{\kappa}y+\mathcal{O}(\xi^2)]\\
&\quad\times[(1+\cos\widehat{\kappa}d_{s,t})\sin\widehat{\kappa}z_{s,t}\sin\widehat{\kappa}z]^{-1}\}\\
&=\{(1+\cos\widehat{\kappa}d_{s,t})[\cos\widehat{\kappa}f_{s,t}+\cos\widehat{\kappa}z]\\
&\quad-[\mu+\widehat{\kappa}(ty)\sin\widehat{\kappa}y][\nu+\widehat{\kappa}(sx)\sin\widehat{\kappa}x]+\mathcal{O}(\xi^2)\}\\
&\quad\times[(1+\cos\widehat{\kappa}d_{s,t})\sin\widehat{\kappa}z_{s,t}\sin\widehat{\kappa}z]^{-1},
\end{aligned}$$

where we keep the notation $\mu=\cos\widehat{\kappa}y+\cos\widehat{\kappa}z$ and $\nu=1+\cos\widehat{\kappa}x$. By invoking the triangle inequality, we see that $\cos\widehat{\kappa}d_{s,t}=\cos\widehat{\kappa}x+\mathcal{O}(\xi)$, whence $1/(1+\cos\widehat{\kappa}d_{s,t})=1/\nu+\mathcal{O}(\xi)$. So, we get:

$$q_{s,t}=\frac{I}{\nu\widehat{\kappa}z_{s,t}\sin\widehat{\kappa}z}[1+\mathcal{O}(\xi)],$$

where

$$\begin{aligned}
I&=(1+\cos\widehat{\kappa}d_{s,t})(\cos\widehat{\kappa}f_{s,t}+\cos\widehat{\kappa}z)\\
&\quad-[\mu+\widehat{\kappa}(ty)\sin\widehat{\kappa}y][\nu+\widehat{\kappa}(sx)\sin\widehat{\kappa}x]+\mathcal{O}(\xi^2).
\end{aligned}$$

Let $K>0$. By the cross-diagonal estimate lemma,

$$\begin{aligned}
I&\le I'\\
&=[\nu+\kappa(ty)\sin\kappa x\cos\alpha_0(s,t)]\\
&\quad\times[\mu+\kappa(sx)\sin\kappa y\cos\alpha_0(s,t)]\\
&\quad-[\mu+\kappa(ty)\sin\kappa y][\nu+\kappa(sx)\sin\kappa x]+\mathcal{O}(\xi^2)\\
&=\kappa\{-\nu\sin\kappa y[(ty)-(sx)\cos\alpha_0(s,t)]\\
&\quad-\mu\sin\kappa x[(sx)-(ty)\cos\alpha_0(s,t)]+\mathcal{O}(\xi^2)\},
\end{aligned}$$

whence by invoking the lower four point cosq_K condition, we have:

$$\frac{-\nu\sin\kappa y[(ty)-(sx)\cos\alpha_0(s,t)]-\mu\sin\kappa x[(sx)-(ty)\cos\alpha_0(s,t)]}{\nu z_{s,t}\sin\kappa z} \geq q_{s,t}\geq -1+\mathcal{O}(\xi),$$

which is equivalent to inequality (6.3). Hence, the inequality of part (i) of the lemma follows. The case of negative K is similar.

The proof of the growth estimate lemma is complete. □

It is well-known that $\alpha_K(s,t)-\alpha_0(s,t)$, $\beta_K(s,t)-\beta_0(s,t)$, and $\gamma_K(s,t)-\gamma_0(s,t)$ are $\mathcal{O}(\sigma(AX_sY_t))=\mathcal{O}(\xi^2)$. Hence, by recalling that $\alpha_0(s,t)+\beta_0(s,t)+\gamma_0(s,t)=\pi$, we get the following:

COROLLARY 6.2. *Under the hypotheses of the growth estimate lemma (Lemma 6.3), the following inequalities hold: (i) If $K>0$, then for every $(s,t)\in\mathcal{A}$,*

$$\frac{\cos\kappa y+\cos\kappa z}{1+\cos\kappa x}\sin\kappa x\cos\beta_K(s,t) - \sin\kappa y[\cos(\alpha_K(s,t)+\beta_K(s,t))] \leq \sin\kappa z+\mathcal{O}(\xi);$$

(ii) If $K<0$, then for every $(s,t)\in\mathcal{A}$,

$$\frac{\cosh\kappa y+\cosh\kappa z}{1+\cosh\kappa x}\sinh\kappa x\cos\beta_K(s,t) - \sinh\kappa y[\cos(\alpha_K(s,t)+\beta_K(s,t))] \leq \sinh\kappa z+\mathcal{O}(\xi),$$

where $\mathcal{O}(\xi)=\mathcal{O}_{\lambda,\eta,\underline{m},\overline{m},K}(\xi)$.

6.5. Existence of Proportional Angles

Let $(\mathcal{M},\rho)$ be a metric space, and let $\mathcal{L}=\mathcal{AB}$ and $\mathcal{N}=\mathcal{AC}$ be shortests in $(\mathcal{M},\rho)$ starting at a common point $A\in\mathcal{M}$. Let $K\in\mathbb{R}$ and $t\in(0,1]$. Set $\alpha_K(t)=\alpha_K(t,t)$, $\beta_K(t)=\beta_K(t,t)$, $\gamma_K(t)=\gamma_K(t,t)$, and $z_t=z_{t,t}$. In this section, we derive from the growth estimate lemma that the proportional angle $\lim_{t\to 0+}\alpha_0(t)$ exists. We begin with the following:

LEMMA 6.4. *Let $K\neq 0$ and $\underline{m}>0$, and let $(\mathcal{M},\rho)$ be a metric space satisfying the one-sided four-point cosq_K condition. Also, suppose that $\operatorname{diam}(\mathcal{M})\leq\pi/(2\sqrt{K})$ when $K>0$. Let $\mathcal{L}=\mathcal{AB}$, $\mathcal{N}=\mathcal{AC}$ be shortests in $(\mathcal{M},\rho)$ starting at a common point $A\in\mathcal{M}$ and $t\in(0,1]$. If $0<\underline{m}\leq z_t/t$ for $0<t<\varepsilon$ for some $\varepsilon\in(0,1)$, then there is $\varepsilon'\in(0,\varepsilon]$ such that for every $\tau\in(0,t^2)\cap(0,\varepsilon')$, the following inequality holds:*

$$z_\tau\leq\frac{\tau}{t}(z_t+\mu t^2),$$

where $\mu=\mu(\lambda,\eta,\underline{m},K)>0$.

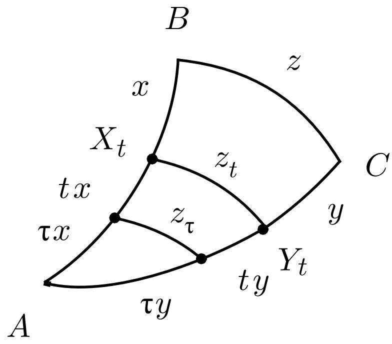

Figure 12 Sketch for Lemma 6.4

Proof. The notation of the lemma is illustrated in Figure 12. Let $0 < \tau < t^2 \leq t \leq 1$. In the growth estimate lemma, take $x := tx$ and $y := ty$. Then $\xi := \xi_t = \max\{tx, ty\}\frac{\tau}{t} = \frac{\tau}{t}t\lambda = \tau\lambda$. Hence, $\mathcal{O}(\xi_t) = \mathcal{O}(\tau)$. By the growth estimate lemma applied to the shortests $\mathcal{AX}_t$ and $\mathcal{AY}_t$, if $K > 0$, then

$$\sin\kappa ty\cos\gamma_0(\tau) + \frac{\cos\kappa ty + \cos\kappa z_t}{1+\cos\kappa tx}\sin\kappa tx\cos\beta_0(\tau) \leq \sin\kappa z_t + \mathcal{O}(\tau), \tag{6.4}$$

and if $K < 0$, then

$$\sinh\kappa ty\cos\gamma_0(\tau) + \frac{\cosh\kappa ty + \cosh\kappa z_t}{1+\cosh\kappa tx}\sinh\kappa tx\cos\beta_0(\tau) \leq \sinh\kappa z_t + \mathcal{O}(\tau)$$

for every $t \in (0, \varepsilon)$, where $\mathcal{O}(\tau) = \mathcal{O}_{\lambda,\eta,\underline{m},K}(\tau)$.

