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Multiple Lerch Zeta Functions and an Idea of Ramanujan

Sanoli Gun & Biswajyoti Saha

Abstract. In this article, we derive meromorphic continuation of multiple Lerch zeta functions by generalizing an elegant identity of Ramanujan. Further, we describe the set of all possible singularities of these functions. Finally, for the multiple Hurwitz zeta functions, we list the exact set of singularities.

1. Introduction and Statements of Theorems

In 1917, Ramanujan [18] introduced a novel idea that enabled him to derive an elegant functional equation of the classical Riemann zeta function. He showed that for $\Re(s) > 1$, the Riemann zeta function $\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}$ satisfies the following formula:

$$1 = \sum_{k\geq 0} (s-1)_k (\zeta(s+k) - 1), \tag{1}$$

where the right-hand side of (1) converges normally on any compact subset of $\Re(s) > 1$, and

$$(s)_k := \frac{s \cdots (s+k)}{(k+1)!}$$

for $k \geq 0$ and $s \in \mathbb{C}$. An elementary proof of this formula, as suggested by Ecalle [5], can be deduced from the identity

$$(n-1)^{1-s} - n^{1-s} = \sum_{k\geq 0} (s-1)_k n^{-s-k},$$

which is valid for natural numbers $n \geq 2$ and $s \in \mathbb{C}$.

In fact, Ecalle [5] also suggested how a formula similar to (1) can be derived for the multiple zeta functions. Following Ecalle's indication, the second author, Mehta, and Viswanadham [13] derived such a formula for the multiple zeta functions and studied the meromorphic continuations and the set of their polar singularities (see [13] and [16] for details).

Meromorphic continuations of the multiple zeta functions was proved first by Zhao [20]. Around the same time, Akiyama, Egami, and Tanigawa [1] gave an alternate proof of meromorphic continuations along with the exact set of polar hyperplanes for these functions. In [13], the second author, Mehta, and Viswanadham introduced a method of matrix formulation to write down the residues of the multiple zeta functions in a computable form and thereby reproved the theorem of Akiyama, Egami, and Tanigawa.

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In this paper, we generalize the identity of Ramanujan to obtain meromorphic continuations and the set of possible singularities of the multiple Lerch zeta functions (defined below). When $r = 1$, it was done by the second author in [19]. Let $r > 0$ be a natural number, and let U_r be the open subset of $\mathbb{C}^r$ defined as

$$U_r := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \mid \Re(s_1 + \cdots + s_i) > i \text{ for all } 1 \le i \le r\}.$$

Then for real numbers $\lambda_1, \ldots, \lambda_r, \alpha_1, \ldots, \alpha_r \in [0, 1)$ and complex r-tuples $(s_1, \ldots, s_r) \in U_r$, the multiple Lerch zeta function of depth r is defined by

$$L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{e(\lambda_1 n_1) \cdots e(\lambda_r n_r)}{(n_1 + \alpha_1)^{s_1} \cdots (n_r + \alpha_r)^{s_r}}, \tag{2}$$

where $e(a) := e^{2\pi \iota a}$ for $a \in \mathbb{R}$. The series on the right-hand side of (2) is normally convergent on compact subsets of U_r (see Proposition 1) and hence defines a holomorphic function there.

Before we state our theorems, let us introduce some more notation. For integers $1 \le i \le r$ and $k \ge 0$, let

$$H_{i,k} := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \mid s_1 + \cdots + s_i = i - k\}.$$

Also, for $1 \le i \le r$, let

$$\mu_i := \sum_{j=1}^{i} \lambda_j,$$

and let $\mathbb{Z}_{\le j}$ denote the set of integers less than or equal to j. In this article, we prove the following theorems.

THEOREM 1. *Assume that $\mu_i \notin \mathbb{Z}$ for all $1 \le i \le r$. Then $L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1, \ldots, s_r)$ can be extended analytically to the whole $\mathbb{C}^r$.*

REMARK 1. If $r = 1$ and $\lambda_1 \notin \mathbb{Z}$, Lerch [11] showed that $L_1(\lambda_1; \alpha_1; s_1)$ can be extended to an entire function on $\mathbb{C}$.

THEOREM 2. *With the notation as before, let $i_1 < \cdots < i_m$ be the only indices for which $\mu_{i_j} \in \mathbb{Z}$, $1 \le j \le m$.*

- *If $i_1 = 1$, then $L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1, \ldots, s_r)$ can be meromorphically continued to $\mathbb{C}^r$ with possible simple poles along the hyperplanes*

$$H_{1,0} \textit{ and } H_{i_j,k} \textit{ for } 2 \le j \le m \textit{ with } (i_j - k) \in \mathbb{Z}_{\le j}.$$

- *If $i_1 \neq 1$, then $L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1, \ldots, s_r)$ can be meromorphically continued to $\mathbb{C}^r$ with possible simple poles along the hyperplanes*

$$H_{i_j,k} \textit{ for } 1 \le j \le m \textit{ with } (i_j - k) \in \mathbb{Z}_{\le j}.$$

REMARK 2. Theorem 2 is well known in the special case where $r = 1$. In this case, if $\lambda_1 \in \mathbb{Z}$, then $L_1(\lambda_1; \alpha_1, s_1)$ is essentially the Hurwitz zeta function and hence can be extended analytically to $\mathbb{C}$, except at 1, where it has a simple pole with residue 1.

Komori [10] considered certain several variable generalizations of the Lerch zeta function and derived meromorphic continuations of these functions through integral representation. He also obtained certain estimation of their possible singularities (see [10], Section 3.6).

Now if we choose $\lambda_i = 0$ for $1 \le i \le r$ in (2), then we get

$$L_r(0,\ldots,0;\alpha_1,\ldots,\alpha_r;s_1,\ldots,s_r) = \zeta_r(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r),$$

the multiple Hurwitz zeta function of depth r, and further, if $\alpha_i = 0$ for $1 \le i \le r$, then we get

$$L_r(0,\ldots,0;0,\ldots,0;s_1,\ldots,s_r) = \zeta_r(s_1,\ldots,s_r),$$

the multiple zeta function of depth r.

Akiyama and Ishikawa [2] obtained the meromorphic continuation of the multiple Hurwitz zeta functions together with their possible polar singularities. In the particular case where $\alpha_i \in \mathbb{Q}$ for $1 \le i \le r$, they also derived the exact set of singularities. This has also been done in [14]. Using the Mellin–Barnes integral formula, Matsumoto [12] showed meromorphic continuation of multiple Hurwitz zeta functions with possible set of singularities. Finally, we refer the interested reader to [7] and [17], where similar topics are addressed. An expression for residues together with possible polar hyperplanes was obtained in [9; 14]. For the multiple Hurwitz zeta functions, we are now able to characterize the exact set of singularities. This complete characterization is new. More precisely, we have the following theorem.

