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Involution and Commutator Length for Complex Hyperbolic Isometries

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Abstract. We study decompositions of complex hyperbolic isometries as products of involutions. We show that PU(2, 1) has involution length 4 and commutator length 1 and that, for all $n \geq 3$, PU$(n, 1)$ has involution length at most 8.

1. Introduction

Riemannian symmetric spaces are characterized by the existence of special isometries called *central involutions:* for each point p of such a space X, there exists an involution $I_p \in \mathrm{Isom}(X)$ such that p is an isolated fixed point of I_p and $d_p I_p = -\mathrm{Id} \in \mathrm{GL}(T_p X)$. The group of displacements of a Riemannian symmetric space X is the subgroup of the isometry group $\mathrm{Isom}(X)$, which is generated by pairwise products of central involutions. It is a classical fact that for connected symmetric spaces, it coincides with the identity component $\mathrm{Isom}^0(X)$ (see, for example, Proposition IV-1.4 of [L]). This means that every isometry in the identity component is a product of a finite (even) number of central involutions.

It is then a natural question to ask, given a symmetric space X, what the central involution length of $\mathrm{Isom}^0(X)$ is, that is, the smallest $n \in \mathbb{N}$ (if any) such that any element of $\mathrm{Isom}^0(X)$ is a product of at most n central involutions. We can also relax the question to more general involutions, which is also of geometric interest since it allows, for example, to consider reflections that have fixed-point loci of maximal (rather than minimal) dimension.

Basmajian and Maskit [BM] investigated the involution length of $\mathrm{Isom}(X)$ when X is a symmetric space of constant (sectional) curvature, that is, one of the model spaces S^n, E^n, or H^n. They found that, allowing orientation-reversing involutions, the involution length is always 2, whereas if we restrict ourselves to orientation-preserving involutions (i.e. involutions in the identity component $\mathrm{Isom}^0(X)$), then it is 2 or 3, depending explicitly on the space and the congruence class of n mod. 4. They deduce from these facts that every element of $\mathrm{Isom}^0(X)$ is a commutator, that is, the commutator length of $\mathrm{Isom}^0(X)$ is 1. Note that this and our analogous result do not follow from general results about linear groups with

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commutator length 1, see the introduction to [D]. Rather, it follows from the involution length and the remark that every square of a triple product of involutions is a commutator. Indeed, for any triple of involutions (I_1, I_2, I_3), we have

$$(I_1I_2I_3)^2 = [I_1I_2, I_3I_2]. \tag{1}$$

In this paper, we study the analogous question in $\mathrm{Isom}(X)$ when X is the complex hyperbolic space $\mathrm{H}^n_{\mathbb{C}}$, the model complex symmetric space of constant negative holomorphic sectional curvature. Here $\mathrm{Isom}(X)$ has two connected components, one consisting of all holomorphic isometries (the identity component, isomorphic to $\mathrm{PU}(n, 1)$) and the other consisting of all antiholomorphic isometries. It is well known that any element of $\mathrm{PU}(n, 1)$ is a product of two antiholomorphic involutions (usually called *real reflections*); this was originally observed by Falbel and Zocca [FZ] when $n = 2$ and then for all values of n by Choi in [C] (see also [GT] and [N] for the elliptic case, corresponding to $\mathrm{U}(n)$). However, only special elements of $\mathrm{PU}(n, 1)$ are products of two holomorphic involutions (see Lemma 4 in the case of $\mathrm{PU}(2, 1)$). The involution length of $\mathrm{PU}(n, 1)$ is thus at least 3 (for $n \geq 2$). Our main result is the following:

THEOREM 1. *The involution length of* $\mathrm{PU}(2, 1)$ *is* 4.

To prove Theorem 1, we use a combination of geometric arguments, describing explicit configurations of triples of involutions (I_1, I_2, I_3) such that the product $I_1I_2I_3$ belongs to a prescribed conjugacy class, and of indirect arguments, using general properties of the product map in the Lie group $\mathrm{PU}(2, 1)$.

Theorem 1 will follow from the precise description of the set of elements in $\mathrm{PU}(2, 1)$ that are products of three involutions. We determine exactly which elements of $\mathrm{PU}(2, 1)$ cannot be written as products of three involutions and prove that all elements are products of at most four involutions. (We also show analogous statements where "involution" is replaced by "central involution" or by "complex reflection of order 2".) More specifically, we show that all loxodromic and parabolic isometries in $\mathrm{PU}(2, 1)$ are triple products of involutions, whereas some elliptic conjugacy classes are not.

Loxodromic and parabolic isometries are relatively easy to handle since their conjugacy class is determined by their trace or, equivalently, by the eigenvalue associated with any boundary fixed point. We provide explicit descriptions of triples of involutions (I_1, I_2, I_3) whose product $I_1I_2I_3$ belongs to any fixed loxodromic conjugacy class (see Sections 4 and 4.2). More precisely, with any fixed point p_2 of $I_1I_2I_3$ in $\overline{\mathrm{H}^2_{\mathbb{C}}}$, we associate a triangle in $\overline{\mathrm{H}^2_{\mathbb{C}}}$, which we call the cycle-triangle (see Section 4.1). This triangle is obtained by considering p_2 and its images I_3p_2, $I_2I_3p_2$, and $I_1I_2I_3p_2 = p_2$. The shape of the cycle-triangle is closely related to the conjugacy class of $I_1I_2I_3$: it determines the eigenvalue of the triple product $I_1I_2I_3$ associated with any lift of p_2 (see Propositions 5 and 7 and the proof of Proposition 6).

The situation for elliptic isometries is more complicated, and cycle-triangles do not provide enough information to completely determine the conjugacy class.

We give in Proposition 10 and Corollary 6 a precise description of those regular elliptic elements that are not products of three involutions. This description is given in terms of the *angle pair* of the elliptic isometry. Elliptic isometries preserve two orthogonal complex lines on which they act by rotation; the angle pair is the pair formed by these two rotation angles (see Section 3.3.3). The angle pair determines the conjugacy class of an elliptic element. The fact that not all elliptic elements are products of three involutions is rather subtle. Indeed, it can be seen using, for instance, the trace formulae derived by [Pra] that the map associating with a triple of involutions $(I_1, I_2, I_3) \in \mathrm{SU}(2,1)^3$ the trace of the product $I_1 I_2 I_3$ is onto $\mathbb{C}$ and thus realizes all possible elliptic traces. However, to a given elliptic trace value, there correspond generically three possible elliptic conjugacy classes. This is because two elliptic elements in SU(2, 1) may have the same eigenvalues, but not the same angle pair, depending on the relative position of their eigenvectors and the light-cone in $\mathbb{C}^{2,1}$ (see Section 3.3.3).

Our approach to elliptic triple products uses properties of the product map restricted to pairs of semisimple conjugacy classes (see e.g. [FW2; P]). More precisely, given two such conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$, we consider the map

$$\begin{aligned} \tilde{\mu} : \mathcal{C}_1 \times \mathcal{C}_2 &\longrightarrow \mathcal{G} \\ (A, B) &\longmapsto [AB], \end{aligned} \tag{2}$$

where $\mathcal{G}$ is the space of conjugacy classes of PU(2, 1) (see Section 3.3.4), and $[AB]$ denotes the conjugacy class of AB. We review the main properties of this map in Section 5. The image by $\tilde{\mu}$ of reducible pairs (A, B) form the so-called *reducible walls* that divide $\mathcal{G}$ into *chambers*. The crucial fact is that when $\mathcal{C}_1$ and $\mathcal{C}_2$ are semisimple classes, these chambers are either full or empty. Indeed, the map $\tilde{\mu}$ in (2) is open (when restricted to irreducible pairs) and closed (see Sections 5.2 and 5.3). In particular, $\mathrm{Im}\,\tilde{\mu}$ is a union of chamber closures. The openness of the map (2) is classical and follows from the description of the image of the differential of the product map (see Lemma 4). Its closedness is a consequence of the Bestvina–Paulin compactness theorem ([Be]).

In our case, we consider this map when $\mathcal{C}_1$ is the conjugacy class of a product of two involutions and $\mathcal{C}_2$ is the conjugacy class of an involution. Applying this method, we are able to determine which elliptic conjugacy classes are triple products of involutions. This method could easily be applied to the case of loxodromic conjugacy classes, but we found it simpler to describe simple geometric configurations that provide the same result. We must consider parabolic conjugacy classes separately as they are not semisimple and cannot be separated from conjugacy classes of complex reflections. To prove that the involution length of PU(2, 1) is 4, we show that the map $\tilde{\mu}$ becomes surjective when both $\mathcal{C}_1$ and $\mathcal{C}_2$ are conjugacy classes of pairwise products of involutions.

We also obtain as a byproduct of these results that PU(2, 1) has commutator length 1 (Theorem 5), but slightly more indirectly than in [BM]. Indeed, we show that even though not every element of PU(2, 1) is a triple product of involutions, any element of PU(2, 1) is the square of a triple product of involutions and hence by (1) is a commutator. We were motivated to study this question by the main result of [PaW] which involves a condition on commutators.

In higher dimensions, that is, in $\mathrm{PU}(n,1)$ with $n \geq 3$, the involution length will be at least 3 for the same reason as before (pairwise products of involutions have special properties). However, the finer methods that we use in this work to improve the lower bound to 4 do not extend easily to higher dimensions. Indeed, they rely on a detailed understanding the chamber structure in the space of elliptic conjugacy classes in $\mathrm{PU}(2,1)$ (see Section 5 for more details), which gets significantly more complicated in higher dimensions. Djokovic and Malzan in [DM1] proved that the length of $\mathrm{SU}(n)$ with respect to complex reflections of order 2 is $2n-1$ and in [DM2] that the corresponding length in $\mathrm{SU}(p,q)$ (with $p,q \geq 1$) is $p+q+2$ or $p+q+3$ (depending on the parity of $p+q$). By combining our results for $n=2$ with results of [GT] (namely their bound on the involution length of $\mathrm{SU}(n)$) we obtain the following result.

THEOREM 2. *For all $n \geq 2$, the involution length of* $\mathrm{PU}(n,1)$ *is at most* 8.

The paper is organized as follows. In Section 2, we present some classical facts on products of isometries in the Poincaré disk for later reference. Section 3 is devoted to the description of conjugacy classes in $\mathrm{PU}(2,1)$. In Section 4, we give the first properties of triples of involutions, define the cycle-triangle, and prove that any loxodromic map is a product of three involutions of any kind. We introduce the product map in Section 5 and describe the general strategy to determine its image. We then apply this strategy in Section 6 to determine which regular elliptic isometries are products of three involutions. We deal with parabolic conjugacy classes separately in Section 7. Finally, in Section 8, we apply these results to study the involution length and commutator length.

2. Some Classical Hyperbolic Geometry

PROPOSITION 1. 1. *Every element of* $\mathrm{PSL}(2,\mathbb{R})$ *is a product of two reflections.*
2. *Every antiholomorphic isometry of the Poincaré disk is a product of three reflections.*
3. *Every element of* $\mathrm{PSL}(2,\mathbb{R})$ *is a product of at most three half-turns.*

Proof. The first part of Proposition 1 is classical (see e.g. Sections 7.32 to 7.35) of [Bear]). The second part follows since any antiholomorphic isometry of the Poincaré disk is the product of an element of $\mathrm{PSL}(2,\mathbb{R})$ and a reflection (e.g. $z \longmapsto -\overline{z}$). For the third part, we proceed by case-by-case analysis.

(a) Any hyperbolic element h is a product of two half-turns with fixed points a distance $\frac{\ell}{2}$ apart on its invariant axis, where ℓ is the translation length of h.
(b) To see that elliptic elements are products of three half-turns, consider a triangle $T=(p_1,p_2,p_3)$ in the Poincaré disk with internal angles $\theta_i \in [0,\pi)$, $i=1,2,3$. Let I_k be the half-turn about the midpoint of the edge $[p_{k+1},p_{k+2}]$ of T, where indices are taken modulo 3 (see Figure 1). Then $I_1I_2I_3$ is elliptic (it fixes p_2), and it is a simple exercise in plane hyperbolic geometry to see

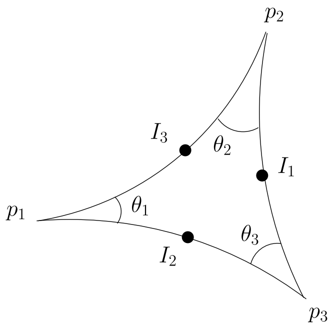


Figure 1 An elliptic triple product of half-turns

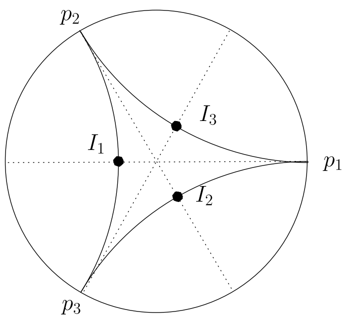


Figure 2 A parabolic triple product of half-turns

that its rotation angle is $\theta = \theta_1 + \theta_2 + \theta_3 \in (0, \pi)$. Changing $I_1 I_2 I_3$ to its inverse $I_3 I_2 I_1$, we see that any nonzero rotation angle in $(-\pi, \pi)$ can be obtained this way. Elliptic elements with angle π are obtained in the case where $I_1 = I_2 = I_3$.

(c) For parabolic elements, consider an ideal triangle $T = (p_1, p_2, p_3)$ in the Poincaré disk, and let I_k be the half-turn fixing the orthogonal projection of p_k onto the opposite edge (see Figure 2). The product $I_1 I_2 I_3$ fixes p_2 and is parabolic. This can be seen, for instance, by considering the orbit of a horosphere based at p_2 (we can also argue that the group $\langle I_1, I_2, I_3 \rangle$ is conjugate to an index 3 subgroup of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ and that $I_1 I_2 I_3$ corresponds to the cube of the parabolic element $z \longmapsto z + 1$ under this conjugation). Since there is only one conjugacy class of parabolic elements in the Poincaré disk, this shows that any parabolic element has this property. $\square$

For later use, we describe the possible conjugacy classes for the product of two isometries of the Poincaré disk lying in certain prescribed conjugacy classes.

Proposition 2. 1. *Let $\mathcal{C}$ be a hyperbolic conjugacy class in* $\mathrm{PSL}(2, \mathbb{R})$.

(a) *The product of an element $h \in \mathcal{C}$ and a half-turn can belong to any nontrivial conjugacy class. In particular, it can be elliptic with arbitrary rotation angle.*

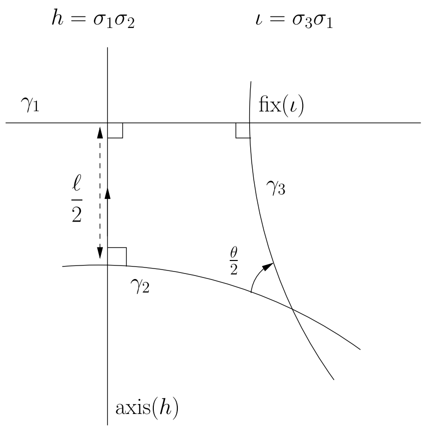
(b) *The product of an element $h \in \mathcal{C}$ and a reflection is a glide reflection with arbitrary translation length.*

2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two elliptic conjugacy classes in* $\mathrm{PSL}(2,\mathbb{R})$ *corresponding to rotation angles θ_1 and θ_2 (with $\theta_i \in [0, 2\pi)$). If $E_1 \in \mathcal{C}_1$ and $E_2 \in \mathcal{C}_2$ are such that E_1E_2 is elliptic, then the rotation angle of E_1E_2 can take any value in $[\theta_1+\theta_2, 2\pi)$ (resp. $(2\pi, \theta_1+\theta_2]$) if $\theta_1+\theta_2 < 2\pi$ (resp. $\theta_1+\theta_2 > 2\pi$).*
3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two hyperbolic conjugacy classes in* $\mathrm{PSL}(2,\mathbb{R})$ *corresponding to translation lengths ℓ_1 and ℓ_2. Then every elliptic isometry is a product h_1h_2 with $h_1 \in \mathcal{C}_1$ and $h_2 \in \mathcal{C}_2$.*

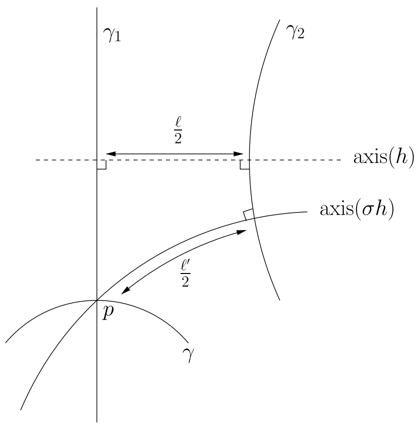
Proof. 1(a). Let $h \in \mathcal{C}$ (with translation length denoted ℓ), and let ι be a half-turn with fixed point p. Let γ_1 be the geodesic orthogonal to the axis of h through the fixed point of ι, and σ_1 the reflection about γ_1 (see Figure 3). Let σ_2 be the unique reflection such that $h = \sigma_1\sigma_2$; it fixes pointwise a geodesic γ_2 that is at distance $\frac{\ell}{2}$ from γ_1. The half-turn ι is the product of σ_1 and the reflection σ_3 about the geodesic γ_3 orthogonal to γ_1 through p. The product ιh is equal to $\sigma_2\sigma_3$. As in Figure 3, we see that as p moves away from the axis of h, the relative position of γ_2 and γ_3 varies continuously from orthogonal (when p is on the axis of h) to disjoint with arbitrarily large distance (as p goes to infinity along γ_1). We thus obtain elliptic classes with any rotation angle $\theta \in [0, \pi[$ (when γ_2 and γ_3 intersect), a parabolic class (when γ_2 and γ_3 are asymptotic), and any hyperbolic class (when γ_2 and γ_3 are ultraparallel). The other elliptic classes are obtained by applying the reflection about the axis of h, which reverses orientation.

