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Hodge Integrals in FJRW Theory

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ABSTRACT. We study higher-genus Fan–Jarvis–Ruan–Witten theory of any chain polynomial with any group of symmetries. Precisely, we give an explicit way to compute the cup product of Polishchuk and Vaintrob's virtual class with the top Chern class of the Hodge bundle. Our formula for this product holds in any genus and without any assumption on the semi-simplicity of the underlying cohomological field theory.

0. Introduction

In 1999, Candelas, de la Ossa, Green, and Parkes [4] proposed a famous formula for the genus-zero invariants enumerating rational curves on the quintic threefold. It has later been proved by Givental [18; 19] and Lian, Liu, and Yau [31; 32; 33], giving a full understanding of Gromov–Witten invariants in genus zero for the quintic threefold. The genus-one case was then completely solved by Zinger [39]. However, we still lack a complete understanding in higher genus.

In fact, even the problem of computing genus-zero Gromov–Witten invariants of projective varieties is not completely solved. One of the techniques is called quantum Lefschetz principle (see, e.g., [12]) and compares Gromov–Witten invariants of a complete intersection with those of the ambient projective space. Thus, we are still missing Gromov–Witten invariants attached to primitive cohomological classes, that is, the classes that do not come from the ambient space.

When considering complete intersections in weighted projective spaces, the theory for genus-zero and with ambient cohomological classes looks as complicated as in the higher genus case because of the lack of a convenient assumption: convexity. Convexity hypothesis roughly turns the virtual fundamental cycle from Gromov–Witten theory into the top Chern class of a vector bundle, making it easier to compute. However, in general, this assumption is not satisfied, and the quantum Lefschetz principle can fail [13].

Fan, Jarvis, and Ruan [17; 16], based on ideas of Witten [38], have switched to another quantum theory, which they define for polynomial singularities. We call it FJRW theory, and it is attached to a Landau–Ginzburg orbifold (W, G), where W is a nondegenerate quasi-homogeneous polynomial singularity, and G is a group of diagonal symmetries of W. The Landau–Ginzburg/Calabi–Yau correspondence conjecture [11] describes, under some Calabi–Yau assumption, the

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relation between this new theory and Gromov–Witten theory of the hypersurface[1] defined by W in the corresponding weighted projective space. In genus zero, this conjecture has been proven in some convex cases in [9].

Therefore, the study of FJRW theory appears as a new point of view toward the study of Gromov–Witten theory. In [23], we described an explicit way to compute FJRW theory in genus zero for polynomials whose Gromov–Witten counterparts are unknown because of the lack of convexity.[2] In the recent work [5], the Landau–Ginzburg/Calabi–Yau correspondence is studied in higher genus for the quintic hypersurface in $\mathbb{P}^4$.

In nonzero genus, both Gromov–Witten and Fan–Jarvis–Ruan–Witten theories are extremely difficult to compute. There are nevertheless some powerful techniques, such as the localization [20] and the degeneration [30] formulas in Gromov–Witten theory and Teleman's reconstruction theorem for conformal generically semisimple cohomological field theories [37]. For instance, the localization formula determines all Gromov–Witten invariants of homogeneous spaces [26; 20]. Also, Teleman's reconstruction theorem takes a major place in the proof of the generalization of Witten conjecture to ADE singularities [15; 17], in the proof of Pixton's relations [35], and more recently in the study of higher-genus mirror symmetry [25] after Costello and Li [14].

The method presented in this paper is quite different from the techniques mentioned and is valid for a range of Landau–Ginzburg orbifolds for which no previous techniques are applicable. More precisely, it works without any semisimplicity assumption and uses instead the K-theoretic vanishing properties of a recursive complex of vector bundles. It is a direct generalization of the results in [23], where recursive complexes are introduced for the first time.

In this introduction, we state our theorem in the chain case with the so-called narrow condition and refer to Theorem 2.2 for a complete statement. Let (W, G) be a Landau–Ginzburg orbifold, where W is a chain polynomial

$$W = x_1^{a_1} x_2 + \cdots + x_{N-1}^{a_{N-1}} x_N + x_N^{a_N},$$

and G is a group of diagonal matrices preserving W and containing the matrix $\mathfrak{j}$ defined in (1).

We take n diagonal matrices $\gamma(1), \dots, \gamma(n)$ in the group G with no entries equal to 1 (narrow condition) and consider the moduli space $\mathcal{S}_{g,n}(W, G)(\gamma(1), \dots, \gamma(n))$ of genus-g (W, G)-spin marked curves with monodromy $\gamma(i)$ at the ith marked point (see Section 1.2 for definitions).

We denote by $\mathcal{L}_1, \dots, \mathcal{L}_N$ the universal line bundles associated with the variables $x_1, \dots, x_N$, by $c_{\mathrm{vir}}^{\mathrm{PV}}(\gamma(1), \dots, \gamma(n))_{g,n}$ the associated virtual class defined by Polishchuk and Vaintrob [36], by π the morphism from the universal curve to the moduli space, and by $\mathbb{E} := \pi_* \omega$ the Hodge vector bundle on the moduli space corresponding to global differential forms on the curves.

[1]More precisely, the Landau–Ginzburg orbifold (W, G) corresponds to the quotient stack $[X/\tilde{G}]$, where X is the hypersurface corresponding to the zero locus of W, $\tilde{G}$ is the group $G/\langle \mathfrak{j} \rangle$, and $\mathfrak{j}$ is the matrix defined in (1).

[2]The corresponding notion in FJRW theory is called concavity.

THEOREM 0.1. *Let* (W, G), $\gamma(1), \ldots, \gamma(n)$, *and* $\mathcal{L}_1, \ldots, \mathcal{L}_N$ *be as before. For any genus* g, *we have*

$$c_{\mathrm{top}}(\mathbb{E}^\vee) c_{\mathrm{vir}}^{\mathrm{PV}}(\gamma(1), \ldots, \gamma(n))_{g,n} = \lim_{t_1 \to 1} \prod_{j=1}^{N} \mathfrak{c}_{t_j}(-R^\bullet \pi_*(\mathcal{L}_j)) \cdot \mathfrak{c}_{t_{N+1}}(\mathbb{E}^\vee),$$

where the variables t_j *satisfy the relations* $t_j^{a_j} t_{j+1} = 1$ *for* $j \leq N$, *and where the function* $\mathfrak{c}_t$ *is the characteristic class introduced in* [23]; *see also equation* (16).

The theorem has several consequences:

- computation of Hodge integrals in FJRW theory via a computer [22; 24],
- computation of double ramification hierarchies [2; 3],
- new method to study nonsemisimple cohomological field theories,
- tautological relations in the Chow ring of the moduli spaces of (W, G)-spin curves, in particular, in the moduli spaces of r-spin and of stable curves.

REMARK 0.2. It is important to stress the fact that FJRW theory is not a generically semisimple cohomological field theory in general, especially when the group G is generated by the element $\mathfrak{j}$ defined in (1). In such cases, Teleman's reconstruction theorem [37] does not apply, and, to our knowledge, the method presented in this paper is the first comprehensive approach in higher genus for these theories, although we only obtain a partial information on the virtual class.

Integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} c_{\mathrm{top}}(\mathbb{E}^\vee)\alpha, \quad \alpha \in H^*(\overline{\mathcal{M}}_{g,n})$$

are called Hodge integrals. Thus Theorem 0.1, together with Mumford's [34] and Chiodo's [8] formulas, yields an explicit way to compute Hodge integrals in FJRW theory in any genus, and it has been implemented into a computer; see [22; 24]. In particular, it is used in [3] to provide a positive answer to Buryak's conjecture [2] on the double ramification hierarchy for r-spin theory with $r \leq 5$.

THEOREM 0.3 (See [3, Theorem 1.1]). *For the* 3*-spin theory, the double ramification hierarchy coincides with the Dubrovin–Zhang hierarchy. For the* 4*- and* 5*-spin theories, the double ramification hierarchy is related to the Dubrovin–Zhang hierarchy by the following Miura transformation*:

$$\begin{cases} w^1 = u^1 + \dfrac{\varepsilon^2}{96} u^3_{xx}, \\ w^2 = u^2, \\ w^3 = u^3, \end{cases} \qquad \text{for } r = 4;$$

$$\begin{cases} w^1 = u^1 + \dfrac{\varepsilon^2}{60} u^3_{xx}, \\ w^2 = u^2 + \dfrac{\varepsilon^2}{60} u^4_{xx}, \\ w^3 = u^3, \\ w^4 = u^4, \end{cases} \quad \textit{for } r = 5.$$

Theorem 0.1 has another remarkable consequence: it provides tautological relations in the Chow ring of the moduli space of (W, G)-spin curves. Indeed, the result of Theorem 0.1 holds in the Chow ring and not only in the cohomology ring. Furthermore, it is a statement on the moduli space of (W, G)-spin curves, obtained before forgetting the spin structure to end in the moduli space of stable curves. Even in the r-spin case, where the underlying cohomological field theory is generically semisimple and conformal, these results are new. The main reason is that Teleman's reconstruction theorem [37] only holds in the cohomology ring and after pushing forward to the moduli space of stable curves.

