Michigan Math. J. **66 (2017), 661–670**

Few Sewings of Certain Crumpled n-Cubes Yield S^n

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Abstract. For $n \geq 5$, we produce a crumpled n-cube C and homeomorphism h of $\operatorname{Bd} C$ to itself such that, for any homeomorphism $H : \operatorname{Bd} C \to \operatorname{Bd} C$ sufficiently close to h, the sewing space $C \cup_H C$ is not a manifold. This contrasts starkly with a classical three-dimensional result that a dense collection of sewings of an arbitrary pair of crumpled 3-cubes yields the 3-sphere. The key new ingredient is recent work by V. Krushkal, providing a Cantor set in the d-sphere, $d \geq 4$, that cannot be slipped off itself with a small ambient adjustment.

The objects of interest here are crumpled n-cubes, namely, any union C of an $(n-1)$-sphere Σ in S^n and one of its complementary domains; the sphere Σ is called the *boundary* of C, denoted $\operatorname{Bd} C$, and $C - \Sigma$ is called the *interior* of C, denoted $\operatorname{Int} C$. Typically the sphere Σ is wildly embedded in S^n.

Roughly 50 years ago, Daverman and Eaton [13] proved that near any homeomorphism between the boundaries of any two crumpled 3-cubes C and D, there is another homeomorphism H, often called a sewing, that yields S^3 as its sewing space (defined in the next section). Later, Eaton [16] characterized the sewings h of C to D that yield S^3 in terms of a mismatch property: $\operatorname{Bd} C$ and $\operatorname{Bd} D$ must contain homotopy taming sets T_C and T_D, respectively, such that $h(T_C) \cap T_D = \emptyset$. For sewings of crumpled n-cubes, $n > 3$, the mismatch property is a sufficient but not necessary condition for a sewing to yield S^n [12; 5]. It implies that a dense set of sewings of any two crumpled 4-cubes yields S^4: a result of Ancel and McMillan [1] established that near any sewing of crumpled 4-cubes, there is another sewing that satisfies the mismatch property. The same is true for sewings of higher-dimensional crumpled cubes, provided that one of them has a homotopy taming set that is a countable union of Cantor sets that are tame (i.e., standardly embedded) relative to the crumpled cube boundary.

Krushkal [21] recently established the existence of a sticky Cantor set Z in S^d, $d \geq 4$, sticky in the sense that no homeomorphism of S^d to itself close to the identity shifts Z off itself. He did this using what is called a spun Bing–Cantor set K and producing a homeomorphism $h : S^d \to S^d$ such that, for any homeomorphism $H : S^d \to S^d$ sufficiently close to h, $H(K) \cap K \neq \emptyset$. His work is the basis for the main result of this paper, Theorem 5.3, which, for $n \geq 5$, exhibits a crumpled n-cube C and sewing $h : \operatorname{Bd} C \to \operatorname{Bd} C$ such that, for any sewing H of C to itself sufficiently close to h, the sewing space $C \cup_H C$ is not a manifold.

Received May 10, 2016. Revision received August 17, 2017.

Krushkal made use of spun Bing–Cantor sets. Needing more complicated gadgets, we work with ramified versions of those examples. Loosely put, the ramification process leads to a Cantor set's worth of pairwise disjoint Cantor sets, each embedded like the source of the ramification process. We construct a crumpled n-cube C in S^n, the boundary of which is locally flat modulo a Cantor set X that is embedded in both $\mathrm{Bd}\, C$ and S^n as a ramified, spun Bing–Cantor set. Moreover, the interior of C will house a loop L such that every contraction of L in S^n will contain a spun Bing–Cantor set as a subset of the Cantor set X in $\mathrm{Bd}\, C$. The sewing $h : \mathrm{Bd}\, C \to \mathrm{Bd}\, C$ of interest will be essentially the same as that used by Krushkal for producing a sticky Cantor set. It will follow that no sewing of C to itself close to h yields a manifold.

This paper relies heavily on results and techniques from [7] about defining sequences for embedded Cantor sets and compatibility of Cantor sets embedded in manifolds of differing dimensions. We will not reprove the results from [7], but we will take shortcuts afforded by the very simple regularity of the defining sequences at play in this work.