1(b). Let $h \in \mathcal{C}$ with translation length denoted ℓ. Write $h = \sigma_1\sigma_2$, where σ_1 and σ_2 are reflections about geodesics γ_1 and γ_2 orthogonal to the axis of h and a distance $\frac{\ell}{2}$ apart. Now consider a geodesic γ orthogonal to γ_1 and the reflection σ about it. The product $\sigma\sigma_1$ is the half-turn about the point $p = \gamma \cap \gamma_1$. Therefore σh is the product of a reflection and a half-turn, which is a glide reflection (since p is not fixed by σ_2). As γ moves away from the axis of h, the translation length ℓ' of σh can take any positive value (see Figure 3).

2. Let γ_3 be the geodesic connecting the fixed points of E_1 and E_2, and σ_3 the associated reflection. Decompose the two elliptics as products $E_1 = \sigma_1\sigma_3$ and $E_2 = \sigma_3\sigma_2$, where σ_1 and σ_2 are reflections about geodesics through the fixed points of E_1 and E_2. The geodesics γ_1 and γ_2 intersect γ_3 with angles $\theta_1/2$ and $\theta_2/2$, as indicated on Figure 4. The product $E_1E_2 = \sigma_1\sigma_2$ is elliptic if and only if γ_1 and γ_2 intersect inside the disk. The result follows by studying the possible angles of the triangle bounded by γ_1, γ_2, and γ_3 when the distance between the fixed points of E_1 and E_2 varies.
3. The argument is about the same as for the previous item. Consider the right-hand side of Figure 4. The two hyperbolic isometries are $h_1 = \sigma_1\sigma$ and



Proposition 2, 1(a). Product of a hyperbolic isometry and a half-turn (elliptic case)



Proposition 2, 1(b). Product of a hyperbolic isometry and a reflection

Figure 3 Product of a hyperbolic isometry and an involution

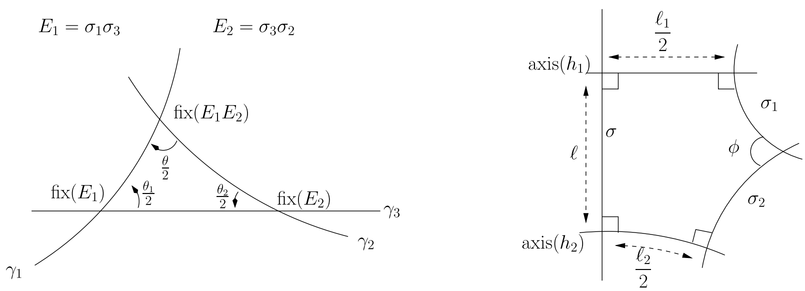


Figure 4 Elliptic product of two elliptic (left) or hyperbolic (right) isometries in given conjugacy classes

$h_2 = \sigma\sigma_2$. As the distance ℓ varies from 0 to ∞, the product $h_1 h_2$ varies from identity (when $\ell = 0$) to hyperbolic with arbitrarily large translation length. In particular, the angle ϕ can take any value between 0 and π. □

3. Complex Hyperbolic Space and Its Isometries

3.1. Basic Definitions

The standard reference for complex hyperbolic geometry is [G1]. For the reader's convenience, we include a brief summary of key definitions and facts. Our main result concerns the case of dimension $n = 2$, but the general setup is identical for higher dimensions, so we state it for all $n \geq 1$.

Distance Function. Consider $\mathbb{C}^{n,1}$, the vector space $\mathbb{C}^{n+1}$ endowed with a Hermitian form $\langle\cdot,\cdot\rangle$ of signature $(n, 1)$. We take the convention that the Hermitian product is given by $\langle X, Y\rangle = X^T H\overline{Y}$, where H is the matrix of the Hermitian product in a given basis of $\mathbb{C}^n$. Let $V^- = \{Z \in \mathbb{C}^{n,1} \mid \langle Z, Z\rangle < 0\}$. Let $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}\mathrm{P}^n$ denote projectivization. Define $\mathrm{H}^n_{\mathbb{C}}$ to be $\pi(V^-) \subset \mathbb{C}\mathrm{P}^n$ endowed with the distance d (Bergman metric) given by

$$\cosh^2\left(\frac{d(\pi(X), \pi(Y))}{2}\right) = \frac{|\langle X, Y\rangle|^2}{\langle X, X\rangle\langle Y, Y\rangle}. \tag{3}$$

Different choices of Hermitian forms of signature $(n, 1)$ give rise to different models of $\mathrm{H}^n_{\mathbb{C}}$. The two most standard choices are the following. First, when the Hermitian form is given by $\langle Z, Z\rangle = |z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2$, the image of V^- under projectivization is the unit ball of $\mathbb{C}^n$ seen in the affine chart $\{z_{n+1} = 1\}$ of $\mathbb{C}P^n$. This model is referred to as the *ball model* of $\mathrm{H}^n_{\mathbb{C}}$. Secondly, when $\langle Z, Z\rangle = 2\,\mathrm{Re}(z_1\overline{z_{n+1}}) + |z_2|^2 + \cdots + |z_n|^2$, we obtain the so-called *Siegel model* of $\mathrm{H}^n_{\mathbb{C}}$, which generalizes the Poincaré upper half-plane.

Isometries. From (3) it is clear that $\mathrm{PU}(n, 1)$ acts by isometries on $\mathrm{H}^n_{\mathbb{C}}$, where $\mathrm{U}(n, 1)$ denotes the subgroup of $\mathrm{GL}(n+1, \mathbb{C})$ preserving $\langle\cdot,\cdot\rangle$, and $\mathrm{PU}(n, 1)$ its image in $\mathrm{PGL}(n+1, \mathbb{C})$. In fact, $\mathrm{PU}(n, 1)$ is the group of holomorphic isometries of $\mathrm{H}^n_{\mathbb{C}}$, and the full group of isometries is $\mathrm{PU}(n, 1) \ltimes \mathbb{Z}/2$, where the $\mathbb{Z}/2$ factor corresponds to a real reflection (see further). Holomorphic isometries of $\mathrm{H}^n_{\mathbb{C}}$ can be of three types, depending on the number and location of their fixed points. Namely, $g \in \mathrm{PU}(n, 1)$ is:

- *elliptic* if it has a fixed point in $\mathrm{H}^n_{\mathbb{C}}$;
- *parabolic* if it has (no fixed point in $\mathrm{H}^n_{\mathbb{C}}$ and) exactly one fixed point in $\partial_\infty\mathrm{H}^n_{\mathbb{C}}$;
- *loxodromic* if it has (no fixed point in $\mathrm{H}^n_{\mathbb{C}}$ and) exactly two fixed points in $\partial_\infty\mathrm{H}^n_{\mathbb{C}}$.

Totally Geodesic Subspaces. A *complex k-plane* is a projective k-dimensional subspace of $\mathbb{C}P^n$ intersecting $\pi(V^-)$ nontrivially (so, it is an isometrically embedded copy of $\mathrm{H}^k_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{C}}$). Complex 1-planes are usually called *complex lines*. If $L = \pi(\tilde{L})$ is a complex $(n-1)$-plane, then any $v \in \mathbb{C}^{n+1} - \{0\}$ orthogonal to $\tilde{L}$ is called a *polar vector* of L. Such a vector satisfies $\langle v, v\rangle > 0$, and we will usually normalize v so that $\langle v, v\rangle = 1$.

A *real k-plane* is the projective image of a totally real $(k+1)$-subspace W of $\mathbb{C}^{n,1}$, that is, a $(k+1)$-dimensional real vector subspace such that $\langle v, w\rangle \in \mathbb{R}$ for all $v, w \in W$. We will usually call real 2-planes simply real planes, or $\mathbb{R}$-planes. Every real n-plane in $\mathrm{H}^n_{\mathbb{C}}$ is the fixed-point set of an antiholomorphic isometry of order 2 called a *real reflection* or $\mathbb{R}$-reflection. The prototype of such an isometry is the map given in affine coordinates by $(z_1, \dots, z_n) \mapsto (\overline{z_1}, \dots, \overline{z_n})$ (this is an isometry provided that the Hermitian form has real coefficients).

In $\mathrm{H}^2_{\mathbb{C}}$, the relative position of complex lines can be determined using using the following lemma.

LEMMA 1. *Let* $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *be linearly independent vectors in* $\mathbb{C}^{2,1}$ *such that* $\langle \mathbf{n}_k, \mathbf{n}_k\rangle \neq 0$. *When* $\mathbf{n}_k$ *has negative type, we denote by* n_k *its projection to* $\mathrm{H}^2_{\mathbb{C}}$; *when it has positive type, we denote by* L_k *its polar complex line. Consider*

$$\kappa = \frac{|\langle \mathbf{n}_1, \mathbf{n}_2\rangle|^2}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle \langle \mathbf{n}_2, \mathbf{n}_2\rangle}. \tag{4}$$

1. *If* $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *both have negative type, then* $\kappa > 1$ *and* $\kappa = \cosh^2(d/2)$, *where* $d = d(n_1, n_2)$.
2. *If* $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *have opposite types, say* $\mathbf{n}_1$ *has positive type and* $\mathbf{n}_2$ *negative type, then* $\kappa \leq 0$ *and* $\kappa = -\sinh^2(d/2)$, *where* $d = d(L_1, n_2)$. *In particular,* $\kappa = 0$ *if and only if* n_2 *belongs to* L_1.
3. *If* $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *both have positive type, then*:
   (a) L_1 *and* L_2 *are ultraparallel* $\iff \kappa > 1$; *in that case,* $\kappa = \cosh^2(d/2)$, *where* $d = d(L_1, L_2)$,
   (b) L_1 *and* L_2 *intersect inside* $\mathrm{H}^2_{\mathbb{C}} \iff 0 \leq \kappa < 1$; *in that case,* $\kappa = \cos^2(\theta)$, *where* θ *is the angle between* L_1 *and* L_2,
   (c) L_1 *and* L_2 *are asymptotic* $\iff \kappa = 1$.

Proof. The first item comes from the distance between two points in $\mathrm{H}^2_{\mathbb{C}}$, which is given by (3). The third one is a reformulation of Section 3.3.2 of [G1]. To prove the second one, we note that if $\mathbf{n}_1$ is polar to L_1, then the orthogonal projection of n_2 on L_1 is given by the vector

$$\mathbf{v} = \mathbf{n}_2 - \frac{\langle \mathbf{n}_2, \mathbf{n}_1\rangle}{\langle \mathbf{n}_2, \mathbf{n}_2\rangle}\mathbf{n}_1.$$

The distance between n_1 and L_2 is then obtained by applying (3) to $\mathbf{v}$ and $\mathbf{n}_2$, and this gives the result. □

REMARK 1. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be two vectors in $\mathbb{C}^{2,1}$. The Hermitian cross product of $\mathbf{n}_1$ and $\mathbf{n}_2$ is defined by

$$\mathbf{n}_1 \boxtimes \mathbf{n}_2 = \overline{H^{-1}} \cdot \overline{\mathbf{n}_1 \wedge \mathbf{n}_2}, \tag{5}$$

where H is the matrix of the Hermitian form. The vector $\mathbf{n}_1 \boxtimes \mathbf{n}_2$ is H-orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Note that in the case of the Siegel and ball models, H has real coefficients and satisfies $H^2 = I$, so that $\mathbf{n}_1 \boxtimes \mathbf{n}_2 = H \cdot \overline{\mathbf{n}_1 \wedge \mathbf{n}_2}$. Geometrically, the Hermitian cross product can be interpreted as follows.

1. If $\mathbf{n}_1$ and $\mathbf{n}_2$ both have negative type, then $\mathbf{n}_1 \boxtimes \mathbf{n}_2$ is a polar vector for the complex line spanned by the projections to $\mathrm{H}^2_{\mathbb{C}}$ of $\mathbf{n}_1$ and $\mathbf{n}_2$.
2. If $\mathbf{n}_1$ has negative type and $\mathbf{n}_2$ positive type, then $\mathbf{n}_1 \boxtimes \mathbf{n}_2$ is a polar vector for the complex line through the projection of $\mathbf{n}_1$, which is orthogonal to the line polar to $\mathbf{n}_2$.
3. If $\mathbf{n}_1$ and $\mathbf{n}_2$ both have positive type, they represent two complex lines L_1 and L_2. Then $\mathbf{n}_1 \boxtimes \mathbf{n}_2$ is a polar vector for their common orthogonal complex line if they a ultraparallel, and it is a lift of their intersection point if they intersect in the closure of $\mathrm{H}^2_{\mathbb{C}}$.

It is sometimes useful to note that if L_1 and L_2 are distinct, then they are asymptotic if and only if the family $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_1 \boxtimes \mathbf{n}_2)$ is linearly dependent. This can be seen easily, for instance, by considering the Gram matrix of this family for $\langle\cdot,\cdot\rangle$. We refer to Section 2.2.7 of [G1] for more details.

DEFINITION 1. For any triple of pairwise distinct points (p_1, p_2, p_3) in $\mathrm{H}^2_{\mathbb{C}}$, we define the angular invariant $\mathbb{A}(p_1, p_2, p_3) = \arg(-\langle \mathbf{p}_1, \mathbf{p}_2\rangle\langle \mathbf{p}_2, \mathbf{p}_3\rangle\langle \mathbf{p}_3, \mathbf{p}_1\rangle)$, where the vectors $\mathbf{p}_k$ are any lifts of the points p_k.

The angular invariant is a well-defined projective invariant. It can be given a geometric meaning as follows. Let Δ be (any filling of) the geodesic triangle with vertices p_1, p_2, p_3. Then the following holds (see Section 7.1.4 of [G1]):

$$\int_\Delta \omega = 2\mathbb{A}(p_1, p_2, p_3), \tag{6}$$

where ω is the Kähler form of $\mathrm{H}^2_{\mathbb{C}}$. When the three points lie on the boundary of $\mathrm{H}^2_{\mathbb{C}}$, this invariant is often referred to as the Cartan invariant. The angular invariant is well adapted to ideal triangles, in fact, it classifies ideal triangles up to the action of $\mathrm{PU}(2,1)$. In that case, its range is $[-\pi/2, \pi/2]$, and it is equal to $\pm\pi/2$ (resp. 0) if and only if the ideal triangle is contained in a complex line (resp. a real plane); see Chapter 7 of [G1].

3.2. *The Two-Dimensional Siegel Model*

We provide a few details about the two-dimensional Siegel model, as we will use it a lot when working with parabolic isometries. It is the one associated with the Hermitian form given by

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{7}$$

The complex hyperbolic plane corresponds to the domain given by $|z_2|^2 - 2\,\mathrm{Re}(z_1) < 0$ for $(z_1, z_2) \in \mathbb{C}^2$ (seen as the affine chart $\{z_3 = 1\}$ of $\mathbb{C}P^2$). Any point $m \in \mathrm{H}^2_{\mathbb{C}}$ lifts to a unique vector in $\mathbb{C}^3$ of the form

$$\mathbf{m}_{z,t,u} = \begin{bmatrix} -|z|^2 - u + it \\ z\sqrt{2} \\ 1 \end{bmatrix} \quad \text{where } z \in \mathbb{C}, t \in \mathbb{R}, \text{and } u > 0. \tag{8}$$

The triple (z, t, u) is called *horospherical coordinates* for m. In these coordinates, the boundary of $\mathrm{H}^2_{\mathbb{C}}$ is formed by the points for which $u = 0$, that is, the projections of the vectors $\mathbf{m}_{z,t,0}$, together with the point at infinity, which is the projection of $\mathbf{q}_\infty = [1\ 0\ 0]^T$. In turn, the boundary $\partial_\infty \mathrm{H}^2_{\mathbb{C}}$ is the one-point compactification of the three-dimensional Heisenberg group $\mathbb{C} \times \mathbb{R}$ with group law

$$[z, t] \cdot [w, s] = [z + w, t + s + 2\,\mathrm{Im}(z\overline{w})]. \tag{9}$$

We will see further that left Heisenberg multiplication corresponds to the action of a unipotent parabolic isometry fixing the point at infinity (see Section 3.3.2). These parabolics preserve each level set of u (which are in fact the horospheres centered at q_∞). We often call $[z, t]$ the Heisenberg coordinates of the point in the boundary of $\mathrm{H}^2_{\mathbb{C}}$ given by $\mathbf{m}_{z,t,0}$.

Note that $\langle \mathbf{m}_{z,t,u}, \mathbf{m}_{z,t,u} \rangle = -2u$; in particular, if $u < 0$, then the vector $\mathbf{m}_{z,t,u}$ is polar to a complex line. In fact, a complex line is either polar to a vector $\mathbf{m}_{z,t,u}$ for some $u < 0$ (if it does not contain q_∞) or polar to a vector of the form $[-z\sqrt{2}\ 1\ 0]$ (if it does). The latter vector is polar to the complex line connecting q_∞ to the boundary point with Heisenberg coordinates $[z, 0]$.

