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The Tangent Space of the Punctual Hilbert Scheme

DORI BEJLERI & DAVID STAPLETON

Introduction

In this paper, we study the Zariski tangent space of the punctual Hilbert scheme parameterizing subschemes of a smooth surface that are supported at a single point. We give a lower bound on the dimension of the tangent space in general and show that the bound is sharp for subschemes of the affine plane cut out by monomials.

Let S be a smooth connected complex surface, and denote by $S^{[n]}$ the Hilbert scheme parameterizing length n subschemes of S. Fogarty [Fog68] showed that $S^{[n]}$ is smooth and irreducible. We write $S^{(n)}$ for the symmetric power of S. The Hilbert–Chow morphism

$$h : S^{[n]} \to S^{(n)}$$

that sends a length n subscheme to its cycle is invaluable in the study of $S^{[n]}$. We denote by P_n the *nth punctual Hilbert scheme,* which is the reduced fiber of h over a multiplicity n cycle in $S^{(n)}$. Thus P_n parameterizes length n subschemes supported at one point. Note that P_n is the same for any smooth surface, so throughout we assume that $S \cong \mathbb{C}^2$.

The Hilbert–Chow morphism and the punctual Hilbert scheme have attracted a great deal of attention. Beauville [Bea83] has shown that if S is a K3 surface, then $S^{[n]}$ is a holomorphic symplectic variety (one of few known examples). Mukai [Muk84] gave a description of the symplectic form in terms of the pairing on $\mathrm{Ext}^1(I, I)$. For general surfaces, h gives a crepant resolution. Briançon [Bri77] has shown that P_n is irreducible, and Haiman [Hai98] has shown that P_n is the scheme-theoretic fiber of h and that P_n is a local complete intersection scheme. Iarrobino [Iar77] and Granger [Gra83] studied a stratification of P_n into loci parameterizing subschemes having some fixed Hilbert–Samuel function. The Betti numbers of the punctual Hilbert scheme were computed by Ellingsrud and Stromme [ES87] by studying the action of $(\mathbb{C}^*)^2$ on $(\mathbb{C}^2)^{[n]}$. Later Yaméogo [Yam94] studied the closure relations between the Hilbert-Samuel strata. Iarrobino [Iar72] showed that the Hilbert scheme of length n subschemes of $\mathbb{A}^k$ is reducible when $k \geq 3$ and n is large, and Erman [Erm12] showed that the (graded) Hilbert–Samuel strata can acquire arbitrary singularities. Huibregtse [Hui79; Hui82] studied questions of irreducibility and smoothness of a variety related to P_n that consists of subschemes of $S^{[n]}$ whose sum in the Albanese variety of S is constant.

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There is a natural tautological vector bundle $(T_S)^{[n]}$ on $S^{[n]}$ whose fiber at a point corresponding to the length n subscheme $\xi \subset S$ is the $2n$-dimensional vector space $H^0(S, T_S|_\xi)$. In [Sta16] the second author showed that there is a natural injection of sheaves

$$\alpha_n : (T_S)^{[n]} \to T_{S^{[n]}}$$

that at a point $[\xi] \in S^{[n]}$ comes from the normal sequence of $\xi \subset S$, that is,

$$\alpha_n|_{[\xi]} : H^0(S, T_S|_\xi) \mapsto \mathrm{Hom}(I_\xi, \mathcal{O}_\xi).$$

Moreover, $(T_S)^{[n]}$ is the log-tangent sheaf of the exceptional divisor of h. Thus it is natural to expect that the degeneracy loci of α_n are connected to the singularities of the exceptional divisor of h. To make this precise, we relate the rank of α_n to the dimension of the Zariski tangent space of the punctual Hilbert scheme.

THEOREM A. *If $\xi \subset \mathbb{C}^2$ is a length n subscheme supported at the origin, then*

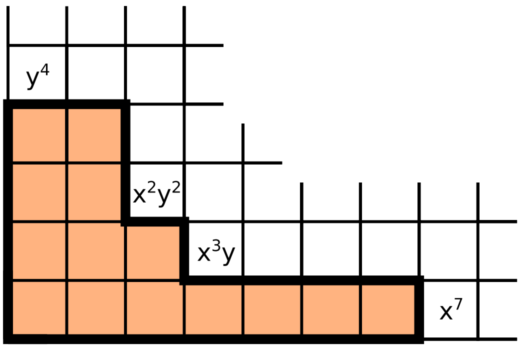
$$\dim(T_{[\xi]} P_n) \geq 2n - \mathrm{rank}(\alpha_n|_{[\xi]}) = \mathrm{corank}(\alpha_n|_{[\xi]}).$$

Moreover, equality holds when the ideal of ξ is generated by monomials.

When ξ is a monomial subscheme, the ideal of ξ (written $I_\xi \subset \mathbb{C}[x, y]$) has an associated Young diagram $\mu_\xi \subset \mathbb{N}^2$ defined as

$$\mu_\xi := \{(i, j) \in \mathbb{N}^2 | x^i y^j \notin I_\xi\}.$$

For example, when $I_\xi = (y^4, x^2y^2, x^3y, x^7)$, the length of ξ is 14, and we associate to ξ the following Young diagram:



An elementary statistic associated with μ_ξ is given by tracing the top perimeter of the Young diagram from the top left to the bottom right and keeping track of the horizontal and vertical steps. For example, in the figure, we have a sequence of horizontal steps $\Delta h = (2, 1, 4)$ and vertical steps $\Delta v = (2, 1, 1)$.

THEOREM B. *If ξ is defined by monomials and μ_ξ is the corresponding Young diagram, then*

$$\operatorname{rank}(\alpha_n|_{[\xi]}) = \begin{pmatrix} \textit{maximum of horizontal} \\ \textit{steps of } \mu_\xi \end{pmatrix} + \begin{pmatrix} \textit{maximum of vertical} \\ \textit{steps of } \mu_\xi \end{pmatrix}.$$

In the pictured example, we have $\operatorname{rank}(\alpha_n|_{[\xi]}) = 4+2 = 6$, so $\dim T_{[\xi]}P_n = 28 - 6 = 22$.