The paper is organized as follows. Section 1 contains basic definitions and notation. Section 2 provides background material about spun Bing decompositions and spun Bing–Cantor sets. Section 3 treats the ramified versions of those two concepts. Section 4 sets forth the construction of the crumpled n-cube promised in the abstract. Finally, Section 5 indicates why sewings close to a specific sewing—namely, the homeomorphism of $\mathrm{Bd}\, C$ to itself exploited by Krushkal—do not yield S^n.

1. Definitions and Basic Properties

The *wild set* W of a crumpled n-cube C is the set of points at which $\mathrm{Bd}\, C$ is not locally collared in C. A subset T of $\mathrm{Bd}\, C$ is a *homotopy taming set* for C if each map $f : I^2 \to C$ can be approximated, arbitrarily closely, by a map f' such that $f'(I^2) \subset T \cup \mathrm{Int}\, C$. If T is one such homotopy taming set for C and W is the wild set of C, then $T \cap W$ is another homotopy taming set for C.

A *sewing* h of crumpled n-cubes C and D is a homeomorphism between their boundaries. The associated *sewing space,* denoted $C \cup_h D$, is the one obtained from the disjoint union of C and D after identification of each $c \in \mathrm{Bd}\, C$ with $h(c) \in \mathrm{Bd}\, D$. The overarching concern here is to understand which sewings h yield (and which do not yield) S^n as its sewing space.

In addition, C is called a *closed n-cell-complement* if $S^n - \mathrm{Int}\, C$ is an n-cell. This is very much a feature of the specific embedding, and it is known that every crumpled n-cube can be embedded in S^n as a closed n-cell-complement [20; 22; 9; 10]. That result is not particularly useful for this effort since the crumpled cube of significance arises as a closed n-cell-complement.

In passing, it is worth noting that, when C is embedded as a closed n-cell-complement, its wild set is precisely the set of points at which $\mathrm{Bd}\, C$ fails to be locally flat in S^n.

A metric space Z satisfies the disjoint disks property if any two maps $\psi_1, \psi_2 : I^2 \to Z$ can be approximated, arbitrarily closely, by maps $\psi_1', \psi_2' : I^2 \to Z$ such that $\psi_1'(I^2) \cap \psi_2'(I^2) = \emptyset$.

For spaces X and Y with $Y \subset X$, we say that Y is 1-LCC embedded in X at a point $y \in Y$ if for each neighborhood N of y, there exists a smaller neighborhood N' of y such that each loop in $N' - Y$ is null homotopic in $N - Y$. The following result is a fundamentally important tool.

THEOREM 1.1. *Let C be a crumpled n-cube, and U an open subset of* $\mathrm{Bd}\, C$ *such that* $\mathrm{Bd}\, C$ *is 1-LCC embedded in C at each point of U. Then U is collared in C.*

Treat C as a closed n-cell-complement in S^n. Then $\mathrm{Bd}\, C$ is 1-LCC embedded in S^n at each point of U. This implies that $\mathrm{Bd}\, C$ is locally flatly embedded in S^n at each point of U (by [8] or [6] when $n \geq 5$, by [19, Theorem 9.3A] when $n = 4$, and by [3] when $n = 3$). It follows easily that $\mathrm{Bd}\, C$ is locally collared in C at each point of U. That U itself has a collar follows from [4].

Finally, we note that all sewing spaces come about as decomposition spaces associated with relatively simple cell-like decompositions of a sphere.

PROPOSITION 1.2. *For any sewing $h : \mathrm{Bd}\, C \to \mathrm{Bd}\, D$ of crumpled n-cubes C and D, there is a cell-like map $f : S^n \to C \cup_h D$, all nondegenerate preimages of which are the fiber arcs of an n-dimensional annulus $A \subset S^n$.*

See [15, Proposition 7.10.2]. Although the stated result is said to apply in case $n \geq 5$, the same construction works equally well when $n = 3$ and $n = 4$ since all crumpled n-cubes can be realized as closed n-cell-complements.

2. Spun Decompositions and Spun Bing–Cantor Sets

Spun decompositions are treated in Section 28 of [11]. In simple form, they involve a decomposition G of the k-cell B^k each nondegenerate element of which meets ∂B^k. All cases treated here will have that simple form, and k will always equal 3. For $n \geq 3$, there exists a surjective map $\psi : B^k \times S^{n-k} \to S^n$ having as its nondegenerate point inverses the sets $b \times S^{n-k}$, $b \in \partial B^k$. The associated spun decomposition $Sp^{n-k}(G)$ of S^n has nondegenerate elements $\psi(g \times S^{n-k})$, where g denotes a nondegenerate element of G. Here $Sp^{n-k}(G)$ is cell-like, provided that G is cell-like and each nondegenerate $g \in G$ meets ∂B^k in a cell-like set [11, Lemma 28.1].