3.3. *Conjugacy Classes in* PU(2, 1)

We denote by $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{E}$ the spaces of loxodromic, parabolic, and elliptic conjugacy classes in PU(2, 1). We will say that an eigenvalue of a transformation $A \in \mathrm{SU}(2, 1)$ has *positive type* (resp. *null type,* resp. *negative type*) if it corresponds to a positive (resp. null, resp. negative) type eigenvector.

3.3.1. Loxodromic Classes. In the Siegel model, any loxodromic isometry is conjugate to that given by the diagonal matrix

$$L_\lambda = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \overline{\lambda}\lambda^{-1} & 0 \\ 0 & 0 & \overline{\lambda}^{-1} \end{bmatrix} \tag{10}$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ (the attracting eigenvalue of L_λ). The parameter λ is uniquely defined up to multiplication by a cube root of 1 (this corresponds to the three lifts to SU(2, 1) of an element in PU(2, 1)). Writing $\lambda = re^{-i\theta/3}$ with $r > 1$, we see that $\mathcal{L}$ is homeomorphic to the cylinder $S^1 \times \mathbb{R}^+$, where S^1 is the interval $[0, 2\pi]$ with endpoints identified; see Figure 5. The parameter θ is the *rotation angle* of L_λ; the translation length of L_λ is given by $\ell = 2\ln|\lambda|$. Note that the unit modulus eigenvalue of a loxodromic element does not determine its conjugacy class, but it determines its rotation angle, in particular, the vertical line of the cylinder $\mathcal{L}$ to which it belongs.

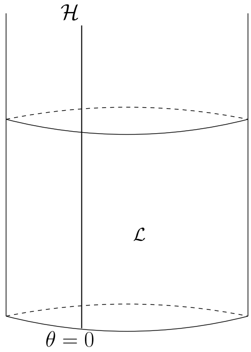


Figure 5 The space of loxodromic conjugacy classes

We call *hyperbolic* any loxodromic isometry with angle $\theta = 0$ (that is, conjugate to L_r for some $r \in (1, +\infty)$). Similarly, we will call *half-turn loxodromics* those with rotation angle $\theta = \pi$ (conjugate to L_{-r} with $r \in (1, +\infty)$). The axis of a loxodromic isometry L is contained in an S^1-family $(P_\alpha)_{\alpha\in[0,\pi)}$ of real planes on which L acts by rotation: $P_\alpha \longmapsto P_{\alpha+\theta}$, where θ is the rotation angle of L. In particular, hyperbolic (resp. half-turn loxodromic) isometries preserve each real plane containing their axis and act on it as a hyperbolic isometry (resp. glide reflection). We denote by $\mathcal{H}$ the space of hyperbolic conjugacy classes. Vertical lines in $\mathcal{L}$ are those with fixed value of θ. Hyperbolic and half-turn loxodromic classes form the vertical lines $\theta = 0$ and $\theta = \pi$. Using (10), it is easy to see that a loxodromic map is hyperbolic (resp. half-turn loxodromic) if and only if it has a lift to SU(2, 1) with real trace larger than 3 (resp. less than -1).

3.3.2. Parabolic Classes. Parabolic isometries are those whose lifts to SU(2, 1) are nondiagonalizable. A parabolic isometry is called *unipotent* if it has a unipotent lift to SU(2, 1); otherwise, it is called *screw-parabolic* (or *ellipto-parabolic;* see e.g. [CG] or [G1]). A unipotent parabolic isometry is called either *2-step* or *3-step,* according to whether the minimal polynomial of its unipotent lift is $(X-1)^2$ or $(X-1)^3$ (see Section 3.4 of [CG]). In the first case (also called *vertical Heisenberg translation*) the unipotent lift is conjugate to one of the following matrices:

$$\begin{bmatrix} 1 & 0 & \pm i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

In the second case (also called *horizontal Heisenberg translation*), the unipotent lift is conjugate to

$$\begin{bmatrix} 1 & -\sqrt{2} & -1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

The terms horizontal and vertical Heisenberg translation refer to the fact that the boundary of complex hyperbolic space can be identified with the Heisenberg group, and unipotent parabolics act on the boundary as left Heisenberg translations. We refer the reader to Chapter 4 of [G1] or to Section 2.3 of [W2]. Screw-parabolic isometries have a lift conjugate to a matrix of the form

$$\begin{bmatrix} 1 & 0 & it \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where } \theta \in [0, 2\pi). \tag{13}$$

Note that the latter matrix does not have determinant 1. The parameter θ is called the *rotation angle* of the screw-parabolic, and t its translation length. Screw-parabolic isometries preserve a complex line, on which they act as a usual parabolic isometry of the Poincaré disk, and they rotate through an angle θ around this line. In particular, we call *half-turn parabolic maps* those screw parabolic maps with rotation angle π. Parabolic isometries having a lift to $\mathrm{SU}(2,1)$ with real trace are either unipotent or half-turn parabolic. Screw parabolics and two-step unipotent parabolics preserve a complex line (in (11) and (13), it is the one polar to the second vector of the canonical basis of $\mathbb{C}^3$). On the other hand, they preserve no real plane. Likewise, three-step unipotent parabolics preserve a real plane (in the case of (12), it is the projection of $\mathbb{R}^3 \subset \mathbb{C}^3$ to $\mathrm{H}^2_{\mathbb{C}}$), but no complex line. Indeed, if a three-step unipotent map P preserved a complex line L, then it would have as an eigenvector of positive type any vector $\mathbf{n}$ polar to L. Taking a lift $\mathbf{p}$ of the fixed point of P, we see that $(\mathbf{n}, \mathbf{p})$ span a two-dimensional eigenspace with eigenvalue 1. This can only happen if P is two-step.

As explained in Section 3.2, the boundary of $\mathrm{H}^2_{\mathbb{C}}$ can be identified to the one-point compactification of the Heisenberg group. All unipotent isometries can be written under the form

$$T_{[z,t]} = \begin{bmatrix} 1 & -\bar{z}\sqrt{2} & -|z|^2 + it \\ 0 & 1 & z\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where } z \in \mathbb{C} \text{ and } t \in \mathbb{R}. \tag{14}$$

It is a direct verification to see that these matrices respect the group multiplication law given in (9):

$$T_{[z,t]} \cdot T_{[w,s]} = T_{[z,t]\cdot[w,s]}. \tag{15}$$

For that reason, unipotent parabolics are often called *Heisenberg translations.* In particular, the representatives of the unipotent conjugacy classes given in (11) and (12) are $T_{[0,\pm 1]}$ and $T_{[1,0]}$.

3.3.3. Elliptic Classes. An elliptic isometry g is called *regular* if any of its matrix representatives $A \in \mathrm{U}(n,1)$ has distinct eigenvalues. The eigenvalues of a matrix $A \in \mathrm{U}(n,1)$ representing an elliptic isometry g have modulus one. Exactly one of these eigenvalues has eigenvectors in V^- (projecting to a fixed point of g in $\mathrm{H}^n_{\mathbb{C}}$), and we call such an eigenvalue *of negative type.* Regular elliptic isometries have an isolated fixed point in $\mathrm{H}^n_{\mathbb{C}}$. A nonregular elliptic isometry is called *special.* Among the special elliptic isometries, there are the following two types (which exhaust all special elliptic types when $n = 2$):

1. A *complex reflection* is an elliptic isometry $g \in \mathrm{PU}(n,1)$ whose fixed-point set is a complex $(n-1)$-plane. In other words, any lift of such an isometry to U(n,1) has a negative-type eigenvalue of multiplicity n.
2. A *complex reflection in a point* is an elliptic isometry whose lifts have a simple eigenvalue of negative type and another eigenvalue of multiplicity n. In other words, such an isometry is conjugate to $\lambda\mathrm{Id} \in \mathrm{U}(n)$ (for some $\lambda \in \mathrm{U}(1)$), where $\mathrm{U}(n)$ is the stabilizer of the origin in the ball model. Complex reflections in a point of order 2 are also called *central involutions;* these are the symmetries that give $\mathrm{H}^n_{\mathbb{C}}$ the structure of a symmetric space.

In the ball model of $\mathrm{H}^2_{\mathbb{C}}$, any lift of an elliptic isometry g is conjugate to a diagonal matrix of the form:

$$\begin{bmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & e^{i\gamma} \end{bmatrix}, \quad \text{where } \alpha, \beta, \gamma \in [0, 2\pi). \tag{16}$$

Here the negative type eigenvalue is $e^{i\gamma}$. The two positive eigendirections correspond to a pair of (orthogonal) preserved complex lines in $\mathrm{H}^2_{\mathbb{C}}$, and the negative one to a fixed point in $\mathrm{H}^2_{\mathbb{C}}$. Projectively, the isometry g acts on its preserved lines as rotations through angles $\theta_1 = \alpha - \gamma$ and $\theta_2 = \beta - \gamma$, respectively. The conjugacy class of an elliptic isometry is determined by this (unordered) pair of angles. In particular, the eigenvalue spectrum of a lift to $\mathrm{SU}(2,1)$ of an elliptic isometry does not determine its conjugacy class. There are generically three possible angle pairs for a given triple of eigenvalues. Conversely, an elliptic conjugacy class with angle pair $\{\theta_1, \theta_2\}$ is represented by the following matrix in $\mathrm{SU}(2,1)$:

$$E_{\theta_1,\theta_2} = \begin{bmatrix} e^{i(2\theta_1-\theta_2)/3} & 0 & 0 \\ 0 & e^{i(2\theta_2-\theta_1)/3} & 0 \\ 0 & 0 & e^{-i(\theta_1+\theta_2)/3} \end{bmatrix}. \tag{17}$$

We denote by $\mathcal{E}$ the space of elliptic conjugacy classes in $\mathrm{PU}(2,1)$. From the previous discussion we may identify $\mathcal{E}$ with the quotient of $S^1 \times S^1$ under the relation $\{\theta_1, \theta_2\} \simeq \{\theta_2, \theta_1\}$ or, in other words, with

$$\Delta/\sim, \quad \text{where } \Delta = \{(\theta_1, \theta_2), 0 \le \theta_2 \le \theta_1 \le 2\pi\}, \tag{18}$$

with identifications $(0,\theta) \sim (\theta, 2\pi)$ for all θ; see Figure 6. An elliptic isometry is said to be *real elliptic with angle* θ if its angle pair is of the form $\{2\pi - \theta, \theta\}$ with $\theta \in [0, \pi]$. One of the lifts to $\mathrm{SU}(2,1)$ of such an isometry has eigenvalues $\{e^{i\theta}, e^{-i\theta}, 1\}$ (with 1 of negative type) and trace equal to $1 + 2\cos\theta \in \mathbb{R}$. The two conditions of having a lift with real trace and negative type eigenvalue equal to 1 characterize real elliptics among elliptics. Moreover, that lift is conjugate to an element of $\mathrm{SO}(2,1) \subset \mathrm{SU}(2,1)$; in particular, real elliptics preserve a real plane, on which they act by rotation through angle θ.

REMARK 2. There are two conjugacy classes of involutions in $\mathrm{PU}(2,1)$:

1. *Central involutions* (or complex reflections in a point of order 2) are the isometries conjugate to $(z_1, z_2) \longmapsto (-z_1, -z_2)$ in the ball model. They have angle

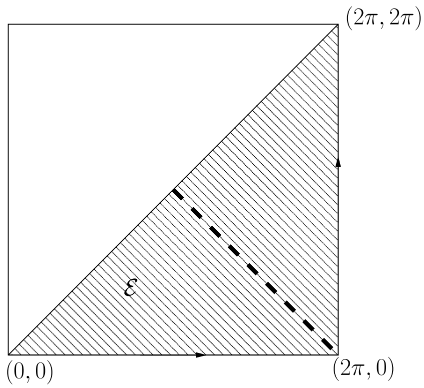


Figure 6 The space of elliptic conjugacy classes. Arrows on the edges of the triangles indicate identifications. The dashed segment represents angle pairs of real elliptics

pair $\{\pi, \pi\}$, that is, are real elliptics with angle π. Central involutions have an isolated fixed point in $\mathrm{H}^2_{\mathbb{C}}$ and preserve every complex line through that fixed point, acting on it as a half-turn.

2. *Complex symmetries* (or complex reflections of order 2) are the isometries conjugate to $(z_1, z_2) \longmapsto (z_1, -z_2)$ in the ball model. They have angle pair $\{\pi, 0\}$. Complex symmetries fix pointwise a unique complex line in $\mathrm{H}^2_{\mathbb{C}}$, called their *mirror*. They preserve every complex line orthogonal to the mirror, acting on it as a half-turn.

Both types of involutions can be lifted to SU(2, 1) as follows. Let n be a point in $\mathbb{C}P^2 \setminus \partial_\infty \mathrm{H}^2_{\mathbb{C}}$. Let $\mathbf{n}$ be a lift of n such that $\langle \mathbf{n}, \mathbf{n} \rangle = 2\epsilon$ with $\epsilon \in \{-1, 1\}$. If $\epsilon = -1$ (resp. 1), then n is a point of $\mathrm{H}^2_{\mathbb{C}}$ (resp. is polar to a complex line in $\mathrm{H}^2_{\mathbb{C}}$). Consider the linear involution of $\mathbb{C}^{2,1}$ defined by

$$I_n(Z) = -Z + \epsilon \langle Z, \mathbf{n} \rangle \mathbf{n}, \quad \text{for } Z \in \mathbb{C}^{2,1}. \tag{19}$$

The involution I_n acts on $\mathrm{H}^2_{\mathbb{C}}$ as the central involution fixing the point $n \in \mathrm{H}^2_{\mathbb{C}}$ when $\epsilon = -1$ and as the complex symmetry across $\mathbf{n}^\perp$ when $\epsilon = 1$.

Definition 2. Let $I \in$ PU(2, 1) be an involution. We call *the standard lift* to SU(2, 1) of I the one given by (19), and we say that the vector $\mathbf{n}$ *represents* I.

Note that given an involution I in PU(2, 1), its standard lift is the unique lift of I that is also an involution. We will often identify a holomorphic involution with its standard lift.

3.3.4. The Space of Conjugacy Classes. We are interested in the space $\mathcal{G}$ of conjugacy classes of the group $G =$ PU(2, 1) (see Section 3.1 for basic definitions). As a topological space (with the quotient topology), this space is not Hausdorff; more specifically, the conjugacy class of complex reflections with a given

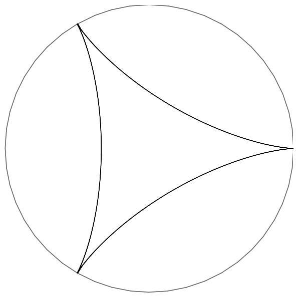


Figure 7 The null-locus of the polynomial f inscribed in the circle of radius 3 centered at the origin

(nonzero) rotation angle has the same neighborhoods as the screw-parabolic class with the same angle, and likewise, the identity and the three unipotent classes all share the same neighborhoods. For most of our purposes, it will be sufficient to consider the set $\mathcal{G}^{\text{reg}}$ of *regular semisimple* classes, that is, those classes of elements whose lifts are semisimple with distinct eigenvalues (so, loxodromic or regular elliptic). However, it will also be useful to consider as in [FW2] the maximal Hausdorff quotient $c(\mathcal{G})$ of the full space of conjugacy classes in G.

Concretely, $c(\mathcal{G})$ consists of the open dense set $\mathcal{G}^{\text{reg}}$, together with the set $\mathcal{B}$ of equivalence classes of complex reflections and screw-parabolics, as well as the identity and unipotents, which are identified in the quotient; we call such classes *boundary classes.* We denote by $\mathcal{L}$ (respectively $\mathcal{E}$, $\mathcal{E}^{\text{reg}}$) the subsets of $\mathcal{G}$ consisting of loxodromic (resp. elliptic, resp. regular elliptic) elements of G; the conjugacy class of an element $A \in G$ is denoted $[A]$. The global topology of $\mathcal{G}$ can be described as follows: $\mathcal{E}$ is closed (in fact, compact), $\mathcal{L}$ and $\overset{\circ}{\mathcal{E}} = \mathcal{E} \setminus \mathcal{B}$ are open and disjoint, and $\mathcal{E} \cap \overline{\mathcal{L}} = \mathcal{B}$. Note that $\mathcal{L}$ and $\overset{\circ}{\mathcal{E}}$ have natural smooth structures (which were used in [FW1; FW2], and [P]), whereas boundary classes are singular points of $\mathcal{G}$ since they have arbitrarily small neighborhoods homeomorphic to three half-disks glued along a common diameter (two of them in $\mathcal{E}$, and one in $\mathcal{L}$).

As in the classical case of the Poincaré disk, the isometry type of an isometry is closely related to the trace of a lift to SU(2, 1). The following proposition can be found in Chapter 7 of [G1]; see Figure 7.

PROPOSITION 3 (Goldman). *Let the function f be defined by*

$$f(z) = |z|^4 - 8\,\text{Re}(z^3) + 18|z|^2 - 27, \quad z \in \mathbb{C}. \tag{20}$$

Then, for any isometry $g \in \text{PU}(2,1)$ with lift $A \in \text{SU}(2,1)$:

- *g is regular elliptic* $\iff f(\text{tr}(A)) < 0$.
- *g is loxodromic* $\iff f(\text{tr}(A)) > 0$.
- *g is special elliptic or screw-parabolic* $\iff f(\text{tr}(A)) = 0$ *and* $\text{tr}(A) \notin 3C_3$, *where C_3 is the set of cube roots of* 1.
- *g is unipotent or the identity* $\iff \text{tr}(A) \in 3C_3$.

