Michigan Math. J. 66 (2017), 451–480

Difference Nevanlinna Theories with Vanishing and Infinite Periods

Yik-Man Chiang & Xu-Dan Luo

Dedicated to the memory of J. Milne Anderson.

Abstract. By extending the idea of a difference operator with a fixed step to a varying-step difference operator, we establish a difference Nevanlinna theory for meromorphic functions with steps tending to zero (vanishing period) and a difference Nevanlinna theory for finite-order meromorphic functions with steps tending to infinity (infinite period). We can recover the classical little Picard theorem from the vanishing period theory, but we require additional finite-order growth restriction for meromorphic functions from the infinite period theory. Then we give some applications of our theories to exhibit connections between discrete equations and and their continuous analogues.

1. Introduction

Halburd and Korhonen [9] established a new Picard-type theorem and Picard values with respect to difference operator $\Delta f(z) = f(z+1) - f(z)$ for finite-order meromorphic functions defined on $\mathbb{C}$ versus the classical Picard theorem and Picard values. More specifically, their theory allows them to show that if there are three points a_j, $j = 1, 2, 3$, in $\widehat{\mathbb{C}}$ such that each preimage $f^{-1}(a_j)$ is an infinite sequence consisting of points lying on a straight line on which any two consecutive points differ by a fixed difference c (but is otherwise arbitrary), then the function must be a periodic function with period c. This result can be considered as a discrete version of the classical little Picard theorem for finite-order meromorphic functions. A crucial tool of their theory, which follows from their difference-type Nevanlinna theory for finite-order meromorphic functions, is the difference logarithmic derivative lemma [8] (see also [3]), that is,

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o(T(r, f)),$$

where c is a fixed nonzero constant. Instead of a fixed c, we define $g(z, c) := f(z+c)$, where $(z, c) \in \mathbb{C}^2$. Then $f(z+c)$ is a meromorphic function in $\mathbb{C}^2$.

Received June 1, 2015. Revision received March 4, 2017.

This research was supported in part by the Research Grants Council of the Hong Kong Special Administrative Region, China (16306315).

Moreover, a difference operator with *varying steps* is defined by

$$\begin{aligned}\Delta f_c &:= f(z+c) - f(z)\\ &= g(z,c) - g(z,0), \quad (z,c) \in \mathbb{C}^2.\end{aligned} \tag{1.1}$$

The first author of the current paper and Ruijsenaars [5] showed that, for a nonzero meromorphic function $f(z)$ and $c \in \mathbb{C}$,

$$m(r, f(z+c)) < \frac{R+2r}{R-2r} m(R, f) + \sum_{l=0}^{L} \frac{1}{2\pi} \int_0^{2\pi} \log\left|\frac{R^2 - \overline{b}_l(re^{i\phi}+c)}{R(re^{i\phi}+c-b_l)}\right| d\phi,$$

where $|c| < r$, and $b_0, \dots, b_L$ are poles of $f(z)$ in $|z| < R$, which implies the following uniform bound:

$$m(r, f(z+c)) \le 5m(3r, f) + \log 4 \cdot n(3r, f)$$

whenever $|c| < r$.

This uniform bound still holds if we restrict c, for example, to such that $0 < |c| < \frac{1}{r}$ or $\sqrt{r} < |c| < r$ when $r > 1$, which will lead to vanishing steps and infinite steps when r is sufficiently large, that is, $0 < |c| < \frac{1}{r}$ and $\sqrt{r} < |c| < r$ will result in $c \to 0$ and $c \to \infty$, respectively, as $r \to \infty$. This motivates us to establish the corresponding difference Nevanlinna theories.

We consider the cases with *vanishing period* and *infinite period,* that is, when $c \to 0$ and $c \to \infty$, respectively. On the one hand, if we denote $c = \eta$ as it tends to zero via a sequence $\eta_n \to 0$, then the period guaranteed by the Halburd–Korhonen theory for each n would tend to zero in a formal manner. Thus, the periodic function with a vanishing period, when suitably defined, would *formally* reduce to a constant. On the other hand, if we denote $c = \omega$ as it tends to infinity via a sequence $\omega_n \to \infty$, then, similarly, the period, as asserted by the Halburd–Korhonen theory, for each n would become infinite, and the distance between any two consecutive points on each of the three preimage infinite sequences would become sparse and *eventually* reduce to a single point at most formally in the limit. In both cases described, we would *formally* recover the original little Picard theorem (namely, the inverse image of each of three Picard values must be at most a finite set).