Corollary 0.4. *Let* (W, G), $\gamma(1), \dots, \gamma(n)$, *and* $\mathcal{L}_1, \dots, \mathcal{L}_N$ *be as before. For any genus* g, *the expression*

$$\prod_{j=1}^{N} \mathfrak{c}_{t_j}(-R^\bullet \pi_*(\mathcal{L}_j)) \cdot \mathfrak{c}_{t_{N+1}}(\mathbb{E}^\vee)$$

from Theorem 0.1 is a Laurent power series in the variable $\varepsilon := t_1^{-1} - 1$ *of the form*

$$C_{-p} \cdot \frac{1}{\varepsilon^p} + C_{-p+1} \cdot \frac{1}{\varepsilon^{p-1}} + \cdots + C_{-1} \cdot \frac{1}{\varepsilon} + C_0 + C_1 \cdot \varepsilon + \cdots,$$

where

$$C_m \in \bigoplus_{k \geq \mathrm{degvir} + g - m} A^k(\mathcal{S}^G_{g,n})$$

and $p = 2g - 3 + n - \mathrm{degvir}$, *the integer* degvir *being the Chow degree of* $c^{\mathrm{PV}}_{\mathrm{vir}}(\gamma(1), \dots, \gamma(n))_{g,n}$. *Thus, we obtain tautological relations*[3]

$$C_m = 0 \quad \textit{for all } m < 0$$

in the Chow ring of the moduli space of (W, G)*-spin curves.*

In a subsequent work, we will address the question to compare the pushforward of these relations to the moduli space of stable curves with other tautological relations, for example, Pixton's relations [35]. For this matter, it could be useful to rephrase the formula of Theorem 0.1 as a sum over dual graphs. It is done in Theorem 3.3.

[3] It is not expected for the coefficients $C_0, C_1, \dots$ to be zero in general; for example, direct computations show that C_1 can be nonzero. Moreover, the coefficient C_0 equals the virtual class and is generally nonzero. However, in special cases, for example, under the so-called Ramond vanishing, the coefficient C_0 yields another tautological relation.

Structure of the Paper

In the first part, we briefly recall the main definitions and constructions in FJRW theory following our previous article [23]. The second part consists of the main Theorem 2.2 together with its proof. In the third part, we give an alternative formula as a sum over dual graphs.

1. Quantum Singularity Theory

In this section, we give a brief summary of the necessary definitions for the quantum singularity (or FJRW) theory of a Landau–Ginzburg (LG) orbifold. We use notations of [23], where we dealt with invertible polynomials, but here we are mainly interested in chain or loop polynomials.

1.1. Conventions and Notations

The quantum singularity theory was first introduced by Fan, Jarvis, and Ruan [17; 16] after ideas of Witten [38]. In particular, Fan, Jarvis, and Ruan constructed a cohomological class, called the virtual class, via an analytic construction from Witten's initial sketched idea [38] formalized for A-singularities by Mochizuki. Polishchuk and Vaintrob [36] provided an algebraic construction, which generalized their previous construction and that of Chiodo [7] in the A-singularity case.

We do not know in general whether the two constructions coincide. In FJRW terminology, there is a decomposition of the state space into narrow and broad states. Chang, Li, and Li [6, Theorem 1.2] proved the match when only narrow entries occur. For almost all LG orbifolds (W, G) where W is an invertible polynomial and G is the maximal group of symmetries, we proved in [23, Theorem 3.25] that the two classes are the same up to a reparameterization of the broad states. Nevertheless, for smaller groups or more general polynomials, we still do not know whether these two classes coincide. Therefore, in the whole paper, by the virtual class we mean the Polishchuk–Vaintrob's version, as soon as we are working with broad states together with nonmaximal group G.

Furthermore, we work in the algebraic category and over $\mathbb{C}$. All moduli functors considered here are represented by proper Deligne–Mumford stacks; we use also the term "orbifold" for this type of stacks. We denote orbifolds by curly letters, for example, $\mathcal{C}$ is an orbifold curve, and the scheme C is its coarse space. We recall that vector bundles are coherent locally free sheaves and that the symmetric power of a two-term complex is the complex

$$\mathrm{Sym}^k([A \to B]) = [\mathrm{Sym}^k A \to \mathrm{Sym}^{k-1} A \otimes B \to \cdots \to A \otimes \Lambda^{k-1} B \to \Lambda^k B]$$

with morphisms induced by $A \to B$.

All along the text, the index i varies from 1 to n and refers exclusively to the marked points of a curve, whereas the index j varies from 1 to N and corresponds to the variables of the polynomial. We represent tuples by overlined notations, for example, $\overline{\gamma} = (\gamma(1), \dots, \gamma(n))$, or by underlined notations, for example, $\underline{p} = (p_1, \dots, p_N)$.

1.2. Landau–Ginzburg Orbifold

Let $w_1, \dots, w_N$ be coprime positive integers, d be a positive integer, and $\mathfrak{q}_j := w_j/d$ for all j. We consider a quasi-homogeneous polynomial W of degree d with weights[4] $w_1, \dots, w_N$ and with an isolated singularity at the origin. We say that such a polynomial W is nondegenerate. In particular, for any $\lambda, x_1, \dots, x_N \in \mathbb{C}$, we have

$$W(\lambda^{w_1} x_1, \dots, \lambda^{w_N} x_N) = \lambda^d W(x_1, \dots, x_N),$$

and the dimension of the Jacobian ring

$$\mathcal{Q}_W := \mathbb{C}[x_1, \dots, x_N]/(\partial_1 W, \dots, \partial_N W)$$

is finite over $\mathbb{C}$.

An admissible group of symmetries for the polynomial W is a group G made of diagonal matrices $\mathrm{diag}(\lambda_1, \dots, \lambda_N)$ satisfying

$$W(\lambda_1 x_1, \dots, \lambda_N x_N) = W(x_1, \dots, x_N) \quad \text{for every } (x_1, \dots, x_N) \in \mathbb{C}^N$$

and containing the grading element

$$\mathfrak{j} := \mathrm{diag}(e^{2i\pi \mathfrak{q}_1}, \dots, e^{2i\pi \mathfrak{q}_N}), \quad \mathfrak{q}_j := \frac{w_j}{d}. \tag{1}$$

The group G is finite and contains the cyclic group μ_d of order d generated by $\mathfrak{j}$. We denote the biggest admissible group by $\mathrm{Aut}(W)$.

DEFINITION 1.1. A Landau–Ginzburg (LG) orbifold is a pair (W, G) with W a nondegenerate (quasi-homogeneous) polynomial and G an admissible group.

The quantum singularity theory developed by Fan, Jarvis, and Ruan [17; 16] is defined for any LG orbifold. In fact, it mostly depends on the weights, the degree, and the group. Precisely, by [17, Theorem 4.1.8.9] the theories for two LG orbifolds (W_1, G) and (W_2, G) where the polynomials W_1 and W_2 have the same weights and degree are isomorphic.

In the context of mirror symmetry, a well-behaved class of polynomials has been introduced by Berglund and Hübsch [1]. We say that a polynomial is invertible when it is nondegenerate with as many variables as monomials. According to Kreuzer and Skarke [29], every invertible polynomial is a Thom–Sebastiani (TS) sum of invertible polynomials, with disjoint sets of variables, of the following three types.

$$\begin{aligned} &\text{Fermat:} \quad x^{a+1}; \\ &\text{chain of length } c\text{:} \quad x_1^{a_1} x_2 + \cdots + x_{c-1}^{a_{c-1}} x_c + x_c^{a_c+1} \quad (c \geq 2); \\ &\text{loop of length } l\text{:} \quad x_1^{a_1} x_2 + \cdots + x_{l-1}^{a_{l-1}} x_l + x_l^{a_l} x_1 \quad (l \geq 2). \end{aligned} \tag{2}$$

REMARK 1.2. In this paper, we consider only polynomials that are of these three types and not a Thom–Sebastiani sum of them.

[4] We assume that a choice of coprime positive weights $w_1, \dots, w_N$ is unique.