Combining the latter proposition with Section 3.3.1 gives the following:

REMARK 3. An element A in SU(2, 1) represents a hyperbolic isometry if and only if $\mathrm{tr}(A) = \omega x$, where $x \in (3, +\infty)$, and ω is a cube root of unity. An element A in SU(2, 1) represents a half-turn loxodromic isometry if and only if $\mathrm{tr}(A) = -\omega x$, where $x \in (-\infty, -1)$, and ω is a cube root of unity.

3.4. Double Products of Involutions

There is one conjugacy class of antiholomorphic involutions in $\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}})$, *real reflections* that fix pointwise an embedded copy $\mathrm{H}^n_{\mathbb{R}} \subset \mathrm{H}^n_{\mathbb{C}}$. This follows directly from the description of real totally geodesic subspaces given in Section 3.1.9 of [G1]. The standard example is the map $Z \longmapsto \overline{Z}$ in the unit ball of dimension n. It is well known that any holomorphic isometry of $\mathrm{H}^n_{\mathbb{C}}$ is a product of two real reflections (see Falbel and Zocca [FZ] in dimension two and Choi [C] in higher dimensions). In contrast, only very few elements of PU(2, 1) are products of two holomorphic involutions.

PROPOSITION 4. *Let* $g \neq \mathrm{Id}$ *be a holomorphic isometry of* $\mathrm{H}^2_{\mathbb{C}}$. *Then,*

1. *g is a product of two central involutions if and only if it is hyperbolic.*
2. *g is a product of a complex symmetry and a central involution if and only if it is half-turn loxodromic or a complex symmetry.*
3. *g is a product of two complex symmetries if and only if it is hyperbolic, three-step unipotent or real elliptic.*

Proof. Let I_1 and I_2 be two involutions, lifted as in (19) to the linear maps $I_k(Z) = -Z + \epsilon_k\langle Z, \mathbf{n}_k\rangle\mathbf{n}_k$ with $k = 1, 2$ and ϵ_k in $\{-1, 1\}$. As in Lemma 1, we denote by n_k the projection to $\mathrm{H}^2_{\mathbb{C}}$ of $\mathbf{n}_k$ when it is negative and by L_k its polar complex line if it is positive. Then

$$I_1 I_2(Z) = Z - \epsilon_1\langle Z, \mathbf{n}_1\rangle\mathbf{n}_1 - \epsilon_2\langle Z, \mathbf{n}_2\rangle\mathbf{n}_2 + \epsilon_1\epsilon_2\langle Z, \mathbf{n}_2\rangle\langle\mathbf{n}_2, \mathbf{n}_1\rangle\mathbf{n}_1. \tag{21}$$

As we have seen in Remark 1, $(\mathbf{n}_1, \mathbf{n}_1 \boxtimes \mathbf{n}_2, \mathbf{n}_2)$ is a basis of $\mathbb{C}^{2,1}$ unless $\mathbf{n}_1$ and $\mathbf{n}_2$ are both positive and represent asymptotic lines. If this is not the case, then we can write the matrix of $I_1 I_2$ in this basis. It is given by

$$I_1 I_2 = \begin{bmatrix} -1 + \epsilon_1\epsilon_2|\langle\mathbf{n}_1, \mathbf{n}_2\rangle|^2 & 0 & \epsilon_1\langle\mathbf{n}_2, \mathbf{n}_1\rangle \\ 0 & 1 & 0 \\ -\epsilon_2\langle\mathbf{n}_1, \mathbf{n}_2\rangle & 0 & -1 \end{bmatrix}. \tag{22}$$

This directly gives

$$\mathrm{tr}(I_1 I_2) = -1 + 4\kappa, \tag{23}$$

where κ was defined by (4). Note that this expression remains valid when the two lines are asymptotic (in which case the above triple of vectors is no longer a basis; see Remark 1). Hence the product of two involutions in SU(2, 1) always has real trace (up to multiplication by a cube root of 1). From Sections 3.3.1 to 3.3.3, such a product can only be hyperbolic, half-turn loxodromic, unipotent, half-turn parabolic, or real elliptic. There are three different cases, depending on the respective types of $\mathbf{n}_1$ and $\mathbf{n}_2$. The results are obtained directly from Lemma 1.

1. If $\epsilon_1 = \epsilon_2 = -1$, then I_1 and I_2 are central involutions, and κ can take any value in $(1, +\infty)$. In turn, $I_1 I_2$ is hyperbolic, and using Proposition 4, we see that $\kappa = \cosh^2(\ell)$, where ℓ is the translation length of $I_1 I_2$. We can thus obtain all hyperbolic classes in this way.
2. If $\epsilon_1 = -\epsilon_2 = 1$, then I_1 is a complex symmetry, and I_2 is a central involution. In this case, κ can take any value in $(-\infty, 0]$. For negative values of κ, we obtain all possible half-turn loxodromic isometries. If $\kappa = 0$, then n_2 belongs to the mirror of I_1, and $I_1 I_2$ is the complex symmetry about the line orthogonal to L_1 through n_2.
3. If $\epsilon_1 = \epsilon_2 = 1$, then the cases where $\kappa > 1$ give all possible hyperbolic classes. In case $0 \le \kappa < 1$, the vector $\mathbf{n}$ orthogonal to $\mathbf{n}_1$ and $\mathbf{n}_2$ has negative type and is an eigenvector with eigenvalue 1 of $I_1 I_2$ (this follows directly from (21)). Therefore, the eigenvalue spectrum of $I_1 I_2$ is $\{1, e^{i\alpha}, e^{-i\alpha}\}$, where $\kappa = \cos^2 \alpha/2$ (see Lemma 1). In particular, $I_1 I_2$ is real elliptic with rotation angle α. Finally, if $\kappa = 1$, then the two complex lines L_1 and L_2 are asymptotic, and the product $I_1 I_2$ is parabolic. To verify that $I_1 I_2$ is three-step unipotent, pick a vector $\mathbf{n}$ such that $\langle \mathbf{n}, \mathbf{n}_2 \rangle = 0$ and $\langle \mathbf{n}, \mathbf{n}_1 \rangle \neq 0$ ($\mathbf{n}$ corresponds to a point in L_2 but not in L_1). The triple $(\mathbf{n}_1, \mathbf{n}, \mathbf{n}_2)$ is a basis of $\mathbb{C}^3$, and the matrix of $I_1 I_2$ in this basis is equal to
$$M = \begin{bmatrix} 3 & -\langle \mathbf{n}, \mathbf{n}_1 \rangle & \langle \mathbf{n}_2, \mathbf{n}_1 \rangle \\ 0 & 1 & 0 \\ -\langle \mathbf{n}_1, \mathbf{n}_2 \rangle & 0 & -1 \end{bmatrix}.$$
A straightforward verification using $|\langle \mathbf{n}_1, \mathbf{n}_2 \rangle|^2 = 4$ shows that $(M - id)^2$ has rank one. $\square$

REMARK 4. Proposition 4, combined with the description of conjugacy classes given in Section 3.3, shows in particular that for products of two involutions of the same type, the eigenvalue associated with any fixed point in the closure of $\mathrm{H}^2_{\mathbb{C}}$ is real and positive (up to multiplication by a cube root of 1).

From Proposition 4 we see that generic holomorphic isometries are not products of two holomorphic involutions or, in other words, that $\mathrm{PU}(2, 1)$ has involution length at least 3.

4. Triples of Involutions

4.1. General Facts

With any involution $I \in \mathrm{PU}(2, 1)$, we associate a sign : $+1$ if it is a complex symmetry, or -1 if it is a central involution (this is the sign of $\langle \mathbf{n}, \mathbf{n} \rangle$ for $\mathbf{n}$ as in (19)). With any triple of involutions (I_1, I_2, I_3), there is associated a triple of signs $(\epsilon_1, \epsilon_2, \epsilon_3)$. We will often shorten this notation by omitting the 1 and only keeping the signs, for example, $(+, +, -)$ will stand for $(1, 1, -1)$. In view of this, we make the following definition.

DEFINITION 3. We say that an isometry $A \in \mathrm{PU}(2, 1)$ is a triple product of type $(\epsilon_1, \epsilon_2, \epsilon_3)$ if $A = I_1 I_2 I_3$ where each I_k is an involution with sign ϵ_k.

To determine which isometries are products of three involutions, we begin by reducing the number of types that need to be considered.

LEMMA 2. *Assume that* $A \in \mathrm{PU}(2, 1)$ *is a triple product of type* $(\epsilon_1, \epsilon_2, \epsilon_3)$. *Then it is also a triple product of type* $(\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}, \epsilon_{\sigma(3)})$ *for any permutation* $\sigma \in \mathfrak{S}_3$.

Proof. To verify this, is suffices to note that conjugating by I_1 amounts to applying the 3-cycle $(1, 2, 3)$ and that two neighboring signs ϵ and ϵ' can always be exchanged. For example, if I_1 is a complex symmetry and I_2 a central involution, then $I_1 I_2 I_1$ is conjugate to I_2 and is thus a central involution, denoted I_2'. We then have $I_1 I_2 = I_2' I_1$ and $I_1 I_2 I_3 = I_2' I_1 I_3$. □

As a consequence of Lemma 2, we observe that studying the four triple types $(+, +, +)$, $(+, +, -)$, $(+, -, -)$, and $(-, -, -)$ is sufficient to determine which elements in $\mathrm{PU}(2, 1)$ are products of three involutions or not. The following observation will be useful to us.

COROLLARY 1. *Let* $A \in \mathrm{PU}(2, 1)$.

1. *If A is a triple product of type* $(+, -, -)$ *and* $(-, -, -)$, *then it is a triple product of any type*.
2. *If A cannot be written as a* $(+, +, +)$ *triple product nor as a* $(+, +, -)$ *triple product, then it is not a product of three holomorphic involutions*.

Proof. This follows directly from Proposition 4 observing that a product of two central involutions is also a product of two complex symmetries. □

Given a triple of involutions (I_1, I_2, I_3) and a fixed point p_2 of the product $I_1 I_2 I_3$ in $\mathrm{H}^2_{\mathbb{C}} \cup \partial_\infty \mathrm{H}^2_{\mathbb{C}}$, we construct a triangle by setting $p_3 = I_1 p_2$ and $p_1 = I_3 p_2$. Then each I_k exchanges p_{k-1} and p_{k+1} (with indices taken mod. 3). If $I_1 I_2 I_3$ is loxodromic or parabolic, then its fixed points are contained in $\partial_\infty \mathrm{H}^2_{\mathbb{C}}$, and this construction produces an ideal triangle. Similarly, if $I_1 I_2 I_3$ has an isolated fixed point inside $\mathrm{H}^2_{\mathbb{C}}$, that is, if it is either regular elliptic or a complex reflection in a point, then the triangle obtained in this way is compact. In case $I_1 I_2 I_3$ is a complex reflection, we may obtain compact or ideal triangles since $I_1 I_2 I_3$ then has fixed points both inside $\mathrm{H}^2_{\mathbb{C}}$ and on $\partial_\infty \mathrm{H}^2_{\mathbb{C}}$.

DEFINITION 4. For any fixed point $p_2 \in \overline{\mathrm{H}^2_{\mathbb{C}}}$ of $I_1 I_2 I_3$, we call the above triangle (p_1, p_2, p_3) the *cycle-triangle* of p_2 relative to the triple of involutions (I_1, I_2, I_3).

Whenever the involutions will be clear from the context, we will shorten the above terminology to the cycle-triangle of p_2. In the proof of Proposition 1, we have considered cycle-triangles in the setting of the Poincaré disk. We refer the reader to Figures 1 and 2 for geometric insight. As we will see, the shape of the cycle-triangle of $I_1I_2I_3$, that is, its angular invariant is closely related to its eigenvalue associated with p_2. Indeed:

PROPOSITION 5. *Let (I_1, I_2, I_3) be a triple of involutions, and (p_1, p_2, p_3) the cycle-triangle associated with the fixed point $p_2 \in \mathrm{H}^2_{\mathbb{C}} \cup \partial_\infty \mathrm{H}^2_{\mathbb{C}}$ of $I_1I_2I_3$. Identifying I_k to its standard lift, let λ be the eigenvalue of $I_1I_2I_3$ associated with p_2. Then* $\arg(\lambda) = -\mathbb{A}(p_1, p_2, p_3) \mod \pi$.

We omit the proof of Proposition 5 at this stage since it follows directly from the proof of Proposition 6 and from Proposition 7.

4.2. Loxodromic Triple Products

With the geometric information provided by cycle-triangles, we obtain the following:

PROPOSITION 6. *Any loxodromic isometry is a triple product of any type.*

Proof. Let $\Delta = (p_1, p_2, p_3)$ be a nondegenerate ideal triangle (meaning that $p_i \neq p_j$ for $i \neq j$), $\mathbb{A}$ be its Cartan invariant (see Definition 1), and σ_i be the (geodesic) edge of Δ connecting p_{i-1} to p_{i+1} (with indices taken $\mod 3$). We choose lifts $\mathbf{p}_i$ of each p_i such that

$$\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = \langle \mathbf{p}_2, \mathbf{p}_3 \rangle = -1 \quad \text{and} \quad \langle \mathbf{p}_3, \mathbf{p}_1 \rangle = -e^{i\mathbb{A}}. \tag{24}$$

The geodesics σ_j are parameterized by arc length as follows (for $t \in \mathbb{R}$):

$$\begin{aligned} &\sigma_1(t) = e^{t/2}\mathbf{p}_2 + e^{-t/2}\mathbf{p}_3, \qquad \sigma_2(t) = e^{t/2}\mathbf{p}_3 + e^{-t/2}e^{i\mathbb{A}}\mathbf{p}_1, \\ &\sigma_3(t) = e^{t/2}\mathbf{p}_1 + e^{-t/2}\mathbf{p}_2. \end{aligned} \tag{25}$$

Note that t is in fact the distance between $\sigma_i(t)$ and the orthogonal projection of p_i onto σ_i (see Figure 8).

For $k = 1, 2, 3$ and any triple $(t_1, t_2, t_3) \in \mathbb{R}^3$, we define the vectors

$$\begin{aligned} &\mathbf{n}_1 = e^{t_1/2}\mathbf{p}_2 - \epsilon_1 e^{-t_1/2}\mathbf{p}_3, \qquad \mathbf{n}_2 = e^{t_2/2}\mathbf{p}_3 - \epsilon_2 e^{-t_2/2}e^{i\mathbb{A}}\mathbf{p}_1, \\ &\mathbf{n}_3 = e^{t_3/2}\mathbf{p}_1 - \epsilon_3 e^{-t_3/2}\mathbf{p}_2. \end{aligned} \tag{26}$$

A direct computation shows that $\langle \mathbf{n}_k, \mathbf{n}_k \rangle = 2\epsilon_k$. We see thus that when $\epsilon_k = -1$, $\mathbf{n}_k$ represents the point $\sigma_k(t_k)$ on σ_k, and when $\epsilon_k = 1$, it is polar to the complex line through $\sigma_k(t_k)$ orthogonal to the complex line spanned by p_{k-1} and p_{k+1}. Denoting by I_k the involution associated to $\mathbf{n}_k$ we see that I_k exchanges the endpoints of σ_k and fixes $\sigma_k(t_k)$.

Using these expressions, it is straightforward to verify that $I_3(\mathbf{p}_2) = -\epsilon_3 e^{t_3}\mathbf{p}_1$, $I_2(\mathbf{p}_1) = -\epsilon_2 e^{t_2 - i\alpha}\mathbf{p}_3$, and $I_1(\mathbf{p}_3) = -\epsilon_1 e^{t_1}\mathbf{p}_2$, which means that Δ is the cycle-triangle for p_2. As a consequence, we directly obtain

$$I_1I_2I_3\mathbf{p}_2 = -\epsilon_1\epsilon_2\epsilon_3 e^{t_1+t_2+t_3-i\mathbb{A}}\mathbf{p}_2.$$

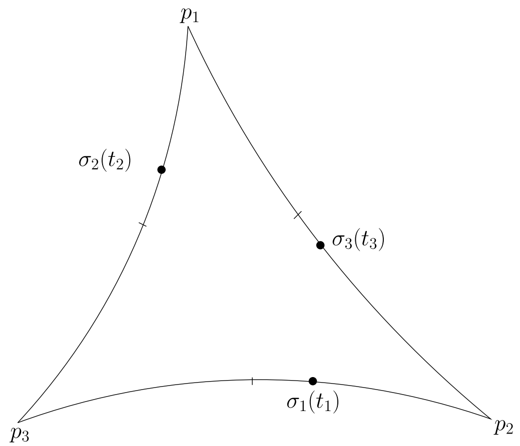


Figure 8 The ideal triangle associated with a fixed point on the boundary of $I_1 I_2 I_3$

To conclude the proof, let λ be a complex number with modulus $|\lambda| > 1$, thought of as the (attracting) eigenvalue of a loxodromic element. First, it is always possible to find three real numbers t_1, t_2, and t_3 such that $|\lambda| = e^{t_1+t_2+t_3}$. Having fixed such values of the t_i, it is possible to find a value of $\mathbb{A} \in [-\pi/2, \pi/2]$ such that $-\epsilon_1\epsilon_2\epsilon_3 e^{t_1+t_2+t_3-i\mathbb{A}}$ is equal to λ up to multiplication by a cube root of unity. In view of Section 3.3, this means that the triple product $I_1 I_2 I_3$ can belong to any loxodromic conjugacy class. $\square$

4.3. Elliptic Triple Products

The above point of view does not give a simple answer for elliptic triple products, as it does for loxodromic ones. However, it is still possible to compute the eigenvalue of $I_1 I_2 I_3$ associated with a fixed point in $\mathrm{H}^2_{\mathbb{C}}$ in terms of the corresponding cycle-triangle.

