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Circle-Valued Morse Theory for Frame Spun Knots and Surface-Links

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Abstract. Let $N^k \subset S^{k+2}$ be a closed oriented submanifold. Denote its complement by $C(N) = S^{k+2} \setminus N$. Denote by $\xi \in H^1(C(N))$ the class dual to N. The Morse–Novikov number of $C(N)$ is by definition the minimal possible number of critical points of a regular Morse map $C(N) \to S^1$ belonging to ξ. In the first part of this paper, we study the case where N is the twist frame spun knot associated with an m-knot K. We obtain a formula that relates the Morse–Novikov numbers of N and K and generalizes the classical results of D. Roseman and E. C. Zeeman about fibrations of spun knots. In the second part, we apply the obtained results to the computation of Morse–Novikov numbers of surface-links in 4-sphere.

1. Introduction

1.1. Overview of the Paper

Let $N^k \subset S^{k+2}$ be a closed oriented submanifold, and let $C(N) = S^{k+2} \setminus N$ be its complement. Alexander duality implies an isomorphism

$$H_k(N, \mathbb{Z}) \approx H^1(C(N), \mathbb{Z}).$$

We denote the image of the fundamental class $[N] \in H_k(N, \mathbb{Z})$ of the manifold N by this isomorphism by

$$\xi \in H^1(C(N), \mathbb{Z}) \approx [C(N), S^1]. \tag{1}$$

This cohomology class takes value 1 on the boundary of each small two-dimensional disc normal to N and positively oriented. We say that N is *fibered* if there is a Morse map $f : C(N) \to S^1$ homotopic to ξ that is regular nearby N (see Definition 1.1) and has no critical points. In general, a Morse map $C(N) \to S^1$ has some critical points; the minimal number of these critical points is called the *Morse–Novikov number of* N and denoted by $\mathcal{MN}(N)$.

In the first part of this paper, we study this invariant in relation with constructions of spinning. The classical Artin's spinning construction [2] associates with each knot $K \subset S^3$ a 2-knot $S(K) \subset S^4$. A twisted version of this construction is due to Zeeman [21]. Roseman [17] introduced a *frame spinning* construction, and Friedman [5] gave a generalization of Roseman's construction to include *twisting*. The input data for frame twist spinning construction are as follows:

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(TFS1) A closed manifold $M^k \subset S^{m+k}$ with trivial (and framed) normal bundle.
(TFS2) An m-knot $K^m \subset S^{m+2}$.
(TFS3) A C^∞ map $\lambda : M \to S^1$.

With these data one associates an n-knot $\sigma(M, K, \lambda)$, where $n = k + m$ (see Section 2). When λ is a constant map, we denote this knot by $\sigma(M, K)$; this is the Roseman's *frame spun knot.* We prove in Section 2 the following formula:

$$\mathcal{MN}(\sigma(M, K, \lambda)) \leq \mathcal{MN}(K) \cdot \mathcal{NN}(M, [\lambda]) \tag{2}$$

(where $\mathcal{NN}(M, [\lambda])$ is the minimal number of critical points of a Morse map $M \to S^1$ homotopic to λ). If λ is null-homotopic, then we have

$$\mathcal{MN}(\sigma(M, K)) \leq \mathcal{MN}(K) \cdot \mathcal{M}(M)$$

(where $\mathcal{M}(M)$ is the Morse number of M). In particular, if K is fibered, then the framed spun knot $\sigma(M, K)$ is fibered (a theorem of Roseman [17]). If in formula (2) the map $\lambda : M \to S^1$ has no critical points, then the knot $\sigma(M, K, \lambda)$ is fibered, and we recover the classical result of Zeeman [21]: for any knot, its twist-spun knot is fibered. In Section 3, we discuss a geometric construction related to spinning, namely *rotation* of a knot $K^m \subset S^{m+2}$ around equatorial sphere Σ of S^{m+2}. The resulting submanifold $R(K)$ is diffeomorphic to $S^1 \times S^m$ and is sometimes called the *spun torus* of K. We prove that

$$\mathcal{MN}(R(K)) \leq 2\mathcal{MN}(K) + 2.$$

Section 4 is about Morse–Novikov theory for surface-links, that is, embeddings of orientable surfaces to S^4. In this case the invariant $\mathcal{MN}(F)$ is related to a simple geometric invariant of the surface-links F, namely *the saddle number.* For a given surface $F \subset \mathbb{R}^4$, let $\mathrm{sdl}(F)$ denote the minimal number of saddle points of the restriction to F of an orthogonal projection $\mathbb{R}^4 \to \mathbb{R}$ (the minimum taken over the nondegenerate restrictions). Define the saddle number of F as the minimum of numbers $\mathrm{sdl}(F')$ over all surfaces $F' \subset \mathbb{R}^4$ ambient isotopic to F. In Section 4.1, we prove the formula

$$\mathcal{MN}(F) \leq 2\,\mathrm{sd}(F) + \chi(F) - 2. \tag{3}$$

In Section 4.2, we discuss the case of spun 2-knots.

Section 4.3 is dedicated to computations. Recall that there is a method of tabulating surface-links, developed in the works of Kawauchi, Shibuya, and Suzuki [11] and Yoshikawa [20]. Kawauchi, Shibuya, and Suzuki [11] introduced a method of representing surface-links by diagrams. Based on this method, Yoshikawa [20] defined a numerical invariant $\mathrm{ch}(F)$ of surface-links F and enumerated all the (weakly prime) surface-links F with $\mathrm{ch}(F) \leq 10$. Yoshikawa's table can be considered as a two-dimensional analog of Rolfsen's knot table. In Section 4.3, we compute the Morse–Novikov numbers of the majority of the oriented surface-links of Yoshikawa's table.

1.2. Basic Definitions and Lower Bounds for Morse–Novikov Numbers

We start with the definition of a regular Morse map.

DEFINITION 1.1. Let $N^k \subset S^{k+2}$ be a closed oriented submanifold. A Morse map $f : C(N) \to S^1$ is said to be *regular* if there is an orientation-preserving C^∞ trivialization

$$\Phi : T(N) \to N \times B^2(0, \varepsilon) \tag{4}$$

of a tubular neighborhood $T(N)$ of N such that the restriction $f \mid (T(N) \setminus N)$ satisfies $f \circ \Phi^{-1}(x, z) = z/|z|$.

Observe that the homotopy class of any regular map f in the set $[C(N), S^1] \approx H^1(C(N), \mathbb{Z})$ equals the class ξ defined in the beginning of the Introduction; see the formula (1). An f-gradient v of a regular Morse map $f : C(N) \to S^1$ is called *regular* if there is a C^∞ trivialization (4) such that $\Phi^*(v)$ equals $(0, v_0)$ where v_0 is the Riemannian gradient of the map $z \mapsto z/|z|$.

If f is a Morse function on a manifold or a Morse map of a manifold to S^1, then we denote by $m_p(f)$ the number of critical points of f of index p. The number of all critical points of f is denoted by $m(f)$.

DEFINITION 1.2. The minimal number $m(f)$ where $f : C(N) \to S^1$ is a regular Morse map is called the *Morse–Novikov number of* N and denoted by $\mathcal{MN}(N)$.