Let $K > 0$. Then we can rewrite (6.4) in the following form:

$$\kappa(ty)[1+\mathcal{O}(t^2)]\cos\gamma_0(\tau) + [1+\mathcal{O}(t^2)]\kappa(tx)\cos\beta_0(\tau) \leq \kappa z_t + \mathcal{O}(t^3) + \mathcal{O}(\tau) = \kappa z_t + \mathcal{O}(t^2),$$

whence

$$y\cos\gamma_0(\tau) + x\cos\beta_0(\tau) \leq \frac{z_t + \mathcal{O}(t^2)}{t}. \tag{6.5}$$

Let $\eta_\tau = z_\tau/\tau$. Recall that

$$\cos\gamma_0(\tau) = \frac{\tau^2y^2 + z_\tau^2 - \tau^2x^2}{2\tau yz_\tau} = \frac{y^2+\eta_\tau^2-x^2}{2y\eta_\tau},$$
$$\cos\beta_0(\tau) = \frac{\tau^2x^2 + z_\tau^2 - \tau^2y^2}{2\tau xz_\tau} = \frac{x^2+\eta_\tau^2-y^2}{2x\eta_\tau}.$$

Hence, by (6.5),

$$\eta_\tau \leq \frac{z_t + \mathcal{O}(t^2)}{t},$$

and the claim of the lemma for positive K follows. The case of negative K is similar.

The proof of Lemma 6.4 is complete. □

By Lemma 6.4,

$$\begin{aligned}\cos\alpha_0(\tau) &= \frac{t^2x^2+t^2y^2-\frac{t^2}{\tau^2}z_\tau^2}{2t^2xy}\\ &\geq \frac{t^2x^2+t^2y^2-(z_t+\mu t^2)^2}{2t^2xy}\\ &= \cos\alpha_0(t)-\frac{\mu z_t}{xy}-\frac{\mu^2t^2}{2xy}.\end{aligned}$$

By the triangle inequality, $z_t \leq (x+y)t$. So, we have the following:

COROLLARY 6.3. *Under the hypothesis of Lemma 6.4, the following inequality holds*:

$$\cos\alpha_0(\tau) \geq \cos\alpha_0(t) - \mu' t,$$

where $\mu' = \mu'(\lambda, \eta, \underline{m}, K) > 0$.

COROLLARY 6.4. *Let* $K \neq 0$, *and let* $(\mathcal{M}, \rho)$ *be a metric space satisfying the one-sided four-point* cosq_K *condition. Also, suppose that* $\operatorname{diam}(\mathcal{M}) \leq \pi/(2\sqrt{K})$ *when* $K > 0$. *Let* $\mathcal{L} = \mathcal{AB}$ *and* $\mathcal{N} = \mathcal{AC}$ *be shortests in* $(\mathcal{M}, \rho)$ *starting at a common point* $A \in \mathcal{M}$. *Then* $\lim_{t\to 0+}\alpha_0(t)$ *exists.*

Proof. Let $\overline{\alpha}_0 = \overline{\lim}_{t\to 0+}\alpha_0(t)$ and $\underline{\alpha}_0 = \underline{\lim}_{t\to 0+}\alpha_0(t)$. Then there are sequences $(t_n)_{n=1}^\infty$ and $(\tau_n)_{n=1}^\infty$ in $(0,1]$ convergent to zero such that $\overline{\alpha}_0 = \lim_{n\to\infty}\alpha_0(\tau_n)$ and $\underline{\alpha}_0 = \underline{\lim}_{n\to\infty}\alpha_0(t_n)$. There is no restriction in assuming that $\tau_n < t_n^2$ for every $n \in \mathbb{N}$. We consider the following two cases.

I. $\lim_{n\to\infty} z_{\tau_n}/\tau_n = 0$. Then

$$\cos\alpha_0(\tau_n) = \frac{x^2+y^2-(z_{\tau_n}/\tau_n)^2}{2xy} \to \frac{x^2+y^2}{2xy} \quad \text{as } n\to\infty.$$

By the triangle inequality, $z_{\tau_n}/\tau_n \geq |x-y|$, whence $x = y$, and we have:

$$\lim_{n\to\infty}\cos\alpha_0(\tau_n) = 1.$$

Hence, $\overline{\alpha}_0 = 0$, and

$$\lim_{t\to 0+}\alpha_0(t) = 0$$

follows.

II. By Corollary 6.3, $\cos\alpha_0(\tau_n) \geq \cos\alpha_0(t_n) + \mathcal{O}(t_n)$ for every $n \in \mathbb{N}$. So, by passing to the limit as $n \to \infty$ in both sides of the last inequality, we get the inequality $\overline{\alpha}_0 \leq \underline{\alpha}_0$. This completes the proof of Corollary 6.4. □

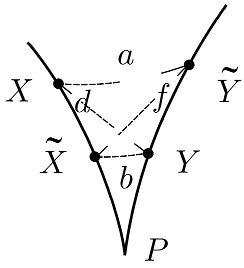

Figure 13 Sketch for Proposition 1

6.6. Existence of Angle

PROPOSITION 1. *Let* $(\mathcal{M}, \rho)$ *be a metric space satisfying the one-sided four-point* cosq_K *condition. Then between any pair of shortests* $\mathcal{L}$ *and* $\mathcal{N}$ *in* $(\mathcal{M}, \rho)$ *starting at a common point* $P \in \mathcal{M}$, *there exists Aleksandrov's angle.*

Proof. Set $\alpha_{av} = (\overline{\measuredangle}(\mathcal{L}, \mathcal{N}) + \underline{\measuredangle}(\mathcal{L}, \mathcal{N}))/2$. If $\overline{\measuredangle}(\mathcal{L}, \mathcal{N}) = 0$ or $\underline{\measuredangle}(\mathcal{L}, \mathcal{N}) = \pi$, then we are done. So, we can assume that $\sin \alpha_{av} > 0$. Contrary to the claim of the proposition, suppose that $\overline{\measuredangle}(\mathcal{L}, \mathcal{N}) - \underline{\measuredangle}(\mathcal{L}, \mathcal{N}) = \varepsilon_0 > 0$.

I. In Step 1 of the proof of Proposition 20 in [8], we showed that for every $0 < \varepsilon < \varepsilon_0$, there are points $\widetilde{X}, X \in \mathcal{L} \backslash \{P\}$ and $Y, \widetilde{Y} \in \mathcal{N} \backslash \{P\}$, or $\widetilde{X}, X \in \mathcal{N} \backslash \{P\}$ and $Y, \widetilde{Y} \in \mathcal{L} \backslash \{P\}$ such that the following conditions are satisfied (for simplicity, we drop ε from our notation for these points):

(i) $\widetilde{X}$ is contained between X and P, and Y is contained between $\widetilde{Y}$ and P, as illustrated in Figure 13, and the points X, $\widetilde{X}$, $\widetilde{Y}$, and Y can be selected arbitrary close to the point P.
(ii) $0 \leq \gamma'' = \measuredangle_0 \widetilde{X} P Y < \underline{\measuredangle}(\mathcal{L}, \mathcal{N}) + \varepsilon/4$.
(iii) $\gamma' = \measuredangle_0 \widetilde{X} P \widetilde{Y} > \underline{\measuredangle}(\mathcal{L}, \mathcal{N}) - \varepsilon/4$.
(iv) $0 \leq \underline{\gamma} = \measuredangle_0 X P \widetilde{Y} < \underline{\measuredangle}(\mathcal{L}, \mathcal{N}) + \varepsilon/4$.
(v) $\overline{\gamma} = \measuredangle_0 X P Y > \overline{\measuredangle}(\mathcal{L}, \mathcal{N}) - \varepsilon/4$.
(vi) $x/\widetilde{x} = \widetilde{y}/y$, where $\widetilde{x} = P\widetilde{X}$ and $y = PY$.

