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Rational Curves on Hypersurfaces

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Abstract. Let (X, D) be a pair where X is a projective variety. We study in detail how the behavior of rational curves on X and the positivity of $-(K_X + D)$ and D influence the behavior of rational curves on D. In particular, we give criteria for uniruledness and rational connectedness of components of D.

1. Introduction

For a projective variety X, the connection between the positivity of $-K_X$ and the behavior of rational curves on X is well understood. Uniruledness and rational connectedness are possibly two birational properties of smooth varieties that have been the most intensively studied. A result of Miyaoka and Mori [MM86] shows that a smooth projective variety X is uniruled if and only if there exists a K_X-negative curve through every general point of X. Later Boucksom, Demailly, Păun, and Peternell [BDPP13] proved that if the canonical divisor of a projective manifold X is not pseudoeffective, then X is uniruled. The rational connectedness of smooth Fano varieties was established by Campana [Cam92] and Kollár, Miyaoka, and Mori [KMM92], and it was later generalized to the log Fano cases by Zhang [Zha06] and Hacon and McKernan [HM07].

A natural question is how the behavior or rational curves on a variety X influences the behavior of rational curves on a hypersurface D. An easy case is where $X = \mathbb{P}^n$; then a general hypersurface of degree $\le n$ is rationally connected. More generally, if (X, D) is a plt pair and $-(K_X + D)$ is ample, then by the adjunction formula we have $(K_X + D)|_D = K_D + \mathrm{Diff}_D(0)$, which is antiample and klt. So by [Zha06, Theorem 1] D is rationally connected and, in particular, uniruled. However, if we assume that $-(K_X + D)$ is big and semiample instead of ample, then the following example shows that D is not necessarily uniruled.

Example 1.1. Let $\pi : X = \mathbb{P}(\mathcal{E}) \to C$ be a ruled surface, where C is an elliptic curve, and $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ is such that $\mathcal{L}$ is a line bundle on C and $\deg(\mathcal{L}) < 0$. Let $e = -\deg(\bigwedge^2 \mathcal{E})$. Then $e > 0$ and $K_X \equiv_{\mathrm{num}} -2C_0 - eF$, where C_0 is the unique section of π with $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$ (see [Har77, Chapter V, Example 2.11.3]), and F is a fiber. So we have

$$-(K_X + C_0) \equiv_{\mathrm{num}} C_0 + eF = \varepsilon C_0 + (1-\varepsilon)\left(C_0 + \frac{e}{1-\varepsilon}F\right),$$

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where $\varepsilon \in (0, 1)$ is any rational number. Now $C_0 + \frac{e}{1-\varepsilon}F$ is ample by [Har77, Chapter V, Proposition 2.20], and C_0 is effective, so $-(K_X + C_0)$ is nef and big. Moreover, by [Gon12, Theorem 1.7] we know that $-(K_X + C_0)$ is semiample. However, C_0 is an elliptic curve and, in particular, not uniruled.

In this paper, we first give a criterion for uniruledness of D. Roughly speaking, we show that if X contains "sufficiently many" rational curves, then as long as $K_X + D$ is not pseudoeffective, the uniruledness of D holds. More precisely, we have the following:

THEOREM A (Theorem 3.1). *Let (X, D) be a pair where $D = \sum_i E_i + \sum_j a_j F_j$ is such that E_i and F_j are distinct prime divisors and $a_j \in (0, 1)$. Suppose that* $\mathrm{rd}(X) \geq 2$ *and $K_X + D$ is not pseudoeffective. Then E_i is uniruled for any i.*

Here $\mathrm{rd}(X)$ is the rational dimension of X, which is the dimension of the general fiber of the maximal rationally connected fibration of X (see Definition 2.1).

The author suspects that Theorem A is already sharp. First, note that we do not have any assumption on the singularities of the pair (X, D) in Theorem A. Next, Example 1.1 shows that the condition $\mathrm{rd}(X) \geq 2$ cannot be weakened even when $K_X + D$ is very negative (e.g. antibig and antisemiample). Finally, the following simple example indicates that the condition that $K_X + D$ is not pseudoeffective cannot be weakened either.