For any $\gamma \in \mathrm{Aut}(W)$, the set of broad variables with respect to γ is

$$\mathfrak{B}_\gamma = \{x_j | \gamma_j = 1\}. \tag{3}$$

DEFINITION 1.3. The state space[5] for the LG orbifold (W, G) is the vector space

$$\mathbf{H}_{(W,G)} = \bigoplus_{\gamma \in G} \mathbf{H}_\gamma = \bigoplus_{\gamma \in G} (\mathcal{Q}_{W_\gamma} \otimes d\underline{x}_\gamma)^G,$$

where W_γ is the γ-invariant part of the polynomial W, $\mathcal{Q}_{W_\gamma}$ is its Jacobian ring, the differential form $d\underline{x}_\gamma$ is $\bigwedge_{x_j \in \mathfrak{B}_\gamma} dx_j$, and the upperscript G stands for the invariant part under the group G.

At last, the quantum singularity theory for an LG orbifold (W, G) is a cohomological field theory, that is, the data of multilinear maps

$$c_{g,n} : \mathbf{H}^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}),$$

which are compatible under gluing and forgetting-one-point morphisms. More precisely, the maps $c_{g,n}$ factor through the cohomology (and even the Chow ring) of another moduli space $\mathcal{S}_{g,n}(W, G)$ attached to the LG orbifold (W, G); the map

$$(c_{\mathrm{vir}})_{g,n} : \mathbf{H}^{\otimes n} \to A^*(\mathcal{S}_{g,n}(W, G))$$

is called the virtual class,[6] where A^* can stand for the cohomology or the Chow ring. Then, via the natural forgetful morphism $\mathrm{o} : \mathcal{S}_{g,n}(W, G) \to \overline{\mathcal{M}}_{g,n}$, we get

$$c_{g,n} := (-1)^{\mathrm{degvir}} \frac{\mathrm{card}(G)^g}{\deg(\mathrm{o})} \cdot \mathrm{o}_*(c_{\mathrm{vir}})_{g,n}, \tag{4}$$

where $(-1)^{\mathrm{degvir}}$ acts as $(-1)^m$ on $A^m(\overline{\mathcal{M}}_{g,n})$.

REMARK 1.4. In the case of r-spin curves, the degree of the forgetful morphism o equals r^{2g-1}. In general, for the maximal group $G = \mathrm{Aut}(W)$, this degree also equals r^{2g-1}, where r is the exponent of the group.

The moduli space $\mathcal{S}_{g,n}(W, G)$ is defined in [17, Section 2] as follows. First, let us fix r to be the exponent of the group G, that is, the smallest integer l such that $\gamma^l = 1$ for every element $\gamma \in G$. We recall that an r-stable curve is a smoothable[7] orbifold curve with markings whose nontrivial stabilizers have fixed order r and are only at the nodes and at the markings. Moreover, its coarse space is a stable curve.

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[5] We refer to [9, equation (4)] or [36, equation (5.12)] for details about the bidegree and the pairing in this space.

[6] For the polynomial x^r, we obtain the moduli space of r-spin structures, and the virtual class is called the Witten r-spin class.

[7] Concretely, smoothable means that the local picture at the node is $[\{xy = 0\}/\mu_r]$ with the balanced action $\zeta_r \cdot (x, y) = (\zeta_r x, \zeta_r^{-1} y)$.

Then, the moduli space $\mathcal{S}_{g,n}(W,G)$ classifies all r-stable curves of genus g with n marked points, together with N line bundles and s isomorphisms

$$(\mathcal{C};\sigma_1,\dots,\sigma_n;\mathcal{L}_1,\dots,\mathcal{L}_N;\phi_1,\dots,\phi_s),$$

where the isomorphisms $\phi_1,\dots,\phi_s$ give some constraints (see further) on the choice of $\mathcal{L}_1,\dots,\mathcal{L}_N$. We call such data a (W,G)-spin curve.

There are two steps to get the constraints $\phi_1,\dots,\phi_s$. First, we choose a Laurent polynomial Z with weights $w_1,\dots,w_N$ and degree d just as W and with the maximal group verifying $\mathrm{Aut}(W+Z)=G$; see [27, Proposition 3.4]. Then, denoting by $M_1,\dots,M_s$ all the monomials of $W+Z$, we have

$$\phi_k : M_k(\mathcal{L}_1,\dots,\mathcal{L}_N)\simeq \omega_{\log}:=\omega_{\mathcal{C}}(\sigma_1+\dots+\sigma_n)\quad \text{for all } k. \tag{5}$$

The moduli space that we obtain does not depend on the choice of the Laurent polynomial Z; see [10].

A line bundle over an orbifold point comes with an action of the isotropy group at that point, that is, locally at a marked point σ_i, we have an action

$$\zeta_r\cdot(x,\xi)=(\zeta_r x,\zeta_r^{m_j(i)}\xi)\quad \text{with } m_j(i)\in\{0,\dots,r-1\} \tag{6}$$

called the monodromy of the line bundle $\mathcal{L}_j$ at the marked point σ_i. Since the logarithmic canonical line bundle $\omega_{\log}$ is a pullback from the coarse curve, its multiplicity is trivial on each marked point, so that equations (5) give

$$\gamma(i):=(e^{2\mathrm{i}\pi m_1(i)/r},\dots,e^{2\mathrm{i}\pi m_N(i)/r})\in \mathrm{Aut}(W+Z)=G.$$

We define the type of a (W,G)-spin curve as $\overline{\gamma}:=(\gamma(1),\dots,\gamma(n))\in G^n$. It yields a decomposition

$$\mathcal{S}_{g,n}(W,G)=\bigsqcup_{\overline{\gamma}\in G^n}\mathcal{S}_{g,n}(W,G)(\gamma(1),\dots,\gamma(n)),$$

where $\mathcal{S}_{g,n}(W,G)(\overline{\gamma})$ is an empty component when the selection rule

$$\gamma(1)\cdots\gamma(n)=\mathfrak{j}^{2g-2+n} \tag{7}$$

is not satisfied; see [17, Proposition 2.2.8].

1.3. Aut(W)-*Invariant States*

From Definition 1.3 of the state space $\mathbf{H}_{(W,G)}$ we see that it always contains the subspace

$$\begin{aligned}\mathbf{H}_{(W,G),\mathrm{Aut}(W)}&=\bigoplus_{\gamma\in G}(\mathcal{Q}_{W_\gamma}\otimes d\underline{x}_\gamma)^{\mathrm{Aut}(W)}\\&\subset \mathbf{H}_{(W,G)}.\end{aligned}$$

DEFINITION 1.5. The subspace $\mathbf{H}_{(W,G),\mathrm{Aut}(W)}$ is called the Aut(W)-invariant part.

For invertible polynomials W and any group G, the Aut(W)-invariant part has a particularly nice and explicit description; see [28; 23]. Using the language from [23], we can attach a graph Γ_W to any invertible polynomial, illustrating its Kreuzer–Skarke decomposition as a Thom–Sebastiani sum of Fermat, chain, and loop polynomials. Then, we consider decorations[8] $\mathfrak{C}_\gamma$ of the graph Γ_W that are admissible and balanced, and with each such decoration, we associate an explicit element $e(\mathfrak{C}_\gamma)$ of $\mathbf{H}_{(W,G),\mathrm{Aut}(W)}$. At last, by [28] and [23, equation (10)], the set of all these elements forms a basis of $\mathbf{H}_{(W,G),\mathrm{Aut}(W)}$.

EXAMPLE 1.6. Let $W = x_1^{a_1}x_2 + \cdots + x_{c-1}^{a_{N-1}}x_N + x_N^{a_N+1}$ be a chain polynomial. For any element $\gamma \in G$, the set of broad variables is of the form

$$\mathfrak{B}_\gamma = \{x_{b+1}, \ldots, x_N\},$$

and there is exactly one admissible decoration $\mathfrak{C}_\gamma$ given by

$$\mathfrak{C}_\gamma = \{x_{N-2j} \mid N - 2j > b\}.$$

This decoration is balanced if and only if $N - b$ is even and the corresponding element is

$$e_\gamma := e(\mathfrak{C}_\gamma) = \Bigg(\prod_{\substack{b<j\leq N\\ N-j \text{ odd}}} a_j x_j^{a_j-1}\Bigg) \cdot dx_{b+1} \wedge \cdots \wedge dx_N.$$

EXAMPLE 1.7. Let $W = x_1^{a_1}x_2 + \cdots + x_{c-1}^{a_{N-1}}x_N + x_N^{a_N}x_1$ be a loop polynomial. For an element $1 \neq \gamma \in G$, the set of broad variables is empty. For the identity element, it is $\mathfrak{B}_1 = \{x_1, \ldots, x_N\}$. Then, if N is odd, then there is no admissible and balanced decoration. But if N is even, then we have two distinct admissible and balanced decorations given by

$$\mathfrak{C}_1^+ = \{x_j \mid j \text{ even}\} \quad \text{and} \quad \mathfrak{C}_1^- = \{x_j \mid j \text{ odd}\};$$

the two corresponding elements are

$$e^+ := e(\mathfrak{C}_1^+) = \Bigg(\prod_{x_j \text{ odd}} a_j x_j^{a_j-1} - \prod_{x_j \text{ even}} -x_j^{a_j-1}\Bigg) \cdot dx_1 \wedge \cdots \wedge dx_N$$

and e^- given by exchanging even and odd.