To obtain lower bounds for numbers $m_p(f)$, we use the *Novikov homology.* Let $L = \mathbb{Z}[t, t^{-1}]$; denote by $\widehat{L} = \mathbb{Z}((t))$ the ring of all series in one variable t with integer coefficients and finite negative part. Consider the infinite cyclic covering $\overline{C(N)} \to C(N)$; the Novikov homology of $C(N)$ is defined as follows:

$$\widehat{H}_*(C(N)) = H_*(\overline{C(N)}) \underset{L}{\otimes} \widehat{L}.$$

Recall that $\widehat{L}$ is a PID (see, e.g., [13], Ch. 10, Thm. 2.4); therefore, for every k, the finitely generated $\widehat{L}$-module $\widehat{H}_k(C(N))$ admits a decomposition into a finite direct sum of cyclic modules. The rank and torsion number of the $\widehat{L}$-module $\widehat{H}_k(C(N))$ are denoted by $\widehat{b}_k(C(N))$ and $\widehat{q}_k(C(N))$, respectively. For any regular Morse map f and a transverse regular f-gradient v, there is *the Novikov complex* $\mathcal{N}_*(f, v)$ over $\widehat{L}$ generated in degree k by critical points of f of index k and such that $H_*(\mathcal{N}_*(f, v)) \approx \widehat{H}_*(C(N))$ (see [13], Ch. 11). Therefore we have the Novikov inequalities

$$\sum_k (\widehat{b}_k(C(N)) + \widehat{q}_k(C(N)) + \widehat{q}_{k-1}(C(N))) \leq \mathcal{MN}(N).$$

These inequalities, which are far from being exact in general, are however very useful in particular in the case of surface-links (see Section 4).

2. Frame Twist-Spun Knots

We start with a recollection of the frame twist spinning construction following [17; 6; 5]. See the input data (TFS1)–(TFS3) for this construction in Section 1.1.

Let $a \in K^m$. Removing a small open disk $D(a)$ from S^{m+2}, we obtain an embedded (knotted) disk K_0 in the disk $D^{m+2} \approx S^{m+2} \setminus D(a)$. We identify D^{m+2} with the standard Euclidean disk of radius 1 and center 0 in $\mathbb{R}^{m+2}$, and then $\partial D^{m+2} = S^{m+1}$. We have the usual diffeomorphism

$$\chi : S^{m+1} \times]0, 1] \xrightarrow{\approx} D^{m+2} \setminus \{0\}, \qquad (x, t) \mapsto tx.$$

We can assume that $K_0 \cap \partial D^{m+2}$ is an equatorial sphere[1] S^{m-1} in $\partial D^{m+2} = S^{m+1}$. Moreover, we can assume that the intersection of K_0 with a neighbourhood of ∂D^{m+2} is also standard, that is,

$$K_0 \cap \chi(S^{m+1} \times [1-\varepsilon, 1]) = \chi(S^{m-1} \times [1-\varepsilon, 1]).$$

We have a framing of M in S^n (recall that $n = m + k$); combining this with the standard framing of S^n in S^{n+2}, we obtain a diffeomorphism

$$\Phi : T(M, S^{n+2}) \xrightarrow{\approx} M \times D^m \times D^2,$$

where $T(M, S^{n+2})$ is a tubular neighbourhood of M in S^{n+2}. We can assume that the restriction of Φ to $T(M, S^n)$ is a diffeomorphism

$$\Phi : T(M, S^n) \xrightarrow{\approx} M \times D^m \times \{0\}$$

induced by the given framing of M. The Euclidean disc D^{m+2} is a subset of $D^m \times D^2$, so that $K_0 \subset D^m \times D^2$.

For $\theta \in S^1$, denote by R_θ the rotation of D^2 around its center. The disc $D^{m+2} \subset D^m \times D^2$ is invariant with respect to this rotation as well as the intersection of K_0 with a small neighbourhood of ∂D^{m+2}. We have $\Phi(S^n \cap T(M, S^{n+2})) = M \times D^m \times \{0\}$. Let

$$Z = \{(x, y, z) \mid (y, z) \in R_{\lambda(x)}(K_0)\}.$$

This is an n-dimensional submanifold of $M \times D^m \times D^2$. We define $\sigma(M, K, \lambda)$ as follows:

$$\sigma(M, K, \lambda) = (S^n \setminus T(M, S^n)) \cup \Phi^{-1}(Z).$$

This is the image of an embedded n-sphere, knotted in general.

Examples and Particular Cases

(1) Let $\dim M = 0$, so that M is a finite set; denote by p its cardinality. Then the n-knot $\sigma(M, K, \lambda)$ is equivalent to the connected sum of p copies of K.
(2) If M is the equatorial circle of the sphere S^2, which is in turn considered as an equatorial sphere of S^4, and $\lambda(x) = 1$, then we obtain the classical Artin's construction. If $\lambda : S^1 \to S^1$ is a map of degree d, then we obtain Zeeman's twist-spinning construction [21].
(3) If $\lambda(x) = 1$ for all $x \in M$, then we obtain Roseman's construction of spinning around the manifold M [17]. In this case, we denote $\sigma(M, K, \lambda)$ by $\sigma(M, K)$.

[1]By *equatorial sphere* in $S^N \subset \mathbb{R}^{N+1}$ we mean the intersection of a linear subspace $L \subset \mathbb{R}^{N+1}$ with S^N; this intersection is a Euclidean sphere of dimension $\dim L - 1$.

THEOREM 2.1.

$$\mathcal{MN}(\sigma(M, K, \lambda)) \le \mathcal{MN}(K) \cdot \mathcal{NN}(M, [\lambda])$$

(*where* $[\lambda] \in H^1(M, \mathbb{Z}) \approx [M, S^1]$ *is the homotopy class of* λ).

Proof. We use the terminology from the previous construction of $\sigma(M, K, \lambda)$. We have the standard fibration

$$\psi_0 : S^{n+2} \setminus S^n \to S^1$$

obtained from the canonical framing of S^n in S^{n+2}. Observe that the map $\alpha = \psi_0 \circ \Phi^{-1}$ is given by the formula

$$\alpha(x, y, z) = \frac{z}{|z|}.$$

Let $f : S^{m+2} \setminus K \to S^1$ be a regular Morse map. We denote the restriction of f to the subset $D^{m+2} \setminus K_0$ by the same letter f. We can assume that the map f equals α in a neighbourhood of $\partial D^{m+2} = S^{m+1}$. In particular, in a neighbourhood of ∂D^{m+2}, we have

$$f(R_\theta(p)) = f(p) + \theta \quad \text{for } p \in S^{m+1} \setminus K_0.$$

(Here and elsewhere we use the additive notation for the group operation on S^1.)

Define the map g on $(M \times D^{m+2}) \setminus Z$ by the formula

$$g(x, \xi) = f(R_{-\lambda(x)}(\xi)) + \lambda(x) \tag{5}$$

(for $x \in M$ and $\xi \in D^{m+2}$). Define the map ψ on the complement of $\sigma(M, K, \lambda)$ by the following formula:

(1) If $p \notin T(M, S^{n+2})$, then $\psi(p) = \psi_0(p)$;
(2) If $p \in T(M, S^{n+2})$, then $\psi(p) = g(\Phi^{-1}(p))$.

We will now prove that if λ is a Morse map (this can be achieved by a small perturbation of λ), then ψ is also a Morse map, and the number $m(\psi)$ of its critical points satisfies

$$m(\psi) = m(\lambda) \cdot m(f).$$

All the critical points of ψ are in $T(M, S^{n+2})$. In this domain the map ψ is diffeomorphic to g, and the count of critical points of g is easily achieved with the help of the next lemma.