To prove the inequality in Theorem A, we remark that the cokernel of the derivative

$$dh : h^*\Omega_{(\mathbb{C}^2)^{(n)}} \to \Omega_{(\mathbb{C}^2)^{[n]}}$$

restricted to $[\xi] \in P_n$ is the cotangent space of P_n. This follows from Haiman's result that P_n is the scheme-theoretic fiber of h. Moreover, $(\mathbb{C}^2)^{[n]}$ is equipped with a holomorphic symplectic form [Nak99, §1.4], which gives an isomorphism $\omega : T_{(\mathbb{C}^2)^{[n]}} \cong \Omega_{(\mathbb{C}^2)^{[n]}}$. So to prove the inequality, it suffices to show that there is a map

$$i : h^*\Omega_{(\mathbb{C}^2)^{(n)}} \to (T_{\mathbb{C}^2})^{[n]}$$

such that $dh = \omega \circ \alpha_n \circ i$. In fact, it suffices to define i away from codimension 2, and away from codimension 2, the map h is (étale locally) a product of the resolution of an A_1 singularity with a smooth variety. So the inequality follows after a computation in the case of an A_1 singularity, using the interpretation of $(T_{\mathbb{C}^2})^{[n]}$ as the log-tangent sheaf of the exceptional divisor of h.

To carry out the computation in Theorem B, we use the description of $\alpha_n|_{[\xi]}$ coming from the normal sequence of $\xi \subset \mathbb{C}^2$ and carry out the calculation on $\mathbb{C}^2$.

To show that equality holds in Theorem A for subschemes of $\mathbb{C}^2$ cut out by monomials, our main computational tool is the affine chart that Haiman introduced in [Hai98] for $(\mathbb{C}^2)^{[n]}$ and the description Haiman gave of the cotangent space at monomial subschemes. Using these tools, we explicitly compute the rank of dh at points in $(\mathbb{C}^2)^{[n]}$ corresponding to monomial subschemes and show that, for $\xi \subset \mathbb{C}^2$ cut out by monomials,

$$\operatorname{rank}(dh|_{[\xi]}) = \begin{pmatrix} \text{maximum of horizontal} \\ \text{steps of } \mu_\xi \end{pmatrix} + \begin{pmatrix} \text{maximum of vertical} \\ \text{steps of } \mu_\xi \end{pmatrix}.$$

Associating this rank to a partition μ gives a new statistic on the set of partitions. We expect that this is an interesting combinatorial statistic, adding to the list of geometrically meaningful statistics constructed using Hilbert schemes (e.g. [Hai98; LW09]).

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1. The Proof of the Inequality in Theorem A

In this section, we prove the inequality in Theorem A. We start by recalling the main properties of the Hilbert scheme of points that we will need. Let $\mathcal{Z}_n$ be the universal family of the Hilbert scheme of points on S. Then $\mathcal{Z}_n$ has two natural projections:

$$\begin{tikzcd} S \times S^{[n]} \supset \mathcal{Z}_n \arrow[r, "p_1"] \arrow[d, "p_2"'] & S. \\ S^{[n]} & \end{tikzcd}$$

If $\mathcal{E}$ is a vector bundle on S, then *the tautological bundle associated with* $\mathcal{E}$ is $\mathcal{E}^{[n]} := p_{2*}(p_1^*\mathcal{E})$. The map α_n is obtained by looking at the normal sequence of the inclusion $\mathcal{Z}_n \subset S \times S^{[n]}$,

$$0 \to T_{\mathcal{Z}_n} \to T_{S\times S^{[n]}}|_{\mathcal{Z}_n} \cong p_1^*(T_S) \oplus p_2^*(T_{S^{[n]}}) \xrightarrow{\beta_n} \mathcal{H}\mathrm{om}(I_{\mathcal{Z}_n}/I_{\mathcal{Z}_n}^2, \mathcal{O}_{\mathcal{Z}_n}).$$

Applying $p_{2*}(-)$, we see that $p_1^*(T_S)$ pushes forward to $(T_S)^{[n]}$ and $\mathcal{H}\mathrm{om}(I_{\mathcal{Z}_n}/I_{\mathcal{Z}_n}^2, \mathcal{O}_{\mathcal{Z}_n})$ pushes forward to $T_{S^{[n]}}$. Then

$$\alpha_n := p_{2*}(\beta_n|_{p_1^*(T_S)}).$$

The symmetric power $(\mathbb{C}^2)^{(n)}$ is the quotient of $(\mathbb{C}^2)^n$ by the permutation action of the symmetric group on n elements: $\mathfrak{S}_n$. The Hilbert–Chow morphism

$$h : (\mathbb{C}^2)^{[n]} \to (\mathbb{C}^2)^{(n)}$$

maps a point corresponding to a subscheme $[\xi]$ to the n-cycle:

$$h([\xi]) = \sum_{p\in \mathrm{Supp}(\xi)} \mathrm{length}_{\mathbb{C}}(\mathcal{O}_{\xi,p}) \cdot [p].$$

The exceptional divisor of h, denoted by B_n, consists of nonreduced subschemes.

REMARK 1. Let $f : X \to Y$ be a map of schemes, $p \in Y$ a point, and $f^{-1}(p)$ the scheme-theoretic fiber over p. If $q \in f^{-1}(p)$ is a point in this fiber, then the Zariski tangent space of $f^{-1}(p)$ at q is

$$T_q f^{-1}(p) \cong \mathrm{Coker}(df|_q : f^*\Omega_Y|_q \to \Omega_X|_q)^\vee.$$

Haiman proved [Hai98, Prop. 2.10] that the variety P_n is the scheme-theoretic fiber of h. Therefore, to compute the tangent space of P_n, it suffices to compute the corank of dh. In particular, to prove the inequality in Theorem A, it suffices to prove

$$\mathrm{corank}(dh|_{[\xi]}) \geq \mathrm{corank}(\alpha_n|_{[\xi]}).$$

Recall that there is a holomorphic symplectic form $\omega_n \in H^0((\mathbb{C}^2)^{[n]}, \wedge^2\Omega_{(\mathbb{C}^2)^{[n]}})$ on $(\mathbb{C}^2)^{[n]}$ [Nak99, §1.4] that gives an isomorphism $\omega_n : T_{(\mathbb{C}^2)^{[n]}} \cong \Omega_{(\mathbb{C}^2)^{[n]}}$. To bound the corank of dh, it suffices to prove that the map dh factors through $\omega_n \circ \alpha_n$:

$$\begin{tikzcd} & (T_{\mathbb{C}^2})^{[n]} \arrow[d, "\omega_n \circ \alpha_n"] \\ h^*\Omega_{(\mathbb{C}^2)^{(n)}} \arrow[r, "dh"] \arrow[ur, dashed, "\exists"] & \Omega_{(\mathbb{C}^2)^{[n]}}. \end{tikzcd}$$

Since $\omega_n \circ \alpha_n$ is injective, is suffices to show that $\omega_n \circ \alpha_n (T_{\mathbb{C}^2})^{[n]}$ contains the image of dh.