THEOREM 3. *The multiple Hurwitz zeta function $\zeta_r(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)$ has meromorphic continuation to $\mathbb{C}^r$. Further, all its poles are simple, and they are along the hyperplanes*

$$H_{1,0} \text{ and } H_{i,k} \text{ for } 2 \le i \le r,\, k \ge 0$$

except for $i = 2$ and $k \in K$, where

$$K := \{n \in \mathbb{N} \mid B_n(\alpha_2 - \alpha_1) = 0\},$$

and $B_n(t)$ denotes the nth Bernoulli polynomial defined by the generating series

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n\ge 0} B_n(t)\frac{x^n}{n!}.$$

Before proceeding further, we indicate, compare, and contrast some of the other existing works vis-à-vis our work. In [15], the authors obtain meromorphic continuation for a multiple Hurwitz zeta function of an arbitrary depth r using binomial expansion. To this end, they deduce a functional equation involving various multiple Hurwitz zeta functions of a fixed depth r (see Theorem 5.2). The novelty of our work is deducing a functional equation involving multiple Hurwitz zeta functions of depth r with multiple Hurwitz zeta functions of depth $r - 1$ (see Theorem 4). This is the crucial ingredient, which enables us to derive information about the poles and residues of such functions, which was not done in [15]. The use of binomial expansion has also been exploited in [6] for proving the meromorphic continuation of multiple Hurwitz zeta functions. More precisely, he uses products of binomial expansions, which we avoid. Also he deals only with the diagonal vectors in the r-dimensional complex plane, whereas we allow arbitrary vectors in $\mathbb{C}^r$. Furthermore, the author does not deal with the poles and residues of these functions.

The paper is distributed as follows. In the next section, we prove some intermediate results and derive functional identities for the multiple Lerch zeta function, which are generalizations of the Ramanujan identity (see Theorem 4). In Section 3, we derive meromorphic continuation of the multiple Lerch zeta functions and a possible set of their singularities using these functional identities. In Section 4, we follow [13] to write down the relevant functional identity for the multiple Hurwitz zeta functions in terms of infinite matrices in order to obtain an expression for residues along the singular hyperplanes (see Theorem 6). Finally in Section 5, we complete the proof of Theorem 3. For this, we need to use some fundamental properties of the zeros of the Bernoulli polynomials. These results are discussed in Section 5.1.

2. Intermediate Results and Generalized Ramanujan's Identity

In this section, we derive an analogue of (1) (see (3)) for the multiple Lerch zeta functions. To establish (3), we need some intermediate results. Before we state our theorem, we start with the notion of normal convergence.

DEFINITION 1. Let X be a set, and let $(f_i)_{i\in I}$ be a family of complex-valued functions defined on X. We say that the family $(f_i)_{i\in I}$ is normally summable on X (or the series $\sum_{i\in I} f_i$ converges normally on X) if

$$\|f_i\|_X := \sup_{x\in X} |f(x)| < \infty \quad \text{for all } i \in I$$

and the family of real numbers $(\|f_i\|_X)_{i\in I}$ is summable.

DEFINITION 2. Let X be an open subset of $\mathbb{C}^r$, and let $(f_i)_{i\in I}$ be a family of meromorphic functions on X. We say that $(f_i)_{i\in I}$ is normally summable (or $\sum_{i\in I} f_i$ is normally convergent on all compact subsets of X) if for any compact subset K of X, there exists a finite set $J \subset I$ such that each f_i for $i \in I \setminus J$ is holomorphic in an open neighborhood of K and the family $(f_i \mid K)_{i\in I\setminus J}$ is normally summable on K. In this case, $\sum_{i\in I} f_i$ is a well-defined meromorphic function on X.

We now have the following theorem.

THEOREM 4. *Let $r \geq 2$ be a natural number, and let $\lambda_1, \ldots, \lambda_r, \alpha_1, \ldots, \alpha_r \in [0, 1)$. Then for any $(s_1, \ldots, s_r) \in U_r$, we have*

$$e(\lambda_1) \sum_{k\geq -1} (s_1)_k (\alpha_2 - \alpha_1)^{k+1} \\ \times L_{r-1}(\mu_2, \lambda_3, \ldots, \lambda_r; \alpha_2, \ldots, \alpha_r; s_1 + s_2 + k + 1, s_3, \ldots, s_r)$$

$$= (1 - e(\lambda_1))L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1, \ldots, s_r) + \sum_{k\geq 0}(s_1)_k L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1 + k + 1, s_2, \ldots, s_r), \tag{3}$$

where $(s)_{-1} := 1$ *and for* $k \geq 0$,

$$(s)_k := \frac{s \cdots (s+k)}{(k+1)!},$$

and the series on both sides of (3) *converge normally on every compact subset of* U_r.

If $\lambda_1 = 0$, then we rewrite (3) as

$$\sum_{k\geq -1}(s_1 - 1)_k(\alpha_2 - \alpha_1)^{k+1} \times L_{r-1}(\lambda_2, \lambda_3, \ldots, \lambda_r; \alpha_2, \ldots, \alpha_r; s_1 + s_2 + k, s_3, \ldots, s_r) = \sum_{k\geq 0}(s_1 - 1)_k L_r(0, \lambda_2, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1 + k, s_2, \ldots, s_r). \tag{4}$$

From now on, we call identities (3) and (4) the generalized Ramanujan identity for the multiple Lerch zeta functions. To prove Theorem 4, we introduce another notation and prove some intermediate results. For any $m \geq 0$, let

$$U_r(m) := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \mid \Re(s_1 + \cdots + s_i) > i - m \text{ for all } 1 \leq i \leq r\}.$$

Note that $U_r = U_r(0)$. We first observe that the series on the right-hand side of (2) is normally convergent on compact subsets of U_r. For this, we need the following lemma from [13].

LEMMA 1. *For an integer* $r \geq 1$, *the family of functions*

$$\left(\frac{1}{n_1^{s_1} \cdots n_r^{s_r}}\right)_{n_1 > \cdots > n_r > 0}$$

converges normally on any compact subset of U_r.

PROPOSITION 1. *For an integer* $r \geq 1$ *and for* $\lambda_1, \ldots, \lambda_r, \alpha_1, \ldots, \alpha_r \in [0, 1)$, *the family of functions*

$$\left(\frac{e(\lambda_1 n_1) \cdots e(\lambda_r n_r)}{(n_1 + \alpha_1)^{s_1} \cdots (n_r + \alpha_r)^{s_r}}\right)_{n_1 > \cdots > n_r > 0}$$

converges normally on any compact subset of U_r.

Proof. The proposition follows immediately from Lemma 1 since

$$\left|\frac{e(\lambda_1 n_1) \cdots e(\lambda_r n_r)}{(n_1 + \alpha_1)^{s_1} \cdots (n_r + \alpha_r)^{s_r}}\right| \leq \left|\frac{1}{n_1^{s_1} \cdots n_r^{s_r}}\right| \quad \text{in } U_r.$$

□

We further need the following propositions.

PROPOSITION 2. *Let $m \geq 0$ and $r \geq 2$ be natural numbers, and let $\lambda_1, \ldots, \lambda_r, \alpha_1, \ldots, \alpha_r \in [0, 1)$. Then the family of functions*

$$\left((s_1)_k \frac{e(\lambda_1 n_1) e(\lambda_2 n_2) \cdots e(\lambda_r n_r)}{(n_1 + \alpha_1)^{s_1+k+1} (n_2 + \alpha_2)^{s_2} \cdots (n_r + \alpha_r)^{s_r}} \right)_{\substack{n_1 > \cdots > n_r > 0, \\ k \geq m-1}}$$

is normally summable on compact subsets of $U_r(m)$.