EXAMPLE 1.2. Let $C \subset \mathbb{P}^2$ be an elliptic curve of degree 3. Then we have $K_{\mathbb{P}^2} + C \sim_{\mathrm{lin}} 0$. Let $f : X \to \mathbb{P}^2$ be the blow-up of X at a point not in C. We have
$$K_X + f_*^{-1}C = f^*(K_{\mathbb{P}^2} + C) + E,$$
where E is the exceptional divisor. Now X is a rational surface, in particular, $\mathrm{rd}(X) = 2$. $K_X + f_*^{-1}C$ is pseudoeffective and yet not nef. In this case, $f_*^{-1}C$ is not a rational curve and hence not uniruled.

The strategy to prove Theorem A is to use the minimal model program in arbitrary dimension developed in [BCHM10] and an induction on the dimension of X.

Note that [LZ15, Theorem 3.7] implies Theorem A in the case where (X, D) is dlt. This was pointed out by De-Qi Zhang after the completion of this paper.

Motivated by Theorem A, we also consider rational connectedness of hypersurfaces and obtain the following:

THEOREM B (Theorems 4.2, 4.5, and 4.10). *Let (X, D) be a pair where $D = E + \sum_j a_j F_j$ is such that $\lfloor D \rfloor = E$ and F_j are discinct prime divisors and $a_i \in [0, 1)$. Assume that (X, D) is plt. Suppose that we are in one of the following cases.*

(1) *X is rationally connected, $\lfloor D \rfloor$ is big, and $K_X + D$ is not pseudoeffective.*
(2) *X is a rationally connected threefold, D is a prime divisor, and $-(K_X + D)$ is Cartier, nef, and big.*
(3) *X is a toric variety, and $-(K_X + D)$ is big and semiample.*

Then $\lfloor D \rfloor$ is rationally connected.

2. Preliminaries

In this paper, we work over the field of complex numbers $\mathbb{C}$. We freely use the standard notation in [HK10, especially 3.G] (e.g. pair, discrepancy, and klt, plt, dlt, and lc singularities). Terms such as uniruled, rationally connected (RC), and rationally chain connected (RCC) are also be used, and their definitions can be found in [Kol96]. The following definition can be found in [Har] by Harris.

DEFINITION 2.1. Let X be a proper smooth variety, and $f : X \dashrightarrow Z$ the maximal rationally connected fibration (see [Kol96, Definition 5.3]). We define the *rational dimension* of X as $\mathrm{rd}(X) := \dim(X) - \dim(Z)$. If X is singular, then we define the rational dimension of X as $\mathrm{rd}(\tilde{X})$ for some resolution $\mu : \tilde{X} \to X$ of X.

Next, we present two theorems, which are essential in the proof of Theorem A. We have the following definition of a minimal dlt model.

DEFINITION 2.2 ([KK10, Definitions and Notation 1.9]). Let (X, D) be a pair, and $f^m : X^m \to X$ a proper birational morphism such that

$$K_{X^m} + (f^m)_*^{-1} D = (f^m)^*(K_X + D) + \sum_i a_i E_i.$$

Let $D^m := (f^m)^{-1} D + \sum_{a_i \leq -1} E_i$. Then (X^m, D^m) is a *minimal dlt model* of (X, D) if it is a dlt pair and the discrepancy of every f^m-exceptional divisor is at most -1.

THEOREM 2.3 (Dlt modification by Hacon ([KK10, Theorem 3.1])). *Let (X, D) be a pair such that X is quasiprojective, D is a boundary, and $K_X + D$ is a $\mathbb{Q}$-Cartier divisor. Then (X, D) admits a $\mathbb{Q}$-factorial minimal dlt model $(X^m, D^m) \to (X, D)$. In particular, if $K_X + D$ is not pseudoeffective, then $K_{X^m} + D^m$ is also not pseudoeffective.*

REMARK 2.4. The reason for the second statement in Theorem 2.3 is the following. We have that f^m only extracts divisors with discrepancy ≤ -1. So by the definition of D^m we can write

$$K_{X^m} + D^m = f^*(K_X + D) + \sum_j b_j E_j,$$

where $b_j \leq 0$. Therefore the second statement holds.

The second theorem is the existence of a Mori fiber space established by Birkar, Cascini, Hacon, and McKernan. For convenience, we give the definition of a Mori fiber space.