The following lemma proves that we can check $\mathrm{im}(dh) \subset \omega_n \circ \alpha_n (T_{\mathbb{C}^2})^{[n]}$ étale locally away from a subvariety of codimension 2.

LEMMA 2. *Suppose that X is a smooth variety and $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{E}$ are subsheaves of a torsion-free sheaf on X. If $\mathcal{F}_2$ is reflexive, then the following are equivalent*:

(1) *$\mathcal{F}_1 \subset \mathcal{F}_2$ as subsheaves of $\mathcal{E}$.*
(2) *There is an open subset $V \subset X$ with codimension 2 complement such that $\mathcal{F}_1|_V \subset \mathcal{F}_2|_V$ as subsheaves of $\mathcal{E}|_V$.*
(3) *There is an étale neighborhood $i : U \to X$ such that the complement of $V := i(U)$ has codimension 2 and $i^*\mathcal{F}_1 \subset i^*\mathcal{F}_2$ as subsheaves of $i^*\mathcal{E}$.*

Proof. It is clear that (1) implies (2). Now we show the reverse. We remark that $\mathcal{F}_1$ is torsion-free, so it includes into its reflexive hull $\mathcal{F}_1 \hookrightarrow \mathcal{F}_1^{\vee\vee}$. The inclusion $\mathcal{F}_1|_V \subset \mathcal{F}_2|_V$ as submodules of $\mathcal{E}$ extends to a map $\mathcal{F}_1|_V^{\vee\vee} \to \mathcal{F}_2|_V$ by taking double duals. This map is injective since it is injective at the generic point. Now an inclusion of reflexive sheaves on a smooth variety outside of a codimension 2 set uniquely extends to an inclusion on the whole variety. This follows from the fact that reflexive sheaves are *normal* (see [OSS11, p. 76]). Thus we have an inclusion $\mathcal{F}_1^{\vee\vee} \subset \mathcal{F}_2$ and consequently $\mathcal{F}_1 \subset \mathcal{F}_1^{\vee\vee} \subset \mathcal{F}_2$.

Flatness of i proves that (2) implies (3). For the reverse, faithful flatness of i mapping onto V gives an inclusion $\mathcal{F}_1|_V \subset \mathcal{F}_2|_V$, and V is an open set whose complement has codimension 2. □

Let $\mathfrak{S}_2 = \langle (12) \rangle \leq \mathfrak{S}_n$ be the subgroup that exchanges 1 and 2. Denote by $\Delta \subset (\mathbb{C}^2)^n$ the big diagonal fixed by $\mathfrak{S}_2$. The quotient map $\sigma : (\mathbb{C}^2)^n \to (\mathbb{C}^2)^{(n)}$ factors as

$$\begin{tikzcd} (\mathbb{C}^2)^n \arrow[d, "\tau"'] \arrow[dr, "\sigma"] & \\ (\mathbb{C}^2)^n/\mathfrak{S}_2 \arrow[r, "j"'] & (\mathbb{C}^2)^{(n)}. \end{tikzcd}$$

After appropriate change of coordinates, $(\mathbb{C}^2)^n/\mathfrak{S}_2 \cong \mathbb{C}^2/(\pm 1) \times (\mathbb{C}^2)^{n-1}$, and the symplectic form on the smooth locus of $(\mathbb{C}^2)^{(n)}$ pulls back and extends to the product symplectic form on the smooth locus of $\mathbb{C}^2/(\pm 1) \times (\mathbb{C}^2)^{n-1}$. Recall that $\mathbb{C}^2/(\pm 1) \times (\mathbb{C}^2)^{n-1}$ admits a symplectic resolution

$$h_0 \times id_{(\mathbb{C}^2)^{n-1}} : T^*\mathbb{P}^1 \times (\mathbb{C}^2)^{n-1} \to \mathbb{C}^2/(\pm 1) \times (\mathbb{C}^2)^{n-1}$$

by blowing up $\tau(\Delta)$. Here $T^*\mathbb{P}^1$ denotes the cotangent bundle of $\mathbb{P}^1$ with the standard symplectic structure $\omega_{T^*\mathbb{P}^1}$, and

$$h_0 : T^*\mathbb{P}^1 \to \mathbb{C}^2/(\pm 1)$$

is the minimal resolution of the A_1 surface singularity with exceptional divisor E.

Let V denote the unramified locus of j; it is the complement of the image of all big diagonals except Δ. Important to us is that the image of V in $(\mathbb{C}^2)^{(n)}$ contains all cycles in $(\mathbb{C}^2)^{(n)}$ where at most two points come together. The following lemma says that if $i : U \to (\mathbb{C}^2)^{[n]}$ is the base change of the étale neighborhood V along h, then U satisfies the conditions of Lemma 2(3), so that we can check the inclusion $\operatorname{im}(dh) \subset \omega_n \circ \alpha_n((T_{\mathbb{C}^2})^{[n]})$ by pulling back to U.

LEMMA 3. *The fiber product*

$$\begin{tikzcd} U \arrow[r, "h'"] \arrow[d, "i"'] & V \arrow[d, "j"] \\ (\mathbb{C}^2)^{[n]} \arrow[r, "h"] & (\mathbb{C}^2)^{(n)} \end{tikzcd}$$

is such that i is étale and the complement of $i(U)$ has codimension 2. Moreover, $U \subset T^*\mathbb{P}^1 \times (\mathbb{C}^2)^{n-1}$ is such that $h_0 \times id_{(\mathbb{C}^2)^{n-1}}|_U = h'$, and the restriction of the symplectic form from $T^*\mathbb{P}^1 \times (\mathbb{C}^2)^{n-1}$ equals $i^*(\omega_n)$.

Proof. This is essentially the proof that $(\mathbb{C}^2)^{[n]}$ admits a holomorphic symplectic form, and we refer the interested reader to [Bea83, p. 766] or [Nak99, §1.4]. □

In [Sta16, Thm. B] the second author proved that the map α_n induces an isomorphism of $(T_{\mathbb{C}^2})^{[n]}$ with the subsheaf $\mathrm{Der}_{\mathbb{C}}(-\log B_n)$, which consists of vector fields tangent to B_n. To set up the proof of the inequality in Theorem A, we consider the symplectic resolution of the A_1-singularity and prove that the log-tangent sheaf $\mathrm{Der}_{\mathbb{C}}(-\log E)$ is isomorphic to the image dh_0 as subsheaves of $\Omega_{T^*\mathbb{P}^1}$.