Proof. Let K be a compact subset of $U_r(m)$, and let $S := \sup_{(s_1,\ldots,s_r)\in K} |s_1|$. Since $r \geq 2$, we have $n_1 \geq 2$, and hence for $k \geq m-1$ and $(s_1, \ldots, s_r) \in U_r(m)$, we have

$$\begin{aligned} &\left\| (s_1)_k \frac{e(\lambda_1 n_1) \cdots e(\lambda_r n_r)}{(n_1 + \alpha_1)^{s_1+k+1} (n_2 + \alpha_2)^{s_2} \cdots (n_r + \alpha_r)^{s_r}} \right\|_K \\ &\leq \frac{(S)_k}{2^{k-m+1}} \left\| \frac{1}{n_1^{s_1+m} n_2^{s_2} \cdots n_r^{s_r}} \right\|_K . \end{aligned}$$

Note that $(s_1, \ldots, s_r) \in U_r(m)$ if and only if $(s_1 + m, s_2, \ldots, s_r) \in U_r$. Now the proof of Proposition 2 follows from Lemma 1 and from the convergence of the series

$$\sum_{k \geq m-1} \frac{(S)_k}{2^{k-m+1}}.$$

□

PROPOSITION 3. *Let $m \geq 0$ and $r \geq 2$ be natural numbers, and let $\lambda_1, \ldots, \lambda_r, \alpha_1, \ldots, \alpha_r \in [0, 1)$. Then the family of functions*

$$\left((s_1)_k (\alpha_2 - \alpha_1)^{k+1} \frac{e(\mu_2 n_2) e(\lambda_3 n_3) \cdots e(\lambda_r n_r)}{(n_2 + \alpha_2)^{s_1+s_2+k+1} (n_3 + \alpha_3)^{s_3} \cdots (n_r + \alpha_r)^{s_r}} \right)_{\substack{n_2 > \cdots > n_r > 0, \\ k \geq m-1}}$$

is normally summable on any compact subset of $U_r(m+1)$ and hence on U_r.

Proof. As before, let K be a compact subset of $U_r(m+1)$, and let

$$S := \sup_{(s_1,\ldots,s_r)\in K} |s_1|.$$

Then, for $k \geq m-1$, $r \geq 2$, and $(s_1, \ldots, s_r) \in U_r(m)$, we have

$$\begin{aligned} &\left\| (s_1)_k \frac{(\alpha_2 - \alpha_1)^{k+1} e(\mu_2 n_2) e(\lambda_3 n_3) \cdots e(\lambda_r n_r)}{(n_2 + \alpha_2)^{s_1+s_2+k+1} (n_3 + \alpha_3)^{s_3} \cdots (n_r + \alpha_r)^{s_r}} \right\| \\ &\leq (S)_k (\alpha_2 - \alpha_1)^{k+1} \left\| \frac{1}{n_2^{s_1+s_2+m} n_3^{s_3} \cdots n_r^{s_r}} \right\|. \end{aligned}$$

Note that

$$\begin{aligned} (s_1, \ldots, s_r) \in U_r(m+1) \quad &\Longrightarrow \quad (s_1 + s_2, s_3, \ldots, s_r) \in U_{r-1}(m) \\ &\Longrightarrow \quad (s_1 + s_2 + m, s_3, \ldots, s_r) \in U_{r-1}. \end{aligned}$$

The proof now follows from Lemma 1 (for $(r-1)$) and from the convergence of

$$\sum_{k \geq m-1} (S)_k (\alpha_2 - \alpha_1)^{k+1}$$

as $|\alpha_2 - \alpha_1| < 1$. □

PROPOSITION 4. *Let $r \geq 2$ be an integer, and let $\lambda_1, \ldots, \lambda_r, \alpha_1, \ldots, \alpha_r \in [0, 1)$. The family of functions*

$$\left(\frac{e(\lambda_1 n_1)\cdots e(\lambda_r n_r)}{(n_1+\alpha_1-1)^{s_1}(n_2+\alpha_2)^{s_2}\cdots(n_r+\alpha_r)^{s_r}}\right)_{n_1>\cdots>n_r>0}$$

is normally summable on any compact subset of U_r.

Proof. Note that

$$\left|\frac{e(\lambda_1 n_1)\cdots e(\lambda_r n_r)}{(n_1+\alpha_1-1)^{s_1}(n_2+\alpha_2)^{s_2}\cdots(n_r+\alpha_r)^{s_r}}\right| \leq \left|\frac{1}{(n_1-1)^{s_1} n_2^{s_2}\cdots n_r^{s_r}}\right|.$$

Also, note that

$$\begin{aligned}\left|\sum_{n_1\geq n_2+1}(n_1-1)^{-s_1}\right| &\leq n_2^{-\Re(s_1)} + \sum_{n_1\geq n_2+1} n_1^{-\Re(s_1)}\\ &\leq n_2^{-\Re(s_1)} + \int_{n_2}^{\infty} x^{-\Re(s_1)}\,dx\\ &= n_2^{-\Re(s_1)} + \frac{1}{\Re(s_1)-1} n_2^{1-\Re(s_1)}.\end{aligned}$$

The proof follows from Lemma 1. □

2.1. *Proof of Theorem 4*

We begin with the following identity for integers $n \geq 2$, real numbers $\alpha \geq 0$, and complex numbers s:

$$(n+\alpha-1)^{-s} = \sum_{k\geq -1} (s)_k (n+\alpha)^{-s-k-1}. \tag{5}$$

It is easily obtained by writing the left-hand side as $(n+\alpha)^{-s}(1-\frac{1}{n+\alpha})^{-s}$ and expanding $(1-\frac{1}{n+\alpha})^{-s}$ as a Taylor series in $\frac{1}{n+\alpha}$.

In (5) we replace n, α, s by n_1, α_1, s_1, respectively, then multiply both sides by

$$\frac{e(\lambda_1 n_1)\cdots e(\lambda_r n_r)}{(n_2+\alpha_2)^{s_2}\cdots(n_r+\alpha_r)^{s_r}},$$

and, finally, sum over $n_1 > \cdots > n_r > 0$. Using Proposition 4, we get that

$$\begin{aligned}&\sum_{n_1>\cdots>n_r>0}\frac{e(\lambda_1 n_1)\cdots e(\lambda_r n_r)}{(n_1+\alpha_1-1)^{s_1}(n_2+\alpha_2)^{s_2}\cdots(n_r+\alpha_r)^{s_r}}\\ &\quad= e(\lambda_1)\sum_{n_1>\cdots>n_r>0}\frac{e(\lambda_1 n_1)\cdots e(\lambda_r n_r)}{(n_1+\alpha_1)^{s_1}(n_2+\alpha_2)^{s_2}\cdots(n_r+\alpha_r)^{s_r}}\\ &\qquad+ e(\lambda_1)\\ &\qquad\times\sum_{n_2>\cdots>n_r>0}\frac{e(\mu_2 n_2)e(\lambda_3 n_3)\cdots e(\lambda_r n_r)}{(n_2+\alpha_1)^{s_1}(n_2+\alpha_2)^{s_2}(n_3+\alpha_3)^{s_3}\cdots(n_r+\alpha_r)^{s_r}}.\end{aligned} \tag{6}$$

Now,

$$(n_2+\alpha_1)^{-s_1} = \sum_{k\geq -1} (s_1)_k (\alpha_2-\alpha_1)^{k+1}(n_2+\alpha_2)^{-s_1-k-1}.$$

Hence, using Proposition 3 (for $m=0$), we obtain that

$$\begin{aligned}\sum_{n_1>\cdots>n_r>0} & \frac{e(\lambda_1 n_1)\cdots e(\lambda_r n_r)}{(n_1+\alpha_1-1)^{s_1}(n_2+\alpha_2)^{s_2}\cdots(n_r+\alpha_r)^{s_r}} \\ &= e(\lambda_1)L_r(\lambda_1,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1,\ldots,s_r) \\ &\quad + e(\lambda_1)\sum_{k\geq -1}(s_1)_k(\alpha_2-\alpha_1)^{k+1} \\ &\quad \times L_{r-1}(\mu_2,\lambda_3,\ldots,\lambda_r;\alpha_2,\ldots,\alpha_r;s_1+s_2+k+1,s_3,\ldots,s_r). \end{aligned} \tag{7}$$

On the other hand, using (5) and Proposition 2 (for $m=0$), we get that

$$\begin{aligned}\sum_{n_1>\cdots>n_r>0} & \frac{e(\lambda_1 n_1)\cdots e(\lambda_r n_r)}{(n_1+\alpha_1-1)^{s_1}(n_2+\alpha_2)^{s_2}\cdots(n_r+\alpha_r)^{s_r}} \\ &= \sum_{k\geq -1}(s_1)_k L_r(\lambda_1,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1+k+1,s_2,\ldots,s_r). \end{aligned} \tag{8}$$

Now, equating the right-hand sides of (7) and (8), we deduce (3). This, together with Proposition 2 and Proposition 3, completes the proof.

