

A Remark on Pin(2)-Equivariant Floer Homology

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ABSTRACT. In this remark, we show how the monopole Frøyshov invariant, as well as the analogues of the Involutive Heegaard Floer correction terms \underline{d}, \bar{d} , are related to the Pin(2)-equivariant Floer homology $SWFH_*^G$. We show that the only interesting correction terms of a Pin(2)-space are those coming from the subgroups $\mathbb{Z}/4$, S^1 , and Pin(2) itself.

1. Introduction

Manolescu [11] resolved the triangulation conjecture, establishing that there exist nontriangulable manifolds in all dimensions at least 5. The proof relies on the construction of Pin(2)-equivariant Seiberg–Witten Floer homology, where Pin(2) is the group consisting of two copies of the complex unit circle, with a map j interchanging the two copies and such that $ij = -ji$ and $j^2 = -1$.

Let \mathbb{F} denote the field with two elements. As S^1 -equivariant monopole Floer homology associates with a three-manifold with spin^c structure an $H^*(BS^1) = \mathbb{F}[U]$ -module, Pin(2)-equivariant Floer homology associates with a rational homology three-sphere with spin structure an $H^*(B\text{Pin}(2)) = \mathbb{F}[q, v]/(q^3)$ -module, where cohomology is taken with \mathbb{F} -coefficients. From the module structure of Pin(2)-equivariant Floer homology we obtain three invariants of homology cobordism

$$\alpha, \beta, \gamma : \theta_H^3 \rightarrow \mathbb{Z},$$

where θ_H^3 is the integral homology cobordism group of integral homology three-spheres. These invariants are related to the Rokhlin invariant $\mu(Y)$ by

$$\alpha(Y) \equiv \beta(Y) \equiv \gamma(Y) \equiv \mu(Y) \pmod{2}$$

and also satisfy

$$\alpha(-Y) = -\gamma(Y), \quad \beta(-Y) = -\beta(Y).$$

In particular, these properties for β show that there is no element $Y \in \theta_H^3$ of order 2 with $\mu(Y) = 1$. Galewski and Stern [3] and Matumoto [12] showed that there exist nontriangulable manifolds in all dimensions at least 5 if and only if there exists an element $Y \in \theta_H^3$ of order 2 with $\mu(Y) = 1$, from which Manolescu’s disproof of the triangulation conjecture follows. Manolescu’s invariants α, β , and γ are also defined for rational homology spheres with spin structure, for which they are \mathbb{Q} -valued.

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Let $G = \text{Pin}(2)$. Since the introduction of Manolescu’s G -equivariant Floer homology, denoted $SWFH_*^G(Y, \mathfrak{s})$, other versions of Floer homologies with symmetries beyond the S^1 -symmetry have become available. Lin [8] constructed a $\text{Pin}(2)$ -equivariant refinement of monopole Floer homology in the setting of Kronheimer and Mrowka [6], applying also to manifolds with $b_1 > 0$. In the setting of Heegaard Floer homology introduced by Ozsváth and Szabó [14; 13], Hendricks and Manolescu [5] point out that naturality questions make it difficult to define a G -equivariant version of Heegaard Floer homology. However, they proceed by considering the subgroup $\mathbb{Z}/4 = \langle j \rangle \subset G$ and define a Heegaard Floer analogue of $SWFH^G$ with respect to this smaller group, denoted $HFI(Y, \mathfrak{s})$. As for the mentioned theories, $HFI(Y, \mathfrak{s})$ is a module over $H^*(B\mathbb{Z}/4) = \mathbb{F}[U, Q]/(Q^2)$. Using $HFI(Y, \mathfrak{s})$, they associate two (rational) homology cobordism invariants $\underline{d}(Y, \mathfrak{s})$ and $\overline{d}(Y, \mathfrak{s})$ from the module structure. However, \underline{d} and \overline{d} do not generally reduce to the Rokhlin invariant mod 2.

The purpose of this note is to relate the homology cobordism invariants obtained using theories equivariant with respect to different groups (especially the groups S^1 , $\mathbb{Z}/4$, and G itself). In particular, we will see that, roughly speaking, all homology cobordism invariants that are constructed from Manolescu’s homotopy type using the Borel homology of a subgroup of $\text{Pin}(2)$ are determined by the invariants defined using $\mathbb{Z}/4$, S^1 , and G . For a precise statement, see Theorem 1.5.

We work in the context of Manolescu’s construction of G -equivariant Floer homology, that is, $SWFH^G$.

We recall that to define $SWFH^G$, Manolescu first associates with a rational homology sphere with spin structure (Y, \mathfrak{s}) a G -equivariant stable homotopy type, denoted $SWF(Y, \mathfrak{s})$. Then $SWFH_*^G(Y, \mathfrak{s})$ is constructed from $SWF(Y, \mathfrak{s})$ by taking the G -equivariant Borel homology

$$SWFH_*^G(Y, \mathfrak{s}) = \tilde{H}_*^G(SWF(Y, \mathfrak{s})).$$

Using $\mathbb{Z}/4 = \langle j \rangle \subset G$, we may also consider the $\mathbb{Z}/4$ -Borel homology of $SWF(Y, \mathfrak{s})$. We define

$$SWFH_*^{\mathbb{Z}/4}(Y, \mathfrak{s}) = \tilde{H}_*^{\mathbb{Z}/4}(SWF(Y, \mathfrak{s})).$$

Then $SWFH_*^{\mathbb{Z}/4}(Y, \mathfrak{s})$ has an $H^*(B\mathbb{Z}/4) = \mathbb{F}[U, Q]/(Q^2)$ -module structure, from which we further define homology cobordism invariants $\underline{\delta}(Y, \mathfrak{s}) \leq \overline{\delta}(Y, \mathfrak{s})$, which should correspond, respectively, to the invariants $\underline{d}(Y, \mathfrak{s})/2$ and $\overline{d}(Y, \mathfrak{s})/2$ of [5]. It is natural to ask to what extent these $\mathbb{Z}/4$ invariants are determined by α , β , and γ , and, more generally, by the $\mathbb{F}[q, v]/(q^3)$ -module structure of $SWFH_*^G(Y, \mathfrak{s})$. We show in Theorem 1.3 how to partially determine $\underline{\delta}(Y, \mathfrak{s})$ and $\overline{\delta}(Y, \mathfrak{s})$ from $SWFH_*^G(Y, \mathfrak{s})$ but that in general $\underline{\delta}(Y, \mathfrak{s})$ and $\overline{\delta}(Y, \mathfrak{s})$ are not determined.

To translate our further statements into the language of Heegaard Floer homology, we note that the S^1 -Frøyshov invariant $\delta(Y, \mathfrak{s})$ agrees with $d(Y, \mathfrak{s})/2$, where d is the Heegaard Floer correction term, according to [7; 1]. Moreover, the invariant δ is conjecturally related to Frøyshov’s h -invariant from instanton homology [2]: for all integral homology spheres Y for which $h(Y)$ has been computed, we have $-h(Y) = \delta(Y)$.