With little effort, the proof of (i)–(vi) for $K = 0$ in [8] can be extended to nonzero K. Indeed, by the definition of the lower angle, for every $\eta > 0$, there are $t_\eta \in (0, 1)$ and $\xi, \zeta \in (0, t_\eta)$ such that

$$\measuredangle_0(\xi, \zeta) < \underline{\measuredangle}(\mathcal{L}, \mathcal{N}) + \eta.$$

By Corollary 6.3,

$$\cos \measuredangle_0(\tau\xi, \tau\zeta) \geq \cos \measuredangle_0(\xi, \zeta) - \mu' t,$$

where $t > 0$ is sufficiently small, and $0 < \tau < t^2$. So,

$$\cos \measuredangle_0(\tau\xi, \tau\zeta) \geq \cos(\underline{\measuredangle}(\mathcal{L}, \mathcal{N}) + \eta) - \mu' t.$$

Hence, given $\varepsilon \in (0, \varepsilon_0)$, there is $t' \in (0,1)$ such that the following inequality holds:

$$\measuredangle_0(\tau\xi, \tau\zeta) \le \underline{\measuredangle}(\mathcal{L}, \mathcal{N}) + \frac{\varepsilon}{4}, \quad 0 < \tau < t'.$$

After this point, the proof of (i)–(vi) is the same as in Step I of the proof of Proposition 20 in [8].

Let $\widehat{\gamma} = \max\{\gamma'', \underline{\gamma}\}$. By (ii) and (iv), for sufficiently small positive ε, the following inequalities hold:

(vii) $\widehat{\gamma} \le \underline{\measuredangle}(\mathcal{L}, \mathcal{N}) + \varepsilon/4 < \pi$.

Now consider $I = 2\cos\widehat{\gamma} - [\cos\overline{\gamma} + \cos\gamma']$. By (iii) and (v),

$$\begin{aligned} I &\ge \cos(\underline{\measuredangle}(\mathcal{L}, \mathcal{N}) + \varepsilon/4) - \cos(\overline{\measuredangle}(\mathcal{L}, \mathcal{N}) - \varepsilon/4) \\ &\quad + \cos(\underline{\measuredangle}(\mathcal{L}, \mathcal{N}) + \varepsilon/4) - \cos(\underline{\measuredangle}(\mathcal{L}, \mathcal{N}) - \varepsilon/4) \\ &= 2\sin\frac{\overline{\measuredangle}(\mathcal{L}, \mathcal{N}) - \underline{\measuredangle}(\mathcal{L}, \mathcal{N}) - \varepsilon/2}{2}\sin\alpha_{av} - 2\sin\frac{\varepsilon}{4}\sin\underline{\measuredangle}(\mathcal{L}, \mathcal{N}) \\ &> 2\sin\frac{\varepsilon_0}{4}\sin\alpha_{av} - 2\sin\frac{\varepsilon}{4}\sin\underline{\measuredangle}(\mathcal{L}, \mathcal{N}). \end{aligned}$$

Hence, for small positive ε, the inequality

$$I = 2\cos\widehat{\gamma} - [\cos\overline{\gamma} + \cos\gamma'] > \sin\frac{\varepsilon_0}{4}\sin\alpha_{av} > 0 \tag{6.6}$$

follows.

By Corollary 6.1, there is no restriction in assuming that $X \ne \widetilde{Y}$ and $\widetilde{X} \ne Y$. In what follows, $t = \widetilde{x}/x$.

II. Let $(\mathcal{M}, \rho)$ **satisfy the upper four-point** cosq_K **condition**. Set

$$p = \operatorname{cosq}_K(\overrightarrow{X\widetilde{Y}}, \overrightarrow{\widetilde{X}Y}).$$

Let $f = \widetilde{X}\widetilde{Y}$ and $d = XY$, as shown in Figure 13. Then

$$\begin{aligned} p &= \frac{\cos\widehat{\kappa}(1-t)\widetilde{y} + \cos\widehat{\kappa}a\cos\widehat{\kappa}b}{\sin\widehat{\kappa}a\sin\widehat{\kappa}b} \\ &\quad - \frac{(\cos\widehat{\kappa}a + \cos\widehat{\kappa}f)(\cos\widehat{\kappa}b + \cos\widehat{\kappa}d)}{[1 + \cos\widehat{\kappa}(1-t)x]\sin\widehat{\kappa}a\sin\widehat{\kappa}b} \\ &= \frac{\cos\widehat{\kappa}\widetilde{y} + \widehat{\kappa}(ty)\sin\widehat{\kappa}\widetilde{y} + \cos\widehat{\kappa}a + \mathcal{O}(\lambda^2t^2)}{\sin\widehat{\kappa}a\sin\widehat{\kappa}b} \\ &\quad - \frac{(\cos\widehat{\kappa}a + \cos\widehat{\kappa}f)[1 + \cos\widehat{\kappa}d + \mathcal{O}(\lambda^2t^2)]}{\sin\widehat{\kappa}a\sin\widehat{\kappa}b} \\ &\quad \times\left[\frac{1}{1+\cos\widehat{\kappa}x} - \frac{\widehat{\kappa}(tx)\sin\widehat{\kappa}}{(1+\cos\widehat{\kappa}x)^2} + \mathcal{O}(\lambda^2t^2)\right]. \end{aligned}$$

Let $K > 0$. Set $\gamma'_K = \measuredangle_K\widetilde{X}P\widetilde{Y}$ and $\overline{\gamma}_K = \measuredangle_K XPY$. By the spherical cosine formula, $\cos\kappa f = \cos\kappa tx\cos\kappa\widetilde{y} + \sin\kappa tx\sin\kappa\widetilde{y}\cos\gamma'_K$. Recall that $\gamma'_K - \gamma' = \mathcal{O}(\sigma(\widetilde{X}P\widetilde{Y})) = \mathcal{O}(\lambda t)$, whence $\cos\gamma'_K = \cos\gamma' + \mathcal{O}(\lambda t)$. So, we get:

$$\cos\kappa f = \cos\kappa\widetilde{y} + \kappa(tx)\sin\kappa\widetilde{y}\cos\gamma' + \mathcal{O}(\lambda^2t^2). \tag{6.7}$$

In a similar way,

$$\cos\kappa d = \cos\kappa x + \kappa(t\widetilde{y})\sin\kappa x\cos\overline{\gamma} + \mathcal{O}(\lambda^2 t^2). \tag{6.8}$$

For brevity, set $\mu = \cos\kappa a + \cos\kappa\widetilde{y}$ and $\nu = 1 + \cos\kappa x$. By (6.7) and (6.8) and by invoking the upper four-point cosq_K condition, we get:

$$\begin{aligned} p &= \frac{\mu + \kappa(t\widetilde{y})\sin\kappa\widetilde{y} + \mathcal{O}(\lambda^2 t^2)}{\sin\widehat{\kappa a}\sin\widehat{\kappa b}} - [\mu + \kappa(tx)\sin\kappa\widetilde{y}\cos\gamma' + \mathcal{O}(\lambda^2 t^2)] \\ &\quad\times \frac{[\nu + \kappa(t\widetilde{y})\sin\kappa x\cos\overline{\gamma} + \mathcal{O}(\lambda^2 t^2)][1 - \frac{\kappa(tx)\sin\kappa x}{\nu} + \mathcal{O}(\lambda^2 t^2)]}{\nu\sin\kappa a\sin\kappa b} \\ &= \kappa t\frac{\sin\kappa\widetilde{y}(\widetilde{y} - x\cos\gamma') + \frac{\mu}{\nu}\sin\kappa x(x - \widetilde{y}\cos\overline{\gamma}) + \mathcal{O}(\lambda^2 t)}{\sin\kappa a\sin\kappa b} \le 1. \end{aligned} \tag{6.9}$$