3. Proofs of Theorem 1 and Theorem 2

In this section, we use the generalized Ramanujan identities (3) and (4) to prove Theorem 1 and Theorem 2. We prove these theorems by induction on depth r. We assume that the multiple Lerch zeta function of depth $(r-1)$ has already been extended to $\mathbb{C}^r$, and then by induction on $m \geq 1$ we extend the multiple Lerch zeta function of depth r to each of $U_r(m)$. Since $(U_r(m))_{m\geq 1}$ is an open covering of $\mathbb{C}^r$, we get the desired result.

3.1. *Proof of Theorem 1*

When $r=1$, Theorem 1 is true by Remark 1. Now let $r\geq 2$ and $\mu_i \notin \mathbb{Z}$ for $1\leq i\leq r$. For any $m\geq 1$, we rewrite (3) as

$$\begin{aligned} e(\lambda_1) & \sum_{k\geq m-2}(s_1)_k(\alpha_2-\alpha_1)^{k+1} \\ &\quad \times L_{r-1}(\mu_2,\lambda_3,\ldots,\lambda_r;\alpha_2,\ldots,\alpha_r;s_1+s_2+k+1,s_3,\ldots,s_r) \\ &\quad + e(\lambda_1)\sum_{-1\leq k\leq m-3}(s_1)_k(\alpha_2-\alpha_1)^{k+1} \\ &\quad \times L_{r-1}(\mu_2,\lambda_3,\ldots,\lambda_r;\alpha_2,\ldots,\alpha_r;s_1+s_2+k+1,s_3,\ldots,s_r) \\ &= (1-e(\lambda_1))L_r(\lambda_1,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1,\ldots,s_r) \end{aligned}$$

$$+\sum_{k\geq m-1}(s_1)_k L_r(\lambda_1,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1+k+1,s_2,\ldots,s_r)$$
$$+\sum_{0\leq k\leq m-2}(s_1)_k L_r(\lambda_1,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1+k+1,s_2,\ldots,s_r).$$

Now by Proposition 2, Proposition 3, and the induction hypothesis for multiple Lerch zeta functions of depth $(r-1)$ we see that all the k-sums in (3) are analytic in $U_r(1)$. Therefore (3) defines an analytic continuation of

$$L_r(\lambda_1,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1,\ldots,s_r)$$

as $e(\lambda_1)\neq 1$. Now suppose that we have an analytic continuation of

$$L_r(\lambda_1,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1,\ldots,s_r)$$

to $U_r(m-1)$ that satisfies (3) in $U_r(m-1)$. Thus we get that the sum

$$\sum_{0\leq k\leq m-2}(s_1)_k L_r(\lambda_1,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1+k+1,s_2,\ldots,s_r)$$

is analytic in $U_r(m)$. Again, we appeal to Proposition 2, Proposition 3, and the induction hypothesis for multiple Lerch zeta functions of depth $(r-1)$ to deduce that all the k-sums in (3) are analytic in $U_r(m)$. Hence we obtain an analytic continuation of

$$L_r(\lambda_1,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1,\ldots,s_r)$$

to $U_r(m)$. Since $(U_r(m))_{m\geq 1}$ is an open covering of $\mathbb{C}^r$, this completes the proof.

3.2. *Proof of Theorem 2*

When $r=1$, Theorem 2 follows from Remark 1 if $\lambda_1\notin\mathbb{Z}$ and from Remark 2 if $\lambda_1\in\mathbb{Z}$. Now suppose that $r\geq 2$ and Theorem 2 is true for a multiple Lerch zeta function of depth $(r-1)$.

3.3. *Case 1: $i_1=1$*

In this case, we have $\lambda_1=0$ and hence use (4). Recall that

$$\begin{aligned}\sum_{k\geq -1}&(s_1-1)_k(\alpha_2-\alpha_1)^{k+1}\\ &\times L_{r-1}(\lambda_2,\lambda_3,\ldots,\lambda_r;\alpha_2,\ldots,\alpha_r;s_1+s_2+k,s_3,\ldots,s_r)\\ &=\sum_{k\geq 0}(s_1-1)_k L_r(0,\lambda_2,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1+k,s_2,\ldots,s_r).\end{aligned}\tag{4}$$

To prove this case, we establish the meromorphic continuation of

$$(s_1-1)L_r(0,\lambda_2,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1,\ldots,s_r)$$

to $\mathbb{C}^r$ using (4) and determine all its possible singularities.

For any $m \geq 1$, we know by Proposition 2 and Proposition 3 that the families of functions

$$\left((s_1-1)_k \frac{e(\lambda_2 n_2)\cdots e(\lambda_r n_r)}{(n_1+\alpha_1)^{s_1+k}(n_2+\alpha_2)^{s_2}\cdots(n_r+\alpha_r)^{s_r}}\right)_{\substack{n_1>\cdots>n_r>0,\\ k\geq m}}$$

and

$$\left((s_1-1)_k(\alpha_2-\alpha_1)^{k+1} \frac{e(\lambda_2 n_2)e(\lambda_3 n_3)\cdots e(\lambda_r n_r)}{(n_2+\alpha_2)^{s_1+s_2+k}(n_3+\alpha_3)^{s_3}\cdots(n_r+\alpha_r)^{s_r}}\right)_{\substack{n_2>\cdots>n_r>0,\\ k\geq m-1}}$$

are normally summable on every compact subset of $U_r(m)$.

Now, for any $m \geq 1$, we rewrite (3) as

$$\begin{aligned}
\sum_{k\geq m-1} & (s_1-1)_k(\alpha_2-\alpha_1)^{k+1} \\
& \times L_{r-1}(\lambda_2,\lambda_3,\ldots,\lambda_r;\alpha_2,\ldots,\alpha_r;s_1+s_2+k,s_3,\ldots,s_r) \\
& + \sum_{-1\leq k\leq m-2} (s_1-1)_k(\alpha_2-\alpha_1)^{k+1} \\
& \times L_{r-1}(\lambda_2,\lambda_3,\ldots,\lambda_r;\alpha_2,\ldots,\alpha_r;s_1+s_2+k,s_3,\ldots,s_r) \\
= {} & (s_1-1)L_r(0,\lambda_2,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1,\ldots,s_r) \\
& + \sum_{k\geq m}(s_1-1)_k L_r(0,\lambda_2,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1+k,s_2,\ldots,s_r) \\
& + \sum_{1\leq k\leq m-1} (s_1-1)_k L_r(0,\lambda_2,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1+k,s_2,\ldots,s_r).
\end{aligned}$$