First, we show that although the S^1 -Frøyshov invariant δ is not determined by $\underline{\delta}$ and $\bar{\delta}$, it is determined by $SWFH^{\mathbb{Z}/4}$.

THEOREM 1.1. *Let (Y, \mathfrak{s}) be a rational homology three-sphere with spin structure. Then*

$$\delta(Y, \mathfrak{s}) = \frac{1}{2}(\min\{m \equiv 2\mu(Y, \mathfrak{s}) + 1 \pmod{2} \mid \exists x \in SWFH_m^{\mathbb{Z}/4}(Y, \mathfrak{s}), x \in \text{Im } U^\ell \text{ for all } \ell \geq 0, x \notin \text{Im } Q\} - 1).$$

We next relate the S^1 - and $\mathbb{Z}/4$ -invariants with those coming from G . Here, even given $SWFH_*^G(Y, \mathfrak{s})$, it is not possible to specify $\delta(Y, \mathfrak{s})$, $\underline{\delta}(Y, \mathfrak{s})$, or $\bar{\delta}(Y, \mathfrak{s})$ although we have the following theorems.

THEOREM 1.2. *Let (Y, \mathfrak{s}) be a rational homology three-sphere with spin structure. Let*

$$\delta_G(Y, \mathfrak{s}) = \frac{1}{2}(\min\{m \equiv 2\mu(Y, \mathfrak{s}) + 2 \pmod{4} \mid \exists x \in SWFH_m^G(Y, \mathfrak{s}), x \in \text{Im } v^\ell \text{ for all } \ell \geq 0, x \notin \text{Im } q\} - 2).$$

Then

$$\delta(Y, \mathfrak{s}) = \delta_G(Y, \mathfrak{s}) \quad \text{or} \quad \delta_G(Y, \mathfrak{s}) + 1.$$

THEOREM 1.3. *Let (Y, \mathfrak{s}) be a rational homology three-sphere with spin structure. Let*

$$\underline{\delta}_G(Y, \mathfrak{s}) = \frac{1}{2}(\min\{m \equiv 2\mu(Y, \mathfrak{s}) + 2 \pmod{4} \mid \exists x \in SWFH_m^G(Y, \mathfrak{s}), x \in \text{Im } v^\ell \text{ for all } \ell \geq 0, x \notin \text{Im } q^2\} - 2)$$

and

$$\bar{\delta}_G(Y, \mathfrak{s}) = \frac{1}{2}(\min\{m \equiv 2\mu(Y, \mathfrak{s}) + 1 \pmod{4} \mid \exists x \in SWFH_m^G(Y, \mathfrak{s}), x \in \text{Im } v^\ell \text{ for all } \ell \geq 0, x \notin \text{Im } q^2\} - 1).$$

Then

$$\underline{\delta}(Y, \mathfrak{s}) = \underline{\delta}_G(Y, \mathfrak{s}) \quad \text{or} \quad \underline{\delta}_G(Y, \mathfrak{s}) + 1$$

and

$$\bar{\delta}(Y, \mathfrak{s}) = \bar{\delta}_G(Y, \mathfrak{s}) \quad \text{or} \quad \bar{\delta}_G(Y, \mathfrak{s}) + 1.$$

To interpret $\delta_G(Y, \mathfrak{s})$, we may think of it just as the invariant γ but with an adjustment coming from the $\mathbb{F}[v]$ -torsion submodule of $SWFH_*^G(Y, \mathfrak{s})$. Similarly, $\underline{\delta}_G(Y, \mathfrak{s})$ is an adjustment of γ as well, whereas $\bar{\delta}_G(Y, \mathfrak{s})$ is an adjustment of β . As an immediate corollary of Theorem 1.3, we have the following:

COROLLARY 1.4. *Let (Y, \mathfrak{s}) be a rational homology three-sphere with spin structure. Then*

$$\alpha(Y, \mathfrak{s}) \geq \bar{\delta}(Y, \mathfrak{s}) \geq \beta(Y, \mathfrak{s}) \geq \underline{\delta}(Y, \mathfrak{s}) \geq \gamma(Y, \mathfrak{s}).$$

Given that the homology cobordism invariants from S^1 and $\mathbb{Z}/4 \subset \text{Pin}(2)$ cannot be determined from the $\text{Pin}(2)$ -equivariant homology, it is natural to ask if there are other subgroups of $\text{Pin}(2)$ that produce new homology cobordism invariants. We show that this is not the case. We call a homology cobordism invariant δ_I a generalized Frøyshov invariant if it is constructed analogously to δ , but perhaps using a different subgroup H of G ; for the precise definition, see Section 3.3.

THEOREM 1.5. *Let $\{\delta_I\}$ be the set of generalized Frøyshov invariants associated with a subgroup $H \subset G$. Then*

$$\{\delta_I\} \subseteq \{\delta, \underline{\delta}, \bar{\delta}, \alpha, \beta, \gamma\},$$

where the generalized Frøyshov invariants are viewed as maps $\theta_H^3 \rightarrow \mathbb{Z}$.

The proof of Theorem 1.5 also gives the next corollary. We note that the closed subgroups of G are precisely $\mathbb{Z}/2 = \langle j^2 \rangle$, $\mathbb{Z}/4 = \langle j \rangle$, S^1 , cyclic subgroups of S^1 , and generalized quaternion groups $Q_{4m} = \langle e^{\pi i/m}, j \rangle$.

COROLLARY 1.6. *Let $H = Q_{4m} \subset G$ for m even, and (Y, \mathfrak{s}) any rational homology three-sphere with spin structure. Then the isomorphism type of $\text{SWFH}_*^G(Y, \mathfrak{s})$ (as an $H^*(BG)$ -module) is specified by the isomorphism type of $\text{SWFH}_*^H(Y, \mathfrak{s})$ as an $H^*(BH)$ -module.*

Organization

In Section 2, we recall what we will need from equivariant topology. In Section 3, we define Gysin sequences and then use these to establish Propositions 3.6–3.8, which form the equivariant topology input for Theorems 1.1–1.3. In Section 3.2, we state the existence of Manolescu’s Seiberg–Witten Floer stable homotopy type $\text{SWF}(Y, \mathfrak{s})$ and show Theorems 1.1–1.3 of the Introduction. In Section 3.3, we prove Theorem 3.12, which is the equivariant topology input for Theorem 1.5.

2. Spaces of Type SWF

2.1. G -CW Complexes

In this section, we recall the definition of spaces of type SWF from [11] and briefly review the basics of equivariant topology, referring to Section 2 of [11] for further details. Spaces of type SWF are the output of the construction of the Seiberg–Witten Floer stable homotopy type of [11] and [10]; see Section 3.2. Throughout, all homology will be taken with $\mathbb{F} = \mathbb{Z}/2$ -coefficients.

Let K be a compact Lie group. A (finite) K -CW decomposition of a space (X, A) with K -action is a filtration $(X_n \mid n \in \mathbb{Z}_{\geq 0})$ of X such that

- $A \subset X_0$, and $X = X_n$ for n sufficiently large.
- The space X_n is obtained from X_{n-1} by attaching K -equivariant n -cells, copies of $(K/H) \times D^n$, for a closed subgroup H of K .