1.4. Sketch of Definition of PV Virtual Class

The Polishchuk–Vaintrob construction [36] of the virtual class $(c_{\mathrm{vir}})_{g,n}$ for an LG orbifold (W, G) uses the notion of matrix factorizations. We briefly recall the main steps.

Consider a component $\mathcal{S}_{g,n}(\overline{\gamma})$ of type $\overline{\gamma} = (\gamma(1), \ldots, \gamma(n)) \in G^n$. We denote by π the projection of the universal curve to this component, and we look at the higher pushforwards $R^\bullet\pi_*\mathcal{L}_j$ of the universal line bundles. We take resolutions

[8] A decoration $\mathfrak{C}_\gamma$ is a subset of the set of broad variables $\mathfrak{B}_\gamma = \{x_j | \gamma_j = 1\}$. Definitions for admissible and balanced decorations are in [23, Definition 1.5].

of $R^\bullet\pi_*\mathcal{L}_j$ by complexes $[A_j \to B_j]$ of vector bundles, and we set

$$X := \operatorname{Spec}\operatorname{Sym}\bigoplus_{j=1}^{N} A_j^\vee \quad \text{and} \quad p : X \to \mathcal{S}_{g,n}(\overline{\gamma}).$$

The differential $[A_j \to B_j]$ induces a section β of the vector bundle $p^*\bigoplus_j B_j$ on X. Polishchuk and Vaintrob show how to construct a section α of the dual vector bundle $p^*\bigoplus_j B_j^\vee$ using the algebraic relations (5) between the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_N$. The choice of the resolutions and the existence of the section α require several steps; see [36, Section 4.2, Steps 1–4].

Using evaluation of the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_N$ at the marked points, they also construct a morphism

$$Z : X \to \mathbb{A}^{\overline{\gamma}} := \prod_{i=1}^{n} (\mathbb{A}^N)^{\gamma(i)}, \tag{8}$$

where $(\cdot)^{\gamma(i)}$ is the fixed locus under the action of $\gamma(i)$. In particular, the set of coordinates of the affine space $\mathbb{A}^{\overline{\gamma}}$ is indexed as

$$\{x_j(i)\}_{(\sigma_i, x_j)\in\mathfrak{B}_{\overline{\gamma}}}, \quad \text{where } \mathfrak{B}_{\overline{\gamma}} = \{(\sigma_i, x_j) \mid \gamma_j(i) = 1\},$$

and we further consider the invertible polynomial $W_{\overline{\gamma}}$ on $\mathbb{A}^{\overline{\gamma}}$ given by

$$W_{\overline{\gamma}} := W_{\gamma(1)}(x_1(1), \ldots, x_N(1)) + \cdots + W_{\gamma(n)}(x_1(n), \ldots, x_N(n)),$$

where $W_{\gamma(i)}$ is the restriction of W to $(\mathbb{A}^N)^{\gamma(i)}$.

At last, the two sections α and β yield a Koszul matrix factorization $\mathbf{PV}$ on X. Polishchuk and Vaintrob checked that the potential of $\mathbf{PV}$ is precisely the function $-Z^*W_{\overline{\gamma}}$ on X. To sum up, we have

$$\begin{tikzcd}[column sep=small] & & \mathbf{PV} \in \mathrm{MF}(X, -Z^*W_{\overline{\gamma}}) \arrow[dl] \\ & X \arrow[dl, "Z"'] \arrow[dr, "p"] & \\ \mathbb{A}^{\overline{\gamma}} & & S \end{tikzcd}$$

The matrix factorization $\mathbf{PV}$ is used as a kernel in the Fourier–Mukaï transform

$$\begin{aligned} \Phi : \mathrm{MF}(\mathbb{A}^{\overline{\gamma}}, W_{\overline{\gamma}}) &\to \mathrm{MF}(S, 0) \\ U &\mapsto p_*(Z^*(U) \otimes \mathbf{PV}), \end{aligned} \tag{9}$$

where the two-periodic complex $Z^*(U) \otimes \mathbf{PV}$ is supported inside the zero section $S \hookrightarrow X$ (see [36, Section 4.2, Step 4; Proposition 1.4.2]), so that the pushforward functor is well defined.

Polishchuk and Vaintrob proved that the Hochschild homology of the category of matrix factorizations on an affine space with polynomial potential $f(y_1, \ldots, y_m)$ is isomorphic to $\mathcal{Q}_f \otimes dy_1 \wedge \cdots \wedge dy_m$. They also give a very explicit description of the Chern character map. We then have the commutative diagram

$$\begin{tikzcd} \mathrm{MF}(\mathbb{A}^{\overline{\gamma}}, W_{\overline{\gamma}}) \arrow[r, "\Phi"] \arrow[d, "\mathrm{Ch}"'] & \mathrm{MF}(S, 0) \arrow[d, "\mathrm{Ch}"] \\ \bigotimes_{i=1}^{n} \mathbf{H}_{\gamma(i)} \arrow[r, "\Phi_*"'] & H^*(S) \end{tikzcd}$$

At last, given states $u_{\gamma(1)}, \ldots, u_{\gamma(n)}$ such that $u_{\gamma(i)} \in \mathbf{H}_{\gamma(i)}$, the virtual class evaluated at these states is

$$(c_{\mathrm{vir}})_{g,n}(u_{\gamma(1)}, \ldots, u_{\gamma(n)}) = \Phi_*(u_{\gamma(1)}, \ldots, u_{\gamma(n)}) \prod_{j=1}^{N} \frac{\mathrm{Td}(B_j)}{\mathrm{Td}(A_j)} \tag{10}$$

and is an element of $A^*(\mathcal{S}_{g,n}(W, G)(\gamma(1), \ldots, \gamma(n)))$. By linearity it is extended to

$$(c_{\mathrm{vir}})_{g,n} : \mathbf{H}^{\otimes n} \to A^*(\mathcal{S}_{g,n}(W, G)).$$

1.5. PV Virtual Class on the Aut(W)*-Invariant State Space*

The evaluation of the virtual class on the states $e(\mathfrak{C}_{\gamma(1)}), \ldots, e(\mathfrak{C}_{\gamma(n)})$ has a beautiful form. In [23, Section 2.4], we find an explicit Koszul matrix factorization

$$\mathbf{K}(e(\mathfrak{C}_{\overline{\gamma}})) \in \mathrm{MF}(\mathbb{A}^{\overline{\gamma}}, W_{\overline{\gamma}})$$

such that its Chern character is the element

$$e(\mathfrak{C}_{\overline{\gamma}}) := e(\mathfrak{C}_{\gamma(1)}) \otimes \cdots \otimes e(\mathfrak{C}_{\gamma(n)}).$$

Then, we reformulate the construction of Polishchuk and Vaintrob as follows.

We start with the line bundles

$$\mathcal{L}_j^{\mathfrak{C}} := \mathcal{L}_j\Bigg(- \sum_{(\sigma_i, x_j) \in \mathfrak{C}_{\overline{\gamma}}} \sigma_i\Bigg) \tag{11}$$

instead of $\mathcal{L}_j$, and we apply the same procedure as Polishchuk and Vaintrob [36, Sections 4.1–4.2] to get resolutions by the vector bundles

$$R^\bullet \pi_* \mathcal{L}_j^{\mathfrak{C}} = [A_j \to \widetilde{B}_j]$$

and morphisms[9]

$$\begin{aligned} &\widetilde{\alpha}_j : \mathcal{O} \to \mathrm{Sym}^{a_{j+1}} A_{j+1}^\vee \otimes \widetilde{B}_j^\vee \oplus (\mathrm{Sym}^{a_j - 1} A_j^\vee \otimes A_{j-1}^\vee) \otimes \widetilde{B}_j^\vee, \\ &\widetilde{\beta}_j : \widetilde{B}_j^\vee \to A_j^\vee. \end{aligned} \tag{12}$$

Here, the convention is $(A_0, A_{N+1}) = (0, A_N)$ for a chain polynomial and $(A_0, A_{N+1}) = (A_N, A_1)$ for a loop polynomial.

[9] The morphism $\widetilde{\beta}_j$ comes from the resolution of $R^\bullet \pi_* \mathcal{L}_j^{\mathfrak{C}}$, and the morphism $\widetilde{\alpha}_j$ arises from the algebraic relations (5).