Using the previous observation, we obtain that both infinite k-sums in this equation are analytic in $U_r(m)$. From the induction hypothesis we deduce that the sum

$$\begin{aligned}
\sum_{-1\leq k\leq m-2} & (s_1-1)_k(\alpha_2-\alpha_1)^{k+1} \\
& \times L_{r-1}(\lambda_2,\lambda_3,\ldots,\lambda_r;\alpha_2,\ldots,\alpha_r;s_1+s_2+k,s_3,\ldots,s_r)
\end{aligned}$$

has a meromorphic continuation to $\mathbb{C}^r$. Now if the function

$$(s_1-1)L_r(0,\lambda_2,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1,\ldots,s_r)$$

has a meromorphic continuation to $U_r(m-1)$ for each $m \geq 1$, then we can deduce that the sum

$$\sum_{1\leq k\leq m-1} (s_1-1)_k L_r(0,\lambda_2,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1+k,s_2,\ldots,s_r)$$

has a meromorphic continuation to $U_r(m)$ for each $m \geq 1$. Therefore we obtain a meromorphic continuation of

$$(s_1-1)L_r(0,\lambda_2,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_r;s_1,\ldots,s_r)$$

to $U_r(m)$ by means of (4). Since $(U_r(m))_{m\geq 1}$ is an open covering of $\mathbb{C}^r$, we obtain a meromorphic continuation of

$$(s_1 - 1)L_r(0, \lambda_2, \dots, \lambda_r; \alpha_1, \dots, \alpha_r; s_1, \dots, s_r)$$

to $\mathbb{C}^r$.

Now for the set of singularities, we see from (4) that the singularities of

$$(s_1 - 1)L_r(0, \lambda_2, \dots, \lambda_r; \alpha_1, \dots, \alpha_r; s_1, \dots, s_r)$$

can only come from that of

$$L_{r-1}(\lambda_2, \dots, \lambda_r; \alpha_2, \dots, \alpha_r; s_1 + s_2 + k, s_3, \dots, s_r)$$

for all $k \geq -1$, and these singularities are known from the induction hypothesis. Finally, we deduce that

$$(s_1 - 1)L_r(0, \lambda_2, \dots, \lambda_r; \alpha_1, \dots, \alpha_r; s_1, \dots, s_r)$$

has possible polar singularities only along the hyperplanes

$$H_{i_j,k} \text{ for } 2 \leq j \leq m \text{ with } (i_j - k) \in \mathbb{Z}_{\leq j}.$$

This completes the proof of this case.

3.4. *Case 2: $i_1 \neq 1$*

Since in this case the applicable generalized Ramanujan identity is (3), a proof of this case follows exactly the line of argument in the proof of Theorem 1. The only difference is that, on each of $U_r(m)$, the depth r multiple Lerch zeta function can only be extended as a meromorphic function. This is because the induction hypothesis implies that the depth $(r-1)$ multiple Lerch zeta functions

$$L_{r-1}(\mu_2, \lambda_3, \dots, \lambda_r; \alpha_2, \dots, \alpha_r; s_1 + s_2 + k + 1, s_3, \dots, s_r)$$

for $k \geq -1$ can only be extended as meromorphic functions to $\mathbb{C}^r$.

Now for the set of singularities, we see from (3) that the singularities of

$$L_r(\lambda_1, \dots, \lambda_r; \alpha_1, \dots, \alpha_r; s_1, \dots, s_r)$$

can only come from that of

$$L_{r-1}(\mu_2, \lambda_3, \dots, \lambda_r; \alpha_2, \dots, \alpha_r; s_1 + s_2 + k + 1, s_3, \dots, s_r)$$

for $k \geq -1$. These singularities are known from the induction hypothesis, and hence we deduce that

$$L_r(\lambda_1, \dots, \lambda_r; \alpha_1, \dots, \alpha_r; s_1, \dots, s_r)$$

has only possible polar singularities along the hyperplanes

$$H_{i_j,k} \text{ for } 1 \leq j \leq m \text{ with } (i_j - k) \in \mathbb{Z}_{\leq j}.$$

This completes the proof of Theorem 2.

4. Explicit Computations of Residues for Multiple Hurwitz Zeta Functions

To get the exact set of singularities, we need to calculate the residues of a multiple Lerch zeta function along its possible polar hyperplanes. For a hyperplane $H_{i,k}$, by residue of

$$L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1, \ldots, s_r)$$

along $H_{i,k}$ we mean the restriction to $H_{i,k}$ of the meromorphic function

$$(s_1 + \cdots + s_i - i + k) L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1, \ldots, s_r).$$

This particular notion of residue was introduced by Oesterlé [16] during a course of lectures on multiple zeta values at Institute of Mathematical Sciences, Chennai. This definition has also been used in [13]. It turns out that to study non-vanishing of these residues, we need information about zero sets of a family of polynomials with two variables (see Remark 3). However, for multiple Hurwitz zeta functions, we only have to deal with the family of Bernoulli polynomials. As the zero set of Bernoulli polynomials is well studied, we just have enough information to determine the exact set of singularities of the multiple Hurwitz zeta functions.

In what follows, we obtain a computable expression for residues of the multiple Hurwitz zeta functions. Note that the applicable generalized Ramanujan identity in this case is (4). Following this process, we can also obtain a similar expression for residues of the multiple Lerch zeta functions. For brevity, we do not include this here. We begin this section with some elementary remarks about infinite triangular matrices.

Let R be a commutative ring with unity. By $\mathbf{T}(R)$ we denote the set of upper triangular matrices of type $\mathbb{N} \times \mathbb{N}$ with coefficients in R. Adding or multiplying such matrices involves only finite sums, and hence $\mathbf{T}(R)$ is a ring and even an R-algebra. The group of invertible elements of $\mathbf{T}(R)$ is the matrices whose diagonal elements are invertible. Now let $\mathbf{P}$ be a matrix in $\mathbf{T}(R)$ with all diagonal elements equal to 0, and let $f = \sum_{n \geq 0} a_n x^n \in R[[x]]$ be a formal power series. Then the series $\sum_{n \geq 0} a_n \mathbf{P}^n$ converges in $\mathbf{T}(R)$, and we denote its sum by $f(\mathbf{P})$. For our purpose, we take R to be the field of rational fractions $\mathbb{C}(t)$ in one indeterminate t over $\mathbb{C}$.

Recall that from Theorem 4 we get that the multiple Hurwitz zeta functions of depth r satisfy the following generalized Ramanujan identity:

$$\sum_{k \geq -1} (s_1 - 1)_k (\alpha_2 - \alpha_1)^{k+1} \zeta_{r-1}(s_1 + s_2 + k, s_3, \ldots, s_r; \alpha_2, \alpha_3, \ldots, \alpha_r)$$
$$= \sum_{k \geq 0} (s_1 - 1)_k \zeta_r(s_1 + k, s_2, \ldots, s_r; \alpha_1, \alpha_2, \ldots, \alpha_r), \tag{9}$$

where both series of meromorphic functions converge normally on compact subsets of $\mathbb{C}^r$. Formula (9), together with the set of relations obtained by applying successively the change of variable $s_1 \mapsto s_1 + n$ for $n \geq 1$ to (9), can be written as

$$\mathbf{A_2}(\alpha_2-\alpha_1;s_1-1)\mathbf{V}_{r-1}(s_1+s_2-1,s_3,\dots,s_r;\alpha_2,\dots,\alpha_r)$$
$$=\mathbf{A_1}(s_1-1)\mathbf{V}_r(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r). \tag{10}$$