Now we approximate (6.9) w.r.t. x and $\widetilde{y}$:

$$\begin{aligned} p &= \kappa^2 t\frac{\widetilde{y}(\widetilde{y} - x\cos\gamma') + x(x - \widetilde{y}\cos\overline{\gamma}) + \mathcal{O}(t\lambda^2) + \mathcal{O}(\lambda^4)}{\sin\kappa a\sin\kappa b} \\ &= \kappa^2 t\frac{x^2 + \widetilde{y}^2 - x\widetilde{y}(\cos\overline{\gamma} + \cos\gamma') + \mathcal{O}(t\lambda^2) + \mathcal{O}(\lambda^4)}{\sin\kappa a\sin\kappa b}. \end{aligned}$$

Let $A = x^2 + \widetilde{y}^2 - x\widetilde{y}(\cos\overline{\gamma} + \cos\gamma') + \mathcal{O}(t\lambda^2) + \mathcal{O}(\lambda^4)$ and $B = x^2 + \widetilde{y}^2 - 2x\widetilde{y}\cos\widehat{\gamma}$. Notice that by (6.6),

$$\begin{aligned} A &> B + x\widetilde{y}\sin\frac{\varepsilon_0}{4}\sin\alpha_{av} + \mathcal{O}(t\lambda^2) + \mathcal{O}(\lambda^4) \\ &> B + \frac{1}{2}x\widetilde{y}\sin\frac{\varepsilon_0}{4}\sin\alpha_{av} \ge \frac{1}{2}x\widetilde{y}\sin\frac{\varepsilon_0}{4}\sin\alpha_{av} > 0 \end{aligned} \tag{6.10}$$

for sufficiently small λ and t. Set

$$a' = \sqrt{x^2 + \widetilde{y}^2 - 2x\widetilde{y}\cos\widehat{\gamma}} \quad\text{and}\quad b' = t\sqrt{x^2 + \widetilde{y}^2 - 2x\widetilde{y}\cos\widehat{\gamma}}.$$

Because $\underline{\gamma}, \gamma'' \le \widehat{\gamma}$, we readily see that $a \le a'$ and $b \le b'$. Hence,

$$\begin{aligned} p &\ge k^2 t\frac{A}{\sin\kappa a'\sin\kappa b'} \\ &= t\frac{A}{a'b'}[1 + \mathcal{O}(\lambda^2)] \\ &= \frac{x^2 + \widetilde{y}^2 - x\widetilde{y}(\cos\overline{\gamma} + \cos\gamma') + \mathcal{O}(t\lambda^2) + \mathcal{O}(\lambda^4)}{x^2 + \widetilde{y}^2 - 2x\widetilde{y}\cos\widehat{\gamma}}. \end{aligned}$$

So, by invoking the upper four-point cosq_K condition, (6.6), and (6.10), for sufficiently small λ and t, we get:

$$\begin{aligned} 1 &< 1 + \frac{\frac{x\widetilde{y}}{2}\sin\frac{\varepsilon_0}{4}\sin\alpha_{av}}{x^2 + \widetilde{y}^2 - 2x\widetilde{y}\cos\widehat{\gamma}} \\ &\le \frac{x^2 + \widetilde{y}^2 - x\widetilde{y}(\cos\overline{\gamma} + \cos\gamma') + \mathcal{O}(t\lambda^2) + \mathcal{O}(\lambda^4)}{x^2 + \widetilde{y}^2 - 2x\widetilde{y}\cos\widehat{\gamma}} \le p \le 1, \end{aligned}$$

a contradiction. The case of negative K is similar.

III. Let $(\mathcal{M}, \rho)$ satisfy the lower four-point cosq_K condition. Set

$$q = \operatorname{cosq}_K(\overrightarrow{XY}, \overrightarrow{YX}).$$

We have:

$$q = \frac{\cos\widehat{\kappa} f + \cos\widehat{\kappa} a \cos\widehat{\kappa} b}{\sin\widehat{\kappa} a \sin\widehat{\kappa} b} - \frac{[\cos\widehat{\kappa} a + \cos\widehat{\kappa}(1-t)\widetilde{y}][\cos\widehat{\kappa} b + \cos\widehat{\kappa}(1-t)x]}{(1+\cos\widehat{\kappa} d)\sin\widehat{\kappa} a \sin\widehat{\kappa} b}.$$

Approximating q relative to t, we get $q = I/(J \sin\widehat{\kappa} a \sin\widehat{\kappa} b)$, where

$$\begin{aligned} I &= [\cos\widehat{\kappa} f + \cos\widehat{\kappa} a + \mathcal{O}(\lambda^2 t^2)][1 + \cos\widehat{\kappa} d] \\ &\quad - [\mu + \widehat{\kappa} t\widetilde{y} \sin\widehat{\kappa}\widetilde{y} + \mathcal{O}(\lambda^2 t^2)][\nu + \widehat{\kappa}(tx)\sin\widehat{\kappa} x + \mathcal{O}(\lambda^2 t^2)], \\ J &= (1 + \cos\widehat{\kappa} d), \end{aligned}$$

and where we set $\mu = \cos\widehat{\kappa} a + \cos\widehat{\kappa}\widetilde{y}$ and $\nu = 1 + \cos\widehat{\kappa} x$.

Let $K > 0$. By recalling (6.7) and (6.8), we get:

$$\begin{aligned} I &= [\mu + \kappa(tx)\sin\kappa\widetilde{y}\cos\gamma'][\nu + \kappa(t\widetilde{y})\sin\kappa x \cos\overline{\gamma}] - [\mu + \kappa t\widetilde{y}\sin\widehat{\kappa}\widetilde{y}] \\ &\quad \times [\nu + \kappa(tx)\sin\kappa x] + \mathcal{O}(\lambda^2 t^2), \\ J &= (1 + \cos\kappa d). \end{aligned}$$

After simplifications, we have:

$$I = -kt[\nu \sin k\widetilde{y}(\widetilde{y} - x\cos\gamma') + \mu \sin\kappa x(x - \widetilde{y}\cos\overline{\gamma}) + \mathcal{O}(\lambda^2 t)].$$

By (6.8), $J^{-1} = \nu^{-1}[1 + \mathcal{O}(\lambda^2 t)]$. By the lower four-point cosq_K condition,

$$-q = kt\frac{\sin k\widetilde{y}(\widetilde{y} - x\cos\gamma') + \frac{\mu}{\nu}\sin\kappa x(x - \widetilde{y}\cos\overline{\gamma}) + \mathcal{O}(\lambda^2 t)}{\sin\kappa a \sin\kappa b} \le 1.$$

So, from the lower four-point cosq condition we derived inequality (6.9). Hence, by using the arguments of part II, we see that the lower four-point cosq condition also implies the existence of Aleksandrov's angle. The case of negative K is similar.

The proof of Proposition 1 is complete. □

6.7. Angle Comparison Theorem

We begin with the following identity in the K-plane.