LEMMA 2.2. *Let* $g_1 : N_1 \to S^1$ *and* $g_2 : N_2 \to S^1$ *be Morse maps on manifolds* N_1 *and* N_2. *Let* $F : N_1 \times N_2 \to N_2$ *be a* C^∞ *map such that, for each* $a \in N_1$, *the map* $x \mapsto F(a, x)$ *is a diffeomorphism* $N_2 \to N_2$. *Define the map* $g : N_1 \times N_2 \to S^1$ *by the formula*

$$g(x_1, x_2) = g_1(x_1) + g_2(F(x_1, x_2)).$$

Then g *is a Morse map, there is a bijection* $B : \mathrm{Crit}(g_1) \times \mathrm{Crit}(g_2) \xrightarrow{\approx} \mathrm{Crit}(g)$, *and for* $a_1, a_2 \in \mathrm{Crit}(g_1) \times \mathrm{Crit}(g_2)$, *we have* $\operatorname{ind} B(a_1, a_2) = \operatorname{ind} a_1 + \operatorname{ind} a_2$.

Proof. Define the map g_0 on $N_1 \times N_2$ by the formula

$$g_0(x_1, x_2) = g_1(x_1) + g_2(x_2).$$

The conclusions of our lemma obviously hold if we replace g by g_0 in the statement of the lemma. Observe now that the map g is diffeomorphic to g_0 via the diffeomorphism

$$(x_1, x_2) \mapsto (x_1, F(x_1, x_2)).$$

The lemma follows. □

The proof of Theorem 2.1 is now complete. □

COROLLARY 2.3. *Let $K \subset S^3$ be a classical knot, and denote by $S(K)$ the spun knot of K. Then*

$$\mathcal{MN}(S(K)) \le 2\mathcal{MN}(K). \tag{6}$$

Proof. In this case, $M = S^1$ and $[\lambda] = 0$. We have $\mathcal{M}(S^1) = 2$, and the result follows. □

The classical theorems concerning fibrations of spun knots follow from Theorem 2.1:

COROLLARY 2.4 (Roseman [17]). *If K is fibered, then $\sigma(M, K)$ is fibered.*

Proof. Since $\mathcal{MN}(K) = 0$, Theorem 2.1 implies $\mathcal{MN}(\sigma(M, K)) = 0$. □

COROLLARY 2.5 (Zeeman [21]). *The d-twist spun knot of any classical knot K is fibered for $d \ge 1$.*

Proof. Let Σ be an equatorial circle in S^2. The d-twist spun knot of K is by definition the 2-knot $\sigma(\Sigma, K, \lambda)$ in S^4, where $\lambda : \Sigma \to \Sigma$ is a map of degree d. The assertion follows, since $\mathcal{NN}(S^1, [\lambda]) = 0$. □

REMARK 2.6. Zeeman's theorem immediately generalizes to the following statement: If $\mathcal{NN}(M, [\lambda]) = 0$, then the knot $\sigma(M, K, \lambda)$ is fibered for any knot K.

3. Rotation

Let Σ be an equatorial n-sphere of S^{n+1}. We can view the sphere S^{n+1} as the union of two $(n+1)$-dimensional discs $D_+ \cup D_-$ intersecting by Σ. Consider S^{n+1} as the equatorial sphere of S^{n+2}. The sphere S^{n+2} can be considered as the result of rotation of the disc D_+ around its boundary Σ. We have the (linear orthogonal) action of S^1 on S^{n+2} such that Σ is the fixed point set of the action, and the action is free on the rest of the sphere S^{n+2}. Let K^{n-1} be an $(n-1)$-knot in S^{n+1}. We can assume that $K^{n-1} \subset \operatorname{Int} D_+$. Rotation of K^{n-1} around Σ gives a submanifold $R(K)$ of codimension 2 in S^{n+2}. The manifold $R(K)$ is diffeomorphic to $S^1 \times K$. We call this construction *rotation.* When $\dim K = 1$, the manifold $R(K)$ is sometimes called the *spun torus* of K. In this section, we relate the Morse–Novikov numbers of $R(K)$ with those of K.

THEOREM 3.1.

$$\mathcal{MN}(R(K)) \leq 2\mathcal{MN}(K) + 2.$$

To prove the theorem, we associate with each given regular Morse map $\phi : S^{n+1} \setminus K^{n-1} \to S^1$ a regular Morse map $R(\phi) : S^{n+2} \setminus R(K^{n-1}) \to S^1$ such that $m(R(\phi)) = 2m(\phi) + 2$. We begin by an outline of this construction for the simplest case where $n = 1$ and K consists of two points in S^2 (Section 3.1). In Section 3.2, we give a detailed proof of the theorem in full generality.

3.1. Rotation of S^0

Let us consider the Euclidean unit sphere S^2 and the 0-knot $K^0 = \{b, c\} \subset S^2$ consisting of two opposite points b, c on S^2. The manifold $S^2 \setminus \{b, c\}$ is fibered over S^1, and the structure of the level lines of this fibration is shown on Figure 1 (left). Endow S^2 with the Riemannian metric induced from $\mathbb{R}^3$. Let a be any point of S^2 equidistant from the points b and c. Let $D_- = D_-(\alpha)$ be the disc of radius α around a of f with respect to this metric. We assume that $\alpha < 1/4$, so that the points b, c are not in D_-. Denote by D_+ the complement $S^2 \setminus \operatorname{Int} D_-$, so that $S^2 = D_+ \cup D_-$ and the discs $D_\pm$ intersect by their common boundary Σ. Removing D_-, we obtain a map $f : D_+ \setminus \{a, b\} \to S^1$. The structure of the level lines of f is shown on Figure 1 (middle).

The restriction $f \mid \Sigma$ has two nondegenerate critical points, N and S. The arrow in the figure depicts the gradient of the map f. Applying the rotation construction to K_0, we obtain a trivial 2-component link $R(K^0)$ in S^3. Let $F_0 : S^3 \setminus R(K_0) \to S^1$ be the unique S^1-invariant map such that $F_0 \mid D_+ = f$. This map is continuous, but not smooth, since its level surfaces have conical singularities in the points of Σ. To repair this, we modify the map f in a neighbourhood of Σ so that the level lines of the modified map $g : D_+ \setminus \{a, b\} \to S^1$ are as depicted on Figure 1 (right). The map g in a neighbourhood of S has a standard quadratic singularity. Namely, it is locally diffeomorphic to the restriction of the function $g_0(x, y) = y^2 - x^2$ to the upper half-plane $H_+ = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. The diffeomorphism sends the point S to the point $(0, 0) \in H_+$; a neighbourhood of S

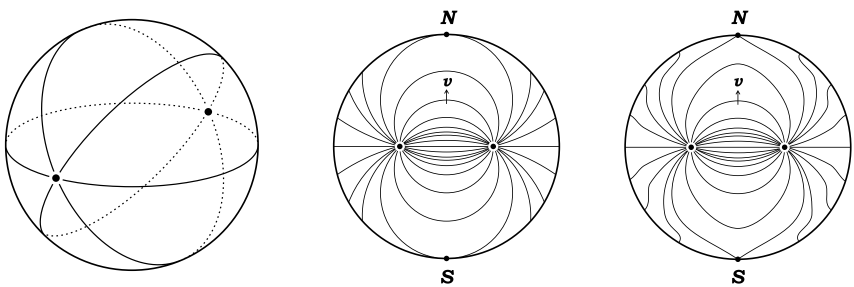

Figure 1

in Σ is sent to a neighbourhood of $(0,0)$ in the real line $\{(x,0) \mid x \in \mathbb{R}\} \subset H_+$. Similar description holds for the map g in a neighbourhood of N.