DEFINITION 2.5 ([BCHM10, Definition 3.10.7]). Let (X, Δ) be a log canonical pair, and $f : X \to Z$ be a projective morphism of normal varieties. Then f is a *Mori fiber space* if

(1) X is $\mathbb{Q}$-factorial and Δ is an $\mathbb{R}$-divisor,
(2) f is a contraction morphism, $\rho(X/Z) = 1$, and $\dim Z < \dim X$, and
(3) $-(K_X + \Delta)$ is f-ample.

THEOREM 2.6 (Existence of a Mori fiber space ([BCHM10, Corollary 1.3.3])). *Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair. Let $\pi : X \to U$ be a projective morphism of normal quasiprojective varieties. Suppose that $K_X + \Delta$ is not π-pseudoeffective. Then we can run a $(K_X + \Delta)$-minimal model program over U that ends with a Mori fiber space over U.*

Finally, in this section, we provide the following lemma, which is known to experts.

LEMMA 2.7. *Let (X, D) be a klt pair. Suppose that we have a morphism $f : X \to Y$ such that* $\dim(Y) < \dim(X)$ *and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Then for a general fiber F of f, $(F, D|_F)$ is klt.*

Proof. We do a log resolution for (X, D), which we denote by $\mu : X' \to X$, and define D' as

$$K_{X'} + D' = \mu^*(K_X + D).$$

We write $D' = \Gamma' - E'$ where Γ' and E' are effective $\mathbb{Q}$-divisors that have no common components. Let $f' = f \circ \mu$, and let F' be a general fiber of f' that maps to a general fiber of f through μ. Then we have the following diagram:

$$\begin{array}{ccc} F' & \xrightarrow{\nu} & F \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\mu} & X \end{array}$$

Since $\Gamma'|_{F'}$ is simple normal crossing, we have that

$$(K_{X'} + \Gamma')|_{F'} = K_{F'} + \Gamma'|_{F'}$$

is klt. So

$$\nu^*(K_F + D|_F) = \mu^*(K_X + D)|_{F'} = (K_{X'} + D')|_{F'} = K_{F'} + \Gamma' - E'$$

is sub-klt. Therefore $K_F + D|_F$ is klt. □

3. Uniruledness of Hypersurfaces

The main theorem of this section is as follows.

THEOREM 3.1. *Let (X, D) be a pair where $D = \sum_i E_i + \sum_j a_j F_j$ is such that E_i and F_j are distinct prime divisors and $a_j \in (0, 1)$. Suppose that* $\mathrm{rd}(X) \geq 2$ *and $K_X + D$ is not pseudoeffective, then E_i is uniruled for any i.*

By Theorem 2.3 and Remark 2.4 we can assume that (X, D) is dlt and $\mathbb{Q}$-factorial by possibly doing a dlt modification. We first consider the case where $\dim(X) = 2$. Note that, in this case, $\mathrm{rd}(X) \geq 2$ is equivalent to that X is RC.

LEMMA 3.2. *Let (X, D) be a dlt pair where* $\dim(X) = 2$. *Suppose that X is RC and $K_X + D$ is not pseudoeffective. Then every component of D with coefficient* 1 *is a rational curve* (*in particular, uniruled*).

Proof. We run a $(K_X + D)$-minimal model program. Since $K_X + D$ is not pseudoeffective, by [Fuj12, Theorem 1.1] the minimal model program ends with a Mori fiber space, which we denote by $g : X' \to Y$. Since X' is an RC surface, Y is either a point or a rational curve. If any component of $f_*^{-1}D$ is contracted during the minimal model program, then by [Kaw91, Theorem 2] that component must be a rational curve. We denote the strict transform of $f_*^{-1}D$ on X' by D' and denote by $D'_1, \dots, D'_m$ the irreducible components of D' with coefficient 1.