When A is a point, we call (X, A) a pointed K -CW complex.

Let X and Y pointed K -CW complexes, at least one of which is a finite complex. We define the smash product $X \wedge Y$ as a K -space by letting K act diagonally. In the case that $X = V^+$ is the one-point compactification of a finite-dimensional K -representation V , we call $\Sigma^V Y = V^+ \wedge Y$ the suspension of Y by V . Define also

$$X \wedge_K Y = (X \wedge Y)/K.$$

Let EK denote a contractible space with free K -action, and let EK_+ denote EK with a disjoint base point added. The reduced Borel homology and cohomology of X are defined by

$$\begin{aligned} \tilde{H}_*^K(X) &= \tilde{H}_*(EK_+ \wedge_K X), \\ \tilde{H}_K^*(X) &= \tilde{H}^*(EK_+ \wedge_K X). \end{aligned} \tag{1}$$

Borel homology and cohomology are invariants of K -equivariant homotopy equivalence.

Furthermore, there is a projection

$$EK_+ \wedge_K X \xrightarrow{\pi_1} EK_+/K = BK_+,$$

which induces a map $\pi_1^* : H^*(BK) \rightarrow \tilde{H}_K^*(X)$. Via π_1^* , $\tilde{H}_K^*(X)$ and $\tilde{H}_*^K(X)$ inherit the structure of $H^*(BK)$ -modules. On $\tilde{H}_K^*(X)$, $H^*(BK)$ increases grading, whereas on $\tilde{H}_*^K(X)$, $H^*(BK)$ decreases grading.

For a subgroup $L \subseteq K$ and a K -CW complex X , we may relate the L -Borel cohomology and K -Borel cohomology of X . Indeed, there is a quotient map

$$EK_+ \wedge_L X \xrightarrow{\pi} EK_+ \wedge_K X,$$

which induces a map in cohomology

$$\tilde{H}_K^*(X) \xrightarrow{\pi^*} \tilde{H}_L^*(X). \tag{2}$$

We call the map π^* the *restriction map* from K to L and write $\pi^* = \text{res}_L^K$.

Setting a point X in (2), we have a restriction map (of algebras) $H^*(BK) \rightarrow H^*(BL)$. Then $\tilde{H}_L^*(X)$ inherits an $H^*(BK)$ -module structure by restriction. We denote $\tilde{H}_L^*(X)$, viewed as a $H^*(BK)$ -module, by $\text{res}_L^K \tilde{H}_L^*(X)$. Then the map

$$\tilde{H}_K^*(X) \xrightarrow{\text{res}_L^K} \text{res}_L^K \tilde{H}_L^*(X) \tag{3}$$

is a map of $H^*(BK)$ -modules.

Let $G = \text{Pin}(2)$ and BG its classifying space. In addition to the definition of G from the Introduction, we may think of G as the set $S^1 \cup jS^1 \subset \mathbb{H}$, where S^1 is the unit circle in the $(1, i)$ plane, with group action on G induced from the group action of the unit quaternions. Thus $S^\infty = S(\mathbb{H}^\infty)$ with its quaternion action is a free G -space. Since S^∞ is contractible, we identify $EG = S^\infty$.

Manolescu [11] shows that

$$H^*(BG) = \mathbb{F}[q, v]/(q^3), \tag{4}$$

where $\text{deg } q = 1$ and $\text{deg } v = 4$.

For convenience, we also record

$$\begin{aligned} H^*(B\mathbb{Z}/2) &= \mathbb{F}[W], \\ H^*(B\mathbb{Z}/4) &= \mathbb{F}[U, Q]/(Q^2 = 0), \\ H^*(BS^1) &= H^*(\mathbb{C}P^\infty) = \mathbb{F}[U], \end{aligned} \tag{5}$$

where $\deg U = 2$ and $\deg W = \deg Q = 1$.

For a subset S of a group K , let $\langle S \rangle$ denote the subgroup generated by S . There are inclusions $\mathbb{Z}/2 = \langle j^2 \rangle \subset S^1 \subset G$ and $\mathbb{Z}/4 \cong \langle j \rangle \subset G$. We describe the corresponding restriction maps in Proposition 3.1.

We will also need to use that Borel cohomology behaves well with respect to suspension.

PROPOSITION 2.1 ([11], Proposition 2.2). *Let V be a finite-dimensional representation of a compact Lie group K . Then, as $H^*(BK)$ -modules,*

$$\begin{aligned} \tilde{H}_K^*(\Sigma^V X) &\cong \tilde{H}_K^{*-\dim V}(X), \\ \tilde{H}_*^K(\Sigma^V X) &\cong \tilde{H}_{*-\dim V}^K(X). \end{aligned} \tag{6}$$

We mention three irreducible representations of G :

- The trivial one-dimensional representation \mathbb{R} .
- The one-dimensional real vector space on which $j \in G$ acts by -1 , and on which S^1 acts trivially, denoted $\tilde{\mathbb{R}}$.
- The quaternionic representation \mathbb{H} , where G acts by left multiplication.

DEFINITION 2.2. Let $s \in \mathbb{Z}$. A *space of type SWF* at level s is a pointed finite G -CW complex X with

- The S^1 -fixed-point set X^{S^1} is G -homotopy equivalent to $(\tilde{\mathbb{R}}^s)^+$, the one-point compactification of $\tilde{\mathbb{R}}^s$;
- The action of G on $X - X^{S^1}$ is free.

We define $\mu(X) \in \mathbb{Q}/2\mathbb{Z}$ by $\mu(X) = \frac{s}{2} \bmod 2$.

We will often have occasion later to work with $2\mu(X)$, which we view as an element of $\mathbb{Q}/4\mathbb{Z}$.

We note that, for a space X of type SWF,

$$\tilde{H}_*^G(X^{S^1}) = \tilde{H}_*^G((\tilde{\mathbb{R}}^s)^+) = \tilde{H}_{*-s}^G(S^0) = H_{*-s}(BG)$$

and

$$\tilde{H}_G^*(X^{S^1}) = H^{*-s}(BG),$$

using Proposition 2.1.