PROPOSITION 2. *Let $K \neq 0$, and let $\mathcal{T} = ABC$ be a triangle in $\mathbb{S}_K$. Set $x = AB$, $y = AC$, $z = BC$, $(x, y, z > 0)$, $\alpha = \measuredangle BAC$, and $\beta = \measuredangle ABC$, as illustrated in Figure 14. Then*

$$\sin\widehat{k}z = \frac{\cos\widehat{k}y + \cos\widehat{k}z}{1 + \cos\widehat{k}x}\sin\widehat{k}x\cos\beta - \sin\widehat{k}y\cos(\alpha+\beta). \tag{6.11}$$

In particular, if $K > 0$, then

$$\sin kz = \frac{\cos ky + \cos kz}{1 + \cos kx}\sin kx\cos\beta - \sin ky\cos(\alpha+\beta),$$

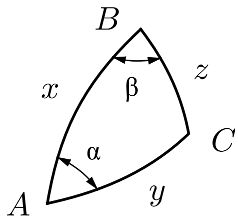

Figure 14 Sketch for Proposition 2

and if $K < 0$, *then*

$$\sinh kz = \frac{\cosh ky + \cosh kz}{1 + \cosh kx} \sinh kx \cos\beta - \sinh ky \cos(\alpha + \beta).$$

Proof. The following cases are possible:

(i) C is between A and B. Then $\alpha = \beta = 0$, $x > y$, and $z = x - y$.
(ii) A is between B and C. Then $\alpha = \pi$, $\beta = 0$, and $z = x + y$.
(iii) B is between A and C. Then $\alpha = 0$, $\beta = \pi$, $y > x$, and $z = y - x$.
(iv) $\mathcal{T}$ is a nondegenerate triangle. Then $\alpha, \beta \in (0, \pi)$.

For example, in case (i), the verification of (6.11) reduces to the direct verification of the elementary trigonometric identity

$$\sin \widehat{k}(x - y) = \frac{\cos \widehat{k}y + \cos \widehat{k}(x - y)}{1 + \cos \widehat{k}x} \sin \widehat{k}x - \sin \widehat{k}y.$$

Cases (ii) and (iii) are similar.

Now we consider case (iv). Let

$$I = \frac{\sin \widehat{k}y}{\sin\beta} \sin(\alpha + \beta) = \frac{\sin \widehat{k}y \sin\alpha \cos\beta}{\sin\beta} + \sin \widehat{k}y \cos\alpha.$$

By the sine formula in $\mathbb{S}_K$,

$$\frac{\sin \widehat{k}y \sin\alpha \cos\beta}{\sin\beta} = \sin \widehat{k}z \cos\beta.$$

By the cosine formula in $\mathbb{S}_K$,

$$\cos\beta = \frac{\cos \widehat{k}y - \cos \widehat{k}x \cos \widehat{k}z}{\sin \widehat{k}x \sin \widehat{k}z},$$

whence

$$\frac{\sin \widehat{k}y}{\sin\beta} \sin\alpha \cos\beta = \frac{\cos \widehat{k}y - \cos \widehat{k}x \cos \widehat{k}z}{\sin \widehat{k}x}.$$

Again, by the cosine formula in $\mathbb{S}_K$,

$$\cos\alpha = \frac{\cos \widehat{k}z - \cos \widehat{k}x \cos \widehat{k}y}{\sin \widehat{k}x \sin \widehat{k}y},$$

whence

$$\sin\widehat{k}y\cos\alpha = \frac{\cos\widehat{k}z - \cos\widehat{k}x\cos\widehat{k}y}{\sin\widehat{k}x}.$$

So,

$$I = \frac{(1-\cos\widehat{k}x)(\cos\widehat{k}y + \cos\widehat{k}z)}{\sin\widehat{k}x} = \frac{\cos\widehat{k}y + \cos\widehat{k}z}{1+\cos\widehat{k}x}\sin\widehat{k}x,$$

whence

$$\frac{\cos\widehat{k}y + \cos\widehat{k}z}{1+\cos\widehat{k}x}\sin\widehat{k}x\cos\beta = \frac{\sin\widehat{k}y}{\sin\beta}\sin(\alpha+\beta)\cos\beta.$$

Hence, if J denotes the right-hand side of (6.11), then

$$J = \frac{\sin\widehat{k}y}{\sin\beta}\sin(\alpha+\beta)\cos\beta - \sin\widehat{k}y\cos(\alpha+\beta).$$

Recall that by the sine formula in $\mathbb{S}_K$, $\sin\widehat{k}y = \sin\widehat{k}z\sin\beta/\sin\alpha$. So,

$$J = \frac{\sin\widehat{k}z}{\sin\alpha}[\sin(\alpha+\beta)\cos\beta - \cos(\alpha+\beta)\sin\beta] = \sin\widehat{k}z,$$

as needed.

The proof of Proposition 2 is complete. $\square$

Let $K \neq 0$, and let $\{A, B, C\}$ be a triple of distinct points in a metric space $(\mathcal{M}, \rho)$ of diameter less than $\pi/(2\sqrt{K})$ if $K > 0$. In what follows, we assume that the points A and B can be joined by a shortest $\mathcal{L} = \mathcal{AB}$ and the points A and C can be joined by a shortest $\mathcal{N} = \mathcal{AC}$. By Proposition 1, there exists an angle α between the shortests $\mathcal{L}$ and $\mathcal{N}$. In what follows, we assume that $0 < \alpha \le \pi$. Set $x = AB$ and $y = AC$.

To state our next lemma, we need the following notation. Let $K' \in \{0, K\}$. Consider a geodesic triangle $\mathcal{T}^{K'} = \widetilde{A}^{K'}\widetilde{B}^{K'}\widetilde{C}^{K'}$ in $\mathbb{S}_{K'}$ such that $\widetilde{A}^{K'}\widetilde{B}^{K'} = x$, $\widetilde{A}^{K'}\widetilde{C}^{K'} = y$, and $\alpha = \measuredangle\widetilde{B}^{K'}\widetilde{A}^{K'}\widetilde{C}^{K'}$. If $K' = K$, then set

$$\widetilde{A}^{K'} = \widetilde{A}, \qquad \widetilde{B}^{K'} = \widetilde{B}, \qquad \widetilde{C}^{K'} = \widetilde{C}, \qquad \widetilde{B}\widetilde{C} = \widetilde{z}, \quad \text{and} \quad \widetilde{\beta} = \measuredangle\widetilde{A}\widetilde{B}\widetilde{C},$$

as illustrated in Figure 15. Suppose that for $t \in (0, 1]$, points $\widehat{X}_t \in \mathcal{L}\backslash\{A\}$ and $\widehat{Y}_t \in \mathcal{N}\backslash\{A\}$ (in the metric space $(\mathcal{M}, \rho)$) have been selected. Consider the Euclidean triangle $\widetilde{\mathcal{T}}_t^0 = \widetilde{A}^0\widetilde{X}_t^0\widetilde{Y}_t^0$ such that $A\widehat{X}_t = \widetilde{A}^0\widetilde{X}_t^0$, $A\widehat{Y}_t = \widetilde{A}^0\widetilde{Y}_t^0$, and $\measuredangle\widetilde{X}_t^0\widetilde{A}^0\widetilde{Y}_t^0 = \alpha$. We claim that given small $t \in (0, 1]$, there is $s_t \in (0, 1]$ such that if $A\widehat{X}_t = s_t x$, $A\widehat{Y}_t = ty$ (and $\measuredangle\widetilde{X}_t^0\widetilde{A}^0\widetilde{Y}_t^0 = \alpha$), then $\measuredangle\widetilde{A}^0\widetilde{X}_t^0\widetilde{Y}_t^0 = \widetilde{\beta}$, as illustrated in Figure 16. Indeed, if $\alpha = \pi$, then $\widetilde{\beta} = 0$. Set $s_t = t$, and we are done. Now let $\alpha \in (0, \pi)$. First, we remark that $\alpha + \widetilde{\beta} < \pi$. It is sufficient to consider $K > 0$. Let $\delta = \measuredangle\widetilde{A}\widetilde{C}\widetilde{B}$. Because $y, \widetilde{z} < \pi/(2\sqrt{K})$, we can extend the shortests $\widetilde{\mathcal{C}}\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}}\widetilde{\mathcal{B}}$ to the shortests $\widetilde{\mathcal{C}}\mathcal{A}'$ and $\widetilde{\mathcal{C}}\mathcal{B}'$ of the lengths $\pi/(2\sqrt{K})$. Consider the spherical triangle $\mathcal{T}' = \widetilde{C}A'B'$. We have $\measuredangle\widetilde{C}A'B' = \measuredangle\widetilde{C}B'A' = \pi/2$. Hence, by recalling the Gauss–Bonnet theorem, we we see that

$$\delta + \alpha + \widetilde{\beta} < \delta + \frac{\pi}{2} + \frac{\pi}{2},$$