Here for an indeterminate t, we have

$$\mathbf{A_1}(t):=\begin{pmatrix} t & \frac{t(t+1)}{2!} & \frac{t(t+1)(t+2)}{3!} & \cdots \\ 0 & t+1 & \frac{(t+1)(t+2)}{2!} & \cdots \\ 0 & 0 & t+2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{11}$$

$$\mathbf{A_2}(\alpha_2-\alpha_1;t):=\begin{pmatrix} 1 & t(\alpha_2-\alpha_1) & \frac{t(t+1)}{2!}(\alpha_2-\alpha_1)^2 & \cdots \\ 0 & 1 & (t+1)(\alpha_2-\alpha_1) & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{12}$$

and

$$\mathbf{V}_r(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r):=\begin{pmatrix} \zeta_r(s_1,s_2,\dots,s_r;\alpha_1,\dots,\alpha_r) \\ \zeta_r(s_1+1,s_2,\dots,s_r;\alpha_1,\dots,\alpha_r) \\ \zeta_r(s_1+2,s_3\dots,s_r;\alpha_1,\dots,\alpha_r) \\ \vdots \end{pmatrix}. \tag{13}$$

Note that the matrix $\mathbf{A_1}(t)$ can be written as

$$\mathbf{A_1}(t)=\mathbf{\Delta}(t)f(\mathbf{M}(t+1)),$$

where f is the formal power series

$$f(x):=\frac{e^x-1}{x}=\sum_{n\ge 0}\frac{x^n}{(n+1)!},$$

and $\mathbf{\Delta}(t)$ and $\mathbf{M}(t)$ are as follows:

$$\mathbf{\Delta}(t):=\begin{pmatrix} t & 0 & 0 & \cdots \\ 0 & t+1 & 0 & \cdots \\ 0 & 0 & t+2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \mathbf{M}(t):=\begin{pmatrix} 0 & t & 0 & \cdots \\ 0 & 0 & t+1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to see that $\mathbf{\Delta}(t)$ and $\mathbf{M}(t)$ satisfy the commuting relation

$$\mathbf{\Delta}(t)\mathbf{M}(t+1)=\mathbf{M}(t)\mathbf{\Delta}(t). \tag{14}$$

Thus, using (14), we have

$$\mathbf{A_1}(t)=f(\mathbf{M}(t))\mathbf{\Delta}(t).$$

Further, it is also possible to write that

$$\mathbf{A_2}(\alpha_2-\alpha_1;t)=h(\mathbf{M}(t)),$$

where h denotes the power series

$$e^{(\alpha_2-\alpha_1)x} = \sum_{n\geq 0} (\alpha_2-\alpha_1)^n \frac{x^n}{n!}.$$

Clearly, the matrix $\mathbf{A_2}(\alpha_2-\alpha_1;t)$ is invertible, and we see that

$$\mathbf{A_2}(\alpha_2-\alpha_1;t)^{-1}\mathbf{A_1}(t) = \frac{f}{h}(\mathbf{M}(t))\boldsymbol{\Delta}(t) = \boldsymbol{\Delta}(t)\frac{f}{h}(\mathbf{M}(t+1)).$$

Hence the inverse of the matrix $\mathbf{A_2}(\alpha_2-\alpha_1;t)^{-1}\mathbf{A_1}(t)$ is given by

$$\begin{aligned}\mathbf{B}(\alpha_2-\alpha_1;t) := \mathbf{A_1}(t)^{-1}\mathbf{A_2}(\alpha_2-\alpha_1;t) &= \frac{h}{f}(\mathbf{M}(t+1))\boldsymbol{\Delta}(t)^{-1}\\ &= \boldsymbol{\Delta}(t)^{-1}\frac{h}{f}(\mathbf{M}(t)),\end{aligned}$$

where $\frac{h}{f}$ is the exponential generating series of the Bernoulli polynomials evaluated at the point $(\alpha_2-\alpha_1)$, that is,

$$\frac{h}{f}(x) = \frac{xe^{(\alpha_2-\alpha_1)x}}{e^x-1} = \sum_{n\geq 0}\frac{B_n(\alpha_2-\alpha_1)}{n!}x^n.$$

More precisely, we have

$$\mathbf{B}(\alpha_2-\alpha_1;t) = \begin{pmatrix} \frac{1}{t} & \frac{B_1(\alpha_2-\alpha_1)}{1!} & \frac{(t+1)B_2(\alpha_2-\alpha_1)}{2!} & \frac{(t+1)(t+2)B_3(\alpha_2-\alpha_1)}{3!} & \cdots \\ 0 & \frac{1}{t+1} & \frac{B_1(\alpha_2-\alpha_1)}{1!} & \frac{(t+2)B_2(\alpha_2-\alpha_1)}{2!} & \cdots \\ 0 & 0 & \frac{1}{t+2} & \frac{B_1(\alpha_2-\alpha_1)}{1!} & \cdots \\ 0 & 0 & 0 & \frac{1}{t+3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{15}$$

However, we cannot express the column vector $\mathbf{V}_r(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)$ as the product of the matrix $\mathbf{B}(\alpha_2-\alpha_1;s_1-1)$ and the column vector $\mathbf{V}_{r-1}(s_1+s_2-1,s_3,\ldots,s_r;\alpha_2,\ldots,\alpha_r)$. This is because the infinite series involved in this product are not convergent. To get around this difficulty, we perform a truncation process.

We first rewrite (10) in the form

$$\begin{aligned}&\boldsymbol{\Delta}(s_1-1)^{-1}\mathbf{V}_{r-1}(s_1+s_2-1,s_3,\ldots,s_r;\alpha_2,\ldots,\alpha_r)\\ &\qquad = \frac{f}{h}(\mathbf{M}(s_1))\mathbf{V}_r(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r).\end{aligned} \tag{16}$$

For notational convenience, let us denote $\frac{f}{h}(\mathbf{M}(s_1))$ by $\mathbf{X}(s_1)$. We then choose an integer $q\geq 1$ and define

$$I := \{k \mid 0\leq k\leq q-1\} \quad \text{and} \quad J := \{k \mid k\geq q\}. \tag{17}$$

Then we write our matrices as block matrices, for example,

$$\mathbf{X}(s_1) = \begin{pmatrix}\mathbf{X}^{II}(s_1) & \mathbf{X}^{IJ}(s_1)\\ \mathbf{0}^{JI} & \mathbf{X}^{JJ}(s_1)\end{pmatrix}.$$

Hence from (16) we get that

$$\begin{aligned}
&\mathbf{\Delta}^{II}(s_1-1)^{-1}\mathbf{V}^{I}_{r-1}(s_1+s_2-1,s_3,\dots,s_r;\alpha_2,\dots,\alpha_r)\\
&\quad=\mathbf{X}^{II}(s_1)\mathbf{V}^{I}_{r}(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)\\
&\qquad+\mathbf{X}^{IJ}(s_1)\mathbf{V}^{J}_{r}(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r).
\end{aligned}\tag{18}$$

Since $\mathbf{X}^{II}(s_1)$ is a finite invertible square matrix, we have

$$\mathbf{X}^{II}(s_1)^{-1}\mathbf{\Delta}^{II}(s_1-1)^{-1}=\mathbf{B}^{II}(\alpha_2-\alpha_1;s_1-1).$$

Therefore we deduce from (18) that

$$\begin{aligned}
&\mathbf{V}^{I}_{r}(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)\\
&\quad=\mathbf{B}^{II}(\alpha_2-\alpha_1;s_1-1)\mathbf{V}^{I}_{r-1}(s_1+s_2-1,s_3,\dots,s_r;\alpha_2,\dots,\alpha_r)\\
&\qquad+\mathbf{Y}^{I}(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r),
\end{aligned}\tag{19}$$

where

$$\begin{aligned}
&\mathbf{Y}^{I}(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)\\
&\quad=-\mathbf{X}^{II}(s_1)^{-1}\mathbf{X}^{IJ}(s_1)\mathbf{V}^{J}_{r}(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r).
\end{aligned}\tag{20}$$