Each nonsingular level line intersecting Σ is orthogonal to Σ at the intersection point. Let $G_0 : S^3 \setminus R(K_0) \to S^1$ be the unique S^1-invariant map such that $G_0 \mid D_+ = g$. Then G_0 is a C^∞ map having two critical points N and S. Observe that the descending disc of the critical point S of the map G_0 is in Σ; therefore the descending disc of the critical point S of G_0 will have the same dimension 1, and $\operatorname{ind}_{G_0} S = 1$. A similar reasoning holds for the *ascending disc* of the critical point N, and therefore $\operatorname{ind}_{G_0} N = 2$.

3.2. The General Case

Consider the Euclidean unit sphere S^{n+1} in $\mathbb{R}^{n+2}$, that is,

$$S^{n+1} = \{(x_0, \dots, x_{n+1}) \mid x_0^2 + \cdots + x_{n+1}^2 = 1\}.$$

Denote by Σ its intersection with the hyperplane $x_{n+1} = 0$. Let $a = (0, \dots, 0, 1)$; for each point $z \in \Sigma$, denote by $C(z)$ the great circle through $a, -a, z$ and by $C'(z)$ the closed semicircle containing these three points. The projection p onto the last coordinate gives the bijection of $C'(z)$ onto the closed interval $[-1, 1]$; this bijection is a diffeomorphism when restricted to $C'(z) \setminus \{a, -a\}$. Let $\beta : [-1, 1] \to [-1, 1]$ be a diffeomorphism such that $\beta(x) = x$ for x in a neighbourhood of ± 1. Then there is a unique diffeomorphism σ_β of S^{n+1} onto itself such that, for every z, the curve $C'(z)$ is σ_β-invariant and $p(\sigma_\beta(v)) = \beta(p(v))$ for every v. The diffeomorphism σ_β is called *the sliding associated with* β. Observe that every sliding is isotopic to the identity map. Let $D_\rho \subset S^{n+1}$ be the geodesic disc of radius ρ centered in $-a$. Let

$$\begin{aligned} D_- &= D_{\pi/2} = \{(x_0, \dots, x_{n+1}) \mid x_{n+1} \le 0\}, \\ D_+ &= \{(x_0, \dots, x_{n+1}) \mid x_{n+1} \ge 0\}. \end{aligned}$$

Put $\Sigma_\rho = \partial D_\rho$. Let $T(\Sigma_\rho, \varepsilon)$ denote the geodesic tubular neighbourhood of Σ_ρ. For given ρ and $\varepsilon > 0$ sufficiently small, there is a sliding σ sending D_ρ to D_- and sending each normal geodesic segment of length 2ε in $T(\Sigma_\rho, \varepsilon)$ isometrically to the corresponding normal geodesic segment in $T(\Sigma, \varepsilon)$. We therefore have a commutative diagram

$$\begin{tikzcd} T(\Sigma_\rho, \varepsilon) \arrow[r, "\sigma"] & T(\Sigma, \varepsilon) \\ \Sigma_\rho \times]-\varepsilon, \varepsilon[\arrow[u, "\Phi"] \arrow[r, "\bar\sigma"] & \Sigma \times]-\varepsilon, \varepsilon[\arrow[u, "\Psi"'] \end{tikzcd}$$

where the vertical arrows are the restrictions of the exponential map of the sphere to the normal bundles of the submanifolds Σ_ρ and respectively Σ. We have $\bar\sigma(x, \tau) = (\sigma(x), \tau)$. It is not difficult to write down explicit formulas for Ψ and Ψ^{-1}. Namely,

$$\Psi(x, t) = (\cos t \cdot x, \sin t), \quad \text{where } x \in \Sigma \subset \mathbb{R}^{n+1}, t \in]-\varepsilon, \varepsilon[. \tag{7}$$

As for Ψ^{-1}, we have

$$\Psi^{-1}(y,u) = \left(\frac{y}{\sqrt{1-u^2}}, \arcsin(u)\right),$$
$$\text{where } (y,u) \in T(\Sigma,\varepsilon) \subset \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}. \tag{8}$$

Let K be an $(n-1)$-knot in S^{n+1}, and $\phi : S^{n+1} \setminus K \to S^1$ a regular Morse map. We can assume that

(1) $K \subset \operatorname{Int} D_+$,
(2) $\phi(-a)$ is a regular value of ϕ.

Choosing ρ sufficiently small, we can assume that $\phi(D_\rho)$ misses at least one point in S^1, and therefore $\phi \mid \Sigma_\rho$ can be considered as a real-valued Morse function. Consider the stereographic projection $P : S^{n+1} \setminus \{a\} \to \mathbb{R}^{n+1}$. The image of the disc D_ρ with respect to P is a Euclidean disc in $\mathbb{R}^{n+1}$ centered in the origin. The function $\phi \circ P^{-1}$ is a C^∞ function on $\mathbb{R}^{n+1}$. For ρ sufficiently small, the restriction of this function to the sphere $P(\Sigma_\rho)$ has exactly two critical points, one minimum and one maximum, as it follows from the next lemma. Therefore the function $\phi \mid \Sigma_\rho$ also has exactly two critical points, one minimum and one maximum, if only ρ is sufficiently small.

LEMMA 3.2. *Denote by $D(0,\alpha)$ the Euclidean ball of radius α in $\mathbb{R}^m$ centered in the origin*; *let $S(0,\alpha)$ be the sphere of radius α. Let $f : D(0,\alpha) \to \mathbb{R}$ be a C^∞ function such that $f'(0) \neq 0$. Then, for $\varepsilon > 0$ sufficiently small, the restriction of f to $S(0,\varepsilon)$ is a Morse function having exactly two critical points, one maximum and one minimum.*

Proof. We can assume that $f(0) = 0$. Put $A = f'(0)$. The restriction to $S(0,1)$ of the function $x \mapsto Ax$ is a Morse function having exactly two critical points. Put $g_\varepsilon(x) = \frac{1}{\varepsilon} f(\varepsilon x)$. It suffices to prove that, for sufficiently small ε, the function $g_\varepsilon : S(0,1) \to \mathbb{R}$ has exactly two critical points. Observe that $g'_\varepsilon(0) = A$. It is not difficult to show that partial derivatives of $g_\varepsilon - A$ converge to zero uniformly as ε converges to zero, that is, $g_\varepsilon - A \to 0$ in C^∞ topology. A classical theorem about C^∞ stability (see, e.g., [7], Ch. 3, §2, Prop. 2.2) implies that, for small ε, the function $g_\varepsilon \mid S(0,1)$ is isotopic to the function $A \mid S(0,1)$. □

Observe that diminishing ε if necessary, we can assume that the sliding σ has the property $\sigma(K) = K$. Indeed, the image $p(K)$ of the knot K with respect to the projection p onto the last coordinate is in $]0,1[$. Therefore if $\varepsilon > 0$ is sufficiently small, then the set $p(K)$ does not intersect $p(T(\Sigma,\varepsilon))$. Then the diffeomorphism $\beta : [-1,1]$ used in the definition of σ can be chosen in such a way that β restricted to $p(K)$ is the identity map, and this implies the required property. Thus we will assume that $\sigma(K) = K$ from now on.