If Y is a point, then $K_{X'} + D'$ is antiample. By the adjunction formula (see [Cor07, Proposition 3.9.2]) we have

$$(K_{X'} + D')|_{D'_i} = (K_{X'} + D'_i)|_{D'_i} + (D' - D'_i)|_{D'_i} = K_{D'_i} + \mathrm{Diff}_{D'_i}(D' - D'_i)$$

and $\mathrm{Diff}_{D'_i}(D' - D'_i) \geq 0$ by [Kol92, Proposition-Definition 16.5]. So $K_{D'_i}$ has negative degree, and hence D'_i is a rational curve. So for the rest of the proof, we assume that Y is a rational curve. For any i, if D' does not dominate Y, then it is a component of a fiber of g, which is a rational curve by [Deb01, Lemma 3.7]. If D'_i dominates Y, then $\deg(g|_{D'_i}) = \deg(D'_i|_F) > 0$. Moreover, we have

$$\begin{aligned} 0 &> \deg((K_{X'} + D')|_F) > \deg((K_{X'} + D'_i)|_F) = \deg(K_{X'}|_F) + \deg((D'_i)|_F) \\ &= \deg(K_F) + \deg((D'_i)|_F) = -2 + \deg((D'_i)|_F), \end{aligned}$$

where the first inequality is by the fact that $-(K_{X'} + D')$ is g-ample, and the last equality is by the fact that $F \cong \mathbb{P}^1$. So we get that $\deg(g|_{D'_i}) = \deg((D'_i)|_F) = 1$, and since Y is rational, we know that D'_i is rational. Hence every component of D with coefficient 1 is rational. □

Proof of Theorem 3.1. By the argument before Lemma 3.2 it suffices to prove the theorem under the hypothesis that (X, D) is dlt. We prove the theorem by induction on the dimension of X. When $\dim(X) = 2$, this is proven in Lemma 3.2. Suppose that the statement holds in any dimension $k \in [2, n-1]$. Then in dimension n, we first run a minimal model program with scaling for $(K_X + D)$. Since $K_X + D$ is not pseudoeffective, there is an effective ample $\mathbb{Q}$-divisor A such that

- no component of A is contained in $\mathrm{Supp}(D)$;
- $K_X + D + A$ is still dlt and not pseudoeffective;
- there exists a $\mathbb{Q}$-divisor D_A such that $D + A \sim_{\mathbb{Q}} D_A$ and (X, D_A) is klt.

We run a $(K_X + D_A)$-MMP, and by Theorem 2.6 it ends with a Mori fiber space as follows:

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \cdots \overset{f_{N-1}}{\dashrightarrow} X_N = X' \overset{g}{\to} Y. \tag{3.1}$$

Denote the strict transform of D, A, and E_i on X_k by D^k, A^k, and E_i^k, respectively. If for a certain i and k, E_i^k is contracted by f_k, then by [Kaw91, Theorem 2] we know that E_i^k is uniruled. By the assumption on $(X, D + A)$ and [KM98, Lemma 3.38] we know that, for any k, $(X_k, D^k + A^k)$, hence (X_k, D^k), is dlt. Moreover, it is easy to see that $\mathrm{rd}(X_i) \geq 2$ for any i. So we can assume that there is a morphism $f : X \to Y$ that is a Mori fiber space. By condition (3) in Definition 2.5, Lemma 2.7, and [Zha06, Theorem 1] we have that a general fiber of f is RC (note that in this step we can in fact work with (X, D_A), which is klt instead of dlt). Now we consider the following three cases.

Case 1. If $\dim(Y) = 0$, then $-(K_X + D)$ is ample. So, for any E_i, by the adjunction formula we have

$$(K_X + D)|_{E_i} = K_{E_i} + \mathrm{Diff}_{E_i}(D - E_i).$$

Hence $K_{E_i} + \mathrm{Diff}_{E_i}(D - E_i)$ is antiample and dlt, and in particular $-K_{E_i}$ is big. Now if we do a K_{E_i}-minimal model program, it would end with a Mori fiber space, and in particular E_i is uniruled.

Case 2. If $1 \leq \dim(Y) \leq n - 2$, then, for any i, we can assume that E_i dominates Y. Indeed, if this is not the case, then for dimensional reasons, E_i is covered by fibers of f, and by the fact that the general fibers of f are RC and [Deb01, Lemma 3.7] we know that every fiber of f is covered by rational curves. So we are done. Now, for a general fiber F of f, we have that F is RC and $2 \leq \dim(F) \leq n - 1$. Suppose that $E_i|_F = \sum_l E^l_{F,i}$, where $E^l_{F,i}$ are the irreducible components of $E_i|_F$. By the adjunction formula we know that

$$(K_X + D)|_F = K_F + D|_F = K_F + E^l_{F,i} + (D|_F - E^l_{F,i})$$

is antiample, so $-(K_F + E^l_{F,i})$ is big for any i. After possibly doing a dlt modification for $(F, E^l_{F,i})$, we can also assume that $(F, E^l_{F,i})$ is dlt. By induction hypothesis we know that $E^l_{F,i}$ is uniruled for any i. Therefore E_i is uniruled.