The spun decompositions that arise here are the spins of a famous example due to Bing [2]. The basis, before any spinning, is the decomposition G of a 3-cell shown in Figure 9-8 of [11]. The (n-3)-spins are known to be shrinkable decompositions of S^n. The shrinking can be done manually, like Bing himself did [2] with the 0-spin of G. Lininger [23] first observed that higher-dimensional spins of Bing are shrinkable. Edwards [18] also depicted how to do the shrinking when $n = 4$. Neuzil [24] has a more general result implying that the 1-spin of Bing is shrinkable. Corollary 28.9B of [11] attests to shrinkability when $n > 4$.

The images of the nondegenerate elements after the shrinking are called spun Bing–Cantor sets.

3. Geometric Defining Sequences for Decompositions and Cantor Sets

Given a Cantor set X in S^n, we are often interested in defining sequences that capture features of the embedding. Alternatively, sometimes we are interested in constructing Cantor sets in a sphere with special properties, and defining sequences often serve as an effective means. We say that a sequence $\mathscr{S} = \{\mathscr{M}_0, \mathscr{M}_1, \mathscr{M}_2, \ldots\}$ is a *geometric defining sequence for* X if each $\mathscr{M}_j$ is a finite collection of pairwise disjoint, compact n-manifolds with boundary, the union of which contains X, each $M \in \mathscr{M}_j$ meets X, each $M' \in \mathscr{M}_{j+1}$ lies in the interior of some $M \in \mathscr{M}_j$, and $d_j \to 0$ as $j \to \infty$, where d_j denotes the diameter of the largest component of $\mathscr{M}_j$. In this work, $\mathscr{M}_0$ will always consist of one element, a compact connected n-manifold with boundary.

There is a closely related notion of a defining sequence $\mathscr{S} = \{\mathscr{M}_0, \mathscr{M}_1, \mathscr{M}_2, \ldots\}$ for a decomposition of S^n. In this more general setting the $\mathscr{M}_j$ are as before, except there is no control on diameters of components as $j \to \infty$. Let $|\mathscr{M}_j|$ denote the union of the elements of $\mathscr{M}_j$. The decomposition of S^n determined by $\mathscr{S}$ consists of the components of $Q = \bigcap_j |\mathscr{M}_j|$ and the singletons from $S^n - Q$.

A spun Bing–Cantor set is a Cantor set $K \subset S^n$ equipped with a very special geometric defining sequence $\mathscr{S} = \{\mathscr{M}_0, \mathscr{M}_1, \mathscr{M}_2, \ldots\}$ in which each element M in each $\mathscr{M}_j$ is homeomorphic to $S^{n-2} \times B^2$, M contains exactly two elements M_1, M_2 of $\mathscr{M}_{j+1}$, and the pair $(M, M_1 \cup M_2)$ is homeomorphic to the $(n-3)$-spin of the pair $(B^3, T_1 \cup T_2)$ shown in Figure 28-3 of [11]; here T_1 and T_2 are the 3-cells that meet the left and right sides, respectively, of the cube appearing in that figure in a pair of disks. We refer to a sequence $\mathscr{S}$ of this type as a *standard (geometric) defining sequence for* K.

4. Ramified Spun Bing–Cantor Sets

We need more complicated Cantor sets than the spun Bing examples. Ramified versions fill that need, generating a Cantor set's worth of spun Bing examples. We say that a geometric defining sequence $\mathscr{S} = \{\mathscr{M}_0, \mathscr{M}_1, \mathscr{M}_2, \ldots\}$ is a *standard defining sequence* for a ramified, spun Bing–Cantor set $X \subset S^n$ if each M in each $\mathscr{M}_j$ is homeomorphic to $S^{n-2} \times B^2$, M contains exactly two elements M_1, M_2 of $\mathscr{M}_{j+1}$, when j is even, the pair $(M, M_1 \cup M_2)$ is homeomorphic to the $(n-3)$-spin of the pair $(B^3, T_1 \cup T_2)$, and when j is odd, $M_1 \cup M_2$ is embedded in $M \approx S^{n-2} \times B^2$ just like $S^{n-2} \times (B_1 \cup B_2)$, where B_1 and B_2 are disjoint subdisks of $\operatorname{Int} B^2$.