Denote the function $\phi \circ \Phi$ by $h : \Sigma_\rho \times]-\varepsilon, \varepsilon[\to \mathbb{R}$. For ρ and ε sufficiently small, this function has no critical points. Consider the restriction of ϕ to the subset $S^{n+1} \setminus (K \cup D_\rho)$. Composing ϕ with σ^{-1}, we obtain a map

$$\phi_0 : D_+ \setminus K \to S^1.$$

This is a Morse map that extends to a geodesic tubular neighbourhood of $\Sigma = \partial D^+$ and can be lifted to a real-valued Morse function in this neighbourhood. The restriction $\phi_0 \mid \Sigma$ has two critical points N and S with $\operatorname{ind} N = n$ and $\operatorname{ind} S = 0$. The map $h_0 = \phi_0 \circ \Psi$ has no critical points. Now we will modify the map ϕ_0 nearby Σ. Let $\lambda : [-\varepsilon, \varepsilon] \to \mathbb{R}$ be a C^∞ function such that $\lambda(t) = |t|$ for t in a neighbourhood of $\{-\varepsilon, \varepsilon\}$, $\lambda(t) = t^2$ for $|t| \le \varepsilon/2$, and $\lambda'(t) \neq 0$ for $t \neq 0$. Define the map h_1 by the formula

$$h_1(x,t) = h_0(x, \lambda(t))$$

and the map

$$\phi_1 : D_+ \setminus K \to S^1$$

(A1) if $v \notin T(\Sigma, \varepsilon)$, put $\phi_1(v) = \phi_0(v)$;
(A2) if $v \in T(\Sigma, \varepsilon)$, $v = \Psi(x,t)$ with $x \in \Sigma$, $t \in]-\varepsilon, \varepsilon[$, put $\phi_1(v) = h_1(x,t)$.

PROPOSITION 3.3. *The map ϕ_1 has two critical points in $T(\Sigma, \varepsilon)$, namely N and S. They are nondegenerate, and their indices are equal, respectively, to n and* 1.

Proof. To find the critical points of ϕ_1, we distinguish two cases:

(1) $t = 0$. We have $\frac{\partial h_1}{\partial t}(x,0) = 0$ for every x. The derivative $\frac{\partial h_1}{\partial x}(x,0) = \frac{\partial h_0}{\partial x}(x,0) =$ vanishes exactly at two points N and S.
(2) $t \neq 0$. In this domain the function h_1 has no critical points, as it follows from the chain rule and the fact that $\lambda'(t) \neq 0$ for $t \neq 0$. □

Now we are ready to construct a Morse map on the complement to $R(K)$. Add one more coordinate x_{n+2} and consider the sphere

$$S^{n+2} = \{(x_0, \dots, x_{n+2}) \mid x_0^2 + \cdots + x_{n+2}^2 = 1\}.$$

We have $D_+ \subset S^{n+2}$. The knot $R(K)$ is defined by the formula

$$R(K) = \left\{(x_0, \dots, x_{n+2}) \mid \left(x_0, \dots, x_n, \sqrt{x_{n+1}^2 + x_{n+2}^2}\right) \subset K\right\}.$$

The circle S^1 acts on S^{n+2} by rotations in the two last coordinates. Define a Morse map ϕ_2 on the complement to $R(K)$ by the two following properties:

(1) $\phi_2 \mid D_+ \setminus K = \phi_1$;
(2) ϕ_2 is S^1-invariant.

The second property implies that

$$\phi_2(x_0, \dots, x_{n+2}) = \phi_1\left(x_0, \dots, x_n, \sqrt{x_{n+1}^2 + x_{n+2}^2}\right).$$

Observe that property (A2) of the map ϕ_1 guarantees that ϕ_2 is C^∞ on the subset $S^{n+2} \setminus R(K)$. Indeed, it suffices to verify that ϕ_2 is C^∞ in a neighbourhood of Σ. In such a neighbourhood, we have

$$\phi_1\left(x_0, \dots, x_n, \sqrt{x_{n+1}^2 + x_{n+2}^2}\right) = h_1\left(\Psi^{-1}\left(x_0, \dots, x_n, \sqrt{x_{n+1}^2 + x_{n+2}^2}\right)\right).$$

We deduce from formula (8) that, for $x_{n+1}^2 + x_{n+2}^2$ sufficiently small, we have

$$\phi_1\big(x_0, \dots, x_n, \sqrt{x_{n+1}^2 + x_{n+2}^2}\big) = h_0\bigg(\frac{(x_0, \dots, x_n)}{\sqrt{1 - x_{n+1}^2 - x_{n+2}^2}}, \big(\arcsin\sqrt{x_{n+1}^2 + x_{n+2}^2}\big)^2\bigg).$$

Using the Taylor development for the function arcsin in a neighbourhood of 0, it is not difficult to show that the function $(x, y) \mapsto (\arcsin\sqrt{x^2 + y^2})^2$ is of class C^∞ in a neighbourhood of $(0, 0)$ in $\mathbb{R}^2$. This completes the proof of the fact that ϕ_2 is of class C^∞.

Proposition 3.4. (1) $\mathrm{Crit}(\phi_2) = S^1 \cdot \mathrm{Crit}(\phi_1) \cup \{N, S\}$.
(2) *The critical points N and S are nondegenerate*, *and*

$$\mathrm{ind}_{\phi_2} N = \mathrm{ind}_{\phi_1} N = n, \qquad \mathrm{ind}_{\phi_2} S = \mathrm{ind}_{\phi_1} S + 1 = 2.$$

Proof. Point (1) is easy to deduce from the definition of ϕ_2. As for the indices of the critical points, observe that the descending disc of the critical point N in $T(\Sigma, \varepsilon)$ belongs to the sphere Σ, which is fixed by the action of S^1. Thus the index of N does not change when we replace ϕ_1 by ϕ_2. A similar argument applies to the *ascending disc* of S, and this implies the rest of the proposition. □

Each critical point of ϕ_1 gives rise to a circle of critical points of ϕ_2. The map ϕ_2 is constant on each of these circles $C_1, \dots, C_k$. Using the classical perturbation techniques for Morse–Bott functions (see, e.g., [3], p. 87), it is not difficult to perturb ϕ_2 in a small neighbourhood of each of C_i so as to obtain a Morse map ψ on the complement of $R(K)$ that has exactly two critical points on each C_i. Thus

$$m(\psi) = 2m(\phi_2) + 2.$$

Theorem 3.1 is proved.