LEMMA 4. *The symplectic isomorphism $\omega_{T^*\mathbb{P}^1} : T_{T^*\mathbb{P}^1} \xrightarrow{\cong} \Omega_{T^*\mathbb{P}^1}$ restricts to an isomorphism of subsheaves*

$$\omega_{T^*\mathbb{P}^1}|_{\mathrm{Der}_{\mathbb{C}}(-\log E)} : \mathrm{Der}_{\mathbb{C}}(-\log E) \to dh_0(h_0^*\Omega_{\mathbb{C}^2/(\pm 1)}).$$

Proof. We have two short exact sequences, and we want to fill in the dashed arrows:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Der}_{\mathbb{C}}(-\log E) \arrow[r] \arrow[d, dashed] & T_{T^*\mathbb{P}^1} \arrow[r] \arrow[d, "\omega_{T^*\mathbb{P}^1}"'] & \mathcal{O}_E(E) \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & dh_0(h_0^*\Omega_{\mathbb{C}^2/(\pm 1)}) \arrow[r] & \Omega_{T^*\mathbb{P}^1} \arrow[r] & \Omega_E \arrow[r] & 0. \end{tikzcd}$$

If $v \in \mathrm{Der}_{\mathbb{C}}(-\log E)(U)$ is any logarithmic vector field, then v is tangent to E, so for any point $p \in E$ with $v|_p \neq 0$, we know that $v|_p$ generates the tangent space of E. On the other hand, the pairing of the 1-form $\omega_{T^*\mathbb{P}^1}(v)|_E$ with $v|_p$ vanishes by skew symmetry of $\omega_{T^*\mathbb{P}^1}$. So the restricted 1-form $\omega_{T^*\mathbb{P}^1}(v)|_E$ vanishes identically. Thus we can fill in the dashed arrows to obtain a commuting diagram with an injection on the left and a surjection on the right. But the surjection on the right is an isomorphism since these are isomorphic line bundles on E, and thus $\omega_{T^*\mathbb{P}^1}|_{\mathrm{Der}_{\mathbb{C}}(-\log E)}$ is also an isomorphism. □

Proof of inequality in Theorem A. According to Remark 1, it suffices to show that, as subsheaves of $\Omega_{(\mathbb{C}^2)^{[n]}}$, we have the containment $dh(h^*\Omega_{(\mathbb{C}^2)^{(n)}}) \subset \omega_n \circ \alpha_n((T_{\mathbb{C}^2})^{[n]})$. If $i : U \to (\mathbb{C}^2)^{[n]}$ is the étale open set from Lemma 3, then by Lemma 2 it suffices to show that $i^*(dh(h^*\Omega_{(\mathbb{C}^2)^{(n)}})) \subset i^*(\omega_n \circ \alpha_n((T_{\mathbb{C}^2})^{[n]}))$ as subsheaves of $i^*\Omega_{(\mathbb{C}^2)^{[n]}} = \Omega_U$.

Let E' denote the exceptional divisor of h'. By Lemma 3 we have a fiber square with i étale and $i^{-1}(B_n) = E'$. It follows that

$$i^*(dh(h^*\Omega_{(\mathbb{C}^2)^{(n)}})) = dh'(h'^*\Omega_V), \quad \text{and} \quad i^*(\alpha_n((T_{\mathbb{C}^2})^{[n]})) = \mathrm{Der}_{\mathbb{C}}(-\log E').$$

For the second equality, we use the interpretation of $\alpha_n((T_{\mathbb{C}^2})^{[n]})$ as the log-tangent sheaf of B_n [Sta16, Thm. B]. On the one hand, the exceptional divisor $E' = U \cap (E \times (\mathbb{C}^2)^{n-1})$ is locally a product, so the log-tangent sheaf of E' splits as a direct sum:

$$\mathrm{Der}_{\mathbb{C}}(-\log E') = (p^*\mathrm{Der}_{\mathbb{C}}(-\log E) \oplus q^*T_{(\mathbb{C}^2)^{n-1}})|_U,$$

where p and q denote projection of $T^*\mathbb{P}^1 \times (\mathbb{C}^2)^{n-1}$ onto $T^*\mathbb{P}^1$ and $(\mathbb{C}^2)^{n-1}$, respectively. Moreover, $h' = (h_0 \times id_{(\mathbb{C}^2)^{n-1}})|_U$, so the subsheaf $dh'(h'^*\Omega_V)$ splits as a direct sum:

$$\begin{aligned} & dh'(h'^*\Omega_V) \\ &\quad = (p^*dh_0(h_0^*\Omega_{\mathbb{C}^2/(\pm 1)}) \oplus q^*\Omega_{(\mathbb{C}^2)^{n-1}})|_U \subset (p^*\Omega_{T^*\mathbb{P}^1} \oplus q^*\Omega_{(\mathbb{C}^2)^{n-1}})|_U \\ &\quad = \Omega_U. \end{aligned}$$

Finally by Lemma 3, $i^*\omega_n$ is the same as the restriction of the product symplectic form on $T^*\mathbb{P}^1 \times (\mathbb{C}^2)^{n-1}$. Therefore it suffices to check that the symplectic form on $T^*\mathbb{P}^1 \times (\mathbb{C}^2)^{n-1}$ identifies $p^*dh_0(h_0^*\Omega_{\mathbb{C}^2/(\pm 1)}) \oplus q^*\Omega_{(\mathbb{C}^2)^{n-1}}$ with $p^*\mathrm{Der}_{\mathbb{C}}(-\log E) \oplus q^*T_{(\mathbb{C}^2)^{n-1}}$. Since $i^*\omega_n$ is a product symplectic form, it respects this direct sum decomposition. The second factors are clearly identified, and the first factors are identified by Lemma 4. □

REMARK 5. This proof actually shows that there is an isomorphism

$$h^*(\Omega_{(\mathbb{C}^2)^{(n)}})^{\vee\vee} \cong (T_{\mathbb{C}^2})^{[n]},$$

that is, $(T_{\mathbb{C}^2})^{[n]}$ is the reflexive hull of $h^*(\Omega_{(\mathbb{C}^2)^{(n)}})$.