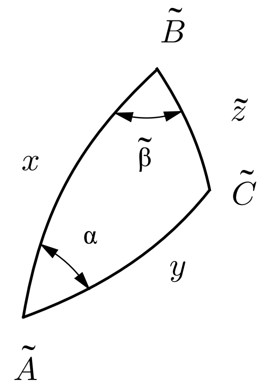

Figure 15 Sketch for Lemma 6.5

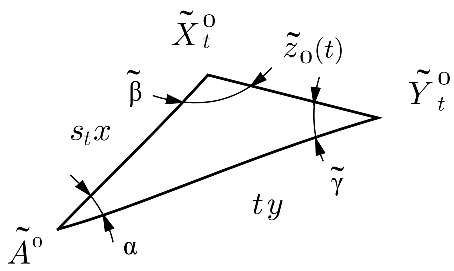

Figure 16 Definition of s_t

whence $\alpha + \widetilde{\beta} < \pi$ follows. In particular, $\alpha \in (0, \pi)$, and setting $\widetilde{\gamma} = \pi - \alpha - \widetilde{\beta}$, we see that $\widetilde{\gamma} \in (0, \pi)$. Hence, we select $s_t = ty \sin \widetilde{\gamma} / (x \sin \widetilde{\beta})$.

Finally, set

$$\widehat{\alpha}_{K'}(t) = \measuredangle_{K'} \widehat{X}_t^{K'} A^{K'} \widehat{Y}_t^{K'}, \qquad \widehat{\beta}_{K'}(t) = \measuredangle_{K'} A^{K'} \widehat{X}_t^{K'} \widehat{Y}_t^{K'},$$
$$\widehat{\gamma}_{K'}(t) = \measuredangle_{K'} A^{K'} \widehat{Y}_t^{K'} \widehat{X}_t^{K'}, \quad \text{and} \quad z(t) = \widehat{X}_t \widehat{Y}_t,$$

as shown in Figure 17.

LEMMA 6.5. *Let* $K \neq 0$. *If* $0 < \alpha \leq \pi$, *then*

$$\lim_{t \to 0+} \widehat{\beta}_K(t) = \widetilde{\beta}$$

(*for the notation*, *see Figures* 15 *and* 17 *for* $K' = K$).

Proof. **I.** Let $\alpha = \pi$; then $\widetilde{\beta} = 0$. We have $\lim_{t \to 0+} \widehat{\alpha}_0(t) = \pi$, whence $\lim_{t \to 0+} \widehat{\beta}_0(t) = 0$. Because $\widehat{\beta}_0(t) - \widehat{\beta}_K(t) = \mathcal{O}(A \widehat{X}_t \widehat{Y}_t) = \mathcal{O}(t^2)$, we have $\lim_{t \to 0+} \widehat{\beta}_K(t) = 0$, as needed.

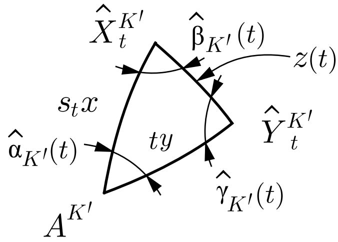

Figure 17 Sketch for Lemma 6.5

II. Now let $\alpha \in (0, \pi)$. Then $\widetilde{\beta}, \widetilde{\gamma} \in (0, \pi)$; see Figure 16. By the Euclidean sine formula applied to the triangle $\widetilde{X}_t^0 \widetilde{A}^0 \widetilde{Y}_t^0$,

$$\sin \widetilde{\beta} = \frac{ty \sin \alpha}{\widetilde{z}_0(t)}.$$

By the Euclidean sine formula applied to the triangle $\widehat{X}_t^0 \widehat{A}^0 \widehat{Y}_t^0$ (see Figure 17 for $K' = 0$),

$$\sin \widehat{\beta}_0(t) = \frac{ty \sin \widehat{\alpha}_0(t)}{z(t)}.$$

So, by recalling Proposition 1 and because $\widehat{\beta}_0(t) - \widehat{\beta}_K(t) = \mathcal{O}(t^2)$, all we have to do is to show that $\lim_{t\to 0+} t/z(t) = \lim_{t \to 0+} t/\widetilde{z}_0(t)$ (in fact, $t/\widetilde{z}_0(t) = \text{const}$). Indeed, by the Euclidean cosine formula applied to the triangle $\widetilde{X}_t^0 \widetilde{A}^0 \widetilde{Y}_t^0$ and $\widehat{X}_t^0 \widehat{A}^0 \widehat{Y}_t^0$ and by recalling that $s_t = ty \sin \widetilde{\gamma}/(x \sin \widetilde{\beta})$, we get:

$$\begin{aligned} \frac{t}{\widetilde{z}_0(t)} &= \frac{1}{y} \frac{\sin \widetilde{\beta}}{\sqrt{(\sin \widetilde{\beta} - \sin \widetilde{\gamma})^2 + 4 \sin \widetilde{\beta} \sin \widetilde{\gamma} \sin^2 \frac{\alpha}{2}}}, \\ \frac{t}{z(t)} &= \frac{1}{y} \frac{\sin \widetilde{\beta}}{\sqrt{(\sin \widetilde{\beta} - \sin \widetilde{\gamma})^2 + 4 \sin \widetilde{\beta} \sin \widetilde{\gamma} \sin^2 \frac{\widehat{\alpha}_0(t)}{2}}}. \end{aligned} \tag{6.12}$$

By Proposition 1, $\lim_{t\to 0+} \widehat{\alpha}_0(t) = \alpha$. Also, recall that $\alpha, \widetilde{\beta}, \widetilde{\gamma} \in (0, \pi)$. Hence, $\lim_{t\to 0+} t/z(t)$ and $\lim_{t\to 0+} t/\widetilde{z}_0(t)$ exist, and they are equal.

The proof of Lemma 6.5 is complete. □

PROPOSITION 3. *Let $K \neq 0$, and let $\{A, B, C\}$ be a triple of distinct points in a metric space $(\mathcal{M}, \rho)$ such that the points A and B can be joined by a shortest $\mathcal{L} = \mathcal{AB}$ and the points A and C can be joined by a shortest $\mathcal{N} = \mathcal{AC}$, and $AB, AC \leq \pi/(6\sqrt{K})$ if $K > 0$. If $(\mathcal{M}, \rho)$ satisfies the one-sided four-point* cosq_K *condition, then $\measuredangle BAC \leq \measuredangle_K BAC$.*

REMARK 6.1. In the hypothesis of Proposition 3, we do not require that $(\mathcal{M}, \rho)$ be a geodesically connected metric space. Also, the bound on AB and AC is not sharp.