PROPOSITION 7. *Let (I_1, I_2, I_3) be a triple of involutions such that $I_1 I_2 I_3$ is elliptic with fixed point p_2 in $\mathrm{H}^2_{\mathbb{C}}$, and let (p_1, p_2, p_3) be the associated cycle-triangle in $\mathrm{H}^2_{\mathbb{C}}$. Then the negative type eigenvalue of $I_1 I_2 I_3$ is equal to $-\epsilon_1\epsilon_2\epsilon_3 e^{-i\mathbb{A}}$, where $\mathbb{A}$ is the angular invariant of (p_1, p_2, p_3).*

Proof. Let p and q be any two points in $\mathrm{H}^2_{\mathbb{C}}$. Then there exists a unique involution in $\mathrm{PU}(2,1)$ of type ϵ, which exchanges p and q. If $\epsilon = -1$, then it is the central involution about the midpoint m of $[p, q]$. If $\epsilon = 1$, then it is the complex symmetry fixing the complex line through m orthogonal to the complex line spanned by p and q. Applying this fact to the three sides of the triangle (p_1, p_2, p_3), we see that for each given type $(\epsilon_1, \epsilon_2, \epsilon_3)$, there exists a unique triple of involutions (I_1, I_2, I_3) of type $(\epsilon_1, \epsilon_2, \epsilon_3)$ for which (p_1, p_2, p_3) is the cycle-triangle.

Now, we choose lifts $(\mathbf{p}_i)_i$ of the points p_i so that

$$\langle \mathbf{p}_i, \mathbf{p}_i \rangle = -2, \qquad \langle \mathbf{p}_1, \mathbf{p}_2 \rangle = r_{12}, \qquad \langle \mathbf{p}_2, \mathbf{p}_3 \rangle = r_{23} \quad \text{and}$$
$$\langle \mathbf{p}_3, \mathbf{p}_1 \rangle = -r_{31} e^{i\mathbb{A}},$$

where $r_{ij} \in (2, +\infty)$, and $\mathbb{A}$ is the angular invariant of (p_1, p_2, p_3). The three involutions (I_1, I_2, I_3) are those respectively associated with the vectors

$$\mathbf{n}_1 = \frac{1}{\sqrt{2}}(\mathbf{p}_2 + \epsilon_1 \mathbf{p}_3), \qquad \mathbf{n}_2 = \frac{1}{\sqrt{2}}(\mathbf{p}_3 - e^{i\alpha}\epsilon_2 \mathbf{p}_1), \quad \text{and}$$
$$\mathbf{n}_3 = \frac{1}{\sqrt{2}}(\mathbf{p}_1 + \epsilon_3 \mathbf{p}_2).$$

Using

$$I_k(Z) = -Z + 2\frac{\langle Z, \mathbf{n}_k \rangle}{\langle \mathbf{n}_k, \mathbf{n}_k \rangle}\mathbf{n}_k,$$

by a direct computation in the spirit of the proof of Proposition 6 we verify that

$$I_1 I_2 I_3(\mathbf{p}_2) = -\epsilon_1\epsilon_2\epsilon_3 e^{-i\mathbb{A}}\mathbf{p}_2. \qquad \square$$

REMARK 5. Proposition 7 provides an expression of the negative type eigenvalue of an elliptic triple product in geometric terms. To use this information to completely determine the conjugacy class of this triple product, we need to determine its positive type eigenvalues (see Section 3.3.3 for details). This can be done in principle as follows. The characteristic polynomial χ of $I_1 I_2 I_3$ is equal to

$$\chi(X) = X^3 - \mathrm{tr}(I_1 I_2 I_3) \cdot X^2 + \overline{\mathrm{tr}(I_1 I_2 I_3)} \cdot X - 1. \tag{27}$$

(See Section 6.2.3 of [G1], more precisely the proof of Lemma 6.2.5.) Identifying the involutions I_k to their standard lifts, it is easy to compute the trace of $I_1 I_2 I_3$ as in the proof of Proposition 4. Knowing the characteristic polynomial and the negative type eigenvalue, we can obtain a factorization of χ of the form

$$\chi(X) = (X + \epsilon_1\epsilon_2\epsilon_3 e^{-i\alpha})(X^2 + aX - \epsilon_1\epsilon_2\epsilon_3 e^{i\alpha}), \tag{28}$$

where the coefficient a can be expressed by comparing (27) and (28). Solving the second factor of (28) provides expressions of the positive type eigenvalues of $I_1 I_2 I_3$. However, the complexity of these expressions makes it unrealistic to determine in this way which elliptic classes are indeed triple products of involutions. For this reason, we need to apply an indirect method in the next section.

The following corollary shows that not all elliptic elements can be $(-,-,-)$ triple products: among complex reflections about points, only central involutions are $(-,-,-)$ triple products. Of course, complex reflections about points could a priori be triple products of other types, but it turns out that not all of them are, as we will see in Section 6. Similar statements can be derived for other involution types, but their statements and proofs are not as simple as this one. For that reason, we omit them. Corollary 2 should be taken as a simple hint that not all elliptic elements are products of three involutions.

COROLLARY 2. *Let (I_1, I_2, I_3) be a $(-,-,-)$ triple of involutions such that $I_1 I_2 I_3$ is a reflection about a point. Then $I_1 I_2 I_3$ is a central involution.*

Proof. We denote by n_k the fixed point of I_k. Then $A = I_2 I_3$ is hyperbolic, and n_2 and n_3 lie on its axis. Since I_1 is a central involution and $I_1 A$ is a complex reflection about a point, the group $\langle I_1, A\rangle = \langle I_1, I_1 A\rangle$ preserves the complex line L containing the fixed points of I_1 and $I_1 A$ (compare with Lemma 3 in Section 5). In particular, L is the complex axis of A, and the three points n_1, n_2, and n_3 are contained in it. Let $\mathbf{n}$ be a vector polar to L.

Now, let p_2 be the fixed point of $I_1 A = I_1 I_2 I_3$, and let $T = (p_1, p_2, p_3)$ be the cycle-triangle associated with p_2, with as usual $\mathbf{p}_k$ a lift of p_k. The points n_k are the midpoints of the sides of the triangle T, and the involutions I_k can be expressed in terms of the vectors $\mathbf{p}_k$ as in the proof of Proposition 7 (with $\epsilon_k = -1$ for all k), so we keep the same notation.

By Proposition 7 the triple product $I_1 I_2 I_3$ fixes p_2 with associated eigenvalue $e^{-i\alpha}$. Since L contains the triangle T, $\mathbf{n}$ satisfies $\langle \mathbf{n}, \mathbf{p}_k\rangle = 0$ and $\langle \mathbf{n}, \mathbf{n}_k\rangle = 0$ for all k. In particular, using standard lifts for the involutions I_k, we see that $I_k(\mathbf{n}) = -\mathbf{n}$ for all k, and thus $I_1 I_2 I_3(\mathbf{n}) = -\mathbf{n}$. Now, let $\mathbf{v}_k$ be the vector $\mathbf{p}_k \boxtimes \mathbf{n}$ (where $\mathbf{n}_k$ is a lift of n_k) for $k = 1, 2, 3$. The vector $\mathbf{v}_k$ is polar to the complex line orthogonal to L through the point n_k. Computing the Hermitian cross-product, we observe that

$$\begin{aligned} I_1 I_2 I_3(\mathbf{v}_2) &= I_1 I_2 I_3(\mathbf{p}_2) \boxtimes I_1 I_2 I_3(\mathbf{n}) \\ &= (e^{-i\alpha}\mathbf{p}_2) \boxtimes (-n) \\ &= -e^{i\alpha}\mathbf{p}_2 \boxtimes \mathbf{n} = -e^{i\alpha}\mathbf{v}_2. \end{aligned}$$

Therefore, in the basis $(\mathbf{n}, \mathbf{v}_2, \mathbf{p}_2)$ the triple product is given by

$$I_1 I_2 I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -e^{i\alpha} & 0 \\ 0 & 0 & e^{-i\alpha} \end{bmatrix}.$$

The angle pair of $I_1 I_2 I_3$ is $\{\pi + \alpha, \pi + 2\alpha\}$. This can only be the angle pair of a complex reflection about a point when $\alpha = 0 \mod 2\pi$, in which case $I_1 I_2 I_3$ is a central involution. □

5. Conjugacy Classes and Products of Isometries

To analyze products of three holomorphic involutions $I_1 I_2 I_3$, we will view them as products of two isometries, one of which is a product of two involutions. As we have seen in Section 3.4, being a product of two holomorphic involutions gives restrictions on the conjugacy class. We are therefore led to study the following product map.

5.1. The Product Map

As in [FW2] and [P], we consider the following question: given two conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$ in $G = \mathrm{PU}(2,1)$, what are the possible conjugacy classes for the product AB as A varies in $\mathcal{C}_1$ and B varies in $\mathcal{C}_2$? More specifically, given two semisimple conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$, the problem is to determine the image of the map

$$\begin{aligned} \tilde{\mu} : \mathcal{C}_1 \times \mathcal{C}_2 &\longrightarrow \mathcal{G} \\ (A, B) &\longmapsto [AB], \end{aligned} \tag{29}$$

where $\mathcal{G}$ is the set of conjugacy classes in PU(2, 1), and $[\cdot]$ denotes the conjugacy class of an element. When studying this question, *reducible pairs* play a crucial role.

DEFINITION 5. We say that a subgroup $\Gamma < \mathrm{PU}(2,1)$ is *reducible* if it fixes a point in $\mathbb{C}P^2$ (so, either all elements of Γ have a common fixed point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$, or they all preserve a common complex line) and *irreducible* otherwise. Likewise, we say that a pair $(A, B) \in \mathrm{PU}(2,1)^2$ is reducible (resp. irreducible) if it generates a reducible (resp. irreducible) group.

The strategy used in [FW2] and [P] consists of the following four parts:

1. Prove that $\mathrm{Im}\, \tilde{\mu}$ is closed;
2. Prove that images of irreducible pairs are interior points of $\mathrm{Im}\, \tilde{\mu}$;
3. Determine the set $W_{\mathrm{red}} = \{[AB] \mid (A, B) \in \mathcal{C}_1 \times \mathcal{C}_2 \text{ reducible}\}$ of *reducible walls;*
4. Determine which *chambers,* that is, connected components of $\mathcal{G} \setminus W_{\mathrm{red}}$, are in the image by parts 1 and 2, $\mathrm{Im}\, \tilde{\mu}$ is a union of chamber closures.

Parts 1 and 2 follow respectively from Sections 5.2 and 5.3. They imply the following crucial fact (see Section 2.5 of [P]), which justifies part 4.

THEOREM 3. *Any chamber of $\mathcal{G} \setminus W_{\mathrm{red}}$ is either full or empty.*

Note that parts 1 and 2 are valid for any choice of semisimple conjugacy classes. In contrast, parts 3 and 4 require a case-by-case analysis.

REMARK 6. 1. In the cases we consider, we will observe that the intersection of the reducible walls with $\mathcal{E}$ (seen as in (18)) consists of a finite collection of linear segments that have slope -1, 2, or $1/2$. We refer to [P] for a general proof of this fact. In particular, this shows that the diagonal segment $\{(\theta, \theta), \theta \in [0, 2\pi)\}$ cannot contain any reducible walls.
2. A useful consequence of Theorem 3 is the following fact. Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ be three conjugacy classes, with $\mathcal{C}_1$, $\mathcal{C}_2$ semisimple. Assume that there exist two pairs (A_1, B_1) and (A_2, B_2) such that $A_i B_i \in \mathcal{C}_3$ for $i = 1, 2$ with (A_1, B_1) reducible and (A_2, B_2) irreducible. Since (A_1, B_1) is reducible, $\mathcal{C}_3$ corresponds to a point on a reducible wall. Since (A_2, B_2) is irreducible, $\mathcal{C}_3$ is interior to the image of the product map. This implies that all chambers having the point $\mathcal{C}_3$ in their closure are full.

We start with two general observations about reducible and irreducible pairs. Recall that a *special elliptic* isometry in PU(2, 1) is one whose lifts have repeated eigenvalues; geometrically, this means that its angle pair has the form $\{\theta, 0\}$ (in which case it is a complex reflection about a line) or $\{\theta, \theta\}$ (in which case it is a complex reflection in a point).

LEMMA 3. *Let* $A, B \in \mathrm{PU}(2,1)$. *If* A *and* B (*resp.* A *and* AB) *are both special elliptic, then the group* $\langle A, B\rangle$ *is reducible.*

Proof. Complex reflections about lines preserve all complex lines perpendicular to their mirror, and complex reflections about points preserve all complex lines containing their isolated fixed point. In all cases, either A and B have a fixed point in common in $\overline{\mathrm{H}^2_{\mathbb{C}}}$, or they preserve a common complex line. □

Lemma 3 is very useful in determining which special elliptic elements are attained as products. It was used in the following form in the proof of Proposition 4.1 of [P]:

COROLLARY 3. *If one of* $\mathcal{C}_1$ *or* $\mathcal{C}_2$ *is a conjugacy class of special elliptic elements, then any chamber of* $\mathrm{Im}\,\tilde{\mu} \cap \mathcal{E}$ *containing an open subinterval of the diagonal in its closure is empty.*

Proof. Assume that we have a chamber C that is full. Then if A is special elliptic and the angle pair of AB lies on the diagonal, AB is special elliptic, and by Lemma 3 the pair (A, AB) is reducible. Since the pair (A, AB) generates the group $\langle A, B\rangle$, this implies that (A, B) is also reducible. This means that the diagonal interval lying in the closure of C is (part of) a reducible wall. This contradicts Remark 6. □

The second observation is that any pair can be deformed to an irreducible pair, unless prohibited by Lemma 3:

PROPOSITION 8. *Let* $\mathcal{C}_1, \mathcal{C}_2$ *be two semisimple conjugacy classes in* $\mathrm{PU}(2,1) \setminus \{\mathrm{Id}\}$, *at least one of which is regular semisimple. Then irreducible pairs form an open dense subset of* $\mathcal{C}_1 \times \mathcal{C}_2$.

Proof. Recall that a pair $(A, B) \in \mathrm{PU}(2,1)^2$ is reducible if (any lifts of) A and B have a common eigenvector in $\mathbb{C}^3$. Therefore reducible pairs form a closed subset of $\mathrm{PU}(2,1)^2$, and irreducible pairs form an open subset.

Now if say $\mathcal{C}_1$ is regular semisimple and $\mathcal{C}_2$ semisimple (and not the identity), then eigenspaces in $\mathbb{C}^3$ of lifts to $\mathrm{U}(2,1)$ of elements of $\mathcal{C}_1$ have (complex) dimension 1, and likewise eigenspaces in $\mathbb{C}^3$ of lifts to $\mathrm{U}(2,1)$ of elements of $\mathcal{C}_2$ have (complex) dimension at most 2. Then, if $(A_0, B_0) \in \mathcal{C}_1 \times \mathcal{C}_2$ is reducible, then let $v \in \mathbb{C}^3$ be a common eigenvector of (lifts of) A_0, B_0, and V_A (resp. V_B) the eigenspace of A_0 (resp. B_0) containing v. Then, in a neighborhood of (A_0, B_0), any pair (A, B) with $V_A \cap V_B = \{0\}$ is irreducible, and since V_A has dimension 1 and V_B dimension at most 2, there exist such pairs arbitrarily close to (A_0, B_0). □

COROLLARY 4. *If $\mathcal{C}_1, \mathcal{C}_2$ are two semisimple conjugacy classes in* $\mathrm{PU}(2,1) \setminus \{\mathrm{Id}\}$, *at least one of which is regular semisimple, then every reducible wall bounds at least one full chamber.*

Proof. By Proposition 8 any reducible pair corresponding to a point on a reducible wall W can be deformed into an irreducible pair. This either gives a point in one of the chambers bounding W, which is therefore full, or a point on the reducible wall, which then also is the image of an irreducible pair. As in Remark 6(2), both chambers bounded by that wall are then full. □

5.2. The Product Map Is Closed

Let G be the identity component of the isometry group of a Riemannian symmetric space with negative sectional curvature. The *translation length* $|g|$ of an isometry $g \in \mathrm{Isom}(X)$ is defined as $|g| = \mathrm{Inf}\{d(x, gx) : x \in X\}$. An isometry g is called *semisimple* if the infimum is attained, that is, if there exists $x \in X$ such that $|g| = d(x, gx)$. In the case of hyperbolic spaces, semisimple isometries are precisely the nonparabolic isometries (in other words, an isometry is semisimple if its matrix representatives are semisimple).

Theorem 4 is the key point in this section. This compactness result, which as stated is Proposition 2 of [FW2] and is essentially Theorem 3.9 of [Be], is sometimes called the Bestvina–Paulin compactness theorem. It is obtained by taking Gromov–Hausdorff limits to get an action on an $\mathbb{R}$-tree.

THEOREM 4. *Let X be a negatively curved Riemannian symmetric space, $G =$* $\mathrm{Isom}^0(X)$, *and (g_i) and (h_i) be two sequences of semisimple elements of G with uniformly bounded translation length. Then either*

(1) *there exists $f_i \in G$ such that $f_i g_i f_i^{-1}$ and $f_i h_i f_i^{-1}$ converge in G (after passing to a subsequence),*

or

(2) *the sequence of translation lengths $|g_i h_i|$ is unbounded.*

We will use the following consequence of this result (Theorem 2 of [FW2]). Recall from the end of Section 3.3 that we denote by $\mathcal{G}$ the space of conjugacy classes of G and by $c(\mathcal{G})$ the maximal Hausdorff quotient of $\mathcal{G}$.