All the series of meromorphic functions involved in the products of matrices in formulas (19) and (20) converge normally on all compact subsets of $\mathbb{C}^r$. Moreover, all entries of the matrices on the right-hand side of (20) are holomorphic on the open set $U_r(q)$, translate of U_r by $(-q,0,\dots,0)$. Therefore the entries of $\mathbf{Y}^{I}(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)$ are also holomorphic in $U_r(q)$. Let $\xi_q(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)$ to be the first entry of the column vector $\mathbf{Y}^{I}(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)$. Then we get from (19) that

$$\begin{aligned}
&\zeta_r(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)\\
&\quad=\frac{1}{s_1-1}\zeta_{r-1}(s_1+s_2-1,s_3,\dots,s_r;\alpha_2,\dots,\alpha_r)\\
&\qquad+\sum_{k=0}^{q-2}\frac{s_1\cdots(s_1+k-1)}{(k+1)!}B_{k+1}(\alpha_2-\alpha_1)\\
&\qquad\times\zeta_{r-1}(s_1+s_2+k,s_3,\dots,s_r;\alpha_2,\dots,\alpha_r)\\
&\qquad+\xi_q(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)
\end{aligned}\tag{21}$$

and that $\xi_q(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)$ is holomorphic in the open set $U_r(q)$. In this formula, whenever empty products and empty sums appear, they are assumed to be 1 and 0, respectively. Formula (21) can also be obtained by using the Euler–Maclaurin summation formula (see [2]).

REMARK 3. A matrix formulation of the generalized Ramanujan's identity (4) would be similar as before. To write down a matrix formulation for identity (3), we encounter a family of polynomials $P_n(a,c)$ defined by the generating series

$$\frac{e^{ax}}{e^x - c} = \sum_{n\geq 0} P_n(a,c)\frac{x^n}{n!}$$

with $c \neq 1$.

We now observe that the following theorem can be deduced as an immediate consequence of Theorem 2.

THEOREM 5. *The multiple Hurwitz zeta function of depth r can be meromorphically continued to $\mathbb{C}^r$ with possible simple poles along the hyperplanes $H_{1,0}$ and $H_{i,k}$, where $2 \leq i \leq r$ and $k \geq 0$. It has at most simple poles along each of these hyperplanes.*

To check if each $H_{i,k}$ is indeed a polar hyperplane, we compute the residue of the multiple Hurwitz zeta function of depth r along this hyperplane using (19) and (21). Recall that it is defined as the restriction of the meromorphic function $(s_1 + \cdots + s_i - i + k)\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ to $H_{i,k}$.

THEOREM 6. *The residue of the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ along the hyperplane $H_{1,0}$ is the restriction of $\zeta_{r-1}(s_2, \ldots, s_r; \alpha_2, \ldots, \alpha_r)$ to $H_{1,0}$, and its residue along the hyperplane $H_{i,k}$, where $2 \leq i \leq r$ and $k \geq 0$, is the restriction to $H_{i,k}$ of the product of $\zeta_{r-i}(s_{i+1}, \ldots, s_r; \alpha_{i+1}, \ldots, \alpha_r)$ with the $(0,k)$th entry of the matrix*

$$\prod_{d=1}^{i-1} \mathbf{B}(\alpha_{d+1} - \alpha_d; s_1 + \cdots + s_d - d).$$

Proof. Let $q \geq 1$ be an integer. As in the proof of Theorem 2, we know from (21) that

$$\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r) - \frac{1}{s_1 - 1}\zeta_{r-1}(s_1 + s_2 - 1, s_3, \ldots, s_r; \alpha_2, \ldots, \alpha_r)$$

has no pole along $H_{1,0}$ inside the open set $U_r(q)$. These open sets cover $\mathbb{C}^r$. Hence the residue of $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ along $H_{1,0}$ is the restriction to $H_{1,0}$ of the meromorphic function $\zeta_{r-1}(s_1 + s_2 - 1, s_3, \ldots, s_r; \alpha_2, \ldots, \alpha_r)$ or equivalently of $\zeta_{r-1}(s_2, \ldots, s_r; \alpha_2, \ldots, \alpha_r)$. This proves the first part of Theorem 6.

Now let i, k be integers with $2 \leq i \leq r$ and $0 \leq k < q$. Also, let I and J be as defined in (17). Now, iterating formula (19) $(i-1)$ times, we get

$$\begin{aligned}
\mathbf{V}_r^I&(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r) \\
&= \left(\prod_{d=1}^{i-1} \mathbf{B}^{II}(\alpha_{d+1} - \alpha_d; s_1 + \cdots + s_d - d)\right) \\
&\quad \times \mathbf{V}_{r-i+1}^I(s_1 + \cdots + s_i - i + 1, s_{i+1}, \ldots, s_r; \alpha_i, \ldots, \alpha_r) \\
&\quad + \mathbf{Y}^{i,I}(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r),
\end{aligned}$$

where $\mathbf{Y}^{i,I}(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ is a column matrix whose entries are finite sums of products of rational functions in $s_1, \ldots, s_{i-1}$ with meromorphic functions holomorphic in $U_r(q)$. These entries therefore have no pole along the hyperplane $H_{i,k}$ in $U_r(q)$. The entries of

$$\prod_{d=1}^{i-1} \mathbf{B}^{II}(\alpha_{d+1} - \alpha_d; s_1 + \cdots + s_d - d)$$

are rational functions in $s_1, \ldots, s_{i-1}$ and hence have no poles along $H_{i,k}$. It now follows from the induction hypothesis that the only entry of $\mathbf{V}^I_{r-i+1}(s_1 + \cdots + s_i - i + 1, s_{i+1}, \ldots, s_r; \alpha_i, \ldots, \alpha_r)$ that can possibly have a pole along $H_{i,k}$ in $U_r(q)$ is that of index k,

$$\zeta_{r-i+1}(s_1 + \cdots + s_i - i + k + 1, s_{i+1}, \ldots, s_r; \alpha_i, \ldots, \alpha_r).$$

Its residue is the restriction of $\zeta_{r-i}(s_{i+1}, \ldots, s_r; \alpha_{i+1}, \ldots, \alpha_r)$ to $H_{i,k} \cap U_r(q)$, where $2 \le i \le r$ and $0 \le k < q$. Since the open sets $U_r(q)$ for $q > k$ cover $\mathbb{C}^r$, the residue of $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ along $H_{i,k}$ is the restriction to $H_{i,k}$ of the product of the $(0, k)$th entry of the matrix

$$\prod_{d=1}^{i-1} \mathbf{B}(\alpha_{d+1} - \alpha_d; s_1 + \cdots + s_d - d)$$

with $\zeta_{r-i}(s_{i+1}, \ldots, s_r; \alpha_{i+1}, \ldots, \alpha_r)$. This proves the last part of Theorem 6. □

5. Proof of Theorem 3

5.1. Zeros of Bernoulli Polynomials

The information about the exact set of poles of multiple Hurwitz zeta functions in Theorem 3 requires knowledge about the zeros of the Bernoulli polynomials. In this section, we discuss the properties of the zeros of the Bernoulli polynomials that are relevant to our study.

Recall that the Bernoulli polynomials $B_n(t)$ are defined by

$$\sum_{n \ge 0} B_n(t) \frac{x^n}{n!} = \frac{x e^{tx}}{e^x - 1}.$$

We have the following theorem by Brillhart [3] and Dilcher [4] about the zeros of Bernoulli polynomials.