Case 3. If $\dim(Y) = n - 1$, then for the same reason as in *Case 2,* we can assume that E_i dominates Y for any i. After shrinking X to its nonsingular locus, by generic smoothness the general fibers of f are isomorphic to $\mathbb{P}^1$. Since $\mathrm{rd}(X) \geq 2$, we know that Y is uniruled. So we only need to show that $f|_{E_i}$ has degree 1. If $\deg(f|_{E_i}) \geq 2$, then for a general fiber F of f, we have

$$\deg((K_X + D)|_F) \geq \deg(K_F + E_i|_F) = \deg(K_F) + \deg(E_i|_F) \geq -2 + 2 = 0,$$

and in particular $-(K_X + D)$ cannot be f-ample. This is a contradiction, so we are done. □

4. Rational Connectedness of Hypersurfaces

Of course, we can also ask whether certain positivity of $-(K_X + D)$ implies rational connectedness of components of D. This seems more complicated than uniruledness. We first point out that we cannot get RC-ness of components of D by simply letting X be RC in Theorem 3.1, even for log-smooth pairs in dimension 3.

EXAMPLE 4.1. Let $g : X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be the $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(d)$ and $d \leq -1$. Then $\omega_{X/\mathbb{P}^2} = g^*(\bigwedge^2 \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2)$. Hence

$K_X \sim_{\text{lin}} (d-3)g^*H - 2h$, where H is a hyperplane in $\mathbb{P}^2$, and h is the divisor class in $\mathbb{P}(\mathcal{E})$ induced by $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. We take a general hypersurface $S \sim_{\text{lin}} 3H$ in $\mathbb{P}^2$, which is an elliptic curve. Let $D := g^{-1}(S)$. Then

$$-K_X - D \sim_{\text{lin}} -dg^*H + 2h,$$

which is big, but obviously $g^{-1}(S)$ is not RC as S is not rational.

However, if we assume the bigness of $\lfloor D \rfloor$, then we have the following result.

THEOREM 4.2. *Let (X, D) be a plt pair. Suppose that X is RC, $\lfloor D \rfloor$ is big, and $K_X + D$ is not pseudoeffective. Then $\lfloor D \rfloor$ is RC.*

LEMMA 4.3. *Let (X, D) be a $\mathbb{Q}$-factorial pair where D is a big prime divisor. Let $\pi : X \to X'$ be a divisorial contraction such that $\rho(X/X') = 1$. Then* $\dim(\pi(D)) = \dim(D)$.

Proof. Suppose that D is contracted to a lower-dimensional variety. If D is π-nef, then by the negativity lemma (see [KM98, Lemma 3.39]) we have $D = 0$, which is a contradiction. If D is not π-nef, then there is a curve $\tilde{C}$, contracted by π, such that $\tilde{C} \cdot D < 0$. We also observe that by the bigness of D, $C' \cdot D \geq 0$ for a general curve C' contracted by π. On the other hand, whenever we choose a very ample divisor H on X, we have that $C \cdot H > 0$ for any curve C in X. This is a contradiction to the assumption $\rho(X/X') = 1$. So we are done. □

Proof of Theorem 4.2. Let $S := \lfloor D \rfloor$ and $B := \{D\}$. We do the same minimal model program as in the proof of Theorem 3.1 as follows:

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \cdots \overset{f_{N-1}}{\dashrightarrow} X_N = X' \overset{g}{\to} Y. \tag{4.1}$$

We denote the strict transform of D, S, and B on X_i by D_i, S_i, and B_i, respectively. Certainly, S_i is big, and hence it cannot be contracted. Moreover, by the adjunction formula we have

$$(K_{X_i} + D_i)|_{S_i} = K_{S_i} + \text{Diff}_{S_i}(B_i),$$

so $(S_i, \text{Diff}_{S_i}(B_i))$ is klt for any i. Therefore we can assume the existence of a morphism $f : X \to Y$ that is a Mori fiber space. Now since X is RC, so is Y, and since S is big, it must dominate Y. Next, we consider the following three cases.