At last, we get a two-periodic complex (T,δ) on the moduli space $\mathcal{S}_{g,n}(W,G)(\overline{\gamma})$, given by the infinite-rank vector bundles

$$T^+ := \mathrm{Sym}(A_1^\vee \oplus \cdots \oplus A_N^\vee) \otimes \bigwedge_{\text{even}} (\widetilde{B}_1^\vee \oplus \cdots \oplus \widetilde{B}_N^\vee),$$
$$T^- := \mathrm{Sym}(A_1^\vee \oplus \cdots \oplus A_N^\vee) \otimes \bigwedge_{\text{odd}} (\widetilde{B}_1^\vee \oplus \cdots \oplus \widetilde{B}_N^\vee),$$

with the differential δ induced by (12). By [23, below equation (34)] and [36, Remark 1.5.1] we have a quasi-isomorphism

$$(T,\delta) \simeq p_*(\mathbf{PV} \otimes \mathbf{K}(\mathfrak{C}_{\overline{\gamma}})).$$

As a consequence, the virtual class evaluated at $e(\mathfrak{C}_{\overline{\gamma}})$ equals

$$(c_{\mathrm{vir}})_{g,n}(e(\mathfrak{C}_{\overline{\gamma}})) = \mathrm{Ch}(H^+(T,\delta) - H^-(T,\delta)) \prod_{j=1}^{N} \frac{\mathrm{Td}(\widetilde{B}_j)}{\mathrm{Td}(A_j)}. \tag{13}$$

In genus zero and for chain polynomials,[10] the main result of [23] provides an explicit expression of the Chern character of the cohomology of (T,δ) in terms of the Chern characters of the higher pushforwards $R^\bullet\pi_*\mathcal{L}_j^{\mathfrak{C}}$. The later are computed by Chiodo's formula [8] using the Grothendieck–Riemann–Roch theorem. Thus the virtual class can be computed as well.

Interestingly, the same method provides an explicit computation of the cup product between the top Chern class of the Hodge bundle and the virtual class in arbitrary genus. We explain it in the following section.

2. Polishchuk and Vaintrob's Virtual Class in Higher Genus

In this section, we prove our main theorem generalizing the computation of the virtual class in genus zero from [23, Theorem 3.21] to Hodge integrals in arbitrary genus; see Theorem 2.2.

2.1. Statement

Let us consider an LG orbifold (W,G) where W is a Fermat monomial, a chain polynomial, or a loop polynomial and G is an admissible group of symmetries. We fix some elements $\gamma(1),\dots,\gamma(n) \in G$ and some admissible decorations $\mathfrak{C}_{\gamma(1)},\dots,\mathfrak{C}_{\gamma(n)}$. We consider the evaluation of the virtual class at the $\mathrm{Aut}(W)$-invariant state

$$e(\mathfrak{C}_{\overline{\gamma}}) := e(\mathfrak{C}_{\gamma(1)}) \otimes \cdots \otimes e(\mathfrak{C}_{\gamma(n)}).$$

[10] It works also with certain invertible polynomials; see [23, Theorem 3.21] for a precise statement.

In the case where W is a loop polynomial, we further assume the existence of a variable x_{j_0} such that

$$\begin{aligned} &\gamma_{j_0}(i) \in \langle e^{2\pi \mathrm{i} w_{j_0}/d}\rangle \quad \forall i, \\ &w_{j_0} \mid d, \\ &\mathcal{L}_{j_0}^{\mathfrak{C}} = \mathcal{L}_{j_0}(-\sigma_1 - \cdots - \sigma_n). \end{aligned} \tag{14}$$

By a cyclic permutation of the indices we can assume that $j_0 = N$.

REMARK 2.1. Conditions (14) are always true for a Fermat monomial and for the last variable x_N of a chain polynomial, even when the group G is nonmaximal. The case of a chain polynomial with nonmaximal group is of great interest as its theory in general is not generically semisimple.

THEOREM 2.2. *Let* (W, G) *and* $e(\mathfrak{C}_{\overline{\gamma}})$ *be as before. For any genus* g, *we have the following equality in the Chow ring of the moduli space of* (W, G)*-spin curves*:

$$\begin{aligned} \lambda_g^{\vee} c_{\mathrm{vir}}^{\mathrm{PV}}(e(\mathfrak{C}_{\overline{\gamma}}))_{g,n} &= \lim_{t\to 1} \prod_{j=1}^{N} \mathfrak{c}_{t_j}(-R^{\bullet}\pi_*(\mathcal{L}_j^{\mathfrak{C}})) \cdot \mathfrak{c}_{t_{N+1}}(\mathbb{E}^{\vee}) \\ &= \lim_{t\to 1} \prod_{\substack{j=1 \\ N-j \text{even}}}^{N} (1-t_j)^{r_j} \\ &\quad \cdot \prod_{j=1}^{N} \mathfrak{c}_{t_j}(-R^{\bullet}\pi_*(\mathcal{L}_j)) \cdot \mathfrak{c}_{t_{N+1}}(\mathbb{E}^{\vee}), \end{aligned} \tag{15}$$

where $\lambda_g^{\vee} := c_g(\mathbb{E}^{\vee})$ *is the top Chern class of the dual of the Hodge bundle*, *the integer* $r_j := \mathrm{card}\{i \mid \gamma_j(i) = 1\}$ *counts broad states*, *and*

$$t_{j+1} = \begin{cases} t & \text{if } j = 0, \\ t_j^{-a_j} & \text{if } 1 \le j \le N-1, \\ t_N^{-d/w_N} & \text{if } j = N. \end{cases}$$

The characteristic class $\mathfrak{c}_t : K^0(S) \to A^*(S)[[t]]$ *is defined by*

$$\mathfrak{c}_t(B-A) = (1-t)^{-\mathrm{Ch}_0(A-B)} \exp\Bigg(\sum_{l\ge 1} s_l(t)\,\mathrm{Ch}_l(A-B)\Bigg), \tag{16}$$

where the functions $s_l(t)$ *are defined in* [23, *equation* (67)] *by*

$$s_l(t) = \begin{cases} -\ln(1-t) & \text{if } l = 0, \\ B_l(0)/l + (-1)^l \sum_{k=1}^{l} (k-1)!(\frac{t}{1-t})^k \gamma(l,k) & \text{if } l \ge 1, \end{cases} \tag{17}$$

with the number $\gamma(l, k)$ *defined by the generating function*

$$\sum_{l\ge 0} \gamma(l,k) \frac{z^l}{l!} := \frac{(e^z - 1)^k}{k!}.$$

REMARK 2.3. As explained in [23], the characteristic class $\mathfrak{c}_t$ naturally appears in K-theory. Indeed, it is defined as

$$\mathfrak{c}_t(x) = \mathrm{Ch}(\lambda_{-t}(x^\vee))\,\mathrm{Td}(x) \in H^*(S)[[t]], \quad \forall x \in K^0(S),$$

where λ_t is the lambda-structure in K-theory defined for a vector bundle V on S by $\lambda_t(V) = \sum_{k\geq 0} \Lambda^k(V)t^k$ and $\lambda_t(-V) = \sum_{k\geq 0} \mathrm{Sym}^k(V)(-t)^k$. Note the classical formula $\lim_{t\to 1} \mathfrak{c}_t(V) = c_{\mathrm{top}}(V)$.

Theorem 2.2 relies on our method developed in [23, Section 3] together with two important observations:

- conditions (14) imply the algebraic relation
$$(\mathcal{L}_N^{\mathfrak{C}})^{\otimes d/w_N} \otimes \mathcal{O} \hookrightarrow \omega_{\mathcal{C}}, \tag{18}$$
which is similar to relations (5),
- the sheaf $\pi_*\omega$ is a vector bundle of rank g. It is called the Hodge bundle, and we denote it by $\mathbb{E}$.

REMARK 2.4. In the particular case of r-spin theory, Theorem 2.2 simplifies as

$$\lambda_g^\vee c_{\mathrm{vir}}^{\mathrm{PV}} = \lim_{t\to 1} \mathfrak{c}_t(-R^\bullet\pi_*(\mathcal{L})) \cdot \mathfrak{c}_{t^{-r}}(\mathbb{E}^\vee).$$

Recall from Section 1.5 that the virtual class comes from the cohomology of a two-periodic complex of infinite-rank vector bundles $T := \mathrm{Sym}(A^\vee) \otimes \Lambda(B^\vee)$, where $[A \to B]$ is a resolution of the derived pushforward $R^\bullet\pi_*(\mathcal{L})$ by vector bundles. The idea of the proof of Theorem 2.2 is to enrich the two-periodic complex T with the Hodge bundle $\mathbb{E}$ into a two-periodic complex $\mathbf{T}$; see Section 2.2. Then, we write the differential of $\mathbf{T}$ as the sum of three differentials such that we obtain two double complexes K_1 and K_2; see Section 2.2. Indeed, the total cohomology of each of these double complexes equals the cohomology of $\mathbf{T}$. To conclude, we use spectral sequences to express the total cohomology of K_1 and K_2, leading naturally to the definition of the virtual class twisted by the Hodge bundle for K_1 and to the formula in terms of the characteristic class $\mathfrak{c}_t$ for K_2. The latter claim follows from [23, Theorem 3.5]: the power series $\mathfrak{c}_t(-R^\bullet\pi_*(\mathcal{L})) \cdot \mathfrak{c}_{t^{-r}}(\mathbb{E}^\vee)$ is a polynomial in t, since the coefficient of t^k is given by a nondegenerate Koszul complex, which is thus exact for $k \gg 0$.