COROLLARY 5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two semisimple conjugacy classes in G, and consider the diagonal action of G on $\mathcal{C}_1 \times \mathcal{C}_2$ by conjugation. Then:*

(a) *the product map $\mu : (A, B) \longrightarrow AB$ descends to a map $\bar{\mu} : \mathcal{C}_1 \times \mathcal{C}_2 / G \longrightarrow c(\mathcal{G})$ that is proper, and*
(b) *the image of $\bar{\mu}$ is closed in $c(\mathcal{G})$.*

Proof. (a) If K is a compact subset of $c(\mathcal{G})$ and $(g_i, h_i) \in G \times G$ is (a choice of representatives of) a sequence in $\bar{\mu}^{-1}(K)$, then the sequence of translation lengths $|g_i h_i|$ is bounded, and therefore by Theorem 4 $\bar{\mu}^{-1}(K)$ is compact.

(b) If (c_i) is a sequence in $\mathrm{Im}\,\bar{\mu}$ converging to $c \in c(\mathcal{G})$, then let, as before, $(g_i, h_i) \in G \times G$ be a choice of representatives of preimages of c_i. Then the sequence of translation lengths $|g_i h_i|$ is bounded, and therefore by Theorem 4 (after conjugating) (g_i) and (h_i) converge in G, say to g and h, respectively. Then, by continuity of $\bar{\mu}$, $c = \bar{\mu}(g, h)$ is in $\mathrm{Im}\,\bar{\mu}$, which is therefore closed. □

5.3. The Product Map Is Open

PROPOSITION 9. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two semisimple conjugacy classes in $G = \mathrm{PU}(2, 1)$, and (A, B) be an irreducible pair in $\mathcal{C}_1 \times \mathcal{C}_2$. Then the differential of $\tilde{\mu}$ at (A, B) is surjective, and thus $\tilde{\mu}$ is locally surjective at that point.*

The key point in the proof is the following lemma (see Lemma 2.4 of [P], the proof of Proposition 4.2 of [FW1], or the final section of [G2] in a different context). Denote by $\mu : \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow G$ the product map and by $\mathfrak{z}(A, B)$ the Lie algebra of the centralizer of the group generated by A and B:

LEMMA 4. *The image of the differential of the product map μ at a pair (A, B) is the subspace of the tangent space $T_{AB}G$ obtained by right-translating the subspace Killing-orthogonal to $\mathfrak{z}(A, B)$ by AB. Namely, $Im(d_{(A,B)}\mu) = \mathfrak{z}(A, B)^{\perp} \cdot AB$, where orthogonality is measured with respect to the Killing form of G.*

Now if (A, B) is irreducible, then $\mathfrak{z}(A, B) = \{0\}$, so μ is a submersion at such a point (the Killing form is nondegenerate). The proposition follows since the projection $\pi : G \longrightarrow \mathcal{G}$ is open as a quotient map.

5.4. Reducible Walls in the Elliptic–Elliptic Case

In this section, we review the case where the two classes $\mathcal{C}_1$ and $\mathcal{C}_2$ are elliptic as well as their product. This situation has been analyzed in detail in [P], but we recall it briefly for self-containedness. Assume that the two classes correspond to angle pairs (θ_1, θ_2) and (θ_3, θ_4) with $0 \le \theta_1 \le \theta_2 < 2\pi$ and $0 \le \theta_3 \le \theta_4 < 2\pi$. The possible reducible configurations for a pair (A, B) in $\mathcal{C}_1 \times \mathcal{C}_2$ fall into three types, which correspond to points and segments in the affine chart Δ (see Section 3.3.3).

Totally reducible pairs. This is when A and B commute. In that case, A and B have a common fixed point in $\mathrm{H}^2_{\mathbb{C}}$ and preserve the same complex lines. The corresponding angle pairs are thus given by the two points $\{\theta_1 + \theta_3, \theta_2 + \theta_4\}$ and $\{\theta_1 + \theta_4, \theta_2 + \theta_3\}$ (the precise order of the coordinates depends on the values of the θ_i).

Spherical reducible pairs. This is when A and B have a common fixed point in $\mathrm{H}^2_{\mathbb{C}}$. In that case, A and B can be lifted to $\mathrm{U}(2, 1)$ as a pair

$$A = \begin{bmatrix} \tilde{A} & \\ & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \tilde{B} & \\ & 1 \end{bmatrix},$$

where $\tilde{A}$ and $\tilde{B}$ are matrices in $\mathrm{U}(2)$ with respective spectra $\{e^{i\theta_1}, e^{i\theta_2}\}$ and $\{e^{i\theta_3}, e^{i\theta_4}\}$. The problem is thus reduced to the similar one in $\mathrm{U}(2)$. The set of angle pairs of spherical reducible pairs is then the segment of slope -1 connecting the two totally reducible vertices. This segment may appear as disconnected in Δ (see [P]).

Hyperbolic reducible pairs. This is when A and B preserve a common complex line L in $\mathrm{H}^2_{\mathbb{C}}$. If A and B are regular, then each preserves two complex lines. When the product AB is elliptic, we denote by θ_C its rotation angle in the line L and by θ_N its rotation angle in the normal direction. There are four families of hyperbolic reducible configurations that correspond to the possible choices of rotation angles of A and B in the common stable complex line. The possible values of the angle pairs for hyperbolic reducible configurations are those lying on the projection to $\mathcal{E}$ of one of the four segments C_{ij} in Δ, where $i \in \{1, 2\}$, $j \in \{3, 4\}$, and C_{ij} is the affine segment defined by the conditions

$$\theta_C = 2\theta_N + (\theta_i + \theta_j) - 2(\theta_k + \theta_l)$$
$$\text{with } \begin{cases} \theta_i + \theta_j < \theta_C < 2\pi & \text{if } \theta_i + \theta_j < 2\pi, \\ 2\pi < \theta_C < \theta_i + \theta_j & \text{if } \theta_i + \theta_j > 2\pi, \end{cases} \tag{30}$$

where we use the convention that $\{k, l\}$ and $\{i, j\}$ are disjoint. The wall C_{ij} corresponds to the case where A and B rotate through angles θ_i and θ_j, respectively, in the complex line L. For example, the segment C_{14} corresponds to the case where A rotates through θ_1 and B through θ_4. Then A and B can be conjugated in $\mathrm{U}(2, 1)$ so that

$$A = \begin{bmatrix} e^{i\theta_2} & \\ & \tilde{A} \end{bmatrix}, \qquad B = \begin{bmatrix} e^{i\theta_3} & \\ & \tilde{B} \end{bmatrix}, \quad \text{and} \quad AB = \begin{bmatrix} e^{i(\theta_2+\theta_3)} & \\ & \tilde{C} \end{bmatrix},$$

where $\tilde{A}$ and $\tilde{B}$ are matrices in $\mathrm{U}(1, 1)$ with respective spectra $\{e^{i\theta_1}, 1\}$ and $\{e^{i\theta_4}, 1\}$. The eigenvalues of $\tilde{C}$ are $e^{i\alpha}$ (positive type) and $e^{i\beta}$ (negative type) for some α and β in $[0, 2\pi)$. The angle pair of AB is given by

$$\theta_C = \alpha - \beta \quad \text{and} \quad \theta_N = \theta_2 + \theta_3 - \beta.$$

On the other hand, the relation $\det(AB) = \det(A)\det(B)$ gives the relation $\alpha + \beta = \theta_1 + \theta_4 \mod 2\pi$. The precise range given in (30) is obtained by applying Proposition 2.

6. Regular Elliptic Triple Products

Our goal in this section is to show that not all elliptic isometries are triple products of involutions and to determine precisely which regular elliptic conjugacy classes cannot be written as triple products of involutions. By Corollary 1 it suffices to determine those classes that can be written neither as a product of type $(+, +, -)$ nor of type $(+, +, +)$. To do so, we will study the products of an involution of any type with the product of two complex symmetries. By Proposition 4 this means we have to study pairs (I, A) where I is an involution of any type and A is either hyperbolic, three-step unipotent or real elliptic (see Section 3.3.3 for definitions). As hyperbolic and real elliptic isometries are semisimple, we will apply the strat-

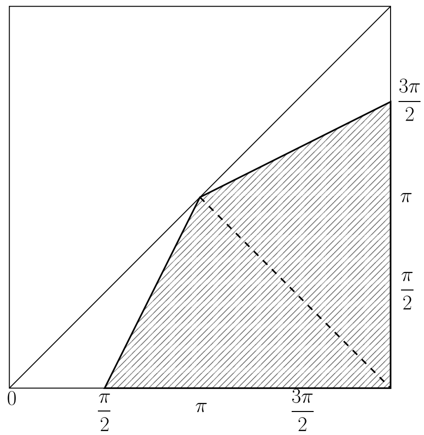


Figure 9 $\mathcal{E}_{++-}$: angle pairs of regular elliptic products of one central involution and two complex symmetries

egy described in Section 5 to determine those elliptic classes that can be written as such products. We will see that these classes correspond to angle pairs lying in a union of polygons in the triangle Δ.

We cannot apply this strategy when the two complex symmetries have three-step unipotent product since parabolic elements are not semisimple. However, three-step unipotent isometries can be seen as limits both of sequences of hyperbolic isometries and of sequences of real elliptics. This can be seen by considering a sequence of pairs of complex lines (L_n, L'_n) with respective complex symmetries I_n and I'_n. If L_n and L'_n are ultraparallel (resp. intersecting) for all n and converge to a pair (L_∞, L'_∞) of asymptotic lines, then the product $I_n I'_n$ is hyperbolic (resp. real elliptic), and $I_\infty I'_\infty$ is three-step unipotent. As a consequence, the elliptic classes that are products of one involution and a three-step unipotent isometry lie in the closure of the set of classes that can be written as products of one involution and a hyperbolic or real elliptic isometry. For that reason, we will only consider pairs (I, A) where I is an involution and A is hyperbolic or real elliptic. The result is the following:

Proposition 10. 1. *An elliptic isometry is the product of one central involution and two complex symmetries if and only if its angle pair lies in the shaded region $\mathcal{E}_{++-}$ depicted on Figure 9.*

2. *An elliptic isometry is the product of three complex symmetries if and only if its angle pair lies in the shaded region $\mathcal{E}_{+++}$ depicted on Figure 10.*

From Proposition 10 we obtain the following by applying Corollary 1.

Corollary 6. *An elliptic isometry $E \in \mathrm{PU}(2, 1)$ is a product of three involutions if and only if its angle pair lies outside the two open triangles T and T' given by*

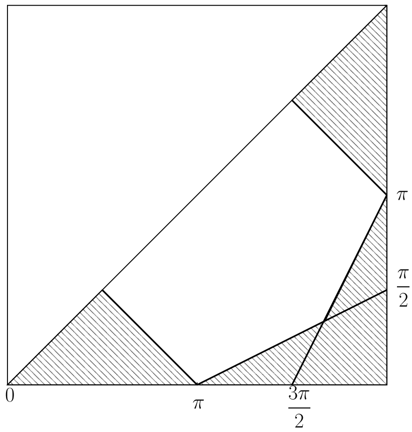


Figure 10 $\mathcal{E}_{+++}$: angle pairs of regular elliptic products of three complex symmetries

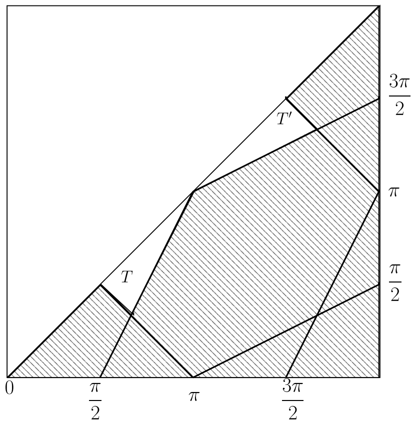


Figure 11 $\mathcal{E}_{++-} \cup \mathcal{E}_{+++}$: Regular elliptic classes that are not products or three involutions are those in the interior of one of the two triangles T and T'

their vertices as follows:

$$T : (\pi, \pi), (2\pi/3, \pi/3), (\pi/2, \pi/2);$$
$$T' : (\pi, \pi), (5\pi/3, 4\pi/3), (3\pi/2, 3\pi/2).$$

The triangles T and T' are shown on Figure 11. To prove Proposition 10, we separate cases, first studying products of an involution an a hyperbolic isometry and then products of an involution and a real elliptic isometry.

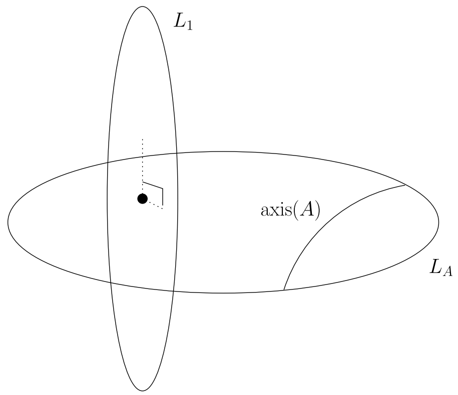


Figure 12 A reducible pair (I_1, A) with I_1 a complex reflection and A hyperbolic

6.1. Products of an Involution and a Hyperbolic Isometry

Applying the strategy of Section 5, we first need to describe the reducible walls. We thus consider reducible pairs (I_1, A), where I_1 is an involution and A is hyperbolic. There are two cases: I_1 can be either a complex symmetry or a central involution (see Remark 2).

- First, assume that I_1 is a complex symmetry with mirror a complex line L_1. If I_1A is regular elliptic, then it has no boundary fixed point, and thus the only possible common fixed point in $\mathbb{C}P^2$ for I_1 and A is the point polar to L_A, the complex axis of A. This means that I_1 preserves L_A, and therefore its mirror is either equal or orthogonal to it. In the first case, I_1 fixes L_A pointwise, so I_1A coincides with A on L_A and is thus loxodromic. Therefore, the mirror of I_1 must be orthogonal to L_A. In particular, the restriction of I_1 to the complex axis of A is a half-turn (see Figure 12 for a schematic picture).
- If I_1 is a central involution, then its eigenvectors are either of positive or negative type (but not of null type), and reducibility means that the fixed point of I_1 belongs to the complex axis of A (which is thus preserved by I_1).

In both cases, we see that I_1 preserves the complex axis of A and acts on it by a half-turn. We use the ball model of $\mathrm{H}^2_{\mathbb{C}}$, with Hermitian form diag$(1, 1, -1)$, and we normalize so that lifts of I_1, A, and I_1A to SU(2, 1) have the form

$$I_1 = \begin{bmatrix} -1 & & \\ & \epsilon & \\ & & -\epsilon \end{bmatrix} \quad \text{with } \epsilon = \pm 1,$$
$$A = \begin{bmatrix} 1 & \\ & \tilde{A} \end{bmatrix}, \quad \text{and} \quad I_1A = \begin{bmatrix} -1 & \\ & \tilde{B} \end{bmatrix}. \tag{31}$$

In (31), I_1 is a central involution when $\epsilon = -1$ and a complex symmetry when $\epsilon = 1$. Since A is hyperbolic, the 2×2 matrix $\tilde{A}$ has spectrum $\{r, 1/r\}$ for some

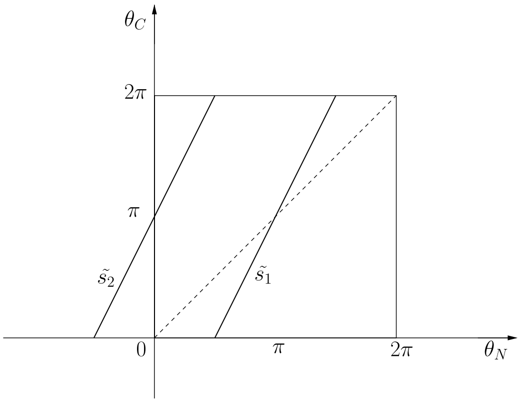


Figure 13 The two segments $\tilde{s}_1$ and $\tilde{s}_2$

$r > 1$. Similarly, $I_1 A$ is elliptic, and thus $\tilde{B}$ has eigenvalues $e^{i\alpha}$ (of positive type) and $e^{i\beta}$ (of negative type) for some $\alpha, \beta \in [0, 2\pi)$. The determinant of $I_1 A$ is equal to 1, and therefore we have $\alpha + \beta = \pi\,[2\pi]$, that is, $\alpha + \beta = \pi$ or $\alpha + \beta = 3\pi$. The angle pair of $I_1 A$ is given by

$$\theta_C = \alpha - \beta \quad \text{and} \quad \theta_N = \pi - \beta,$$

where θ_C is the rotation angle of $I_1 A$ in the complex axis of A (the common preserved complex line), and θ_N is the rotation angle in the normal direction. Using the conditions on the sum $\alpha + \beta$, we see that the angle pair $\{\theta_C, \theta_N\}$ of $I_1 A$ satisfies one of the following two relations:

$$\theta_C = 2\theta_N - \pi \quad \text{if } \alpha + \beta = \pi, \tag{32}$$

$$\theta_C = 2\theta_N + \pi \quad \text{if } \alpha + \beta = 3\pi. \tag{33}$$

We denote by $\tilde{s}_1$ and $\tilde{s}_2$ the segments given by (32) and (33) for $0 < \theta_C < 2\pi$ (see Figure 13).