Case 1. If $\dim(Y) = 0$, then $-(K_X + D)$ is ample, so by the adjunction formula we have that $-(K_S + \text{Diff}_S(B))$ is ample and $(S, \text{Diff}_S(B))$ is klt. Then by [Zha06, Theorem 1] S is RC.

Case 2. If $1 \leq \dim(Y) \leq n-2$, then we denote a general fiber of f by F. Then $(K_X + D)|_F = K_F + D_F$ is antiample. By the Kollár–Shokurov connectedness lemma (see [Pro01, Theorem 2.3.1]) we see that $S|_F$ is connected. Now we do a Stein factorization of $f|_S$ and denote it as

$$S \overset{g}{\to} Z \overset{h}{\to} Y.$$

Since $S|_F$ is connected, we know that h is birational. So Z is RC since RC-ness is a birational invariant (cf. [Kol96, Chapter IV, Proposition 3.3]).

On the other hand, since $-(K_X+D)$ is f-ample, $K_S+\mathrm{Diff}_S(B)$ is $f|_S$-ample and hence g-ample. So if we denote the fiber of $f|_S$ over a general point z of Z by S_z, then by Lemma 2.7 we know that $K_{S_z}+\mathrm{Diff}_S(B)|_{S_z}$ is klt and antiample. Hence S_z is RC.

Finally, by [GHS03, Corollary 1.3] we know that S is RC.

Case 3. If $\dim(Y)=n-1$, then by the same argument as in the proof of Theorem 3.1 we have that $\deg(f|_S)=1$. Moreover, since X is RC, we know that Y is RC, and hence S is RC. □

Going back to Example 1.1, we see that $-(K_X+D)$ being big and semiample does not imply the RC-ness of components of D. Nevertheless, we can ask what happens if we assume in addition the RC-ness of X. Clearly, we cannot expect that every component of D with coefficient 1 is RC. For example, if we take $D=g^{-1}(S)+h$ in Example 4.1, then $-K_X-D=-dg^*H+h$, which is big and semiample, but $g^{-1}(S)$ is still not RC. However, on the other hand, if (X, D) is dlt, then by the Kollár–Shokurov connectedness lemma the union of all the components of D with coefficient 1 is connected. So we can still ask whether such locus is rationally chain connected.

QUESTION 4.4. Let (X, D) be a dlt pair where $D=\sum_i E_i+\sum_j a_jF_j$ is such that E_i and F_j are prime divisors and $a_j\in(0,1)$. Suppose that X is RC and $-(K_X+D)$ is big and semiample, then is $\bigcup_i E_i$ RCC?

Unfortunately, we do no have an answer to *Question* 4.4 in general so far. Nevertheless, we are able to show that the answer is positive for certain cases of threefolds and toric varieties.

THEOREM 4.5. *Let (X, D) be three-dimensional plt pair where D a prime divisor on X. Suppose that X is RC and $-(K_X+D)$ is Cartier, nef, and big. Then D is RC.*

LEMMA 4.6. *Let S be a normal surface with rational singularities. If S is birational to a ruled surface and $H^1(S,\mathcal{O}_S)=0$, then S is a rational surface.*

Proof. We do a resolution $f: S'\to S$ for S. Since S is birational to a ruled surface, so is S'. In particular, $H^0(S',\mathcal{O}_{S'}(2K_{S'}))=0$. On the other hand, we have

$$h^1(S',\mathcal{O}_{S'})=h^1(S',f^*\mathcal{O}_S)=h^1(S,f_*f^*\mathcal{O}_S)=h^1(S,\mathcal{O}_S)=0,$$

where the second equality is by the assumption that S has rational singularities. So by a theorem of Castelnuovo (see [Bea96, Theorem V.1]) we know that S is a rational surface. □

LEMMA 4.7. *Let (X, D) be a plt pair, where X has dimension $n\geq 2$, and D is prime divisor on X. Suppose that X is RC and $-(K_X+D)$ is Cartier, nef, and big. Then $H^1(D,\mathcal{O}_D)=0$.*