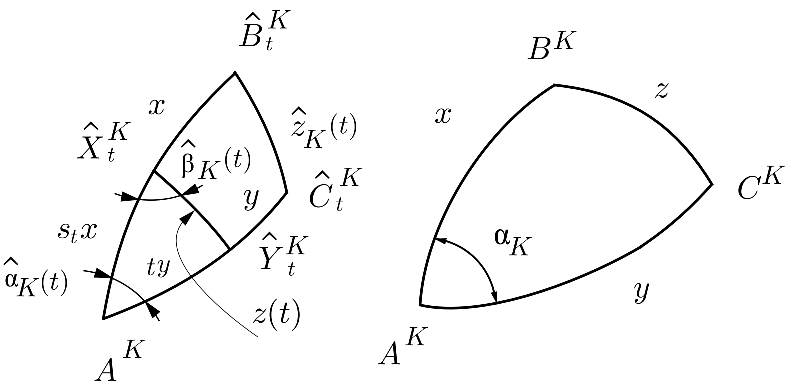

Figure 18 Sketch for Proposition 3

Proof. Let $\alpha = \measuredangle BAC$ and $\alpha_K = \measuredangle_K BAC$. There is no restriction in assuming that $\alpha \in (0, \pi]$. To prove the inequality $\alpha \le \alpha_K$, we consider a geodesic triangle $\widehat{T}_t^K = A^K \widehat{B}_t^K \widehat{C}_t^K$ in $\mathbb{S}_K$ such that $A^K \widehat{B}_t^K = x$, $A^K \widehat{C}_t^K = y$, and $\measuredangle \widehat{B}_t^K A^K \widehat{C}_t^K = \widehat{\alpha}_K(t)$. Set $\widehat{z}_K(t) = \widehat{B}_t^K \widehat{C}_t^K$, as illustrated in Figure 18. It is readily seen that $\alpha \le \alpha_K$ if and only if $\widetilde{z} = \lim_{t\to 0+} \widehat{z}_K(t) \le z$ (for the notation, see Figures 15 and 17 for $K' = K$). So, our goal is to derive the inequality $\widetilde{z} \le z$.

By Proposition 1, $\alpha = \lim_{t\to 0+} \widehat{\alpha}_K(t)$. It is readily seen that if $\alpha = \pi$, then $z(t)/t = x + y$, that is, it is bounded above and below by positive constants. Let $\alpha \in (0, \pi)$. Because $\widehat{\alpha}_0(t) \to \alpha$ as $t \to 0+$,

$$\sin \frac{\widehat{\alpha}_0(t)}{2} \ge \frac{1}{2} \sin \frac{\alpha}{2}$$

for small t. Then by recalling (6.12), it is not difficult to see that

$$\frac{t}{z(t)} \le \frac{1}{2y\sqrt{\sin \widetilde{\gamma}} \sin \frac{\widehat{\alpha}_0(t)}{2}} \le \frac{1}{y\sqrt{\sin \widetilde{\gamma}} \sin \frac{\alpha}{2}} < +\infty.$$

So, the hypotheses of Corollary 6.2 are satisfied.

Let $K > 0$. By Corollary 6.2,

$$\begin{aligned} &\frac{\cos \kappa y + \cos \kappa z}{1 + \cos \kappa x} \sin \kappa x \cos \widehat{\beta}_K(t) - \sin \kappa y \cos(\widehat{\alpha}_K(t) + \widehat{\beta}_K(t)) \\ &\quad \le \sin \kappa z + \mathcal{O}(t), \end{aligned}$$

By Proposition 1, $\lim_{t\to 0+} \widehat{\alpha}_K(t) = \alpha$, and by Lemma 6.5, $\lim_{t\to 0+} \widehat{\beta}_K(t) = \widetilde{\beta}$. Let $K > 0$. By letting $t \to 0+$, we get

$$\begin{aligned} &\sin \kappa z - \frac{\cos \kappa z}{1 + \cos \kappa x} \sin \kappa x \cos \widetilde{\beta} \\ &\quad \ge \frac{\cos \kappa y}{1 + \cos \kappa x} \sin \kappa x \cos \widetilde{\beta} - \sin \kappa y \cos(\alpha + \widetilde{\beta}), \end{aligned}$$

By Proposition 2,

$$\sin\kappa\widetilde{z} - \frac{\cos\kappa\widetilde{z}}{1+\cos\kappa x}\sin\kappa x\cos\widetilde{\beta} = \frac{\cos\kappa y}{1+\cos\kappa x}\sin\kappa x\cos\widetilde{\beta} - \sin\kappa y\cos(\alpha+\widetilde{\beta}),$$

whence

$$\sin\kappa z - \frac{\cos\kappa z}{1+\cos\kappa x}\sin\kappa x\cos\widetilde{\beta} \geq \sin\kappa\widetilde{z} - \frac{\cos\kappa\widetilde{z}}{1+\cos\kappa x}\sin\kappa x\cos\widetilde{\beta}. \tag{6.13}$$

By the triangle inequality, $z, \widetilde{z} \leq \pi/(3\kappa)$. By Corollary 6.1, there is no restriction in assuming that $z > 0$. So, we can also assume that $\widetilde{z}$ is also positive. Consider the function

$$f(u) = \sin\kappa u - \frac{\cos\kappa u}{1+\cos\kappa x}\sin\kappa x\cos\widetilde{\beta}, \quad u \in \left(0, \frac{\pi}{3\kappa}\right].$$

It is readily seen that $f(u)$ is a strictly increasing function if $u \in (0, \pi/(3\kappa)]$. So, the inequality $\widetilde{z} \leq z$ for positive K follows from inequality (6.13), as needed.

In a similar way, for $K < 0$, we have:

$$\sinh\kappa z - \frac{\cosh\kappa z}{1+\cosh\kappa x}\sinh\kappa x\cos\widetilde{\beta} \geq \sinh\kappa\widetilde{z} - \frac{\cosh\kappa\widetilde{z}}{1+\cosh\kappa x}\sinh\kappa x\cos\widetilde{\beta}. \tag{6.14}$$

It is easy to see that the function

$$g(u) = \sinh\kappa u - \frac{\cosh\kappa u}{1+\cosh\kappa x}\sinh\kappa x\cos\widetilde{\beta}, \quad u \in (0, +\infty),$$

is an increasing function if $u \in (0, +\infty)$. Hence, (6.14) implies the inequality $\widetilde{z} \leq z$ for negative K, as claimed.

The proof of Proposition 3 is complete. □

COROLLARY 6.5. *Let* $K > 0$, *and let* $(\mathcal{M}, \rho)$ *be a geodesically connected metric space such that* $\operatorname{diam}(\mathcal{M}) \leq \pi/(2\sqrt{K})$ *when* $K > 0$. *If* $(\mathcal{M}, \rho)$ *satisfies the one-sided four-point* cosq_K *condition, then it is an* $\Re_K$ *domain with the same diameter restriction.*

Proof. Theorem 9 in [3, Section 3] states that a metric space $(\mathcal{M}, \rho)$ such that

(i) $(\mathcal{M}, \rho)$ is geodesically connected,
(ii) the perimeter of every geodesic triangle in $(\mathcal{M}, \rho)$ is less than $2\pi/\sqrt{K'}$ if $K' > 0$,
(iii) every point of $(\mathcal{M}, \rho)$ has a neighborhood which is an $\Re_{K'}$ domain, and
(iv) shortests in $(\mathcal{M}, \rho)$ depend continuously on their end points is an $\Re_{K'}$ domain.

By the hypothesis of Corollary 6.5, (i) and (ii) for $K' = K$ are satisfied; (iii) for $K' = K$ is satisfied by Proposition 3; and (iv) is satisfied by Lemma 6.1. Hence, $(\mathcal{M}, \rho)$ is an $\Re_K$ domain. □

Finally, Theorem 1.1 follows from Theorem 4.1, Proposition 3 ($K < 0$), and Corollary 6.5 ($K > 0$).

7. Proof of Theorem 1.2

In this section, we consider an extremal case of Theorem 1.1 when $|\operatorname{cosq}_K| = 1$. We will need a rigidity lemma on geodesic convex hulls of quadruples.

In [3, Section 4, Thm. 6], Aleksandrov established the following rigidity result: if $\mathcal{T} = \mathcal{ABC}$ is a triangle in an $\Re_K$ domain and $\measuredangle ABC = \measuredangle_K ABC$, then $BX = B^K X^K$ for every $X \in \mathcal{AC}$ and $X^K \in \mathcal{A}^K\mathcal{C}^K$ such that $AX = A^K X^K$. Aleksandrov's proof also implies the converse: if $BX_0 = B^K X_0^K$ for at least one point $X_0 \in \mathcal{AC}\backslash\{A, C\}$, then $\measuredangle ABC = \measuredangle_K ABC$. In [11, Prop. 2.9], Bridson and Haefliger slightly improved Aleksandrov's theorem by proving isometry of the convex hulls of the triangles (see also (1) and (2) of Section 2.10 in [11]). The following rigidity lemma is close to Aleksandrov's rigidity theorem in its spirit and in the method of the proof.