PROPOSITION 11. *Let A be a hyperbolic isometry with complex axis L_A, and I_1 an involution such that the pair (I_1, A) is reducible and $I_1 A$ is elliptic.*

1. *If I_1 is a central involution, then the possible angle pairs for $I_1 A$ are the points of $\tilde{s}_1$ (see Figure 13).*
2. *If I_1 is a complex symmetry, then the possible angle pairs for the product $I_1 A$ are the points of $\tilde{s}_2$ (see Figure 13).*

Proof. From the discussion at the beginning of this section we know that in both cases, the restriction of I_1 to L_A is a half-turn. As a consequence, we can apply Proposition 2 in L_A.

1. If I_1 is a central involution, then decompose A as a product $I_2 I_3$ of two central involutions with fixed points on the (real) axis of A. First, if I_1 coincides with I_2 or I_3, then the product $I_1 A$ is a central involution, and in this case $\theta_C =$

$\theta_N = \pi$. This angle pair corresponds to the midpoint of $\tilde{s}_1$. Deforming this configuration, Proposition 2 shows that any point on $\tilde{s}_1$ can be obtained by a reducible product of three central involutions.

2. If I_1 is a complex symmetry, then we decompose A as a product $I_2 I_3$ of two complex symmetries. Again, if I_1 coincides with either one of I_2 or I_3, then $I_1 A$ is a complex symmetry. This gives the angle pair $\theta_C = \pi$ and $\theta_N = 0$, which corresponds to the midpoint of $\tilde{s}_2$. By a similar argument as before, any point on (32) with $0 < \theta_C < 2\pi$ can be obtained by a reducible product of three complex symmetries.

To conclude the proof, we need to show that no point of $\tilde{s}_1$ (resp. $\tilde{s}_2$) corresponds to a product of three complex symmetries (resp. three central involutions).

To do so, consider three central involutions (I_1, I_2, I_3) with fixed points in a common complex line L. It follows from part (b) in the proof of Proposition 1 that the triple product $I_1 I_2 I_3$ acts on L as a half-turn if and only if at least two of the I_k are equal. In that case, $I_1 I_2 I_3$ is a central involution. This proves in particular that a complex symmetry cannot be a product $I_1 A$ where the the pair (I_1, A) is reducible, I_1 is a central involution, and A is hyperbolic.

By a similar argument, a central involution cannot be a product $I_1 A$ where (I_1, A) is reducible, I_1 is a complex reflection of order two, and A is hyperbolic.

Now, if a point of $\tilde{s}_1$ were a reducible product of a complex symmetry and a hyperbolic, then by Proposition 2 we could deform it continuously to obtain the midpoint of $\tilde{s}_1$. This contradicts the previous discussion. A similar argument shows that no point of $\tilde{s}_2$ can be a reducible product of a central involution and a hyperbolic map. □

The following corollary describes the reducible walls. It is obtained in a straightforward way from Proposition 11 by projecting the two segments $\tilde{s}_1$ and $\tilde{s}_2$ onto the lower half of the square by reduction modulo 2π and symmetry about the diagonal.

Corollary 7. *Let $\mathcal{C}_1$ be a conjugacy class of involutions, $\mathcal{C}_2$ be a hyperbolic conjugacy class, and $(I_1, A) \in \mathcal{C}_1 \times \mathcal{C}_2$ be a reducible pair such that $I_1 A$ is elliptic.*

1. *If $\mathcal{C}_1$ is the class of central involutions, then the angle pair of $I_1 A$ can take any value on the two segments $[(\frac{\pi}{2}, 0), (\pi, \pi)]$ and $[(\pi, \pi), (2\pi, \frac{3\pi}{2})]$.*
2. *If $\mathcal{C}_1$ is the class of complex symmetries, then the angle pair of $I_1 A$ can take any value on the two segments $[(\pi, 0), (2\pi, \frac{\pi}{2})]$ and $[(\frac{3\pi}{2}, 0), (2\pi, \pi)]$.*

These segments are the thicker ones on Figures 14 and 15. We can now describe all elliptic classes that are obtained as a product of an involution and a hyperbolic map.

Proposition 12. 1. *An elliptic isometry is the product of a central involution and a hyperbolic isometry if and only if its angle pair lies in the dashed polygon depicted on Figure 14.*

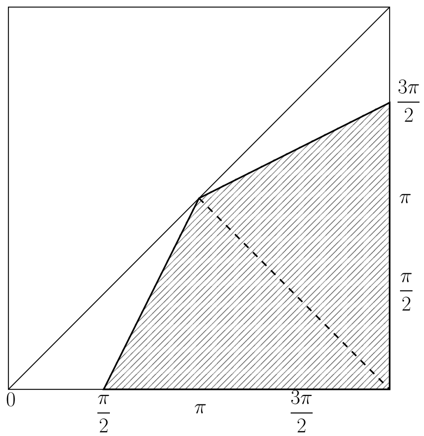


Figure 14 Elliptic classes that are products of one central involution and a hyperbolic map

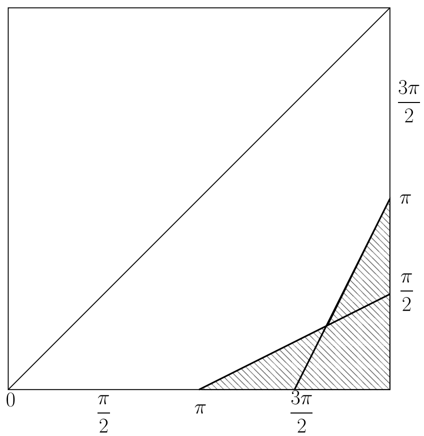


Figure 15 Elliptic classes that are products of one complex symmetry and a hyperbolic map

2. *An elliptic isometry is the product of a complex symmetry and a hyperbolic isometry if and only if its angle pair lies in the dashed polygon depicted on Figure* 15.

Proof. For each of the reducible walls, Corollary 4 tells us that at least one of the chambers bounded by this wall is full. In the case where I_1 is a central involution, Corollary 3 tells us that the two chambers bounded by a piece of the diagonal are empty (see Figure 14). Therefore the third chamber must be full. In the case where I_1 is a complex symmetry, Corollary 3 tells us that the chamber bounded by the

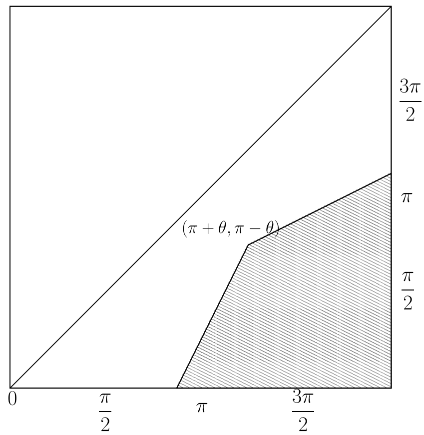


Figure 16 Elliptic classes that are products of one central involutions and a real elliptic map with angle pair $\{2\pi - \theta, \theta\}$

diagonal is empty. Applying Corollary 4 at the intersection point of the reducible walls tells us that the three other chambers are full. □

6.2. Products of an Involution and a Real Elliptic Isometry

We now consider products of an involution and a real elliptic map, (i.e.) triple products of types $(-, +, +)$ and $(+, +, +)$, where two of the complex symmetries have intersecting mirrors. These cases belong to those treated in [P], where products of two elliptic elements are considered. For that reason, we omit the proofs and only state the result and provide pictures of the full chambers. The reader may also want to follow the method from the previous section. It should be noted that in each case, we obtain a description of the image of the product map for products of one involution and a real elliptic map of angle pair $\{2\pi - \theta, \theta\}$ for a fixed value of $\theta \in [0, \pi]$. In turn, the set of elliptic conjugacy classes that are products of one involution and one real elliptic appears as the union of these polygons for all values of θ.

We begin by the case where the involution is central.

Proposition 13. *The possible angle pairs for a product of one central involution and a real elliptic isometry with angle pair* $\{2\pi - \theta, \theta\}$ *for some* θ *in* $[0, \pi]$ *are those laying in the convex hull of the four points with coordinates*

$$\left(\frac{\pi + 3\theta}{2}, 0\right), (\pi + \theta, \pi - \theta), \left(2\pi, \frac{3(\pi - \theta)}{2}\right), \textit{and } (2\pi, 0) \quad (\textit{see Figure } 16).$$

Taking the union over all values of $\theta \in [0, \pi]$, we obtain directly the following:

Corollary 8. *An elliptic element is the product of a central involution and two complex symmetries with intersecting mirrors if and only if its angle pairs belong to the convex polygon with vertices* $(\pi/2, 0)$, (π, π), $(2\pi, 3\pi/2)$ *and* $(2\pi, 0)$.

We now deal with the case where the involution is a complex symmetry.

PROPOSITION 14. *Let* (R_1, A) *be a pair, where* R_1 *is a complex symmetry, and* A *is a real elliptic with angle pair* $\{2\pi - \theta, \theta\}$ *for some fixed* $\theta \in [0, \pi]$. *The possible angle pairs for the product* $R_1 A$ *(when it is elliptic) are as follows.*

1. *The two totally reducible vertices are the projections to* $\mathcal{E}$ *of the points in* $\mathbb{R}^2$ *with coordinates* $(3\pi - \theta, \theta)$ *and* $(2\pi - \theta, \pi + \theta)$.
2. *If* (R_1, A) *is spherical reducible, then the possible angle pairs are the points on the slope* -1 *segments in* $\mathcal{E}$ *connecting the projections to* $\mathcal{E}$ *of the two points* $(\theta, \pi - \theta)$ *and* $(2\pi - \theta, \pi + \theta)$.
3. *If* (R_1, A) *is hyperbolic reducible, then the possible angle pairs are the segments* s_3 *and* s_4 *that are respectively the projections to* $\mathcal{E}$ *of two segments* $\tilde{s}_3$ *and* $\tilde{s}_4$ *in* $\mathbb{R}^2$ *given by*

$$\begin{aligned}\tilde{s}_3 &= \left[(\theta, 3\pi - \theta), \left(\frac{3\theta - \pi}{2}, 2\pi\right)\right] \quad \textit{and} \\ \tilde{s}_4 &= \left[(2\pi - \theta, \pi + \theta), \left(\frac{5\pi - 3\theta}{2}, 2\pi\right)\right].\end{aligned} \tag{34}$$

4. *The full chambers are exactly those not containing an open segment of the diagonal in their closure.*

The full chambers in Proposition 14 are depicted on Figure 17 depending on the value of θ. Again, we can now take the union over all values of $\theta \in [0, \pi]$ and obtain the following:

COROLLARY 9. *An elliptic element is the product of a complex symmetry and two complex symmetries with intersecting mirrors if and only if its angle pairs belong to the convex polygon with vertices* $(\pi/2, 0)$, (π, π), $(2\pi, 3\pi/2)$ *and* $(2\pi, 0)$.

7. Parabolic Triple Products

In this section, we examine which parabolic isometries are obtained as triple products of involutions. Table 1 sums up the results of this section.

7.1. Screw-Parabolic Triple Products as Limits of Elliptic Triple Products

The strategy exposed in Section 5 is restricted to semisimple conjugacy classes. For that reason, we cannot apply it directly to obtain parabolic triple products. However, it is possible to show that certain parabolic conjugacy classes are triple products by passing to the limit in the full chambers of the product map. Assume that (E_n) is a sequence of elliptic elements with angle pairs (α_n, β_n) converging to a limit $E_\infty \neq \mathrm{Id}$. Then E_∞ is either parabolic or elliptic. In the case where $\lim \alpha_n = 0$ and $\lim \beta_n = \beta_\infty \neq 0$, E_∞ is either special elliptic with angle pair $(0, \beta_\infty)$ or a screw-parabolic map with rotation angle β_∞. If $\beta_\infty = 0$, then the limit is unipotent parabolic, but it can be of any unipotent type.

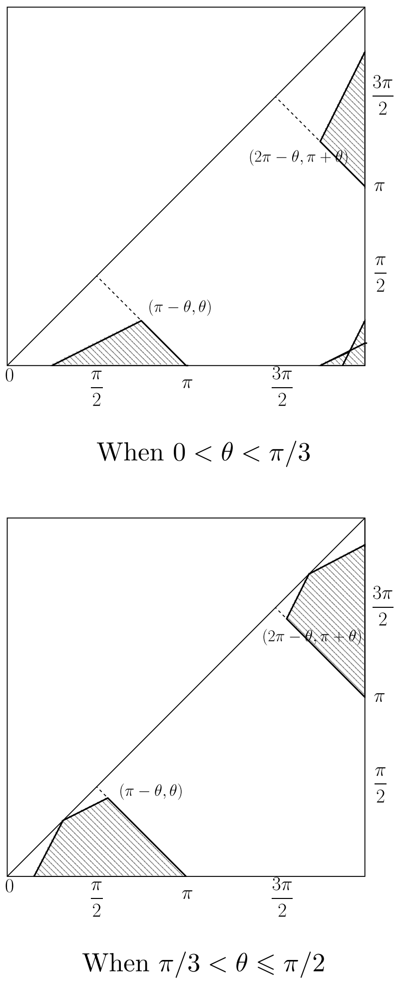


Figure 17 Elliptic products of one complex symmetry and a real elliptic with angle pair $\{2\pi-\theta,\theta\}$ with $0<\theta<\pi$

Table 1 Triple product types of parabolic isometries

Parabolic conjugacy class	$(+,+,+)$	$(+,+,-)$	$(+,-,-)$	$(-,-,-)$
Screw-parabolic with $\theta\neq\pi$	yes	yes	yes	yes
Half-turn parabolic	yes	yes	no	yes
two-step unipotent	yes	no	no	no
three-step unipotent	yes	yes	yes	yes

PROPOSITION 15. 1. *Every screw-parabolic isometry is a* $(-,-,-)$ *triple product and hence also a* $(-,+,+)$ *triple product.*

2. *Every screw-parabolic isometry that is not half-turn parabolic is a* $(+,-,-)$ *triple product and hence also a* $(+,+,+)$ *triple product.*

Proof. 1. Fix a hyperbolic conjugacy class $\mathcal{C}$. In Section 6.1, we described the possible elliptic conjugacy classes of a product HI, with $H \in \mathcal{C}$ and I an involution.

Assume that I is a central involution, so that HI is a product of three central involutions (recall that H can be written as a product of two central involutions). The possible elliptic conjugacy classes for the product HI are depicted on Figure 14. The boundary segments of this chambers are of two types:

- Reducible walls correspond to reducible pairs (H, I).
- One horizontal segment and one vertical one on the boundary of the square, given respectively by $h = \{(\theta, 0), \pi/2 \leq \theta \leq 2\pi\}$ and $v = \{(2\pi, \theta), 0 \leq \theta \leq 3\pi/2\}$ (see Figure 14).

Consider a point on one of the two segments h and v that is not a reducible point, that is, not an intersection point of one of the reducible walls with h or v. Since the image of the product map is closed, this point represents the conjugacy class of a product HI as before. However, if it corresponded to an elliptic conjugacy class, it would be special elliptic, and thus by Lemma 3 the pair (H, I) would be reducible. Therefore the product HI can only be parabolic in that case. Moreover, its rotation angle can take any value θ such that $0 < \theta < 2\pi$.

2. To prove the second item, we proceed along the same lines. We fix a hyperbolic class, so that any element in it is a product of two central involutions, and then we consider the polygon that is the image of the product map of one hyperbolic element and a complex symmetry. This polygon is depicted on Figure 15. By the same argument as for the first item, every screw-parabolic isometry of which rotation angle appears on the nonreducible boundary of the image polygon can be written as a product of two central involution and a complex symmetry. The nonreducible boundary of the image polygon is formed by the two segments $\{(\theta, 0), \pi < \theta < 2\pi\}$ and $\{(2\pi, \theta), 0 < \theta < \pi\}$. In turn, we obtain every screw-parabolic element this way, except for half-turn ones for which we cannot decide yet. □

7.2. *Half-Turn and Unipotent Parabolic Isometries*

We now study separately the remaining parabolic conjugacy classes, unipotent and half-turn parabolic isometries. To decide whether or not a given parabolic isometry is a product of three involutions, we consider pairs (P, I) where P is parabolic and I an involution and decide if PI is a product of two involutions using the results of Section 3.4.

A. Two-Step Unipotent and Half-Turn Parabolic Isometries.

PROPOSITION 16. 1. *A two-step unipotent isometry can be written as a product of three complex symmetries but cannot be written as a triple product of any other kind.*

2. *A half-turn parabolic isometry can be written as a product of three complex symmetries but cannot be written as a product of a complex symmetry and two central involutions.*

By Proposition 15 we already know that a half-turn parabolic map can be a $(-,-,-)$ triple product, and thus a $(-,+,+)$ triple product. The proposition describes the remaining two possible types of triple products. To prove Proposition 16, we will use the following lemma.