Associated with a space X of type SWF at level s , we take the Borel cohomology $\tilde{H}_G^*(X)$, from which Manolescu [11] defines $a(X)$, $b(X)$, and $c(X)$:

$$\begin{aligned} a(X) &= \min\{r \equiv 2\mu(X) \pmod 4 \mid \exists x \in \tilde{H}_G^r(X), v^l x \neq 0 \text{ for all } l \geq 0\}, \\ b(X) &= \min\{r \equiv 2\mu(X) + 1 \pmod 4 \mid \exists x \in \tilde{H}_G^r(X), v^l x \neq 0 \\ &\quad \text{for all } l \geq 0\} - 1, \\ c(X) &= \min\{r \equiv 2\mu(X) + 2 \pmod 4 \mid \exists x \in \tilde{H}_G^r(X), v^l x \neq 0 \\ &\quad \text{for all } l \geq 0\} - 2. \end{aligned} \tag{7}$$

Using S^1 -Borel cohomology, Manolescu [11] also defines

$$d(X) = \min\{r \mid \exists x \in \tilde{H}_{S^1}^r(X), U^l x \neq 0 \text{ for all } l \geq 0\}. \tag{8}$$

The well-definedness of a, b, c , and d follows from the equivariant localization theorem. We list a version of this theorem for spaces of type SWF:

THEOREM 2.3 ([17], III (3.8)). *Let X be a space of type SWF. Then the inclusion $X^{S^1} \rightarrow X$, after inverting v , induces an isomorphism of $\mathbb{F}[q, v, v^{-1}]/(q^3)$ -modules:*

$$v^{-1} \tilde{H}_G^*(X^{S^1}) \cong v^{-1} \tilde{H}_G^*(X).$$

Furthermore,

$$\begin{aligned} U^{-1} \tilde{H}_{S^1}^*(X^{S^1}) &\cong U^{-1} \tilde{H}_{S^1}^*(X), \\ U^{-1} \tilde{H}_{\mathbb{Z}/4}^*(X^{S^1}) &\cong U^{-1} \tilde{H}_{\mathbb{Z}/4}^*(X), \\ W^{-1} \tilde{H}_{\mathbb{Z}/2}^*(X^{S^1}) &\cong W^{-1} \tilde{H}_{\mathbb{Z}/2}^*(X). \end{aligned}$$

For a space X of type SWF, X is a finite G -complex and so we have that $\tilde{H}_G^*(X)$ is finitely generated as an $\mathbb{F}[v]$ -module. In particular, the $\mathbb{F}[v]$ -torsion part of $\tilde{H}_G^*(X)$ is bounded above in grading. Similarly, the $\mathbb{F}[U]$ -torsion parts of $\tilde{H}_{S^1}^*(X)$ and $\tilde{H}_{\mathbb{Z}/4}^*(X)$, as well as the $\mathbb{F}[W]$ -torsion of $\tilde{H}_{\mathbb{Z}/2}^*(X)$, are bounded above in grading.

Following (7), we define analogues of a, b , and c for $\mathbb{Z}/4$. In this definition, recall that $\mu(X)$ need not be an integer.

DEFINITION 2.4. For X a space of type SWF, we define $\bar{d}(X)$ and $\underline{d}(X)$ by

$$\begin{aligned} \bar{d}(X) &= \min\{r \equiv 2\mu(X) \pmod 2 \mid \exists x \in \tilde{H}_{\mathbb{Z}/4}^r(X), U^l x \neq 0 \text{ for all } l \geq 0\}, \\ \underline{d}(X) &= \min\{r \equiv 2\mu(X) + 1 \pmod 2 \mid \exists x \in \tilde{H}_{\mathbb{Z}/4}^r(X), U^l x \neq 0 \\ &\quad \text{for all } l \geq 0\} - 1. \end{aligned} \tag{9}$$

The well-definedness of $\bar{d}(X)$ and $\underline{d}(X)$ follows from Theorem 2.3.

2.2. *Stable G-Equivariant Topology*

Here we define stable equivalence for G -spaces and define the Manolescu invariants α , β , and γ and their $\mathbb{Z}/4$ -analogues.

DEFINITION 2.5 (see [10]). Let X and X' be spaces of type SWF, $m, m' \in \mathbb{Z}$, and $n, n' \in \mathbb{Q}$. We say that the triples (X, m, n) and (X', m', n') are *stably equivalent* if $n - n' \in \mathbb{Z}$ and there exists a G -equivariant homotopy equivalence for some $r \gg 0$ and some nonnegative $M \in \mathbb{Z}$ and $N \in \mathbb{Q}$:

$$\Sigma^r \mathbb{R} \Sigma^{(M-m)\tilde{\mathbb{R}}} \Sigma^{(N-n)\mathbb{H}} X \rightarrow \Sigma^r \mathbb{R} \Sigma^{(M-m')\tilde{\mathbb{R}}} \Sigma^{(N-n')\mathbb{H}} X'. \tag{10}$$

Let \mathfrak{E} be the set of equivalence classes of triples (X, m, n) for a space X of type SWF, $m \in \mathbb{Z}$, $n \in \mathbb{Q}$, under the equivalence relation of stable G -equivalence.¹ The set \mathfrak{E} may be considered as a subcategory of the G -equivariant Spanier–Whitehead category [11] by viewing (X, m, n) as the formal desuspension of X by m copies of $\tilde{\mathbb{R}}$ and n copies of \mathbb{H} . We define Borel cohomology for $(X, m, n) \in \mathfrak{E}$, as an isomorphism class of $H^*(BK)$ -modules, by

$$\tilde{H}_K^*((X, m, n)) = \tilde{H}_K^*(X)[m + 4n] \tag{11}$$

for any closed subgroup K of G . The well-definedness of (11) follows from Proposition 2.1.

Finally, we define the invariants $\alpha, \beta, \gamma, \delta, \underline{\delta}$, and $\bar{\delta}$ associated with an element of \mathfrak{E} .

DEFINITION 2.6. For $[(X, m, n)] \in \mathfrak{E}$, we set

$$\begin{aligned} \alpha((X, m, n)) &= \frac{a(X)}{2} - \frac{m}{2} - 2n, & \beta((X, m, n)) &= \frac{b(X)}{2} - \frac{m}{2} - 2n, \\ \gamma((X, m, n)) &= \frac{c(X)}{2} - \frac{m}{2} - 2n, \\ \delta((X, m, n)) &= \frac{d(X)}{2} - \frac{m}{2} - 2n, \\ \bar{\delta}((X, m, n)) &= \frac{\bar{d}(X)}{2} - \frac{m}{2} - 2n, & \underline{\delta}((X, m, n)) &= \frac{\underline{d}(X)}{2} - \frac{m}{2} - 2n. \end{aligned}$$

These invariants do not depend on the choice of representative of the class $[(X, m, n)] \in \mathfrak{E}$.

For notational convenience later, we also define

$$\mu((X, m, n)) = \mu(X) - \frac{m}{2} - 2n \pmod{2}.$$

¹This convention is slightly different from that of [10]. The object (X, m, n) in the set of stable equivalence classes \mathfrak{E} , as defined before, corresponds to $(X, \frac{m}{2}, n)$ in the conventions of [10].

3. Gysin Sequences

In this section, we recall some facts about Gysin sequences and then provide the topological input for the results in the Introduction. Gysin sequences were first applied to the study of Pin(2)-equivariant Floer theory in [9].