Instead of first spinning, then ramifying, as just described, we could produce the same standard defining sequence structure by first ramifying in the 3-cell and then spinning. We chose the former approach due to its closer alignment with the methods of [7].

The next result justifies speaking of a standard defining sequence for a ramified, spun Bing–Cantor set rather than simply for a ramified, spun Bing decomposition.

PROPOSITION 4.1. *Every ramified, spun Bing decomposition G (arising from a standard geometric defining sequence) of S^n is shrinkable.*

Proof. For $n \geq 5$, it suffices to show that S^n/G has the disjoint disks property [17] [11] [14]. To this end, consider maps $\psi_1, \psi_2 : I^2 \to S^n/G$. Using approximate lifting properties of the decomposition map $\pi : S^n \to S^n/G$ [11, Theorem 16.8], we can focus on maps $\Psi_1, \Psi_2 : I^2 \to S^n$, where $\pi\Psi_e$ is close to ψ_e ($e \in \{1, 2\}$).

Choose an integer $m > 0$ such that the diameter of each $\pi(M)$, $M \in \mathscr{M}_{2m}$, is small. Note that each such M contains precisely two elements $M_1, M_2 \in \mathscr{M}_{2m+1}$, and there exist (nondisjoint) n-cells E_1 and E_2 in M containing M_1 and M_2, respectively, in their interiors. (They can be obtained by spinning 3-cells in B^3.) Since the complement of any interior point of an n-cell retracts to its boundary, we can modify Ψ_1 over each E_1 lying in $M \in \mathscr{M}_{2m}$ so that the new map Ψ_1' satisfies

$$\begin{aligned} \Psi_1'(I^2) \cap \operatorname{Int} E_1 &= \emptyset, \\ \Psi_1' | \Psi_1^{-1}(S^n - |\mathscr{M}_{2m}|) &= \Psi_1 | \Psi_1^{-1}(S^n - |\mathscr{M}_{2m}|), \quad \text{and} \\ \Psi_1'(\Psi_1^{-1}(|\mathscr{M}_{2m}|)) &\subset |\mathscr{M}_{2m}|. \end{aligned}$$

Modify Ψ_2 similarly so that the image of Ψ_2' misses $\operatorname{Int} E_2$. Using general position in the complement of the E_i, we can also achieve $\Psi_1'(I^2) \cap \Psi_2'(I^2) = \emptyset$. As a result, $\pi\Psi_1'(I^2) \cap \pi\Psi_2'(I^2) = \emptyset$, confirming that the disjoint disks property holds.

For $n = 3$ and $n = 4$, the shrinking can be done manually. The idea is simply to ramify the shrinking of the spun Bing decomposition in that dimension. That is, in the $n = 3$ case, we measure shrinking progress by a collection of disks that chop what can be treated as the initial stage of the standard defining sequence into small pieces, as shown in Figure 9-8 of [11, p. 70]. The elements of the (spun) defining sequence are rearranged in that initial stage so as to meet fewer and fewer of these dividing disks, one fewer per each successive defining sequence stage until all solid tori at some deep stage meet at most one of these dividers. All that it takes is to add in parallel ramifications, doubling up the linked strings of solid tori when proceeding from one stage to the next. A similar strategy works in the $n = 4$ case, since four-dimensional spun Bing decompositions are shrinkable [23; 18; 24]. □

Let $\mathscr{S} = \{\mathscr{M}_0, \mathscr{M}_1, \mathscr{M}_2, \ldots\}$ be a standard defining sequence for a ramified spun Bing–Cantor set X in S^n. A nonempty compact subset A of X is *admissible* (with respect to $\mathscr{S}$) if, whenever A meets an element M of an odd-numbered stage of $\mathscr{S}$, then it meets all components M' of the next stage that are contained in M, and whenever it meets an element M' of an even-numbered stage, then it meets at least one of those M'' from the next stage that lie in M'. Moreover, such a compact A is a *minimal* admissible subset of K if, whenever it meets an element M' of an even-numbered stage, then it intersects exactly one of those components from the next stage that lie in M'. It should be clear that an admissible subset A is minimal if and only if no smaller compact subset is admissible.