We now proceed to the proof of Theorem 2.2.

2.2. *Modified Two-Periodic Complex and Recursive Complex*

The two previous observations suggest us to introduce the line bundle

$$\mathcal{L}_{N+1} := \mathcal{O}$$

and to choose a resolution $R^\bullet\pi_*\mathcal{L}_{N+1} = [\mathcal{O} \xrightarrow{0} \mathbb{E}^\vee]$ together with a morphism

$$\widetilde{\alpha}_{N+1} : \mathcal{O} \to \mathrm{Sym}^{d/w_N} A_N^\vee \otimes \mathbb{E}. \tag{19}$$

Now, we consider the two-periodic complex $(\mathbf{T},\widetilde{\delta})$ with

$$\begin{aligned}\mathbf{T}^+ &= \mathrm{Sym}(A_1^\vee \oplus \cdots \oplus A_N^\vee) \otimes \Lambda_{\mathrm{even}}(\widetilde{B}_1^\vee \oplus \cdots \oplus \widetilde{B}_N^\vee \oplus \mathbb{E})\\ &= T^+ \otimes \Lambda_{\mathrm{even}}\mathbb{E} \oplus T^- \otimes \Lambda_{\mathrm{odd}}\mathbb{E}\end{aligned}$$

and similarly for $\mathbf{T}^-$ by exchanging odd and even, and with the differential

$$\widetilde{\delta} = \delta_0 + \delta_1 + \delta_2,$$

where

- δ_0 is induced by $\widetilde{\alpha}_1 + \cdots + \widetilde{\alpha}_{N-1} + \widetilde{\beta}_1 + \cdots + \widetilde{\beta}_N$,
- δ_1 is induced by $\widetilde{\alpha}_N$,
- δ_2 is induced by $\widetilde{\alpha}_{N+1}$.

Precisely, we use the natural maps

$$\begin{aligned}\mathrm{Sym}^{p_j}(A_j^\vee) \otimes A_j^\vee &\to \mathrm{Sym}^{p_j+1}(A_j^\vee),\\ \Lambda^{q_j}(\widetilde{B}_j^\vee) \otimes \widetilde{B}_j^\vee &\to \Lambda^{q_j+1}(\widetilde{B}_j^\vee),\\ \Lambda^{q_j}(\widetilde{B}_j^\vee) &\to \Lambda^{q_j-1}(\widetilde{B}_j^\vee) \otimes \widetilde{B}_j^\vee,\end{aligned}$$

composed with the maps

$$\begin{aligned}&\mathcal{O} \xrightarrow{\widetilde{\alpha}_j} \mathrm{Sym}^{a_{j+1}} A_{j+1}^\vee \otimes \widetilde{B}_j^\vee \oplus (\mathrm{Sym}^{a_j-1} A_j^\vee \otimes A_{j-1}^\vee) \otimes \widetilde{B}_j^\vee,\\ &\widetilde{B}_j^\vee \xrightarrow{\widetilde{\beta}_j} A_j^\vee,\\ &\mathcal{O} \xrightarrow{\widetilde{\alpha}_{N+1}} \mathrm{Sym}^{d/w_N} A_N^\vee \otimes \mathbb{E}.\end{aligned}$$

For instance, the differential δ_2 is

$$\bigotimes_{j=1}^N \mathrm{Sym}^{p_j}(A_j^\vee) \otimes \Lambda^{q_j}(\widetilde{B}_j^\vee) \otimes \Lambda^k(\mathbb{E}) \to \bigotimes_{j=1}^N \mathrm{Sym}^{p'_j}(A_j^\vee) \otimes \Lambda^{q_j}(\widetilde{B}_j^\vee) \otimes \Lambda^{k+1}(\mathbb{E})$$

with $p'_j := p_j$ for $j < N$ and $p'_N := p_N + d/w_N$. We refer to [7, Section 2.5] for a detailed description.

Observe that the differential of the two-periodic complex (T,δ) is closely related to the differential $\delta_0 + \delta_1$. Furthermore, note that we have the anticommutation relations

$$\delta_k \circ \delta_l + \delta_l \circ \delta_k = 0$$

for $0 \leq k, l \leq 2$. It comes from the antisymmetric property of the exterior power and, by [7, Lemma 3.2.3], from the equalities

$$\widetilde{\beta}_k \circ \widetilde{\alpha}_l = 0$$

for $1 \leq k \leq N$ and $1 \leq l \leq N+1$. For $l \neq N+1$, it is the same equality as that used to prove that (T,δ) is a two-periodic complex. For $l = N+1$, it is obvious since $\widetilde{\beta}_{k|\mathbb{E}} = 0$.

Therefore, we obtain two double complexes

$$(K_1 = \mathbf{T}, \delta_0 + \delta_1, \delta_2) \quad \text{and} \quad (K_2 = \mathbf{T}, \delta_0 + \delta_2, \delta_1).$$

The double complex K_1 is very explicit, and we can write in particular

$$(K_1)^{\pm,q} = T^{\pm} \otimes \Lambda^q \mathbb{E},$$

whereas the double complex K_2 is more involved. Nevertheless, the cohomology groups of their associated two-periodic complexes agree and equal

$$H^{\pm}(\mathbf{T}, \delta_0 + \delta_1 + \delta_2).$$

We can abut to the total cohomology by looking at the spectral sequences given by the filtration by rows of these two double complexes. In fact, the first page of the spectral sequence is even enough to compute the total cohomology in K-theory, as we show below.

On one side, we have

$$(H^{\pm}(K_1, \delta_0 + \delta_1), \delta_2)^{\bullet} = (H^{\pm}(T, \delta) \otimes \Lambda^{\bullet}\mathbb{E}, \delta_2),$$

which is represented by a bounded complex of vector bundles by [23, equation (61)] and [7, Theorem 3.3.1], or by [36, equation (1.20)]. Observe indeed that $\Lambda^{\bullet}\mathbb{E}$ is a vector bundle (of finite rank) and that $H^{\pm}(T, \delta)$ can be represented by a bounded complex of vector bundles, since the matrix factorization from which it arises has proper support; see the comment right after equation (9). Moreover, note that this property of $H^{\pm}(T, \delta)$ is crucial in order to define the virtual class. As a consequence, we have the following equalities in K-theory:

$$\begin{aligned}
H^{+}(\mathbf{T}, \delta_0 + \delta_1 + \delta_2) &= \bigoplus_{q \geq 0} (H^{+}(K_1, \delta_0 + \delta_1), \delta_2)^{2q} \\
&\quad \oplus (H^{-}(K_1, \delta_0 + \delta_1), \delta_2)^{2q+1} \\
&= H^{+}(T, \delta) \otimes \Lambda_{\mathrm{even}}\mathbb{E} \oplus H^{-}(T, \delta) \otimes \Lambda_{\mathrm{odd}}\mathbb{E}, \\
H^{-}(\mathbf{T}, \delta_0 + \delta_1 + \delta_2) &= H^{+}(T, \delta) \otimes \Lambda_{\mathrm{odd}}\mathbb{E} \oplus H^{-}(T, \delta) \otimes \Lambda_{\mathrm{even}}\mathbb{E}.
\end{aligned}$$

Therefore, by the definition of the virtual class and by the equality

$$\sum_{q \geq 0} (-1)^q \operatorname{Ch}(\Lambda^q V^{\vee}) \operatorname{Td}(V) = c_{\mathrm{top}}(V)$$

for any vector bundle V, we obtain

$$\operatorname{Ch}(H^{+}(\mathbf{T}, \widetilde{\delta}) - H^{-}(\mathbf{T}, \widetilde{\delta})) \prod_{j=1}^{N} \frac{\operatorname{Td}(\widetilde{B}_j)}{\operatorname{Td}(A_j)} \operatorname{Td}(\mathbb{E}^{\vee}) = c_{\mathrm{vir}}^{\mathrm{PV}}(e(\mathfrak{C}_{\overline{\gamma}}))_{g,n} c_{\mathrm{top}}(\mathbb{E}^{\vee}). \tag{20}$$

On the other side, we look at the cohomology groups

$$H^{\pm}(K_2, \delta_0 + \delta_2).$$

The main point is that the two-periodic complex associated with $(K_2, \delta_0 + \delta_2)$ is a nondegenerate recursive complex with the vanishing condition;[11] see [23,

[11] The vanishing condition comes from the fact that we can choose the resolution of $\mathbb{E}$ by vector bundles to be $[0 \to \mathbb{E}]$ since $\mathbb{E}$ is already a vector bundle.