3.1. Gysin Sequences for Change-of-Groups

Let K and L be compact Lie groups such that there is a fiber sequence of Lie groups

$$L \subset K \rightarrow S^n. \tag{12}$$

Then there is a Gysin sequence, as in [17, §III.2], given by

$$\begin{aligned} \dots \longrightarrow H^*(BK) \xrightarrow{e(K,L)\cup-} H^{*+n+1}(BK) \xrightarrow{p^*} H^{*+n+1}(BL) \\ \longrightarrow H^{*+1}(BK) \longrightarrow \dots, \end{aligned} \tag{13}$$

where $e(K, L)$ is the Euler class of the sphere bundle $S^n \rightarrow BL = EK/L \rightarrow BK$, and $p : BL \rightarrow BK$ is the projection (note that $p^* = \text{res}_L^K$).

PROPOSITION 3.1. *We specify the Euler classes associated with some sequences of Lie groups as in (12).*

- (1) *Associated with $S^1 \rightarrow G \rightarrow S^0$, we have $e(G, S^1) = q$. Further, $p^*v = U^2$.*
- (2) *Associated with $\mathbb{Z}/4 \rightarrow G \rightarrow S^1$, we have $e(G, \mathbb{Z}/4) = q^2$. Further, $p^*q = Q$ and $p^*v = U^2$.*
- (3) *Associated with $\mathbb{Z}/2 = \langle j^2 \rangle \rightarrow \mathbb{Z}/4 \rightarrow \langle j \rangle / \langle j^2 \rangle = \mathbb{Z}/2$, we have $e(\mathbb{Z}/4, \mathbb{Z}/2) = Q$. Further, $p^*Q = 0$, $p^*U = W^2$.*
- (4) *Associated with $\mathbb{Z}/2 = \langle j^2 \rangle \rightarrow S^1 \rightarrow S^1 / \langle j^2 \rangle = S^1$, we have $e(S^1, \mathbb{Z}/2) = 0$. Further, $p^*U = W^2$.*

We call these Gysin sequences Types (1)–(4), respectively.

Proof. In each case (1)–(4), it is straightforward to see that the Euler class is specified by the algebraic structure of the entries of the exact sequence (13). For example, we prove (1). Since $H^1(BS^1) = 0$, we have that $p^*q = 0$. By exactness of (13), q is in the image of the Euler class, and since the Euler class $e(G, S^1)$ is of degree 1, we have $e(G, S^1) = q$. The other cases are similar. □

More generally, for a K -CW complex X , we have a sphere bundle

$$S^n \rightarrow EK \times_L X \rightarrow EK \times_K X \tag{14}$$

and a Gysin sequence

$$H_K^*(X) \xrightarrow{e(X)\cup-} H_K^{*+n+1}(X) \xrightarrow{p^*} H_L^{*+n+1}(X) \longrightarrow \dots, \tag{15}$$

$\tilde{H}_K^d(X)$ must be isomorphic to $H^{d+2\mu(X)}(BK)$, and $\tilde{H}_L^d(X)$ must be isomorphic to $H^{d+2\mu(X)}(BL)$ using that X is of type SWF. Recall from the proof of Proposition 3.1 that there is only one choice of maps p^* and ι^* that make the triples $H^*(BK)$, $H^*(BK)$, and $H^*(BL)$ into exact triangles. Since $\tilde{H}_K^*(X)$ and $\tilde{H}_L^*(X)$ are isomorphic to $H^{*+2\mu(X)}(BK)$ and $H^{*+2\mu(X)}(BL)$, the same reasoning as in the proof of Proposition 3.1 shows that there is only one choice of maps p^* and ι^* that make (18) exact in high degrees (namely, the p^* and ι^* listed). This establishes property (3) of Definition 3.3. \square

To prove Proposition 3.6, the precursor to Theorem 1.1, we need to compare S^1 - and $\mathbb{Z}/4$ -invariants despite there being no Gysin sequence relating S^1 and $\mathbb{Z}/4$ homology. As an intermediate step, we define

$$\delta_{\mathbb{Z}/2}(X) = \frac{1}{2}(\min\{m \mid \exists x \in H_{\mathbb{Z}/2}^m(X), W^\ell x \neq 0 \text{ for all } \ell \geq 0\}) \tag{19}$$

for $X \in \mathfrak{E}$.

A priori, $\delta_{\mathbb{Z}/2}(X) - \mu(X)$ may be a half-integer (the next lemma implies that it is, in fact, an integer).

LEMMA 3.5. *Let $X \in \mathfrak{E}$. Then $\delta_{\mathbb{Z}/2}(X) = \delta(X)$.*

Proof. We will use the Gysin sequence of type (4) associated with X . For now, fix p^* and ι^* to refer to the Gysin sequence maps of that type.

First, we establish $\delta(X) \geq \delta_{\mathbb{Z}/2}(X)$. Let $x \in \tilde{H}_{S^1}^m(X)$ be U -nontorsion. For sufficiently large ℓ , by property (3) of Definition 3.3, $p^*(U^\ell x) \neq 0$. By the equivariance property (2) of the same definition, $p^*(U^\ell x) = W^{2\ell} p^*x$, and so p^*x is a W -nontorsion element of $\tilde{H}_{\mathbb{Z}/2}^m(X)$. For notational convenience, define

$$\underline{d}_{\mathbb{Z}/2}(X) = \min\{m \equiv 2\mu(X) \pmod 2 \mid \exists x \in H_{\mathbb{Z}/2}^m(X), W^\ell x \neq 0 \text{ for all } \ell \geq 0\}.$$

We have then

$$\min\{m \equiv 2\mu(X) \pmod 2 \mid \exists x \in H_{S^1}^m(X), U^\ell x \neq 0 \text{ for all } \ell \geq 0\} \geq \underline{d}_{\mathbb{Z}/2}(X). \tag{20}$$

We note that the only U -nontorsion elements of $\tilde{H}_{S^1}^*(X)$ are in degree $d \equiv 2\mu(X) \pmod 2$ by Proposition 3.4. So the left-hand side of (20) is $2\delta(X)$. We also define

$$\overline{d}_{\mathbb{Z}/2}(X) = \min\{m \equiv 2\mu(X) + 1 \pmod 2 \mid \exists x \in H_{\mathbb{Z}/2}^m(X), W^\ell x \neq 0 \text{ for all } \ell \geq 0\}.$$

By definition, $\delta_{\mathbb{Z}/2}(X) = (\min\{\underline{d}_{\mathbb{Z}/2}(X), \overline{d}_{\mathbb{Z}/2}(X)\})/2$.

We next show the inequality opposite to (20).

Let $x \in \tilde{H}_{\mathbb{Z}/2}^m(X)$ be a W -nontorsion element with $m \equiv 2\mu(X) + 1 \pmod 2$. Then, by property (3) of Definition 3.3, $\iota^*(W^{2\ell}x) = U^\ell \iota^*x$ must be nonzero. In particular, $\iota^*x \in \tilde{H}_{S^1}^{m-1}(X)$ is U -nontorsion. Then we obtain

$$2\delta(X) \leq \overline{d}_{\mathbb{Z}/2}(X) - 1. \tag{21}$$

Furthermore,

$$\overline{d}_{\mathbb{Z}/2}(X) = \underline{d}_{\mathbb{Z}/2}(X) \pm 1, \tag{22}$$

since for x in one of the sets corresponding to $\underline{d}_{\mathbb{Z}/2}$ or $\overline{d}_{\mathbb{Z}/2}$, Wx is in the other.