The reader is invited to check the following:

LEMMA 4.2. *Let $\mathscr{S} = \{\mathscr{M}_0, \mathscr{M}_1, \mathscr{M}_2, \ldots\}$ be a standard defining sequence for a ramified, spun Bing–Cantor set X in a sphere S. Let M be an element of $\mathscr{M}_{2m}$, M_1 and M_2 the elements of $\mathscr{M}_{2m+1}$ contained in M, and N_e either of the two elements of $\mathscr{M}_{2m+2}$ contained in M_e, $e \in \{1, 2\}$. Then the pair $(M, N_1 \cup N_2)$ is homeomorphic to the pair determined by Stage 0 and Stage 1 of the defining sequence for a spun Bing–Cantor set.*

The following result is essentially a specialization of [7, Lemma 4.2].

PROPOSITION 4.3. *Suppose X is a ramified spun Bing–Cantor set in a sphere S, $\mathscr{S} = \{\mathscr{M}_0, \mathscr{M}_1, \ldots\}$ is a standard defining sequence for X, and $f : I^2 \to S$ is a map such that $f(\partial I^2) \subset S - |\mathscr{M}_0|$ and $f|\partial I^2$ is not null homotopic there. Then $f(I^2)$ contains an admissible subset of X.*

PROPOSITION 4.4. *Suppose X and X' are ramified spun Bing–Cantor sets in S^n and ∂B^n, respectively, L is a loop in $S^n - X$, and $\lambda : X' \to X$ is a homeomorphism. Then there exists an embedding $e : B^n \to S^n$ such that $e|X' = \lambda$, $e(B^n) \cap L = \emptyset$ and $e(\partial B^n)$ is locally flat modulo $X = e(X')$.*

Proof. See [7, Theorem 2.2]. □

PROPOSITION 4.5. *Let $\mathscr{S}$ and $\mathscr{S}'$ denote standard defining sequences for ramified spun Bing–Cantor sets X and X'. There exists a homeomorphism $\eta : X \to X'$ that mixes their admissible subsets, in the sense that for any two admissible subsets A of X and A' of X', $\eta(A) \cap A' \neq \emptyset$.*

Proof. This is proved in greater generality as Lemma 3.1 of [7]. Because mixing is so essential to this endeavor, we briefly describe a construction of η in what follows.

For notational simplicity, we ignore X' and $\mathscr{S}'$and describe a mixing homeomorphism $\eta : X \to X$. Our choice of η depends on a labeling of the manifolds listed in the various stages $\mathscr{M}_m$ of $\mathscr{S}$. The 2^m elements of $\mathscr{M}_m$ are labeled as $M_{i(1),\ldots,i(m)}$, where $i(j) \in \{1, 2\}$, subject to the constraint that the two elements of $\mathscr{M}_{m+1}$ contained in $M_{i(1),\ldots,i(m)} \in \mathscr{M}_m$ are labeled as $M_{i(1),\ldots,i(m),1}$ and $M_{i(1),\ldots,i(m),2}$.

Points of X are uniquely determined by an infinite sequence of 1s and 2s, where the point corresponding to $i(1), \ldots, i(m), \ldots$ is the intersection of

$$M_{i(1)} \supset M_{i(1),i(2)} \supset \cdots \supset M_{i(1),i(2),\ldots,i(m)} \supset \cdots .$$

The mixing homeomorphism η is an infinitely iterated transpose, sending the point corresponding to $i(1), i(2), \ldots, i(2k-1), i(2k), \ldots$ to the one corresponding to $i(2), i(1), \ldots, i(2k), i(2k-1), \ldots$. □

Since the elements of $\mathscr{M}_m$ are all thickened codimension 2 spheres, η can be obtained as a limit of homeomorphisms $\eta_{2m} : |\mathscr{M}_{2m}| \to |\mathscr{M}_{2m}|$.

PROPOSITION 4.6. *Let X and X' be ramified spun Bing–Cantor sets in spheres S and S'. There exists a mixing homeomorphism $\eta : X \to X'$ such that, for every minimal admissible subset A of X, $\eta(A)$ is a spun Bing–Cantor set in S'.*

Proof. As before, we suppress X' and S' and seek a spun Bing–Cantor set in S.