Definitions 3.1 and 3.4 and equation (40)]. Roughly, this means that the complex $(K_2, \delta_0 + \delta_2)$ looks like a direct sum of Koszul complexes of the form

$$0 \to \mathrm{Sym}^p(U) \to \mathrm{Sym}^{p+k}(U) \otimes V \to \cdots \to \mathrm{Sym}^{p+k\,\mathrm{rk}(V)}(U) \otimes \Lambda^{\mathrm{rk}(V)}(V) \to 0,$$

where U and V are two vector bundles, and the map is induced by a nondegenerate map $\mathcal{O} \to \mathrm{Sym}^k(U) \otimes V$. A theorem of Green [21, Theorem 2] on Koszul cohomology claims that the complex is exact providing $p \gg 0$, leaving only a finite number of bounded Koszul complexes. Note also that the above Koszul complex evaluated in K-theory gives the coefficient of t^p in the power series $\mathfrak{c}_t(V^\vee - U^\vee)$. As a consequence, [23, Theorem 3.5] implies that the cohomology groups are finite-rank vector bundles, so that

$$H^+(K_2, \delta_0 + \delta_2) - H^-(K_2, \delta_0 + \delta_2) = H^+(\mathbf{T}, \widetilde{\delta}) - H^-(\mathbf{T}, \widetilde{\delta}).$$

Furthermore, [23, Theorem 3.19] yields an explicit computation of this difference in K-theory yielding

$$\begin{aligned} &\mathrm{Ch}(H^+(\mathbf{T}, \widetilde{\delta}) - H^-(\mathbf{T}, \widetilde{\delta})) \prod_{j=1}^{N} \frac{\mathrm{Td}(\widetilde{B}_j)}{\mathrm{Td}(A_j)} \mathrm{Td}(\mathbb{E}^\vee) \\ &\quad = \lim_{t \to 1} \prod_{j=1}^{N} \mathfrak{c}_{t_j}(-R^\bullet \pi_*(\mathcal{L}_j^{\mathfrak{C}})) \cdot \mathfrak{c}_{t_{N+1}}(\mathbb{E}^\vee) \end{aligned} \tag{21}$$

with t_j and $\mathfrak{c}_t$ as in the statement of Theorem 2.2. Equality between equations (21) and (20) proves the theorem. □

2.3. *Some Remarks*

Theorem 2.2, together with Chiodo's expression [8, Theorem 1.1.1] of the Chern characters of $R^\bullet \pi_* \mathcal{L}_j$ and Mumford's formula [34, equation (5.2)], leads to explicit numerical computations of Hodge integrals that we have encoded into a MAPLE program [24; 22]. Moreover, since the rank of the Hodge bundle is zero in genus zero, we easily recover [23, Theorem 3.21].

In particular, formula (15) gives some information in every genus on the Polishchuk–Vaintrob virtual class for every Landau–Ginzburg orbifold (W, G) with W of chain type and G any admissible group, provided that we evaluate the virtual class at $\mathrm{Aut}(W)$-invariant states. In general, there are more broad states, and we still need further work to understand how to deal with them (just as in genus zero).

In the generically semisimple case, for example, where $G = \mathrm{Aut}(W)$, Teleman's result [37] gives a method to compute the pushforward (4) of the virtual class to the moduli space $\overline{\mathcal{M}}_{g,n}$ as a cohomology class. Precisely, it determines higher-genus information from genus-zero information via the so-called R-matrix action. Nevertheless, the answer is only in cohomology and not in the Chow ring. Moreover, it is not on the virtual class itself but only on its pushforward (4). Interestingly, the general situation in FJRW theory is not generically semisimple. Indeed, in general, it is not generically semisimple for the minimal group $G = \mu_d$, and Theorem 2.2 is the only result at hand to treat the virtual cycle directly on its moduli space, as well as after pushing-forward on the moduli space of stable curves.

An important application to the computation of Hodge integrals comes from [2]. Indeed, Hodge integrals naturally appear in the definition of the double ramification hierarchy introduced by Buryak [2], and Theorem 2.2 is then a useful tool to compute the equations of this integrable hierarchy. Precisely, we wrote a specific computer program for r-spin theories [24; 22], and we proved a conjecture of Buryak when $r \leq 5$; see [3, Theorem 1.1].

At last, as already mentioned in the Introduction, Theorem 2.2 yields some tautological relations in the Chow ring of the moduli space of (W, G)-spin curves and therefore of the moduli space of stable curves. Indeed, the right-hand side of formula (15) is the limit of a power series with coefficients in the Chow ring of the moduli space of the theory. We can develop it and express it as a Laurent series in $\varepsilon := t^{-1} - 1$ to find an expression like

$$C_{-p} \cdot \frac{1}{\varepsilon^p} + C_{-p+1} \cdot \frac{1}{\varepsilon^{p-1}} + \cdots + C_{-1} \cdot \frac{1}{\varepsilon} + C_0 + C_1 \cdot \varepsilon + \cdots .$$

According to the discussion before [23, Corollary 3.20], this expression has the property that

$$C_m \in \bigoplus_{k \geq \mathrm{degvir} + g - m} A^k(\mathcal{S}_{g,n}(W, G)(\overline{\gamma}))$$

and $p = 2g - 3 + n - \mathrm{degvir}$, where the integer $\mathrm{degvir} := -\sum_j \mathrm{Ch}_0(R^\bullet \pi_* \mathcal{L}_j^{\mathfrak{C}})$ is the Chow degree of the virtual class. As a consequence of the existence of the limit in (15) as $\varepsilon \to 0$, we obtain the relations

$$C_m = 0 \quad \text{for } m < 0.$$

3. Sum over Dual Graphs

In this section, we give another expression of formula (15) as a sum of tautological classes over dual graphs.

We consider an LG orbifold (W, G) where W is a Fermat monomial, a chain polynomial, or a loop polynomial and G is an admissible group of symmetries, and we take an $\mathrm{Aut}(W)$-invariant state $e(\mathfrak{C}_{\overline{\gamma}})$. In the case where W is a loop polynomial, we impose condition (14).

As in Theorem 2.2, we define the variables

$$t_{j+1} = \begin{cases} t & \text{if } j = 0, \\ t_j^{-a_j} & \text{if } 1 \leq j \leq N-1, \\ t_N^{-d/w_N} & \text{if } j = N. \end{cases}$$

We take the convention

$$\gamma = \exp(2i\pi\Gamma), \quad \Gamma \in [0, 1[,$$

for any complex number γ with modulus one, and we further define the integer

$$\begin{aligned}\deg_j &:= \sum_{i=1}^{n}(\Gamma_j(i)-\mathfrak{q}_j)+(1-2\mathfrak{q}_j)(g-1)+\delta_{N-j\ \text{even}}\cdot\mathrm{card}\{i \mid \gamma_j(i)=1\}\\ &= -\mathrm{Ch}_0(R^\bullet\pi_*\mathcal{L}_j^{\mathfrak{C}}).\end{aligned}$$

As a convention, we take $\mathcal{L}_{N+1}^{\mathfrak{C}} := \mathcal{O}$ and then $\gamma_{N+1} := 1$ and $\mathfrak{q}_{N+1} := 0$.

DEFINITION 3.1. A G-decorated dual graph is the dual graph of a stable curve enhanced with an element $\gamma(h)$ of the group G on each half-edge h satisfying two conditions:

- for each edge $e=(h,h')$, where h and h' are two half-edges, we have

$$\gamma(h)\cdot\gamma(h') = 1 \in G,$$

- for each vertex v of genus g_v and valence n_v, the product of the matrices attached to half-edges incident to v is

$$\prod_{h\to v}\gamma(h) = \mathfrak{j}^{2g_v-2+n_v}.$$

We say that the G-decorated dual graph is of type $(\gamma(1),\dots,\gamma(n)) \in G^n$ if the matrix attached to the ith leg is $\gamma(i)$.

REMARK 3.2. The moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ has a natural stratification by the dual graphs. Similarly, the moduli space $\mathcal{S}_{g,n}(W,G)$ of (W,G)-spin curves is stratified by the G-decorated dual graphs.