Lemma 7.1 [9, Lemma 5.1]. *Let $K \in \mathbb{R}$, and let $\mathfrak{Q} = \{A, P, Q, B\}$ be a quadruple of distinct points in an $\Re_K$ domain. Let $\mathcal{R}$ be a convex quadrangle in $\mathbb{S}_K$ bounded by the closed polygonal curve $\mathcal{L}' = \mathcal{A}'\mathcal{P}'\mathcal{Q}'\mathcal{B}'\mathcal{A}'$ with the vertices at A', P', Q', and B'. Suppose that there is an isometry f from $\mathfrak{Q}$ onto the quadruple $\mathfrak{Q}' = \{A', P', Q', B'\}$ such that $f(A) = A'$, $f(P) = P'$, $f(Q) = Q'$, and $f(B) = B'$. Then the geodesic convex hull of $\mathfrak{Q}$ is isometric to $\mathcal{R}$.*

Finally, we complete the proof of Theorem 1.2. By Theorem 1.1, $(\mathcal{M}, \rho)$ is an $\Re_K$ domain. Let $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = 1$. Because $\operatorname{diam}(A, P, Q, B) < \pi/(2\sqrt{K})$ if $K > 0$, we have $AP + PQ + BQ + AB < 2\pi/\sqrt{K}$, and Reshetnyak's majorization theorem is applicable to the closed curve $\mathcal{L} = \mathcal{APQBA}$. So, as in the proof of Theorem 4.1, consider the closed polygonal curve $\mathcal{L}$ and a convex domain $\mathcal{V} \subseteq \mathbb{S}_K$ ($\partial\mathcal{V} = \mathcal{L}' = \mathcal{A}'\mathcal{P}'\mathcal{Q}'\mathcal{B}'\mathcal{A}'$) majorizing the curve $\mathcal{L}$ and satisfying (4.1). Then, as we showed in the proof of Theorem 4.1,

$$\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) \le \operatorname{cosq}_K(\overrightarrow{A'P'}, \overrightarrow{B'Q'}) \le 1.$$

If either $d = PB < d' = P'B'$ or $f = AQ < f' = A'Q'$, then $1 = \operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) < \operatorname{cosq}_K(\overrightarrow{A'P'}, \overrightarrow{B'Q'})$, a contradiction. So, $f = f'$ and $d = d'$ follow.

Let $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = -1$. By the hypothesis, Reshetnyak's majorization theorem is applicable to the closed curve $\mathcal{N} = \mathcal{AQBPA}$. The reader should follow the proof of Theorem 4.2 in [7] to arrive at the same conclusion $f = f'$ and $d = d'$.

Table 6 Example 8.1, $K = 1$

Case	I	II	III	IV	V	VI
cosq_1	0.0012	0.2048	−0.2865	0.6466	−0.2865	0.2841

Case	VII	VIII	IX	X	XI	XII
cosq_1	0.0012	0.2048	0.6466	0.2841	−0.4756	−0.4756

So, if $|\mathrm{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})| = 1$, then the quadruple $\{A, P, B, Q\}$ in $(\mathcal{M}, \rho)$ is isometric to the quadruple $\{A', P', B', Q'\}$ in $\mathbb{S}_K$. Hence, the statement of Theorem 1.2 follows from Lemma 7.1.

EXAMPLE 7.1. Theorem 1.2 need not be true if we allow $\mathrm{diam}(\mathcal{M}) = \pi/2$. Indeed, consider the metric space $(\mathcal{M}, \rho) = (\mathcal{M}_\varepsilon, \rho_\varepsilon)$ of Example 4.1 for $\varepsilon = 0$. Notice that $(\mathcal{M}, \rho)$ is an $\mathfrak{R}_1$ domain, $\mathrm{diam}(\mathcal{M}) = \pi/2$, and $\mathrm{cosq}_1(\overrightarrow{PO}, \overrightarrow{BQ}) = 1$, whereas $\mathcal{GC}[\{B, Q, O, P\}] = \mathcal{M}$ cannot be isometric to a convex domain in the half-sphere $\mathbb{S}_1$.

8. Remarks

In Section 7, part I, Example 21 in [8], we showed that, for an individual triangular quadruple of points, the four-point cosq_0 condition need not imply 0-concavity, Berestovskii's embeddability condition or Reshetnyak's majorization condition for $K = 0$. It is not difficult to construct a similar example for nonzero K.

EXAMPLE 8.1. Let $\mathfrak{Q} = \{A, B, C, O\}$ be a four-element set. The six (symmetric) distances between the pairs of points in $\mathfrak{Q}$ are given by

$$\rho(A, B) = 0.8, \qquad \rho(B, C) = 1, \qquad \rho(C, O) = 0.95,$$
$$\rho(A, O) = 0.4, \qquad \rho(B, O) = 0.4, \quad \text{and} \quad \rho(A, C) = 1.$$

It is easy to see that ρ is a metric. If we take $A := A$, $P := B$, $B := O$, and $Q := C$, then in the notation of Section 5, all 12 main (approximate) values of cosq_1 and cosq_{-1} for the four-point metric space $(\mathcal{M}, \rho)$ are given in Tables 6 and 7.

Hence, $(\mathfrak{Q}, \rho)$ satisfies the upper four-point cosq_K condition and the lower four-point cosq_K condition for $K = \pm 1$. Notice, that $\mathfrak{Q}$ is a triangular quadruple: O is between A and B. The quadruple $\mathfrak{O}$ is not a nonrectilinear quadruple satisfying Case A in [4], as is required in Theorem 5 in [4, Section 3.]. Let $\mathcal{T}'_+ = A'_+B'_+C'_+$ be a triangle in $\mathbb{S}_1$ and $\mathcal{T}'_- = A'_-B'_-C'_-$ in $\mathbb{S}_{-1}$ be such that the triple $\{A, B, C\}$ is isometric to $\{A'_+, B'_+, C'_+\}$ and $\{A'_-, B'_-, C'_-\}$. Let O'_+ be the midpoint of the shortest $\mathcal{A}'_+\mathcal{B}'_+$, and let O'_- be the midpoint of the shortest $\mathcal{A}'_-\mathcal{B}'_-$. By Lemma 3.1 and a similar formula for $K = -1$, both approximate values for

Table 7 Example 8.1, $K = -1$

Case	I	II	III	IV	V	VI
$\cos q_{-1}$	-0.0106	-0.1647	-0.6208	0.3287	-0.6208	0.6406

Case	VII	VIII	IX	X	XI	XII
$\cos q_{-1}$	-0.0106	-0.1647	0.3287	0.6406	-0.4887	-0.4887

$C'_+O'_+$ and $C'_-O'_-$ are easy to calculate:

$$C'_+O'_+ = \arccos\left(\frac{\cos 1}{\cos 0.4}\right) \approx 0.9439 < 0.95 = CO \quad \text{and}$$

$$C'_-O'_- = \operatorname{arccosh}\left(\frac{\cosh 1}{\cosh 0.4}\right) \approx 0.8944 < 0.95 = CO.$$

Thus, the K-concavity condition fails for the triangular quadruple $\mathfrak{Q}$, and, as a corollary, both Berestovskii's embeddability condition and Reshetnyak's majorization condition for $K = \pm 1$ fail.

In (c) of Part I in [8, Section 7], we erroneously omitted the condition that the triangular quadruple cannot be a nonrectilinear quadruple satisfying case A in [4, Section 1]. We thank Professor Berestovskii for pointing this out in a personal communication.

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