But, combining (20) and (21) we have $\underline{d}_{\mathbb{Z}/2}(X) \leq \overline{d}_{\mathbb{Z}/2}(X) - 1$. So, from (22) we obtain $\underline{d}_{\mathbb{Z}/2}(X) = \overline{d}_{\mathbb{Z}/2}(X) - 1$. It follows that $2\delta(X) = \underline{d}_{\mathbb{Z}/2}(X)$.

Using the definition of $\delta_{\mathbb{Z}/2}(X)$, the proof is complete. □

The following statement corresponds to Theorem 1.1 of the Introduction.

PROPOSITION 3.6. *Let $X \in \mathfrak{E}$. Then*

$$\delta(X) = \frac{1}{2}(\min\{m \equiv 2\mu(X) + 1 \pmod{2} \mid \exists x \in H_{\mathbb{Z}/4}^m(X), U^\ell x \neq 0 \text{ for all } \ell \geq 0, Qx = 0\} - 1). \tag{23}$$

Proof. We denote the right-hand side of (23) by $\delta_{\mathbb{Z}/4}(X)$. We consider the abstract Gysin sequence of type (3) associated with X ; fix p^* and ι^* to refer to the maps in this type of Gysin sequence. Using Lemma 3.5, we need only show

$$\delta_{\mathbb{Z}/2}(X) = \delta_{\mathbb{Z}/4}(X). \tag{24}$$

We start by showing $\delta_{\mathbb{Z}/4}(X) \leq \delta_{\mathbb{Z}/2}(X)$. We note that any W -nontorsion element x of $\tilde{H}_{\mathbb{Z}/2}^*(X)$ with $\deg x \equiv 2\mu(X) + 1 \pmod{2}$ must have $U^\ell \iota^* x = \iota^* W^{2\ell} x \neq 0$ for ℓ sufficiently large. However, $Q\iota^* x = 0$ by exactness of (17). Thus, if $x \in \tilde{H}_{\mathbb{Z}/2}^m(X)$ with $m \equiv 2\mu(X) + 1 \pmod{2}$ is W -nontorsion, then there exists an element $\iota^* x \in \tilde{H}_{\mathbb{Z}/4}^m(X)$ that is U -nontorsion and is annihilated by Q . Thus

$$\min\{m \equiv 2\mu(X) + 1 \pmod{2} \mid \exists x \in \tilde{H}_{\mathbb{Z}/4}^m(X), U^\ell x \neq 0 \text{ for all } \ell \geq 0, Qx = 0\} - 1 \leq \overline{d}_{\mathbb{Z}/2}(X) - 1.$$

By the proof of Lemma 3.5, $(\overline{d}_{\mathbb{Z}/2}(X) - 1)/2 = \delta_{\mathbb{Z}/2}(X)$. Then

$$\delta_{\mathbb{Z}/4}(X) \leq \delta_{\mathbb{Z}/2}(X).$$

Next, we show

$$\delta_{\mathbb{Z}/2}(X) \leq \delta_{\mathbb{Z}/4}(X). \tag{25}$$

Indeed, fix $m \equiv 2\mu(X) + 1 \pmod{2}$ and $x \in \tilde{H}_{\mathbb{Z}/4}^m(X)$ so that x is U -nontorsion and satisfies $Qx = 0$. Then $x \in \text{Im } \iota$, say $x = \iota y$, by exactness of (17). However, since x is U -nontorsion, y is nontorsion as well, so we see that

$$\min\{m \mid \exists x \in \tilde{H}_{\mathbb{Z}/2}^m(X), W^\ell x \neq 0 \text{ for all } \ell \geq 0\} \leq 2\delta_{\mathbb{Z}/4}(X).$$

Recalling the definition of $\delta_{\mathbb{Z}/2}(X)$, this is precisely $\delta_{\mathbb{Z}/2}(X) \leq \delta_{\mathbb{Z}/4}(X)$, completing the proof. □

The following proposition corresponds to Theorem 1.2 from the Introduction.

PROPOSITION 3.7. *Let $X \in \mathfrak{E}$. Fix $N \in \mathbb{Z}$. Then $\delta(X) \leq 2N + \mu(X) + 1$ if and only if*

$$\frac{\min\{m \equiv 2\mu(X) + 2 \pmod{4} \mid \exists x \in \tilde{H}_G^m(X), v^\ell x \neq 0 \text{ for all } \ell \geq 0, qx = 0\} - 2}{2} \leq 2N + \mu(X). \tag{26}$$

Proof. Denote the left-hand side of (26) by $\delta_G(X)$. First, we show that $\delta_G(X) \leq 2N + \mu(X)$ implies $\delta(X) \leq 2N + \mu(X) + 1$.

Say $\delta_G(X) \leq 2N + \mu(X)$, that is, there exists a v -nontorsion element $x \in \tilde{H}_G^{4N+2\mu(X)+2}(X)$, so that $qx = 0$. Then, by exactness of (17), $x \in \text{Im } \iota^*$, say $x = \iota^*y$. Since x is v -nontorsion, y must be U -nontorsion. Thus, $\delta(X) \leq \frac{\deg y}{2} = 2N + \mu(X) + 1$.

Next, say that $\delta(X) \leq 2N + \mu(X) + 1$. Let $x \in \tilde{H}_{S^1}^{4N+2\mu(X)+2}(X)$ be U -nontorsion. Then by (1) of Proposition 3.1, ι^*x must be v -nontorsion. In particular, ι^*x is a v -nontorsion element of $\tilde{H}_G^{4N+2\mu(X)+2}(X)$ with $q(\iota^*x) = 0$. Thus $\delta_G(X) \leq 2N + \mu(X)$, as needed. \square

The following proposition corresponds to Theorem 1.3 from the Introduction.

PROPOSITION 3.8. *Let $X \in \mathfrak{E}$. Then $\underline{\delta}(X) \leq 2N + \mu(X) + 1$ for some integer N if and only if*

$$\frac{\min\{m \equiv 2\mu(X) + 2 \pmod{4} \mid \exists x \in \tilde{H}_G^m(X), v^\ell x \neq 0 \text{ for all } \ell \geq 0, q^2x = 0\} - 2}{2} \leq 2N + \mu(X). \tag{27}$$

Further, $\bar{\delta}(X) \leq 2N + \mu(X) + 1$ if and only if

$$\frac{\min\{m \equiv 2\mu(X) + 1 \pmod{4} \mid \exists x \in \tilde{H}_G^m(X), v^\ell x \neq 0 \text{ for all } \ell \geq 0, q^2x = 0\} - 1}{2} \leq 2N + \mu(X). \tag{28}$$

Proof. We will consider Gysin sequences of Type (2). Denote the left-hand side of (27) by $\underline{\delta}_G(X)$ and that of (28) by $\bar{\delta}_G(X)$.