Look at Stages 0, 1, and 2 of the defining sequence $\mathscr{S}$ for X. The initial $S^{n-2} \times B^2$ at Stage 0 contains two elements M_1 and M_2 from Stage 1, and in turn each M_e contains $M_{e,1}, M_{e,2}$ $(e = 1, 2)$ from the Stage 2 ramification process. Given a minimal admissible set A of X, we then have a unique $i(1) \in \{1, 2\}$ such that A meets $M_{i(1)}$ at Stage 1. By the definition of admissibility, A meets both $M_{i(1),1}$ and $M_{i(!),2}$. The usual mixing homeomorphism η for this labeling associates $M_{i(1),1}$ with $M_{1,i(1)}$ and associates $M_{i(1),2}$ with $M_{2,i(1)}$. Observe (Lemma 4.2) that $M_{1,i(1)}$ and $M_{2,i(1)}$ lie in $S^{n-2} \times B^2$ just like the first stage $\mathscr{N}_1$ of a spun Bing decomposition (unramified).

Look next at Stages 3 and 4 of $\mathscr{S}$. The element $M_{i(1),1}$ from the second stage contains two elements $M_{i(1),1,1}$ and $M_{i(1),1,2}$ from the third stage, and, by admissibility, there exists $i(3) \in \{1, 2\}$ such that A meets $M_{i(1),1,i(3)}$. Again by admissibility, A meets each of the elements $M_{i(1),1,i(3),1}$ and $M_{i(1),1,i(3),2}$ from Stage 4 that are contained in $M_{i(1),1,i(3)}$. Similarly, A meets some $M_{i(1),2,i'(3)} \subset M_{i(1),2}$ from the third stage and meets both elements $M_{i(1),2,i'(3),1}$ and $M_{i(1),2,i'(3),2}$ from the fourth stage that are contained in $M_{i(1),1,i'(3)}$. The mixing homeomorphism η associates these four elements from the fourth stage with

$$\mathscr{N}_2 = \{M_{1,i(1),1,i(3)}, M_{1,i(1),2,i(3)}, M_{2,i(1),1,i'(3)}, M_{2,i(1),2,i'(3)}\}.$$

Observe (Lemma 4.2) that $\{\mathscr{M}_0, \mathscr{N}_1, \mathscr{N}_2\}$ are arranged just like Stages 0, 1, 2 of a spun Bing decomposition.

In the same manner, we can spell out the elements from the first $2m$ stages of $\mathscr{S}$ that meet A and see that the mixing homeomorphism associates them with elements arranged like the first m stages of a spun Bing decomposition. □

COROLLARY 4.7. *Let X and X' be ramified spun Bing–Cantor sets in spheres S and S'. Then there exists a mixing homeomorphism $\eta : X \to X'$ such that, for any admissible subset A of X, $\eta(A)$ contains a spun Bing–Cantor set in X'.*

The culmination of this section is the following:

EXAMPLE 4.1. A closed n-cell-complement $C \subset S^n$ whose wild set W lies in $\operatorname{Bd} C$ as a ramified spun Bing–Cantor set, plus a loop L in $\operatorname{Int} C$ such that the image of every contraction of L in S^n contains a spun Bing–Cantor set in W. The construction starts with ramified spun Bing–Cantor sets X and X' in S^n and ∂B^n, respectively. Let $\eta : X \to X'$ be the mixing homeomorphism promised by Corollary 4.7. Name a loop $L \subset S^n$ linking the initial sage of the defining sequence for X. Application of Proposition 4.4 with $\lambda = \eta^{-1}$ yields an embedding $e : B^n \to S^n - L$ for which $e|X' = \eta^{-1}$ and $e(\partial B^n)$ is locally flat modulo $X = e(X')$. Here $C = S^n - e(\operatorname{Int} B)$. Proposition 4.5 assures that every contraction of L in S^n contains an admissible subset of X, and Corollary 4.7 indicates that every such contraction of L contains a spun Bing–Cantor set in $W = e(X') \subset e(\partial B^n) = \operatorname{Bd} C$.

Remark. The wild set W of C can be embedded in S^n as a ramified spun Bing–Cantor set. However, this is a convenience, not a necessity. What is most relevant is for the Cantor sets in $\operatorname{Bd} C$ and in S^n to have compatible (i.e., each element at each stage of the defining sequence contains exactly two elements from the next stage) ramified defining sequences and for the one in $\operatorname{Bd} C$ to be of ramified spun Bing type.