3.3. 4-Thread Spinning

In this subsection, we give a brief description of one more construction of surface-links. Let $L \subset S^3$ be a classical link, and $\phi : S^3 \setminus L \to S^1$ a Morse map. Let $p, q \in L$, and let $\gamma : [0, 1] \to S^3$ be a C^∞ curve joining p and q and belonging entirely to one of the regular level surfaces $\phi^{-1}(\lambda)$ of the map ϕ. We assume moreover that $\mathrm{Im}\,\gamma \cap L = \{p, q\}$ and that $\gamma'(0)$ and $\gamma'(1)$ are not tangent to L. Let D be a small neighbourhood of $\mathrm{Im}\,\gamma$ diffeomorphic to a 3-disc. Denote by Σ its boundary. We can assume that $L \cap \Sigma$ consists of four points and that the tangent space of L is orthogonal to the tangent space of Σ at each of these points. Denote by S_0^2 the 2-sphere with four points removed. Recall that there is a standard circle-valued Morse map ϕ_0 on S_0^2 having two critical points both of index 1. We can assume that the restriction of ϕ to $\Sigma \setminus L$ is diffeomorphic to ϕ_0. Remove the interior of D from S^3 and rotate the remaining manifold $S^3 \setminus \mathrm{Int}\, D$ around Σ. We obtain the sphere S^4; the subset that is spun by $L \setminus \mathrm{Int}\, D$ during the rotation is an embedded 2-surface in S^4. We call this construction *4-thread spinning* to distinguish it from the usual spinning and denote the resulting surface-link by $S'(L)$. If p and q are on different connected components of L, then the number of connected components of $S'(L)$ is the same as for L. If p and q are in the same connected component of L, then the number of connected components of $S'(L)$ equals that of L increased by 1. Applying the same method as in the Section 3.2, we can construct a circle-valued Morse map $\widetilde{\phi}$ on $S^4 \setminus S'(L)$ such that $m(\widetilde{\phi}) = 2m(\phi) + 2$.

COROLLARY 3.5.

$$\mathcal{MN}(S'(L)) \leq 2\mathcal{MN}(L) + 2.$$

4. Surface-Links

In this section, we develop circle-valued Morse theory for surface-links.

4.1. Motion Pictures and Saddle Numbers

Let F be a surface-link, that is, a closed oriented two-dimensional C^∞ submanifold of S^4. We can assume that $F \subset \mathbb{R}^4$.

Choose a projection p of $\mathbb{R}^4$ onto a line. Assume that the critical points of the function $p \mid F$ are nondegenerate. Denote by $\mathrm{sdl}(F)$ the minimal number of saddle points of $p \mid F$ over all the projections p.

DEFINITION 4.1. *The saddle number* $\mathrm{sd}(F)$ is the minimum of numbers $\mathrm{sdl}(F')$ where F' ranges over all surface-links F' ambiently isotopic to F.

The invariant $\mathrm{sd}(F)$ is closely related to the ch-*index* of F, introduced and studied by Yoshikawa [20]. In particular, we have $\mathrm{sd}(F) \leq \mathrm{ch}(F)$. To relate the number $\mathrm{sd}(F)$ to $\mathcal{MN}(F)$, we will reformulate the definition of the saddle number.

Let $F \subset S^4$ be a surface-link. The equatorial 3-sphere Σ^3 of the standard Euclidean sphere S^4 divides S^4 into two parts:

$$S^4 = D^4_+ \cup D^4_- \quad \text{with } D^4_+ \cap D^4_- = \Sigma^3.$$

We assume that F is included in $\mathrm{Int}(D^4_-)$ and F does not contain the center of D^4_-. Perturbing the embedding $F \subset D^4_-$ if necessary, we can assume that the restriction $\rho = r|_F$ of the radius function $r : D^4_- \to [0, 1]$ is a Morse function. The family $\{(r^{-1}(t), \rho^{-1}(t))\}_{t\in[0,1]}$ of possibly singular links can be drawn as a *motion picture* (see [10], Ch. 8). Each singularity of a link in the family corresponds to a critical point of ρ. A critical point of ρ of index 0 (1, 2, respectively) is called a *minimal point* (*saddle point, maximal point,* respectively) of ρ, which is represented by a *minimal band* (*saddle band, maximal band,* respectively) in (a modification of) the motion picture.

It is clear that the minimal number of the saddle points for all such Morse functions ρ is equal to $\mathrm{sd}(F)$.

THEOREM 4.2. $\mathcal{MN}(F) \leq 2\,\mathrm{sd}(F) + \chi(F) - 2.$

Proof. Choose a trivialization

$$\Phi\ : U(F) \xrightarrow{\approx} F \times D^2(0, \varepsilon)$$

of a neighbourhood $U(F)$ of F and let $T(F) = \Phi^{-1}(F \times D^2(0, \varepsilon/2))$. Since ρ is a Morse function, the manifold $D^4_- \setminus \operatorname{Int} T(F)$ admits a handle decomposition with one 0-handle and $m_i(\rho)$ handles of index $(i+1)$ for $i \in \{0, 1, 2\}$ (see [9] and also [8], Proposition 6.2.1). The exterior $E(F) = S^4 \setminus \operatorname{Int} T(F)$ of F is obtained by attaching a 4-handle D^4_+ to $D^4_- \setminus \operatorname{Int} T(F)$. Since $D^4_- \setminus \operatorname{Int} T(F)$ is connected, there is a 3-handle in $D^4_- \setminus \operatorname{Int} T(F)$ that connects $\partial T(F)$ with ∂D^4_-. Thus the 3-handle cancels the 4-handle D^4_+ (see [12], Section 5). Turning the handlebody upside down, we obtain a dual handle decomposition of $E(F)$ and a corresponding Morse function $f : E(F) \to \mathbb{R}$, which is constant on $\partial E(F)$, and the following Morse numbers:

$$\begin{aligned}
m_0(f) &= 0,\\
m_1(f) &= m_2(\rho) - 1,\\
m_2(f) &= m_1(\rho),\\
m_3(f) &= m_0(\rho),\\
m_4(f) &= 1.
\end{aligned}$$

We can assume that the restriction of f to $\Phi^{-1}(F \times (D^2(0, \varepsilon) \setminus D^2(0, \varepsilon/2)))$ equals the function $\mu \circ p_2 \circ \Phi$ where $p_2 : F \times D^2(0, \varepsilon) \to D^2(0, \varepsilon)$ is the projection onto the second argument and $\mu(z) = |z|$. Applying the argument from the work of the second author [14], p. 629, we can deform the real-valued Morse function f to a circle-valued regular Morse map $\phi : E(F) \to S^1$ such that $m_k(f) = m_k(\phi)$ for every k. It is not difficult to show using a partition of unity argument that we can choose the map ϕ in such a way that its restriction to $\Phi^{-1}(F \times (D^2(0, 3\varepsilon/4) \setminus D^2(0, \varepsilon/2)))$ equals the map $\nu \circ p_2 \circ \Phi$ where $\nu(z) = z/|z|$. Thus the map ϕ extends to a regular Morse map on the complement to F; we will keep the symbol ϕ to denote this extension as well.