First, we show that $\underline{\delta}(X) \leq 2N + \mu(X) + 1$ implies $\underline{\delta}_G(X) \leq 2N + \mu(X)$. Indeed, say $\underline{\delta}(X) \leq 2N + \mu(X) + 1$ and $x \in \tilde{H}_{\mathbb{Z}/4}^{4N+2\mu(X)+3}(X)$, so that x is U -nontorsion. Then by (2) of Proposition 3.1, ιx is v -nontorsion. Further, $q^2 \iota x = 0$ by exactness of (17). Thus

$$\begin{aligned} \min\{m \equiv 2\mu(X) + 2 \pmod{4} \mid \exists x \in \tilde{H}_G^m(X), v^\ell x \neq 0 \text{ for all } \ell \geq 0, q^2x = 0\} \\ \leq 4N + 2\mu(X) + 2. \end{aligned}$$

Then $\underline{\delta}_G(X) \leq 2N + \mu(X)$, as needed.

Next, suppose $\underline{\delta}_G(X) \leq 2N + \mu(X)$; we will show $\underline{\delta}(X) \leq 2N + \mu(X) + 1$. Choose $x \in \tilde{H}_G^{4N+2\mu(X)+2}(X)$ so that x is v -nontorsion and $q^2x = 0$. Then $x \in \text{Im } \iota$, say $x = \iota y$, and y is U -nontorsion in grading $4N + 2\mu(X) + 3$. We then obtain $\underline{\delta}(X) \leq 2N + \mu(X) + 1$, as needed.

The proof for $\bar{\delta}(X)$ is completely analogous. □

Propositions 3.7 and 3.8 cannot be improved, as we will see in Example 3.13, in that there exist spaces X_1, X_2 with isomorphic $\tilde{H}_G^*(X_i)$ but different $\delta, \underline{\delta}$, and $\bar{\delta}$ invariants.

3.2. Seiberg–Witten Floer Homology

In this section, we convert the results of the previous section into statements for three-manifolds. First, we recall the existence of the Seiberg–Witten Floer stable homotopy type.

THEOREM 3.9 (Manolescu [11; 10]). *There is an invariant $SWF(Y, \mathfrak{s})$, the Seiberg–Witten Floer spectrum class, of rational homology three-spheres with spin structure (Y, \mathfrak{s}) , taking values in \mathfrak{E} . A spin cobordism (W, \mathfrak{t}) , with $b_2(W) = 0$, from Y_1 to Y_2 , induces a map $SWF(Y_1, \mathfrak{t}|_{Y_1}) \rightarrow SWF(Y_2, \mathfrak{t}|_{Y_2})$. The induced map is a homotopy-equivalence on S^1 -fixed-point sets.*

Manolescu constructs $SWF(Y, \mathfrak{s})$ by using finite-dimensional approximation to the Seiberg–Witten equations on the Coulomb slice. From $SWF(Y, \mathfrak{s})$ we extract homology cobordism invariants as in the following definition.

DEFINITION 3.10. For a spin rational homology three-sphere (Y, \mathfrak{s}) , the Manolescu invariants $\alpha(Y, \mathfrak{s})$, $\beta(Y, \mathfrak{s})$, $\gamma(Y, \mathfrak{s})$, and $\delta(Y, \mathfrak{s})$ are defined by $\alpha(SWF(Y, \mathfrak{s}))$, $\beta(SWF(Y, \mathfrak{s}))$, $\gamma(SWF(Y, \mathfrak{s}))$, and $\delta(SWF(Y, \mathfrak{s}))$, respectively. We further define

$$\underline{\delta}(Y, \mathfrak{s}) = \underline{\delta}(SWF(Y, \mathfrak{s})) \quad \text{and} \quad \bar{\delta}(Y, \mathfrak{s}) = \bar{\delta}(SWF(Y, \mathfrak{s})).$$

All these quantities are invariant under homology cobordism.

Proof of Theorems 1.1–1.3. These theorems follow by applying Propositions 3.6–3.8 to $SWF(Y, \mathfrak{s})$ and dualizing.

3.3. Equivariant Homology of Subgroups of G

Here we make precise and prove Theorem 1.5. We first define generalized Frøyshov invariants.

Let $H \subseteq G$ be a Lie subgroup of G . Note that $H^*(BG)$ is periodic, that is, cup product with $v \in H^*(BG)$ defines an isomorphism of \mathbb{F} -modules

$$H^n(BG) \rightarrow H^{n+4}(BG)$$

for all $n \geq 0$. It turns out that $H^*(BH)$ is also periodic; fix $P \in H^*(BH)$ so that cup product with P induces an isomorphism $H^*(BH) \rightarrow H^{*+\deg P}(BH)$.

For a space X of type SWF at level s , let $\iota : X^{S^1} \rightarrow X$ denote the inclusion map of the S^1 -fixed point set, and let ι^* denote the induced map in Borel cohomology

$$\iota^* : \tilde{H}_H^*(X) \rightarrow \tilde{H}_H^*(X^{S^1}) = H^{*+s}(BH).$$

DEFINITION 3.11. For a homogeneous element e of $H^*(BH)/P$ (with $\mathbb{Z}/\text{deg } P$ -grading) and $X \in \mathfrak{E}$, we define the *generalized Frøyshov invariant* $\delta_{H,e}(X)$ by

$$\begin{aligned} &(\min\{m \equiv 2\mu(X) + \text{deg } e \pmod{\text{deg } P} \mid \\ &\exists x \in \tilde{H}_H^m(X), \iota^*x = P^k e \text{ for some } k\} - \text{deg } e)/2. \end{aligned} \tag{29}$$

The well-definedness of the quantity $\delta_{H,e}(X)$ is guaranteed by the equivariant localization theorem. It is apparent that all of $\alpha, \beta, \gamma, \delta$ and $\underline{\delta}$ and $\bar{\delta}$ are particular cases of generalized Frøyshov invariants.

THEOREM 3.12. *Let $H \subset G$ a Lie subgroup, and let $\{\delta_{H,e}\}$ be the set of generalized Frøyshov invariants associated with H . Then*

$$\{\delta_{H,e}\} \subseteq \{\delta, \underline{\delta}, \bar{\delta}, \alpha, \beta, \gamma\},$$

where the generalized Frøyshov invariants are viewed as maps $\mathfrak{E} \rightarrow \mathbb{Z}$. Moreover, $\underline{\delta}(X)$ and $\bar{\delta}(X)$ are not generally determined by $\tilde{H}_*^G(X)$.

Proof. We refer to Example 3.13 for the last assertion, so we need only determine $\delta_{H,e}$.

First, consider strict subgroups $\mathbb{Z}/n = H \subset S^1$.