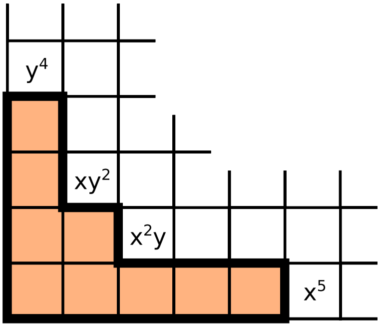

Figure 1 The Young diagram associated with our example $\xi \subset \mathbb{C}^2$

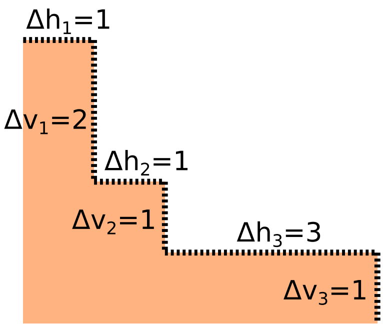

Figure 2 In our example ξ, we have $\Delta h = (1, 1, 3)$ and $\Delta v = (2, 1, 1)$

2. Computing the Rank of α_n at Monomial Subschemes

In this section, we show how to compute the rank of α_n at monomial subschemes, proving Theorem B. During the proof, we exhibit the computation on an example subscheme $\xi \subset \mathbb{C}^2$ with $I_\xi = (y^4, xy^2, x^2y, x^5)$. Throughout we let $\mathbb{C}[\xi] := \mathbb{C}[x, y]/I_\xi$.

Proof of Theorem B. Let $\xi \subset \mathbb{C}^2$ be a length n subscheme whose ideal I_ξ is defined by monomials. As in the Introduction, we associate with ξ the Young diagram (see Figure 1) $\mu = \mu_\xi \subset \mathbb{N}^2$ defined as

$$\mu := \{(i, j) \in \mathbb{N}^2 | x^i y^j \notin I_\xi\}.$$

We associate with μ the elementary statistic given by tracing the top perimeter of μ from the top left to the bottom right and recording the horizontal steps Δh and the vertical steps Δv (see Figure 2).

Our aim is to compute $\operatorname{rank}(\alpha_n|_{[\xi]})$. There are natural isomorphisms $(T_{\mathbb{C}^2})^{[n]}|_{[\xi]} \cong H^0(T_{\mathbb{C}^2}|_\xi) \cong \mathbb{C}[\xi]\frac{\delta}{\delta x} \oplus \mathbb{C}[\xi]\frac{\delta}{\delta y}$ and $T_{(\mathbb{C}^2)^{[n]}}|_{[\xi]} \cong \operatorname{Hom}(I_\xi, \mathbb{C}[\xi])$. Moreover, the map $\alpha_n|_{[\xi]}$ is the map in the normal sequence associated with

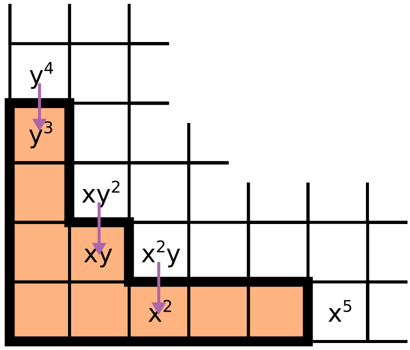

Figure 3 A schematic of $\frac{\delta}{\delta y}$ up to scaling

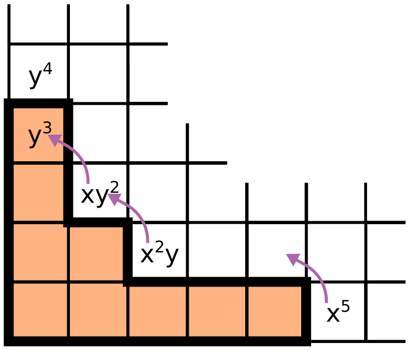

Figure 4 A schematic of $y\frac{\delta}{\delta x}$ up to scaling

$\xi \subset \mathbb{C}^2$ that maps any restricted derivation $\delta \in T_{\mathbb{C}^2}$ to a homomorphism by

$$\alpha_n|_{[\xi]} : \mathbb{C}[\xi]\frac{\delta}{\delta x} \oplus \mathbb{C}[\xi]\frac{\delta}{\delta y} \to \operatorname{Hom}(I_\xi, \mathbb{C}[\xi]), \qquad \delta \mapsto \left(\begin{array}{c} I_\xi \xrightarrow{\alpha_n|_{[\xi]}(\delta)} \mathbb{C}[\xi] \\ f \mapsto \delta(f)|_\xi \end{array} \right).$$

Since I_ξ is generated by monomials, we can decompose $I_\xi = \bigoplus_{\mathbb{N}^2\setminus\mu} \mathbb{C} \cdot x^i y^j$ as a $\mathbb{C}$-vector space. Moreover, the ring of functions on ξ has a monomial $\mathbb{C}$-vector space basis $\mathbb{C}[\xi] = \bigoplus_\mu \mathbb{C} \cdot x^i y^j$. Observe that, for any *monomial* derivation $x^i y^j \frac{\delta}{\delta x}$ or $x^i y^j \frac{\delta}{\delta y}$, the associated homomorphism in $\operatorname{Hom}(I_\xi, \mathbb{C}[\xi])$ maps our basis of I_ξ to our basis of $\mathbb{C}[\xi]$ up to possible scaling. This makes it possible to understand these homomorphisms combinatorially. For example, $\frac{\delta}{\delta y}$ acts (up to scaling) by decreasing the power of y by 1, which on $\mathbb{N}^2$ is a shift down operator annihilating any (i, j) of the form $(i, 0)$ (see Figure 3). The derivation $y\frac{\delta}{\delta x}$ acts by shifting left by 1 and shifting up by 1 (see Figure 4).

More importantly, all monomials are eigenvectors for $x\frac{\delta}{\delta x}$ and $y\frac{\delta}{\delta y}$. Therefore, the homomorphisms associated to $x\frac{\delta}{\delta x}$ and $y\frac{\delta}{\delta y}$ are 0 in $\mathrm{Hom}(I_\xi, \mathbb{C}[\xi])$, and any multiple $x^{i+1}y^j\frac{\delta}{\delta x}$ or $x^iy^{j+1}\frac{\delta}{\delta y}$ for $i, j > 0$ is 0 as a homomorphism $I_\xi \to \mathbb{C}[\xi]$. So the only possible nonzero homomorphisms coming from monomial derivations are of the form $y^j\frac{\delta}{\delta x}$ or $x^i\frac{\delta}{\delta y}$.