LEMMA 4.3. *Let $f : S^4 \setminus F \to S^1$ be a regular Morse map. Then there is a regular Morse map $g : S^4 \setminus F \to S^1$ such that $m_i(g) \le m_i(f)$ for every i and g has no local maxima or minima.*

This lemma is proved in [19], Lemmas 3.1 and 3.2, for the case of Morse maps $f : S^3 \setminus K \to S^1$ where K is a classical knot in S^3. The proofs carry over readily to the present case. Applying Lemma 4.3, we obtain a regular Morse map $g : S^4 \setminus F \to S^1$ with

$$m(g) \le m_2(\rho) - 1 + m_1(\rho) + m_0(\rho) = m(f) - 1$$

and $m_0(g) = m_4(g) = 0$. Observe that the parity of $m(g)$ equals the parity of $m(f)$ (both being equal to the parity of $\chi(S^4 \setminus F)$); therefore we actually have

$$m(g) \le m(f) - 2 = m_2(\rho) - 1 + m_1(\rho) + m_0(\rho) - 1.$$

Recall that $m_0(\rho) - m_1(\rho) + m_2(\rho) = \chi(F)$, and therefore the total number of critical points of g equals $2m_1(\rho) + \chi(F) - 2$. Choosing the function ρ with $m_1(\rho) = \mathrm{sd}(F)$, we accomplish the proof of Theorem 4.2. $\square$

COROLLARY 4.4. *Let $K \subset S^4$ be a 2-knot. Then $\mathcal{MN}(K) \le 2\,\mathrm{sd}(K)$.*

PROPOSITION 4.5. *Let $F \subset S^4$ be the trivial k-component surface-link. Then $\mathcal{MN}(F) = 4k - 2 - \chi(F)$.*

Proof. It is not difficult to show that $\widehat{b}_1(C(F)) \ge k - 1, \widehat{b}_3(C(F)) \ge k - 1$. (Indeed, there is a map of $S^4 \setminus F$ to the wedge W_k of k circles, inducing an epimorphism in π_1. The corresponding map of the infinite cyclic coverings induces an epimorphism in H_1, and therefore it induces an epimorphism in the Novikov homology modules in degree 1. The Novikov homology module of W_k in degree 1 is free of rank $k - 1$. This proves the first inequality, and the second follows from the Poincaré duality.) Therefore, for every regular Morse map $f : C(F) \to S^1$, we have $m_1(f) + m_3(f) \ge 2(k - 1)$. Assuming that $m_0(f) = m_4(f) = 0$, we have $m_1(f) - m_2(f) + m_3(f) = 2 - \chi(F)$ and $\mathcal{MN}(F) \ge 4k - 2 - \chi(F)$; this lower bound coincides with the upper bound derived from Theorem 4.2. $\square$

4.2. Spun Knots

Recall that the spun knot of a knot K is denoted by $S(K)$. Corollary 2.3 says that $\mathcal{MN}(S(K)) \le 2\mathcal{MN}(K)$. In this subsection we show that if K is a classical knot with tunnel number 1, then $\mathcal{MN}(S(K)) = 2\mathcal{MN}(K)$. This equality is obviously true when K is fibered, so it remains to consider the case of nonfibered classical knots with tunnel number 1.

This is the subject of the Proposition 4.8. The proof makes use of some more techniques from the Morse–Novikov theory, which we will recall now. We refer to [13; 18; 15], and [4] for a detailed exposition and proofs.

The construction of the Novikov complex mentioned in Section 1.2 can be generalized so as to obtain a chain complex over a special completion of the group ring of the fundamental group of the knot. This completion is usually called the *Novikov ring*. To recall its definition, let G be any group, and $\xi : G \to \mathbb{Z}$ be a homomorphism. Consider the set of all infinite series of the form $\sum_{k=1}^{\infty} n_k g_k$ where $n_k \in \mathbb{Z}$, $g_k \in G$, and $\xi(g_k) \to -\infty$ as $k \to \infty$. This set obviously contains the group ring $\Lambda = \mathbb{Z}G$ of G. It is denoted by $\widehat{\Lambda}_\xi$. It is easy to see that it has a natural ring structure and the inclusion $\Lambda \subset \widehat{\Lambda}_\xi$ is a ring homomorphism. For a connected topological space X with $\pi_1(X) \approx G$, we define the Novikov completion of the module $C_*(\widetilde{X})$ of the chains of the universal covering of X as follows:

$$\widehat{C}_*(X, \xi) = C_*(\widetilde{X}) \underset{\Lambda}{\otimes} \widehat{\Lambda}_\xi.$$

Its homology is called the *Novikov homology* of X with respect to ξ, and we denote it by $\widehat{H}_*(X, \xi)$. The next simple proposition is certainly known to experts, but we could not find it in the literature, so we include a proof.

PROPOSITION 4.6. *Let X, Y be connected CW-complexes such that their fundamental groups are both isomorphic to G. Let $\chi : \pi_1(X) \to \pi_1(Y)$ be an isomorphism, and let $\xi : \pi_1(Y) \to \mathbb{Z}$ be a homomorphism. Then $\widehat{H}_1(Y, \xi) \approx \widehat{H}_1(X, \xi \circ \chi)$.*

Proof. It suffices to prove the proposition for the case where $Y = K(G, 1)$. In this case, we have a map $\phi : X \to Y$ inducing an isomorphism in π_1 and epimorphism in π_2. Therefore the chain map $\phi_* : C_*(\widetilde{X}) \to C_*(\widetilde{Y})$ induces an isomorphism in H_1 and epimorphism in H_2 (here C_* denotes the module of cellular chains). Therefore the homology of the chain cone $C(\phi_*)$ vanishes up to degree 2. Denote by $\widehat{\phi}_*$ the chain map induced by ϕ in the Novikov completions. Then the homology of the chain cone $C(\widehat{\phi}_*)$ equals zero up to degree 2, and this completes the proof of the proposition. $\square$

Let $K^n \subset S^{n+2}$ be an n-knot, G the fundamental group of the complement to K, and $\xi : G \to \mathbb{Z}$ the canonical homomorphism (see (1)). For a regular transverse gradient of a regular Morse map, there is a chain complex $\widetilde{\mathcal{N}}_*(f, v)$ of free $\widehat{\Lambda}_\xi$-modules such that the number of its free generators in degree k equals $m_k(f)$, and there exists a chain equivalence

$$\widetilde{\mathcal{N}}_*(f, v) \sim \widehat{C}_*(\widetilde{S^3 \setminus K}) \underset{\Lambda}{\otimes} \widehat{\Lambda}_\xi.$$

The reason to consider this complicated object is that in some cases the vanishing of homology of this complex implies that the knot K is fibered. In particular, we have the following theorem.

THEOREM 4.7 (Sikorav [18]). *Let $K \subset S^3$ be a classical knot. Then the condition $\widehat{H}_1(S^3 \setminus K, \xi) = 0$ implies that K is fibered.*

Now we can continue our study of spun knots.

Let K be a classical knot in S^3; denote by $S(K)$ the corresponding spun knot.

PROPOSITION 4.8. *If K is a nonfibered knot of tunnel number* 1, *then* $\mathcal{MN}(S(K)) = 4$.

Proof. Recall that $\mathcal{MN}(S(K)) \le 2\mathcal{MN}(K)$ (Corollary 2.3). In the paper [14] of the second author, it is shown that $\mathcal{MN}(K) \le 2t(K)$, and hence $\mathcal{MN}(S(K)) \le 4$ by Corollary 2.3. Put $G = \pi_1(S^3 \setminus K)$. Then $\pi_1(S^4 \setminus S(K)) \approx G$ (see [16], Ch. 3, Section J, Section 6). Let $f : S^4 \setminus S(K) \to S^1$ be a regular Morse map without local minima and maxima. We claim that $m_1(f) \ge 1$. Indeed, assume that $m_1(f) = 0$. Then $\widehat{H}_1(S^4 \setminus S(K), \xi) = 0$, and by Proposition 4.6 we also have $\widehat{H}_1(S^3 \setminus K, \eta) = 0$, where η is one of generators of the group $H^1(S^3 \setminus K) \approx \mathbb{Z}$. By the preceding theorem the knot K is fibered, which contradicts to our assumptions. Therefore $m_1(f) \ge 1$. Observe that the Euler characteristic of the Novikov complex associated with the map f equals the Euler characteristic of $S^4 \setminus S(K)$, which equals 0. Therefore, $m_1(f) + m_3(f) = m_2(f)$. Hence $m_2(f) \ge 2$, and the proposition is proved. $\square$

4.3. *Surface-Links of Yoshikawa's Table*

Yoshikawa [20] suggested a method for enumerating surface-links. With each surface-link F, he associated a natural number $\mathrm{ch}(F)$. His methods allowed him to make a list of all (weakly prime) surface-links F with $\mathrm{ch}(F) \le 10$. It is clear from the definition of the invariant $\mathrm{ch}(F)$ that we have $\mathrm{sd}(F) \le \mathrm{ch}(F)$. In the rest of this section, we assume that the reader is familiar with Yoshikawa's work and his terminology. There are six two-knots in Yoshikawa's table, namely

$$0_1, 8_1, 9_1, 10_1, 10_2, 10_3.$$

The trivial 2-knot 0_1 is obviously fibered. The knots 8_1 and 10_1 are spun knots of the trefoil knot and respectively of the figure 8 knot, and thus both 8_1 and 10_1 are fibered by [1].