If Ω is a G-decorated dual graph, then we denote by $\gamma(h)$ the matrix attached to the half-edge h, by $V(\Omega)$ and $E(\Omega)$ the sets of vertices and edges of Ω, and by $\mathrm{Aut}(\Omega)$ the group of automorphisms of Ω. Furthermore, a G-decorated dual graph of genus g and type $\overline{\gamma}$ induces the natural gluing morphism[12]

$$[\Omega]_* : A^*\Bigg(\prod_{v\in V(\Omega)}\mathcal{S}_{g,n_v}(W,G)(\overline{\gamma}_v)\Bigg) \to A^*(\mathcal{S}_{g,n}(W,G)(\overline{\gamma})),$$

where n_v is the number of half-edges incident to the vertex v, and $\overline{\gamma}_v \in G^{n_v}$ is the uplet of matrices attached to half-edges incident to v. On the above product of moduli spaces, we define as usual the class ψ_h as the first Chern class of the cotangent line bundle at the marking corresponding to the half-edge h. Therefore, any polynomial in the psi-classes ψ_h associated to half-edges h can be pushforward to a well-defined Chow class of the moduli space $\mathcal{S}_{g,n}(W,G)(\overline{\gamma})$. The Chow degree is then increased by the number $\#E(\Omega)$ of edges of Ω. Note also that the degree of the map $[\Omega]_*$ is exactly the cardinal $|\mathrm{Aut}(\Omega)|$ of its automorphism group.

[12] This morphism is only defined at the level of Chow rings and not on the moduli spaces; see [17, Theorem 4.1.8, (6)].

THEOREM 3.3. *With the above assumptions and notations, we have the following equality in the Chow ring of the moduli space of* (W, G)*-spin curves*

$$\begin{aligned}
&\lambda_g^\vee c_{\mathrm{vir}}^{\mathrm{PV}}(e(\mathfrak{C}_{\overline{\gamma}}))_{g,n} \\
&\quad = \lim_{t\to 1}(1-t_{N+1})^g \cdot \prod_{j=1}^{N}(1-t_j)^{\deg_j} \cdot \sum_{\substack{k\geq 0\\ \Omega_k}} \frac{1}{k!} \cdot \frac{r^{\#E(\Omega_k)}}{|\mathrm{Aut}(\Omega_k)|} \\
&\qquad \cdot [\Omega_k]_*\Bigg(\psi_{n+1}\cdots\psi_{n+k} \cdot \prod_{i=1}^{k}\sum_{j=1}^{N+1} g(t_j, \mathfrak{q}_j, \psi_{n+i}) \\
&\qquad \cdot \prod_{i=1}^{n+k}\exp\Big(-\sum_{j=1}^{N+1} g(t_j, \Gamma_j(i), \psi_i)\Big) \\
&\qquad \cdot \prod_{\substack{e\in E(\Omega_k)\\ e=(h,h')}} \frac{1-e^{-\sum_{j=1}^{N+1} g(t_j,\Gamma_j(h),\psi_h)}e^{-\sum_{j=1}^{N+1} g(t_j,\Gamma_j(h'),\psi_{h'})}}{\psi_h+\psi_{h'}}\Bigg),
\end{aligned}\tag{22}$$

where the sum is taken over all G*-decorated dual graphs* Ω_k *of genus* g *and of type* $\overline{\gamma}\cup(\mathfrak{j},\dots,\mathfrak{j})$ *with the matrix* $\mathfrak{j}$ *taken* k *times, where the integer* r *is the exponent of the group* G, *and where*

$$g(t,x,z) := \sum_{l\geq 1} s_l(t)\frac{B_{l+1}(x)}{(l+1)!}z^l.$$

Proof. Formula (22) is a straightforward computation from formula (15) using the Givental action by the symplectic transformation $R_\Gamma(z) := \exp(-g(t,\Gamma,z))$ on the state space. □

REMARK 3.4. A similar formula in the Chow ring of $\overline{\mathcal{M}}_{g,n}$ can be obtained by replacing the morphism $[\Omega_k]_*$ with

$$A^*\Big(\prod_{v\in V(\Omega)}\overline{\mathcal{M}}_{g,n_v}\Big) \to A^*(\overline{\mathcal{M}}_{g,n})$$

and by replacing the exponent $\#E(\Omega_k)$ with the number of loops $h_1(\Omega_k)$ of the graph.

Observe the following interesting property:

$$g(t,x,z) = \sum_{l\geq 1}\sum_{k\leq l} g_k^l(x)\frac{z^l}{q^k}, \quad \text{with } t = e^{-q},$$

that is, the polynomial degree in z is always greater than the polynomial degree in q^{-1}. Therefore, the three last lines of formula (22) is a Laurent polynomial in q of the form

$$\sum_{k\leq M}\frac{1}{q^k}\cdot\sum_{l\geq k} C_{k,l}$$

with the coefficient $C_{k,l}$ of pure Chow degree l. In particular, since we have

$$g_l^l(x) = (-1)^l (l-1)! \frac{B_{l+1}(x)}{(l+1)!},$$

the projection

$$\sum_{k \leq M} \frac{C_{k,k}}{q^k}$$

is obtained by replacing the function g with the truncated function

$$\hat{g}(q, x, z) := \sum_{l \geq 1} \frac{(-1)^l}{l} \cdot \frac{B_{l+1}(x)}{l+1} \cdot \left(\frac{z}{q}\right)^l.$$

The truncated formula is thus converging as $q \to 0$. Moreover, since the left-hand side of formula (22) is of pure degree equal to $\delta := g + \sum_{j=1}^N \deg_j$, it equals the coefficient $C_{\delta,\delta}$. Therefore, we obtained the following truncated formula.

COROLLARY 3.5. *With the same assumptions and notations as in Theorem 3.3, we have the following equality in* $A^*(\mathcal{S}_{g,n}(W,G)(\overline{\gamma}))$:

$$\begin{aligned}
&\lambda_g^\vee c_{\mathrm{vir}}^{\mathrm{PV}}(e(\mathfrak{C}_{\overline{\gamma}}))_{g,n} \\
&\quad = \lim_{q \to 0} q_{N+1}^g \cdot \prod_{j=1}^N q_j^{\deg_j} \cdot \sum_{\substack{k \geq 0 \\ \Omega_k}} \frac{1}{k!} \cdot \frac{r^{\# E(\Omega_k)}}{|\operatorname{Aut}(\Omega_k)|} \\
&\qquad \cdot [\Omega_k]_* \Bigg(\psi_{n+1} \cdots \psi_{n+k} \cdot \prod_{i=1}^k \sum_{j=1}^{N+1} \hat{g}(q_j, \mathfrak{q}_j, \psi_{n+i}) \\
&\qquad \cdot \prod_{i=1}^{n+k} \exp\Bigg(- \sum_{j=1}^{N+1} \hat{g}(q_j, \Gamma_j(i), \psi_i) \Bigg) \\
&\qquad \cdot \prod_{\substack{e \in E(\Omega_k) \\ e=(h,h')}} \frac{1 - e^{-\sum_{j=1}^{N+1} \hat{g}(q_j, \Gamma_j(h), \psi_h)} e^{-\sum_{j=1}^{N+1} \hat{g}(q_j, \Gamma_j(h'), \psi_{h'})}}{\psi_h + \psi_{h'}} \Bigg),
\end{aligned} \tag{23}$$

where we use the truncated function

$$\hat{g}(q, x, z) := \sum_{l \geq 1} \frac{(-1)^l}{l} \cdot \frac{B_{l+1}(x)}{l+1} \cdot \left(\frac{z}{q}\right)^l,$$

and where $q_j := (-a_1) \cdots (-a_{j-1}) q$.

REMARK 3.6. We can use formula (23) to find tautological relations by extracting, for instance, the coefficient in q^{-1}. The first interesting case is to consider the polynomial $W = x^5$ in genus 1 with five markings of monodromy 2. Then we obtain a tautological relation in $A_2(\overline{\mathcal{M}}_{1,5})$, and we can try to express it in terms of relations from $\overline{\mathcal{M}}_{0,n}$ and from Getlzer's relation. More generally, we will address in a subsequent work the question to compare the relations from Theorem 3.5 to Pixton's relations [35].

REMARK 3.7. The truncated function $\hat{g}$ is closely related to the equivariant Euler class of a vector bundle V:

$$\begin{aligned} e_q^\star(V) &:= q^{\mathrm{rk}(V)}\left(1+\frac{c_1(V)}{q}+\frac{c_2(V)}{q^2}+\cdots\right) \\ &= q^{\mathrm{rk}(V)}\cdot\exp\left(-\sum_{l\geq 1}(-1)^l\frac{(l-1)!}{q^l}\mathrm{Ch}_l(V)\right). \end{aligned}$$

More precisely, we see that formula (23) translates into

$$\lambda_g^\vee c_{\mathrm{vir}}^{\mathrm{PV}}(e(\mathfrak{C}_{\overline{\gamma}}))_{g,n} = \lim_{q\to 0}\prod_{j=1}^{N} e_{q_j}^\star(-R^\bullet\pi_*(\mathcal{L}_j^{\mathfrak{C}}))\cdot e_{q_{N+1}}^\star(\mathbb{E}^\vee). \tag{24}$$

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