LEMMA 5. 1. *Any pair (P, I) where P is two-step unipotent and I is an involution is conjugate in* $\mathrm{Isom}(\mathrm{H}^2_{\mathbb{C}})$ *to a pair given in the Siegel model by (P, I_u) or (P, I_∞), where*

$$P = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad I_u = \begin{bmatrix} 0 & 0 & u \\ 0 & -1 & 0 \\ u^{-1} & 0 & 0 \end{bmatrix} \quad \textit{for some } u \neq 0, \quad \textit{and}$$
$$I_\infty = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{35}$$

2. *Any pair (P, I) where P is half-turn parabolic and I is an involution is conjugate in* $\mathrm{Isom}(\mathrm{H}^2_{\mathbb{C}})$ *to a pair given in the Siegel model by $(P, J_{(x,u)})$ or $(P, J_{(x,\infty)})$, where*

$$P = \begin{bmatrix} -1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$
$$J_{(x,u)} = \frac{1}{u}\begin{bmatrix} -x^2 & x\sqrt{2}(u-x^2) & (u-x^2)^2 \\ x\sqrt{2} & 2x^2-u & x\sqrt{2}(u-x^2) \\ 1 & x\sqrt{2} & -x^2 \end{bmatrix} \quad \textit{for some } u \neq 0, \quad \textit{and} \tag{36}$$

$$J_{(x,\infty)} = \begin{bmatrix} -1 & x\sqrt{2} & x^2 \\ 0 & 1 & x\sqrt{2} \\ 0 & 0 & -1 \end{bmatrix}. \tag{37}$$

Note that the involution I_u is a complex symmetry (resp. a central involution) when $u > 0$ (resp. $u < 0$). The involution I_∞ is a complex symmetry that fixes the fixed point of P.

Proof of Lemma 5. The strategy for proving Lemma 5 is to conjugate the pair (P, I) so that the parabolic map P is upper triangular. Geometrically, this means that P fixes the point q_∞ of the Siegel model and preserves the complex line polar to the vector $[0,1,0]^T$. Then to normalize the involution I, we still have the freedom of conjugating I be an element normalizing P in SU(2, 1).

1. Assume first P is two-step unipotent. In particular, P is central in the Heisenberg group.
   (a) First, assume that I does not fix q_∞, the fixed point of P. Writing $I(q_\infty) = [w,s]$ in Heisenberg coordinates and conjugating (P,I) by $T_[-w,-s]$ leaves P unchanged (because it is central) and transforms I into an involution that exchanges q_∞ and the origin of the Heisenberg group. Such an involution has the form I_u with $u>0$ (resp. $u<0$) if it is a complex symmetry (resp. a central involution).
   (b) If I fixes q_∞, then it has to be a complex symmetry. Let $[w,s]$ be another fixed point of I in $\partial_\infty \mathrm{H}^2_{\mathbb{C}}$. Then conjugating by $T_{[-w,-s]}$ gives a complex symmetry with mirror connecting the complex line q_∞ to the origin of the Heisenberg group. This is I_∞.
2. When P is half-turn parabolic, it is no longer central, and we may thus only conjugate the stabilizer of the complex line spanned by q_∞ and the origin of the Heisenberg group.
   (a) If I does not fix the point q_∞, then again, write $I(q_\infty)=[w,s]$. Conjugating the pair (P,I) by a rotation and an vertical translation that leave P unchanged allows us to assume that w is real and $s=0$. All involutions exchanging q_∞ and $[x,0]$ are of the form $J_{(x,u)}$ as before with $u>0$ (resp. $u<0$) if it is a complex symmetry (resp. a central involution).
   (b) If I fixes q_∞, then again it is a complex symmetry, and rotating around the vertical axis of the Heisenberg group as before its mirror connects q_∞ to a point $[x,0]$ with $x\in\mathbb{R}$. This gives $J_{(x,\infty)}$. □

Proof of Proposition 16. 1. We begin with the case where P is unipotent. Showing that P is a product of three involutions $I_1I_2I_3$ is equivalent to to finding an involution I such that PI is a product of two involutions. In view of Lemma 5, we only need to consider the pairs (P,I_u) or (P,I_∞) as in (35). By a straightforward computation we have

$$\mathrm{tr}(PI_u) = -1 + \frac{i}{u} \quad \text{and} \quad \mathrm{tr}(PI_\infty) = -1. \tag{38}$$

Any lift to SU(2, 1) of a hyperbolic isometry has trace of the form $xe^{2ik\pi/3}$, where $x>3$ and $k\in\{0,1,2\}$. Thus none of the quantities given by (38) can be the trace of a hyperbolic isometry. Since hyperbolic isometries are products of two central involutions, this proves that P is neither a $(-,-,-)$ triple product nor a $(+,-,-)$ one.

We still need to consider the $(+,+,+)$ and $(+,+,-)$ types, that is, triple products where at least two of the involutions are complex symmetries. If the mirrors of the complex symmetries are ultraparallel, then their product is hyperbolic, and we fall into the previous case. We thus need to determine when a product PI as before can be real elliptic or three-step unipotent. First, the previous discussion applies, and the trace of PI still has real part equal to -1. In turn, PI cannot be unipotent since any lift to SU(2, 1) of a unipotent isometry has trace 3ω, where ω is a cube root of 1.

If PI is real elliptic, then its trace must be of the form $xe^{2ik\pi/3}$ with $x \in [-1, 3)$ and $k = 0, 1, 2$. Considering (38), we see that the only possible pairs are

$$(P, I_u) \quad \text{with } u = \pm 1/\sqrt{3} \quad \text{or} \quad (P, I_\infty).$$

- If $u = -1/\sqrt{3}$, then I_u is a central involution, and the angle pair of PI_u is easily seen to be $\{5\pi/3, 4\pi/3\}$. Thus PI_u is not real elliptic and cannot be a product of two complex symmetries.
- Assume that $u = 1/\sqrt{3}$. Now I_u is a complex symmetry. Similarly, we see that the angle pair of the product is $\{5\pi/3, \pi/3\}$. This means that PI_u is real elliptic and thus can be written as a product of two complex symmetries.
- Consider now the pair (P, I_∞). In this case we see that

$$PI_\infty = \begin{bmatrix} -1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

  which is half-turn parabolic. By Proposition 4 it is not a product of two involutions.

The only possibility is thus that P is a product of three complex symmetries.

2. When P is half-turn parabolic, then as before we compute

$$\operatorname{tr}(PJ_{(x,u)}) = -1 + \frac{4x^2}{u} + \frac{i}{u}.$$

We see directly that when u is positive, (i.e. $J_{(x,u)}$ is a complex symmetry), then $\operatorname{Re}(\operatorname{tr}(PJ_{(x,u)})) > -1$, and $PJ_{(x,u)}$ cannot be hyperbolic because its trace is never real. On the other hand, it is easily verified that

$$PJ_{x,\infty} = \begin{bmatrix} 1 & -x\sqrt{2} & x^2 \\ 0 & 1 & x\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

The latter matrix is two-step unipotent when $x = 0$ and three-step otherwise. In turn, we see that $PJ_{0,\infty}$ is a product of two complex symmetries (see Proposition 4), and thus P is a product of three complex symmetries. □

B. Three-Step Unipotent Isometries.

PROPOSITION 17. *A three-step unipotent map is the product of three holomorphic involutions of any type.*

Proof. By Corollary 1 it suffices to prove that a three-step unipotent is both a triple product of type $(-,-,-)$ and $(+,-,-)$.

We first consider the case of three central involutions, that is, $(-,-,-)$. We know from Proposition 1 that a parabolic map in the Poincaré disk is a product of three half-turns. Consider such a configuration of half-turns and embed the Poincaré disk into $\mathrm{H}^2_{\mathbb{C}}$ as a real plane, mapping the three half-turns to central involutions. Each of the central involutions preserves the real plane. As a result, we obtain a parabolic element in PU(2, 1) that preserves a real plane. It is thus three-step unipotent (see Section 3.3.2).

For the second case, consider the two elements given in the Siegel model on $\mathrm{H}^2_{\mathbb{C}}$ by

$$P = T_{[1,0]} = \begin{bmatrix} 1 & -\sqrt{2} & -1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$I = \frac{\omega^{-1}}{8}\begin{bmatrix} -6 & i\sqrt{6} & 1/2 \\ -4i\sqrt{6} & 4 & -i\sqrt{6} \\ 8 & 4i\sqrt{6} & -6 \end{bmatrix}, \quad \text{where } \omega = e^{2i\pi/3}.$$

The map P is three-step unipotent parabolic, and I is the symmetry about the complex line polar to the vector $[1\ -2i\sqrt{6}\ 4]^T$. The product PI satisfies $\mathrm{tr}(PI) = 4$. It is thus hyperbolic and a product of two central involutions. As a consequence, $P = (PI)I$ is a $(-,-,+)$ triple product. □

8. Involution and Commutator Length

8.1. Involution Length

We now prove Theorem 1, stated in the introduction: the involution length of PU(2, 1) is 4.

Proof of Theorem 1. It only remains to prove that those elements in PU(2, 1) that are not products of two or three central involutions are products of four central involutions. This leaves:

1. The regular elliptics whose angle pair does not lie in the shaded polygon of Figure 14.
2. Nonregular elliptic isometries (complex reflections about lines and about points with arbitrary rotation angles).
3. Two-step unipotent parabolic isometries.

For the first part, it suffices to prove that any regular elliptic map is a product of two hyperbolic isometries. To do so, we fix two hyperbolic conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$ and apply the strategy of Section 5. If (A, B) is a reducible pair in $\mathcal{C}_1 \times \mathcal{C}_2$ with elliptic product, then only a positive type vector can be a common eigenvector for A and B. In particular, the product AB has a lift to SU(2, 1) of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{C} \end{bmatrix},$$

where $\tilde{C}$ has eigenvalues $\{e^{i\alpha}, e^{i\beta}\}$, where $e^{i\alpha}$ has positive type, $e^{i\beta}$ has negative type, and α, β lie in $[0, 2\pi)$. The rotation angles of AB are θ_C (in the complex line preserved by A and B) and θ_N (in the normal direction). Applying the same

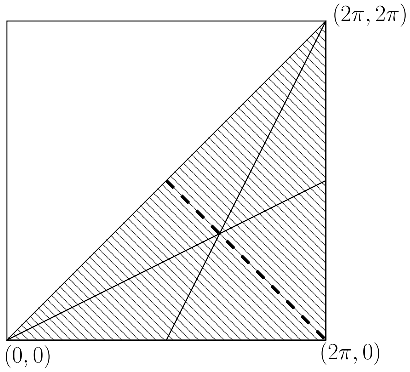


Figure 18 Every regular elliptic isometry is the product of two hyperbolic isometries

arguments as in Sections 4.2 and 6, we see that the two rotation angles of AB satisfy

$$\theta_C = 2\theta_N \mod 2\pi \quad \text{with } \theta_N \in [2, 2\pi). \tag{39}$$

This implies by projecting to the lower triangle of the square $[0, 2\pi]^2$ that the reducible walls are the two segments given by

$$r_1 = [(0,0), (2\pi, \pi)] \quad \text{and} \quad r_2 = [(\pi, 0), (2\pi, 2\pi)]. \tag{40}$$

Considering configurations of two hyperbolic isometries preserving a common real plane, we see that all angle pairs $(\theta, 2\pi - \theta)$ are obtained by irreducible configurations (these pairs form the dashed segment on Figure 18). This implies that all regular elliptic isometries are products of four central involutions.

Let us now consider nonregular elliptics. First, complex reflections about points have angle pairs of the form $\{\theta, \theta\}$ lying on the diagonal of Figure 18. As the image of the product map is closed, they are obtained as limits of regular elliptic products of two hyperbolic maps. Secondly, we know from Proposition 10 that for every regular elliptic element E with angle pair $\{\pi + \theta, \pi\}$, there exists a triple of involutions (I_1, I_2, I_3) such that $E = I_1 I_2 I_3$ (note that the pair $\{\pi + \theta, \pi\}$ lies in $\mathcal{E}_{++-}$) on Figure 14). Now, consider the central involution I_4 about the fixed point of E. The product $I_1 I_2 I_3 I_4$ has angle pair $\{2\pi + \theta, 2\pi\} \sim \{\theta, 0\}$. This shows that $I_1 I_2 I_3 I_4$ is a complex reflection and that any complex reflection can be obtained this way. Finally, we consider two-step parabolics. We know from Sections 7.1 and 7.2 that any half-turn parabolic P is a product of three involutions. Writing $P = I_1 I_2 I_3$, call I_4 the complex symmetry about the complex line preserved by P. Then the product $I_1 I_2 I_3 I_4$ is a two-step parabolic; for example, when

$$P = \begin{bmatrix} -1 & 0 & -it \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad I_4 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

we obtain $P I_4 = T_{[0,1]}$. $\square$

Note that the first part of the proof showed the following result:

PROPOSITION 18. *Let* $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ *be three conjugacy classes in* PU(2, 1), *two of them hyperbolic and one regular elliptic. Then there exists* $(A, B, C) \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ *such that* $ABC = \mathrm{Id}$.

In fact, the following stronger statement follows by combining this with Proposition 9.

PROPOSITION 19. *Let* $\mathcal{C}_3$ *be a regular elliptic conjugacy class in* PU(2, 1). *There exists an open subset of* $\mathcal{L} \times \mathcal{L}$ *containing* $\mathcal{H} \times \mathcal{H}$ *(and depending explicitly on* $\mathcal{C}_3$*) such that for any* $\mathcal{C}_1$, $\mathcal{C}_2$ *in this subset, there exists* $(A, B, C) \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ *such that* $ABC = \mathrm{Id}$.

We can now prove Theorem 2 stated in the introduction: the involution length of $\mathrm{PU}(n, 1)$ is at most 8 for all $n \geq 3$. The proof is done by combining the ingredients of Theorem 1 and Theorem 3.1 of [GT].

Proof of Theorem 2. Let $A \in \mathrm{PU}(n, 1)$ be a holomorphic isometry of $\mathrm{H}^n_{\mathbb{C}}$. First assume that A is elliptic, that is, belongs to a copy of $\mathrm{U}(n)$ (identified to $\mathrm{P}(\mathrm{U}(n) \times \mathrm{U}(1))$, for example, via the embedding $U \mapsto \mathrm{P}((U, 1))$). Given any element $\tilde{B} \in \mathrm{U}(2)$ with $\det(B) = \det(A)^{-1}$, we extend $\tilde{B}$ to an element B of $\mathrm{U}(n,1)$ as follows:

$$B = \begin{bmatrix} \tilde{B} & 0 \\ 0 & I_{n-1} \end{bmatrix}.$$

Then AB belongs to $\mathrm{SU}(n) \times \{1\}$, so by Theorem 3.1 and Lemma 3.3 of [GT] it is a product of at most four involutions of $\mathrm{U}(n) \times \{1\}$.

The matrix B corresponds to an elliptic isometry preserving a copy of $\mathrm{H}^2_{\mathbb{C}}$ in $\mathrm{H}^n_{\mathbb{C}}$. Its rotation angles are $\{\theta_1, \theta_2, 0, \dots, 0\}$, where θ_1 and θ_2 are the rotation angles of $\tilde{B}$. The only constraint on θ_1 and θ_2 is that $e^{i(\theta_1+\theta_2)} = \det(A)^{-1}$. But every line of the form $\theta_1 + \theta_2 = \mathrm{C}$ intersects the region $\mathcal{E}_{++-} \cup \mathcal{E}_{+++}$ representing elliptic conjugacy classes that are triple products of involutions (see Figure 11 and Proposition 10). Therefore we can choose θ_1, θ_2 in such a way that $\tilde{B}$, resp. B, is a product of three involutions in PU(2, 1), resp. in $\mathrm{PU}(n, 1)$ (again, under the embedding of U(2) as $\mathrm{P}(U(2) \times \{1\})$. Therefore AB is a product of at most seven involutions.

Now if A is not elliptic, then there exists an involution $I \in \mathrm{PU}(n, 1)$ such that IA is elliptic. Indeed, pick any point $x_0 \in \mathrm{H}^n_{\mathbb{C}}$, so that $Ax_0 \neq x_0$, and let I be the central involution about the midpoint of (x_0, Ax_0). Then IA fixes x_0, and therefore IA is a product of at most seven involutions, and A is a product of at most eight involutions. □

8.2. Commutator Length

THEOREM 5. *Every holomorphic isometry of* $\mathrm{H}^2_{\mathbb{C}}$ *is a commutator of holomorphic isometries.*

In fact, we get a slightly more precise statement, Proposition 20, using the following definition.

DEFINITION 6. A pair $(A, B) \in \mathrm{PU}(2, 1)^2$ is $\mathbb{C}$-decomposable if there exist three involutions $I_1, I_2, I_3 \in \mathrm{PU}(2, 1)$ such that $A = I_1 I_2$ and $B = I_3 I_2$.

Note that this definition is slightly more general than in [W1], where the involutions were required to be complex symmetries.

PROPOSITION 20. *For any element C in* $\mathrm{PU}(2, 1)$, *there exists a $\mathbb{C}$-decomposable pair (A, B) such that $[A, B] = C$.*

Proof. It suffices to show that every element in PU(2, 1) has a square root, which is a product of three involutions. Indeed, if I_1, I_2, and I_3 are involutions, then we have $(I_1 I_2 I_3)^2 = [I_1 I_2, I_3 I_2]$.

1. This is clear for loxodromic isometries, as the square root of a loxodromic map is loxodromic and thus is a product of three complex symmetries.
2. Every screw- or two-step unipotent parabolic isometry has a square root that is screw-parabolic, and thus a product of thee complex symmetries. The square root of a three-step unipotent isometry is also three-step unipotent and thus a product of three complex symmetries.
3. Let E be an elliptic element with angle pair $\{\theta_1, \theta_2\}$. Its square roots are those elliptic elements with angle pairs $\{\theta_1/2 + n\pi, \theta_2/2 + m\pi\}$, where m and n are 0 or 1. This implies, in particular, that every elliptic element has a square root that is regular elliptic with angle pair in $\mathcal{E}_{+++} \cup \mathcal{E}_{++-}$ (see Figures 14 and 15) and hence is a triple product of involutions. □

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