If n is odd, then $H^*(B\mathbb{Z}/n; \mathbb{Z}/2) \cong \mathbb{F}$, concentrated in degree 0, and so there are no generalized Frøyshov invariants. For n even, we first note that

$$H^*(B\mathbb{Z}/n; \mathbb{Z}/2) = H^*(B\mathbb{Z}/2^r; \mathbb{Z}/2)$$

for $n = 2^r n'$ and n' odd (see, e.g., Proposition 3G.1 of [4]). In fact,

$$H^*(B\mathbb{Z}/2^r; \mathbb{Z}/2) = \mathbb{F}[Q, U]/Q^2 \tag{30}$$

for $r \geq 2$, where $\text{deg } Q = 1$ and $\text{deg } U = 2$ as usual. To see this, we recall (see Example 3G.5 in [4]) that

$$H^*(B\mathbb{Z}/2^{r+1}; \mathbb{Z}/2) \rightarrow H^*(\mathbb{Z}/2^r; \mathbb{Z}/2)$$

is an isomorphism in even degrees and zero in odd degrees. Combining this with the fact that $H^1(B\mathbb{Z}/2^r; \mathbb{Z}/2) \cong \mathbb{F}$ for all r and the fact that the cup product with U (for any r) is an isomorphism

$$H^*(B\mathbb{Z}/2^r; \mathbb{Z}/2) \rightarrow H^{*+2}(B\mathbb{Z}/2^r; \mathbb{Z}/2),$$

we obtain (30).

Using the ring structure as before, an argument as in Lemma 3.5 shows that, for n even, the only associated generalized Frøyshov invariant is $\delta_{\mathbb{Z}/n,1}$ and that $\delta_{\mathbb{Z}/n,1} = \delta$ (in principle, there are two separate Frøyshov invariants when $r \geq 2$; however, they turn out to agree). Thus the generalized Frøyshov invariants associated with a subgroup of S^1 are determined by δ and determine δ .

Next, consider a strict subgroup $H \subset G$ not contained in S^1 and not equal to $\mathbb{Z}/4$. Then H is a generalized quaternion group $Q_{4m} = \langle e^{\pi i/m}, j \rangle$ with $m \geq 2$.

First, let m be even. We note $H^1(BQ_{4m}) = \text{Hom}(Q_{4m}, \mathbb{F}) = \mathbb{F}^2$. Then, since $H^1(BG) = \mathbb{F}$, we see that the Gysin sequence associated to the sphere bundle

$$S^1 \rightarrow BQ_{4m} \rightarrow BG$$

splits:

$$H^*(BQ_{4m}) = H^*(BG) \oplus H^*(BG)[-1] \tag{31}$$

as an $H^*(BG)$ -module. Recall that $H^*(BG)$ acts on $H^*(BQ_{4m})$ by the map $p : H^*(BG) \rightarrow H^*(BQ_{4m})$.

Let r generate $H^1(BG)[-1]$ in the decomposition (31). Then the homogeneous elements of $H^*(BQ_{4m})/(v)$ are

$$1, q, q + r, r, q^2, q^2 + qr, qr, \text{ and } q^2r.$$

Furthermore, we note from the definition of $\delta_{H,e}$ that, for $X \in \mathfrak{E}$,

$$\delta_{H,f}(X) \geq \delta_{H,e}(X) \text{ if } f \text{ divides } e. \tag{32}$$

Repeating the argument in the proof of Propositions 3.7 and 3.8, we see that

$$\begin{aligned} \delta_r(X) &\geq \alpha(X), & \delta_{q+r}(X) &\geq \alpha(X), & \delta_{qr}(X) &\geq \beta(X), \\ \delta_{qr+q^2}(X) &\geq \beta(X), & \delta_{q^2r}(X) &\geq \gamma(X), \end{aligned} \tag{33}$$

$$\delta_1(X) \leq \alpha(X), \quad \delta_q(X) \leq \beta(X), \quad \delta_{q^2}(X) \leq \gamma(X). \tag{34}$$

However, $\delta_r(X) \leq \delta_1(X)$ by (32), so $\delta_r(X) \leq \delta_1(X) = \alpha(X)$. Similarly, we obtain that all the inequalities in (33) and (34) are equalities. Thus $\delta_{H,e}(X)$ are in fact determined by $\alpha(X)$, $\beta(X)$, and $\gamma(X)$. This completes the proof for the even m case.

If m is odd, then we have $H^*(BQ_{4m}) \cong H^*(B\mathbb{Z}/4)$. The argument of Lemma 3.5 then adapts to show that the generalized Frøyshov invariants of Q_{4m} and $\mathbb{Z}/4$ agree. □

Theorem 1.5 follows from Theorem 3.12, whereas Corollary 1.6 is a consequence of the proof of Theorem 3.12. We close with an example showing that $SWFH_*^G(Y, \mathfrak{s})$, as an $H^*(BG)$ -module, does not determine $\delta, \bar{\delta}$, or $\underline{\delta}$.

EXAMPLE 3.13. There are pointed G -stable homotopy types X_1 and X_2 such that

$$\tilde{H}_G^*(X_1) = \tilde{H}_G^*(X_2) = \mathcal{V}_8^+ \oplus \mathcal{V}_1^+ \oplus \mathcal{V}_2^+ \oplus \mathbb{F}_3^2 \oplus \mathbb{F}_4,$$

where \mathcal{V}_n^+ denotes the $\mathbb{F}[v]$ -module $\mathbb{F}[v]$ with grading shifted up by n , and \mathbb{F}_n denotes a copy of \mathbb{F} concentrated in degree n . Furthermore,

$$\bar{\delta}(X_1) = \delta(X_1) = 2, \quad \underline{\delta}(X_i) = 0 \quad \text{and} \quad \bar{\delta}(X_2) = \delta(X_2) = 3.$$

To specify the q -action, let t_8, t_1 , and t_2 be $\mathbb{F}[v]$ -generators of $\mathcal{V}_8^+, \mathcal{V}_1^+$, and \mathcal{V}_2^+ respectively, whereas y_3, y'_3 generate \mathbb{F}_3^2 , and y_4 generates \mathbb{F}_4 . Then $qt_8 = v^2t_1, qt_1 = t_2, qt_2 = y_3$, and $qy'_3 = y_4$.

We give a description of the chain complexes of X_1 and X_2 over $C_*^{CW}(G) = \mathbb{F}[s, j]/(sj = j^3s, s^2 = j^4 + 1 = 0)$ where $\deg s = 1$ and $\deg j = 0$. Indeed, $C_*^{CW}(X_1)$ is $\mathbb{F}[f, x_1, x_3, x_4, x_5, y_3]$ with $\partial(x_1) = f, \partial(x_3) = (1 + j)^3sx_1, \partial(x_4) = (1 + j)x_3, \partial(x_5) = (1 + j)x_4 + sx_3$, and $\partial(y_3) = (1 + j)^2sx_1$. We have $C_*^{CW}(X_2) = \mathbb{F}[f, x_1, x_3, x_4, y_3, y_5]$, where the differentials are as before, and $\partial(y_5) = (1 + j)^2sy_3$. The calculation of the Manolescu invariants for both examples is an application of the techniques of [16; 15]. It is not known to the author

if the spaces X_1 and X_2 are realized as $SWF(Y, \mathfrak{s})$ for some rational homology sphere Y .

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