Proof. We have the short exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D) \to \mathcal{O}_D(K_D) \to 0,$$

which yields the following long exact sequence:

$$\begin{aligned} \cdots &\to H^{n-2}(X, \mathcal{O}_X(K_X + D)) \\ &\to H^{n-2}(D, \mathcal{O}_D(K_D)) \to H^{n-1}(X, \mathcal{O}_X(K_X)) \to \cdots . \end{aligned}$$

Since X is klt and RC, we know thst $H^{n-1}(X, \mathcal{O}_X(K_X)) = H^1(X, \mathcal{O}_X) = 0$. By Kawamata–Viehweg vanishing we also have

$$\begin{aligned} H^{n-2}(X, \mathcal{O}_X(K_X + D)) &= H^2(X, \mathcal{O}_X(-D)) \\ &= H^2(X, \mathcal{O}_X(K_X + (-K_X - D))) = 0 \end{aligned}$$

since $-K_X - D$ is nef and big by assumption. So we get

$$H^{n-2}(D, \mathcal{O}_D(K_D)) = H^1(D, \mathcal{O}_D) = 0.$$ □

Proof of Proposition 4.5. By Lemma 4.7 we have $H^1(D, \mathcal{O}_D) = 0$. On the other hand, by Theorem 3.1 we know that D is birational to a ruled surface. So by Lemma 4.6 we are done. □

Before showing the result for toric varieties, we present the following proposition, which we hope to be of independent interest.

PROPOSITION 4.8. *Suppose that we have a pair (X, D) where $D = \sum_i E_i + \sum_j a_j F_j$ is such that E_i and F_j are prime divisors and $a_j \in (0, 1)$. Suppose that (X, D) is dlt, $-(K_X + D)$ is big and semiample, and there is no lc center (or equivalently, non-klt center) of (X, D) that is contained in $\mathbf{B}_+(-(K_X + D))$. Then E_i is RC for any i.*

To prove this, we need the following lemma, which is a slight modification of [Zha06, Theorem 1].

LEMMA 4.9. *Let (X, D) be a dlt pair and suppose that $-(K_X + D)$ is ample. Then X is RC.*

Proof. By [KM98, Proposition 2.43] we can perturb D so that (X, D) is klt and $-(K_X + D)$ still stays ample. So by [Zha06, Theorem 1] we are done. □

Proof of Proposition 4.8. By assumption there exists an effective $\mathbb{Q}$-divisor H such that $H \sim_{\mathbb{Q}} -(K_X + D)$ and $H \sim_{\mathbb{Q}} A + G$, where A is ample, and G is effective. We have

$$0 \sim_{\mathbb{Q}} K_X + D + H \sim_{\mathbb{Q}} K_X + D + (1 - \varepsilon)H + \varepsilon(A + G).$$

Moreover, we can arrange ε, H, and G such that

- $E_j \not\subseteq \operatorname{Supp}(G)$ for any j;
- $(X, D + (1-\varepsilon)H + \varepsilon G)$ is dlt, and the only components of $D + (1-\varepsilon)H + \varepsilon G$ with coefficient 1 are the E_j.

Now by adjunction there exists an effective $\mathbb{Q}$-divisor D_{E_i} such that

$$(K_X + D + (1-\varepsilon)H + \varepsilon G)|_{E_i} \sim_{\mathbb{Q}} K_{E_i} + D_{E_i}$$

and (K_{E_i}, D_{E_i}) is dlt. By construction we have $K_X + D + (1-\varepsilon)H + \varepsilon G \sim_{\text{lin}} -\varepsilon A$ is antiample, so $K_{E_i} + D_{E_i}$ is antiample as well. Then, by Lemma 4.9, E_i is RC. □

THEOREM 4.10. *Let (X, D) be a plt pair where X is a toric variety. Suppose that $-(K_X + D)$ is big and semiample. Then $\lfloor D \rfloor$ is RC.*

Proof. If $\lfloor D \rfloor$ is toric invariant, then we are done. If not, then by [CLS11, Lemma 15.1.8] we know that $\lfloor D \rfloor$ is $\mathbb{Q}$-linearly equivalent to a linear combination of Cartier toric invariant divisors with nonnegative coefficients, and in particular $\lfloor D \rfloor \nsubseteq \mathbf{B}_+(-(K_X + D))$. So by Proposition 4.8 we are done. □

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