5. Sewings That Do Not Yield S^n

LEMMA 5.1. *Suppose $C \subset S^n$ is a closed n-cell complement, D is a crumpled n-cube, $\theta : \operatorname{Bd} C \to \operatorname{Bd} D$ is a sewing, and $j : C \to C \cup_\theta D$ is the natural embedding of C onto its image in the sewing space. Then there exists a map $\mu : C \cup_\theta D \to S^n$ such that $\mu | j(C) = j^{-1}$ and μ is 1–1 over $\mu j(C) = C \subset S^n$.*

Proof. Set μ equal to j^{-1} on $j(C)$. Let B denote the n-cell $S^n - \operatorname{Int} C$. Extend $\mu | j(C)$ over the copy of D in $C \cup_\theta D$ so as to send D to B, and then modify slightly so as to send $\operatorname{Int} D$ to $\operatorname{Int} B$. □

Krushkal's methods actually prove the following theorem, which is of the form we shall use.

THEOREM 5.2. *Let $M = S^{k-2} \times B^2$ be an unknotted thickened $(k-2)$-sphere in S^k, $k \geq 4$, X a ramified spun Bing–Cantor set in $\operatorname{Int} M$, and K_1 and K_2 any two spun Bing–Cantor sets in X, where each has a standard defining sequence with M as its initial stage. Then there exists a homeomorphism $h : S^k \to S^k$ such that, for any homeomorphism $H : S^k \to S^k$ sufficiently close to h, $H(K_1) \cap K_2 \neq \emptyset$.*

THEOREM 5.3. *For $n \geq 5$, there exist a crumpled n-cube C and a sewing h : $\operatorname{Bd} C \to \operatorname{Bd} C$ such that no sewing H sufficiently close to h yields S^n.*

Proof. For notational clarity, we let each of C_1 and C_2 be copies of the crumpled n-cube described in Example 4.1. Let $M \approx S^{n-3} \times B^2$ be the initial stage of the defining sequence $\mathscr{S}$ for the wild set W of C relative to $\operatorname{Bd} C$; that is, $\mathscr{S}$ is a defining sequence in $\operatorname{Bd} C$ for the ramified, spun Bing–Cantor set W.

Let $h : \operatorname{Bd} C \to \operatorname{Bd} C$ be the homeomorphism described by Krushkal [21]. (Essentially, h repositions the core $(n-3)$-sphere Σ for Stage 0 of $\mathscr{S}$ in $\operatorname{Bd} C$ so that $h(\Sigma)$ meets Σ transversely in an $(n-5)$-sphere.) The claim is that, if $H : \operatorname{Bd} C \to \operatorname{Bd} C$ is another homeomorphism close to h, then the sewing space $C_1 \cup_H C_2$ fails to satisfy the disjoint disks property and, consequently, cannot be a manifold.

Each of C_1 and C_2 in the sewing space $C_1 \cup_H C_2$ contains a copy of the special loop L. Let $\mu : C_1 \cup_H C_2 \to S^n$ denote the map promised by Lemma 5.1 for C_1. By the construction of C, for any singular disk $f_1(I^2)$ in $C_1 \cup_H C_2$ bounded by the copy of L in $C_1 \subset C_1 \cup_H C_2$, $\mu f_1(I^2)$ contains a spun Bin–Cantor set K in the wild set $W \subset \mathrm{Bd}\, C$. Lemma 5.1 implies that $f_1(I^2)$ itself actually contains such a Cantor set K_1 in the wild set $W_1 \subset \mathrm{Bd}\, C_1 \subset C_1 \cup_H C_2$. In that sewing space, H equates $K_1 \subset \mathrm{Bd}\, C_1$ with $H(K_1) \subset \mathrm{Bd}\, C_2$. For the same reasons, any singular disk $f_2(I^2)$ in $C_1 \cup_H C_2$ bounded by the copy of L in C_2 contains a spun Bing–Cantor set K_2 in the wild set W_2 of $C_2 \subset C_1 \cup_H C_2$. By Theorem 5.2, $H(K_1)$ and K_2 must intersect. Hence, $C_1 \cup_H C_2$ does not have the disjoint disks property. □

With techniques like these, we can produce a pair of crumpled n-cubes for which the sewings that yield S^n are contained in a closed, nowhere dense subset of the space of all sewings, under the usual sup-norm metric. However, we have no answer for the following:

Question. Does there exist a pair of crumpled n-cubes C and D such that no sewing $h : \mathrm{Bd}\, C \to \mathrm{Bd}\, D$ yields S^n?

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