Finally, we must determine which powers $y^j\frac{\delta}{\delta x}$ and $x^i\frac{\delta}{\delta y}$ give rise to nonzero homomorphisms. The derivation $y^j\frac{\delta}{\delta x}$ acts on $\mathbb{N}^2$ by shifting to the left 1 and up j. This implies that when $j \geq \max(\Delta v)$, $y^j\frac{\delta}{\delta x}$ maps all x^iy^j for $(i, j) \in \mathbb{N}^2 \setminus \mu$ (the ideal) to other points in $\mathbb{N}^2 \setminus \mu$ (back into the ideal). Thus the associated homomorphism in $\mathrm{Hom}(I_\xi, \mathbb{C}[\xi])$ is 0. Likewise, if $i \geq \max(\Delta h)$, then the derivation $x^i\frac{\delta}{\delta y}$ is in the kernel of $\alpha_n|_{[\xi]}$. Lastly, it is clear that distinct monomial homomorphisms $y^j\frac{\delta}{\delta x}$ for $0 \leq j < \max(\Delta v)$ and $x^i\frac{\delta}{\delta y}$ for $0 \leq i < \max(\Delta h)$ give rise to linearly independent homomorphisms in $\mathrm{Hom}(I_\xi, \mathbb{C}[\xi])$, proving that $\mathrm{rank}(\alpha_n|_{[\xi]}) = \max(\Delta h) + \max(\Delta v)$. □

3. Computing the Dimension of Tangent Spaces at Monomial Subscheme of $\mathbb{C}^2$

In this section, we prove equality in Theorem A when $\xi \subset \mathbb{C}^2$ is cut out by monomials. By Remark 1 and Theorem B it suffices to show the following:

PROPOSITION 6. *If $\xi \subset \mathbb{C}^2$ is a monomial subscheme, then*

$$\mathrm{rank}(dh|_{[\xi]}) = \begin{pmatrix} \textit{maximum of horizontal} \\ \textit{steps of } \mu_\xi \end{pmatrix} + \begin{pmatrix} \textit{maximum of vertical} \\ \textit{steps of } \mu_\xi \end{pmatrix}.$$

Our main computational tool is Haiman's affine charts centered at $[\xi]$. We review without proof the properties of the Haiman chart that we will need and refer the interested reader to [Hai98, §2]. If $\mu = \mu_\xi \subset \mathbb{N}^2$ is the Young diagram associated with a monomial subscheme $\xi \subset \mathbb{C}^2$, then we associate with μ the set of monomials

$$\mathcal{B}_\mu := \{x^iy^j \mid (i, j) \in \mu\}.$$

Clearly, the set $\mathcal{B}_\mu$ forms an (unordered) $\mathbb{C}$-basis for $\mathbb{C}[\xi] := \mathbb{C}[x, y]/I_\xi$. Moreover, for all $[\chi] \in (\mathbb{C}^2)^{[n]}$ sufficiently close to $[\xi]$, the monomials in $\mathcal{B}_\mu$ give a $\mathbb{C}$-basis for $\mathbb{C}[\chi]$.

DEFINITION 7. There is an open set U_μ of these $[\chi] \in (\mathbb{C}^2)^{[n]}$, which we call the *Haiman chart centered at ξ*, that is,

$$U_\mu := \left\{ [\chi] \in (\mathbb{C}^2)^{[n]} \,\middle|\, \begin{array}{l} \mathbb{C}[\chi] \text{ is spanned as a } \mathbb{C}\text{-vector} \\ \text{space by monomials in } \mathcal{B}_\mu \end{array} \right\}.$$

In fact, U_μ is affine [Hai98, Prop. 2.2], and its ring of functions $\mathbb{C}[U_\mu]$ is generated by functions denoted by $c_{i,j}^{r,s}$ (with (r, s) and $(i, j) \in \mathbb{N}^2$). The value of

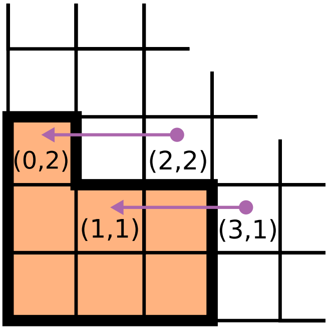

Figure 5 In $\mathfrak{m}_{[\xi]}/\mathfrak{m}_{[\xi]}^2$, we have $c_{02}^{22} \neq c_{11}^{31}$ although the arrows have the same slope

$c_{i,j}^{r,s}([\chi])$ at $[\chi] \in (\mathbb{C}^2)^{[n]}$ is defined by the relation

$$x^r y^s = \sum_{(i,j) \in \mu} c_{i,j}^{r,s}([\chi]) x^i y^j \mod I_\chi.$$

As suggested in [Hai98, p. 210], it is convenient to represent $c_{i,j}^{r,s}$ by an arrow in $\mathbb{N}^2$ that points from $(r,s) \in \mathbb{N}^2 \setminus \mu$ to $(i,j) \in \mu$.

Let $\mathfrak{m}_{[\xi]} \subset \mathbb{C}[U_\mu]$ denote the maximal ideal of the monomial subscheme $[\xi] \in U_\mu$. The cotangent space $\mathfrak{m}_{[\xi]}/\mathfrak{m}_{[\xi]}^2$ is generated by classes of functions $c_{i,j}^{r,s}$ corresponding to arrows with heads in μ and tails in $\mathbb{N}^2 \setminus \mu$. We now state the key *Haiman relations* for these arrows modulo $\mathfrak{m}_{[\xi]}^2$:

HR1 (see [Hai98, eq. 2.18]). Translating an arrow horizontally or vertically does not change the class it represents modulo $\mathfrak{m}_{[\xi]}^2$, provided that the head remains in μ and the tail remains outside of μ. Moreover, two nonzero arrows represent the same class modulo $\mathfrak{m}_{[\xi]}^2$ if and only if one can be taken to the other by a series of such translations.

HR2 (see [Hai98, eq. 2.18]). An arrow represents 0 modulo $\mathfrak{m}_{[\xi]}^2$ if and only if it can be translated so that its head crosses the x-axis or y-axis and the tail remains in $\mathbb{N}^2 \setminus \mu$.

HR3 (see [Hai98, p. 211]). In particular, any strictly southwest pointing arrow vanishes modulo $\mathfrak{m}_{[\xi]}^2$.