The case of 9_1 is more complicated. The saddle number of this 2-knot is 2. Therefore $\mathcal{MN}(9_1) \le 4$. Using the presentation of the fundamental group of the complement to 9_1 (see [20]) and the Poincaré duality properties, it is easy to compute the Novikov torsion numbers of 9_1. Namely we have $\widehat{q}_1 = 1, \widehat{q}_2 = \widehat{q}_3 = 0$. Therefore

$$2 \le \mathcal{MN}(9_1) \le 4.$$

The 2-knot 10_2 is the 2-twist-spun knot of the trefoil knot and hence fibered by Zeeman's theorem [21]. Similarly, 10_3 is fibered, being the 3-twist spun of the trefoil knot.

The surface-link $6_1^{0,1}$ is the result of spinning the Hopf link, which is fibered (see the left of Figure 2), and therefore $\mathcal{MN}(6_1^{0,1}) = 0$.

The surface-link $8_1^{1,1}$ is the spun torus of the Hopf link. Applying Theorem 3.1, we get the upper bound $\mathcal{MN}(8_1^{1,1}) \le 2$. Computing the Euler characteristic implies the inverse inequality, so that $\mathcal{MN}(8_1^{1,1}) = 2$.

The same argument applies to the surface-link 10_1^1, which is the spun torus of the trefoil knot, so that $\mathcal{MN}(10_1^1) = 2$.

The surface-link $10_1^{0,1}$ is the result of spinning of the link 4_1^2; see Figure 2 (middle). The link 4_1^2 is fibered, and therefore $\mathcal{MN}(10_1^{0,1}) = 0$.

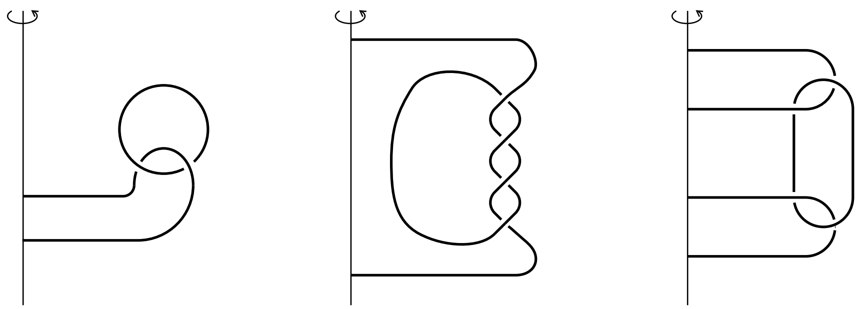

Figure 2

The case of the surface-link $F = 10_1^{0,0,1}$ is more complicated. This surface-link is the result of 4-thread spinning of the connected sum L of two copies of the Hopf link; see Figure 2 (right). Applying Corollary 3.5, we deduce $\mathcal{MN}(F) \leq 2$. The computation of Euler characteristic gives the lower bound 2 for the Morse–Novikov number, and thus $\mathcal{MN}(10_1^{0,0,1}) = 2$.

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References

[1] J. J. Andrews and D. W. Sumners, *On higher-dimensional fibered knots,* Trans. Amer. Math. Soc. 153 (1971), 415–426.

[2] E. Artin, *Zur Isotopie zweidimensionalen Flächen im R_4,* Abh. Math. Semin. Univ. Hambg. 4 (1926), 174–177.

[3] A. Banyaga and D. Hurtubise, *Lectures on Morse homology,* Kluwer, 2004.

[4] S. Friedl, *Novikov homology and non-commutative Alexander polynomials,* J. Knot Theory Ramif. 26 (2017), no. 02, Article ID 1740013.

[5] G. Friedman, *Alexander polynomials of non-locally-flat knots,* Indiana Univ. Math. J. 52 (2003), 1479–1578.

[6] ———, *Knot Spinning* (W. Menasco, M. Thistlethwaite, eds.), Handbook of Knot Theory, pp. 187–208, Elsevier, Amsterdam, 2005.

[7] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities,* Springer, 2004.

[8] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus,* Grad. Stud. Math., 20, American Mathematical Society, Providence, RI, 1999.

[9] C. M. Gordon, *Homology of groups of surfaces in the 4-sphere,* Math. Proc. Cambridge Philos. Soc. 89 (1981), 113–117.

[10] S. Kamada, *Braid and knot theory in dimension four,* Math. Surveys Monogr., 95, Amer. Math. Soc., Providence, RI, 2002.

[11] A. Kawauchi, T. Shibuya, and S. Suzuki, *Descriptions on surfaces in four-space. I. Normal forms,* Math. Sem. Notes Kobe Univ. 10 (1982), 75–125.

[12] J. Milnor, *Lectures on h-cobordism theorem* (Notes by L. Siebenmann and J.Sondow), Princeton Univ. Press, Princeton, NJ, 1965.

[13] A. Pajitnov, *Circle-valued Morse theory,* de Gruyter Studies in Mathematics, 32, de Gruyter, Berlin, 2006.

[14] ———, *On the tunnel number and the Morse–Novikov number of knots,* Algebr. Geom. Topol. 10 (2010), 627–635.

[15] A. Pazhitnov, *On the Novikov complex for rational Morse forms,* Ann. Fac. Sci. Toulouse Math. (6) 4 (1995), 297–338.

[16] D. Rolfsen, *Knots and links,* Publish or Perish, 1976.

[17] D. Roseman, *Spinning knots about submanifolds; spinning knots about projections of knots,* Topology Appl. 31 (1989), 225–241.

[18] J.-Cl. Sikorav, *Points fixes de difféomorphismes symplectiques, intersections de sous-variétés lagrangiennes, et singularités de un-formes fermées,* Thése de Doctorat d'Etat Es Sciences Mathématiques, Université Paris-Sud, Centre d'Orsay, 1987.

[19] K. Veber, A. Pazhitnov, and L. Rudolf, *Morse–Novikov number for knots and links,* Algebra i Analiz 13 (2001), no. 3, 105–118 (in Russian), English translation: St. Petersburg Math. J. 13 (2002), no. 3, 417–426.

[20] K. Yoshikawa, *An enumeration of surfaces in four-space,* Osaka J. Math. 31 (1994), 497–522.

[21] E. C. Zeeman, *Twisting spun knots,* Trans. Amer. Math. Soc. 115 (1965), 471–495.

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