Theorem 7 (Brillhart–Dilcher). *Bernoulli polynomials have no multiple roots.*

This theorem was first proved for the odd Bernoulli polynomials by Brillhart [3] and later extended for the even Bernoulli polynomials by Dilcher [4]. Theorem 7 amounts to say that the Bernoulli polynomials $B_{n+1}(t)$ and $B_n(t)$ are relatively prime as they satisfy the relation

$$B'_{n+1}(t) = (n + 1) B_n(t) \quad \text{for all } n \ge 1,$$

where $B'_{n+1}(t)$ denotes the derivative of the polynomial $B_{n+1}(t)$. With the theorem of Brillhart and Dilcher in place, we can now describe the exact set of singularities of the multiple zeta functions. For that, it is convenient to have some intermediate lemmas.

5.2. *Some Intermediate Lemmas*

LEMMA 2. *Let x, y be two indeterminates, and let the matrix $\mathbf{B}$ be as in (15). Then all the entries in the first row of the matrix*

$$\mathbf{B}(\beta-\alpha; x)\mathbf{B}(\gamma-\beta; y),$$

where $0 \le \alpha, \beta, \gamma < 1$, are nonzero rational functions in x, y with coefficients in $\mathbb{R}$.

Proof. Since entries of these matrices are indexed by $\mathbb{N} \times \mathbb{N}$, the entries of the first row are written as the $(0, k)$th entry for $k \ge 0$. Let us denote the $(0, k)$th entry by $a_{0,k}$. Then we have the formula

$$x(y+k)a_{0,k} = \sum_{i=0}^{k} (x)_{i-1}(y+i+1)_{k-i-1} B_i(\beta-\alpha) B_{k-i}(\gamma-\beta)$$

for all $k \ge 0$. As the Bernoulli polynomial $B_0(t)$ is equal to 1, we get $a_{0,0} = \frac{1}{xy}$ and hence nonzero. For $k \ge 1$, we first note that the set of polynomials

$$P := \{(x)_{i-1}(y+i+1)_{k-i-1} : 0 \le i \le k\}$$

is linearly independent over $\mathbb{R}$.

Now suppose that $B_1(\beta-\alpha) \ne 0$. We know by Theorem 7 that at least one of $B_k(\gamma-\beta)$ and $B_{k-1}(\gamma-\beta)$ is nonzero. It now follows from the linear independence of the set of polynomials in P that $a_{0,k} \ne 0$.

Next suppose that $B_1(\beta-\alpha) = 0$, i.e. $\beta-\alpha = 1/2$. Then $\gamma-\beta \ne 1/2$ as $0 \le \alpha, \gamma < 1$. Hence $B_1(\gamma-\beta) \ne 0$. Again by Theorem 7 we know that at least one of $B_k(\beta-\alpha)$ and $B_{k-1}(\beta-\alpha)$ is nonzero. Now by linear independence of the set of polynomials in P we get $a_{0,k} \ne 0$. This completes the proof of Lemma 2. □

LEMMA 3. *Let $n \ge 0$ be an integer, and let $x, x_1, \ldots, x_n$ be $(n+1)$ indeterminate. Let $\mathbf{D}$ be an infinite square matrix whose entries are indexed by $\mathbb{N} \times \mathbb{N}$ and is in the ring $\mathbb{R}(x_1, \ldots, x_n)$. Further, suppose that all the entries in the first row of $\mathbf{D}$ are nonzero. Then, for any $\alpha, \beta \in \mathbb{R}$, all the entries in the first row of the matrix $\mathbf{DB}(\beta-\alpha; x)$ are nonzero, where the matrix $\mathbf{B}$ is as in (15).*

Proof. We first note that each column of $\mathbf{B}(\beta-\alpha; x)$ has at least one nonzero entry and the nonzero entries of each of these columns are linearly independent over $\mathbb{R}$ as rational functions in x with coefficients in $\mathbb{R}$. Since all the entries in the first row of $\mathbf{D}$ are nonzero, the proof is complete by the previous observation. □

We are now ready to prove Theorem 3.

5.3. Proof of Theorem 3

When $1 \leq i \leq r$ and $k \geq 0$, the restriction of

$$\zeta_{r-i}(s_{i+1}, \ldots, s_r, \alpha_{i+1}, \ldots, \alpha_r)$$

to $H_{i,k}$ is a nonzero meromorphic function. Hence, to prove Theorem 3, we need to show that when $2 \leq i \leq r$ and $k \geq 0$, the $(0, k)$th entry of the matrix

$$\prod_{d=1}^{i-1} \mathbf{B}(\alpha_{d+1} - \alpha_d; s_1 + \cdots + s_d - d)$$

is identically zero if and only if $i = 2$ and $k \in J$. By changing coordinates this is equivalent to say that when $t_1, \ldots, t_{i-1}$ are indeterminate, the $(0, k)$th entry of the matrix

$$\prod_{d=1}^{i-1} \mathbf{B}(\alpha_{d+1} - \alpha_d; t_d)$$

is nonzero in $\mathbb{R}(t_1, \ldots, t_{i-1})$ except when $i = 2$ and $k \in J$.

For $i = 2$, our matrix is $\mathbf{B}(\alpha_2 - \alpha_1; t_1)$, and hence our assertion follows immediately. Now assume that $i \geq 3$. By Lemma 2 we know that all the entries in the first row of the matrix

$$\mathbf{B}(\alpha_2 - \alpha_1; t_1)\mathbf{B}(\alpha_3 - \alpha_2; t_2)$$

are nonzero in $\mathbb{R}(t_1, t_2)$. Hence the theorem follows from Lemma 2 if $i = 3$ and from repeated application of Lemma 3 if $i > 3$.

5.4. A Particular Case

Theorem 3 shows that precise knowledge about zeros of Bernoulli polynomials determines the exact set of singularities of the multiple Hurwitz zeta functions. Now we have precise knowledge about the rational zeros of the Bernoulli polynomials (see Inkeri [8]).

THEOREM 8 (Inkeri). *The rational zeros of a Bernoulli polynomial $B_n(t)$ can only be* 0, 1/2, *and* 1. *This happens only when n is odd and precisely in the following cases*:

(1) $B_n(0) = B_n(1) = 0$ *for all odd* $n \geq 3$,
(2) $B_n(1/2) = 0$ *for all odd* $n \geq 1$.

Using Theorem 8, we deduce the following corollary of Theorem 3. A particular case of this corollary, namely when $\alpha_i \in \mathbb{Q}$ for all $1 \leq i \leq r$, was proved in [2].

COROLLARY 1. *If $\alpha_2 - \alpha_1 = 0$, then the exact set of singularities of the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ is given by the hyperplanes*

$$H_{1,0}, H_{2,1}, H_{2,2k} \text{ and } H_{i,k} \text{ for all } k \geq 0 \text{ and } 3 \leq i \leq r.$$

If $\alpha_2 - \alpha_1 = 1/2$, then the exact set of singularities of the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ is given by the hyperplanes

$$H_{1,0},\, H_{2,2k} \text{ and } H_{i,k} \text{ for all } k \geq 0 \text{ and } 3 \leq i \leq r.$$

If $\alpha_2 - \alpha_1$ is a rational number $\neq 0, 1/2$, then the exact set of singularities of the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ is given by the hyperplanes

$$H_{1,0} \text{ and } H_{i,k} \text{ for all } k \geq 0 \text{ and } 2 \leq i \leq r.$$

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S. Gun
Institute of Mathematical Sciences
HBNI, C.I.T. Campus
Taramani, Chennai, 600 113
India

sanoli@imsc.res.in

B. Saha
Tata Institute of Fundamental Research
Homi Bhabha Road
Navy Nagar, Colaba
Mumbai, 400 005
India

biswa@math.tifr.res.in