Moreover, Haiman proved that the set of equivalence classes of nonvanishing arrows under the Haiman relations form a basis for the cotangent space. (see Figures 5 and 6 for examples of these relations).

Let $R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{\mathfrak{S}_n}$ be the coordinate ring of $(\mathbb{C}^2)^{(n)} = (\mathbb{C}^2)^n/\mathfrak{S}_n$. This ring is generated by the *polarized power sums* [Wey97]:

$$p_{r,s} = \sum_{i=1}^{n} x_i^r y_i^s.$$

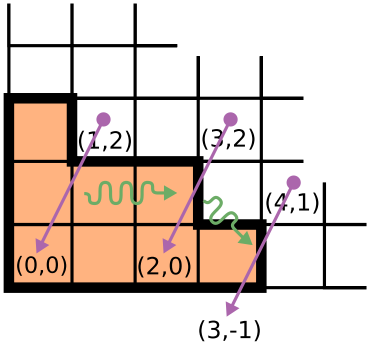

Figure 6 We have the Haiman relations $c_{00}^{12} = c_{20}^{32} = c_{3,-1}^{41} = 0$ in $\mathfrak{m}_{[\xi]}/\mathfrak{m}_{[\xi]}^2$, verifying HR3

Fixing a monomial $x^r y^s \in \mathbb{C}[x, y]$, we can define a function $\operatorname{Tr}(x^r y^s)$ on $(\mathbb{C}^2)^{[n]}$ whose value at $[\chi] \in (\mathbb{C}^2)^{[n]}$ is the trace $\operatorname{Tr}(x^r y^s : \mathbb{C}[\chi])$ of the endomorphism $x^r y^s$ thought of as a $\mathbb{C}$-linear operator on $\mathbb{C}[\chi]$. Haiman [Hai98, p. 208] shows that

$$\operatorname{Tr}(x^r y^s) = h^*(p_{r,s}).$$

Thus, at a point $[\xi] \in (\mathbb{C}^2)^{[n]}$, the image of the map $dh|_{[\xi]}$ between cotangent sheaves is spanned by the classes of $\operatorname{Tr}(x^r y^s) \mod \mathfrak{m}_{[\xi]}^2$.

Proof of Proposition 6. Let U_μ be the Haiman chart centered at ξ. We need to compute the derivative of $\operatorname{Tr}(x^r y^s)$ in $\mathfrak{m}_{[\xi]}/\mathfrak{m}_{[\xi]}^2$. For all $[\chi] \in U_\mu$, we can write $x^r y^s \in \operatorname{End}(\mathbb{C}[\chi])$ as a matrix using the basis $\mathcal{B}_\mu$. Thus we compute the trace

$$\operatorname{Tr}(x^r y^s) = \sum_{(h,k)\in\mu} c_{h,k}^{r+h,s+k}$$

as an element of $\mathbb{C}[U_\mu]$. By the discussion proceeding the proof, the image of $dh|_{[\xi]}$ is generated by

$$d(h^*(p_{r,s})) = d\operatorname{Tr}(x^r y^s) \equiv \sum_{(h,k)\in\mu} c_{h,k}^{r+h,s+k} \mod \mathfrak{m}_{[\xi]}^2.$$

Using the description of the cotangent space as linear combinations of equivalence classes of arrows on the Young diagram μ, $d(h^*(p_{r,s}))$ is a sum of arrows of slope s/r. Whenever both s and r are nonzero, these arrows are pointing southwest, and so by (HR3) they vanish modulo $\mathfrak{m}_{[\xi]}^2$.

When $s = 0$, $d(h^*(p_{r,0}))$ is a sum of horizontal arrows of length r. If $r > \max(\Delta h)$, then by (HR1) we can slide each horizontal arrow up and to the right until the head of the arrow leaves the first quadrant (see Figures 7 and 8). Therefore, by (HR2), $d(h^*(p_{r,0})) = 0 \mod \mathfrak{m}_{[\xi]}^2$. For $1 \le r \le \max(\Delta h)$, we get that at least one of these arrows is nonzero since we cannot slide any arrow of length r past the max horizontal jump in the diagram while still keeping the head in μ. By

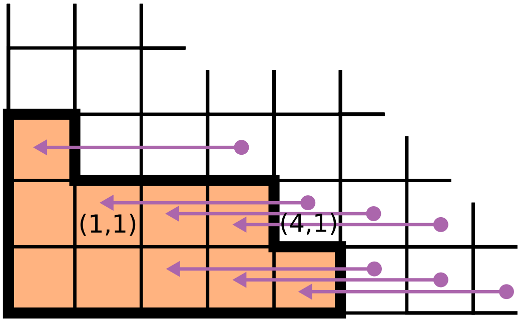

Figure 7 These arrows depict $d(h^*(p_{3,0}))$ modulo $\mathfrak{m}_{[\xi]}^2$. Applying (HR1), we have $d(h^*(p_{3,0})) = 6c_{11}^{41} \neq 0$

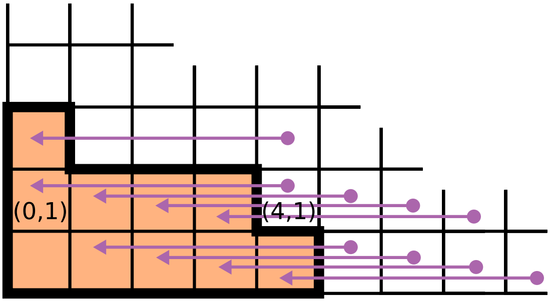

Figure 8 These arrows depict $d(h^*(p_{4,0}))$ modulo $\mathfrak{m}_{[\xi]}^2$. By applying (HR1) and (HR2) to shift up and to the left we see that $d(h^*(p_{4,0})) = 0$

the same argument we see that $d(h^*(p_{0,s}))$ is a sum of vertical arrows of length s and is nonzero if and only if $1 \leq s \leq \max(\Delta v)$.

Now the functions in the set

$$\{d(h^*(p_{r,0})) : 1 \leq r \leq \max(\Delta h)\} \cup \{d(h^*(p_{0,s})) : 1 \leq s \leq \max(\Delta v)\}$$

are all linearly independent modulo $\mathfrak{m}_{[\xi]}^2$. To prove this, we expand any function f in this set in terms of Haiman's basis. All the nonzero terms that appear in the expansion of f correspond to arrows with the same magnitude and direction. However, we can easily check that if f and g are different, then the magnitude or direction of the nonzero arrows that appear in their Haiman basis expansion are different. Since translation preserves magnitude and direction, it follows from Haiman's rules that this set is linearly independent, has size $\max(\Delta v) + \max(\Delta h)$, and generates $\mathrm{im}(dh|_{[\xi]}) \subset \mathfrak{m}_{[\xi]}/\mathfrak{m}_{[\xi]}^2$. Therefore $\mathrm{rank}(dh|_{[\xi]}) = \max(\Delta h) + \max(\Delta v) = \mathrm{rank}(\alpha_n|_{[\xi]})$. $\square$

REMARK 8. Theorem A gives a lower bound on the dimension of the tangent space of P_n. On the other hand, we can obtain upper bounds by taking torus degenerations. In particular,

$$2n - \operatorname{rank}(\alpha_n|_{[\xi]}) \leq \dim T_{[\xi]}P_n \leq \min\{\dim(T_{[\chi]}P_n) | [\xi] \text{ degenerates to } [\chi]\},$$

where I_χ is a monomial ideal.

REMARK 9. In fact, Proposition 6 holds more generally for *formally monomial* subschemes, that is, ξ such that there exist formal coordinates around $0 \in \mathbb{C}^2$ for which I_ξ is a monomial ideal.

It follows from Theorem A that the maximally singular points of P_n are precisely the kth-order neighborhoods of the origin.

COROLLARY 10. *Let* $\mathfrak{m}$ *be the maximal ideal of* $0 \in \mathbb{C}^2$. *If* $\dim T_{[\xi]}P_n = 2n - 2$, *then* $I_\xi = \mathfrak{m}^k$ *for some* k.

Proof. We have an induced action of $(\mathbb{C}^*)^2$ on $(\mathbb{C}^2)^{[n]}$ with fixed points corresponding to monomial subschemes and fixing P_n. Consider a one-parameter subgroup $\sigma : \mathbb{C}^* \to (\mathbb{C}^*)^2$ that acts on $(\mathbb{C}^2)^{[n]}$ with the same fixed points. The limits

$$\lim_{t\to 0} \sigma(t) \cdot [\xi] = [\chi] \quad \text{and} \quad \lim_{t\to\infty} \sigma(t) \cdot [\xi] = [\zeta]$$

exist by properness of P_n, and they are monomial subschemes. Then by Remark 8 we have

$$2n - 2 = \dim T_{[\xi]}P_n \leq \dim T_{[\chi]}P_n \quad \text{and} \quad 2n - 2 = \dim T_{[\xi]}P_n \leq \dim T_{[\zeta]}P_n.$$

Thus $\operatorname{rank}(\alpha_n|_{[\chi]}) = \operatorname{rank}(\alpha_n|_{[\zeta]}) \leq 2$. This is only possible if the Young diagrams μ_χ and μ_ζ are staircases, that is, $I_\chi = I_\zeta = \mathfrak{m}^k$ and $\zeta = \chi$. Thus both of the degenerations occur in the Haiman chart U_χ, and so the degeneration gives a map from $\mathbb{P}^1$ to U_χ. Since U_χ is affine, the map is constant, and $I_\xi = I_\chi = \mathfrak{m}^k$. □

COROLLARY 11. *If* $xy \in I_\xi$, *then* $\dim T_{[\xi]}P_n \leq n + 1$.

Proof. The fact that $xy \in I_\xi$ implies that the only Haiman charts that contain $[\xi]$ correspond to Young diagrams that are hooks. Therefore ξ can only degenerate to monomial schemes with hooks for Young diagrams. An easy computation using Theorem A and Theorem B shows that if $\chi \subset \mathbb{C}^2$ is a monomial subscheme with a hook for a Young diagram, then $\dim T_{[\chi]}P_n = n - 1$ or $n + 1$. Then we are done by Remark 8. □

EXAMPLE 12. Let $\xi \subset \mathbb{C}^2$ be the length n subscheme supported at the origin and defined by the ideal $I_\xi := (x^p + y^q, xy) + \mathfrak{m}^n$, where we assume that $p, q \geq 2$, $p + q = n$, and $\mathfrak{m}$ is the maximal ideal of $0 \in \mathbb{C}^2$. Granger [Gra83, Prop. III.4] shows that analytically locally around $[\xi] \in P_n$, the punctual Hilbert scheme is a product:

$$P_n \cong_{\text{anal.loc.}} Z \times C_{p,q},$$

where $C_{p,q} \subset \mathbb{C}^2$ is the curve defined by $x^p + y^q = 0$, and Z is smooth of dimension $n-2$. In particular, the tangent space is n-dimensional. In this setting, an explicit calculation shows that $\operatorname{rank}(\alpha_n|_{[\xi]}) = n$. Therefore the corank of $\alpha_n|_{[\xi]}$ computes the dimension of $T_{[\xi]}P_n$.

REMARK 13. In particular, Example 12 shows that the upper bounds from torus degeneration coming from Remark 8 are not sharp. Indeed, a simple combinatorial argument shows that there are no length n monomial subschemes $\chi \subset \mathbb{C}^2$ such that $\operatorname{rank}(\alpha_n|_{[\chi]}) = n$.

EXAMPLE 14. Recall that a subscheme $\xi \subset \mathbb{C}^2$ is *curvilinear* if it is contained in a smooth curve. A short calculation shows that the only possible monomial subschemes that are smooth points of the punctual Hilbert scheme correspond to Young diagrams that have dimensions $1 \times n$ or $n \times 1$, that is, the curvilinear subschemes.

In fact, Granger [Gra83, Thm. III-1] shows that the smooth locus of P_n is precisely the locus $[\xi] \in P_n$ such that $\xi \subset \mathbb{C}^2$ is curvilinear. To do this, Granger proves that the locus of subschemes that are isomorphic to the subschemes from Example 12 are dense in the locus of noncurvilinear subschemes (assuming that $n \geq 4$), and the analytic local picture from Example 12 shows that these are singular points. We expect that by applying the methods of Granger and Iarrobino we might be able to expand on the results and examples in our paper. Indeed, we might be able to verify that the dimension of the tangent space of P_n equals the corank of α_n in new cases or, perhaps more interestingly, find an example where they are different.

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D. Bejleri
Department of Mathematics
Brown University
151 Thayer St
Providence, RI 02912
USA

dbejleri@math.brown.edu

D. Stapleton
Department of Mathematics
Stony Brook University
Stony Brook, NY 11794-3651
USA

david.stapleton@stonybrook.edu