In this paper, we rigorously establish that the formal considerations indeed hold under certain senses. The upshot is that we can recover the classical little Picard theorem as $\eta \to 0$ *without* the finite-order restriction and *with* the finite-order restriction as $\omega \to \infty$. In fact, our argument for our vanishing-period results is *independent* of the Halburd–Korhonen theory; we apply methods similar to our earlier works [3; 4] and the Halburd–Korhonen theory [9; 2] in the infinite-period results. We remark that the previous finite-order restriction in the infinite-period case is necessary because it is unlikely that such results would hold for general meromorphic functions. However, the rates at which $\eta \to 0$ and $\omega \to \infty$ in the consideration of vanishing and infinite periods, respectively, generally depend on the growth of f.

Hitherto we shall use the notation η for c when we consider the vanishing-period case, and use the notation ω for c when we consider the infinite-period case. Thus, when the rates at which $\eta \to 0$ and $\omega \to \infty$ are suitably chosen and the *Picard exceptional values* are suitably defined, we have obtained:

(1) when a meromorphic function f has three Picard exceptional values with respect to a *varying-step difference operator with vanishing period,* then f is a constant (Theorem 4.7);

(2) when a finite-order meromorphic function f with three Picard exceptional values with respect to a *varying-step difference operator of infinite period,* then f is a constant (Theorem 5.5).

Case (1) gives an alternative proof of the original little Picard theorem. Case (2) requires an additional finite-order restriction.

Let $f(z)$ be a meromorphic function, $\eta \in \mathbb{C}$ be a variable. For each fixed $r := |z|$. we introduce the symbols $m_\eta(r, f(z+\eta))$, $N_\eta(r, f(z+\eta))$, and $T_\eta(r, f(z+\eta))$ instead of $m(r, f(z+\eta))$, $N(r, f(z+\eta))$, and $T(r, f(z+\eta))$ when we want to emphasize that they are also functions of η. But we still have $m_\eta(r, f(z+\eta)) = m(r, f(z+\eta))$, $N_\eta(r, f(z+\eta)) = N(r, f(z+\eta))$, etc. Our main estimates are as follows.

Let f be an arbitrary meromorphic function, and $0 < |\eta| < \alpha_1(r)$, where $r = |z|$,

$$\alpha_1(r) = \min\{\log^{-1/2} r, 1/(n(r+1))^2\}, \qquad n(r) = n(r, f) + n(r, 1/f).$$

Then, *for each fixed* r, we obtain

$$m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) = o(1)$$

as $\eta \to 0$. If, in addition, f has no pole in $\overline{D}(0, h) \setminus \{0\}$ for some positive h and $0 < |\eta| < \alpha_2(r)$, where

$$\alpha_2(r) = \min\left\{r, \log^{-1/2} r, \frac{h}{2}, \frac{1}{\sum_{0<|b_\mu|<r+\frac{1}{2}} 1/|b_\mu|}\right\},$$

where $(b_\mu)_{\mu\in N}$ is the sequence of poles of $f(z)$, then, for each fixed r,

$$N_\eta(r, f(z+\eta)) = N(r, f(z)) + \varepsilon_1(r),$$

where $|\varepsilon_1(r)| \le n(0, f(z)) \log r + 3$.

Although these results hold without the finite-order restriction, the upper bounds of $|\eta|$, that is, $\alpha_1(r)$ and $\alpha_2(r)$, which tend to zero as $r \to \infty$, are related to the growth of f.

When f has positive finite order σ and ω is suitably restricted by $0 < |\omega| < r^\beta$ with $0 < \beta < 1$, then we have

$$m\left(r, \frac{f(z+\omega)}{f(z)}\right) = O(r^{\sigma-(1-\beta)(1-\varepsilon)+\varepsilon})$$

and

$$N(r, f(z+\omega)) = N(r, f(z)) + O(r^{\sigma-(1-\beta)+\varepsilon})$$

when $\sigma \geq 1$ and

$$N(r, f(z+\omega)) = N(r, f(z)) + O(r^{\beta})$$

when $0 < \sigma < 1$ for all r outside a set of finite logarithmic measure. We have also obtained corresponding estimates for meromorphic functions with finite logarithmic order.

Finally, we show a different kind of vanishing-period result for finite-order meromorphic function:

$$\lim_{r\to\infty}\lim_{\eta\to 0} m_{\eta}\left(r, \frac{1}{\eta}\left(\frac{f(z+\eta)}{f(z)} - 1\right)\right) = O(\log r),$$

thus recovering Nevanlinna's original logarithmic derivative estimate for finite-order functions via another approach independent of previous methods, although we do not have an immediate application of this result.

The paper is organized as follows. We state the main theorems in Sections 2 and 3. We establish Nevanlinna theory for difference operator in terms of vanishing and infinite periods in Sections 4 and 5, respectively. We recall some known results in Section 6. The proofs of main results are given in Sections 7–12. We exhibit some applications of our results to obtain classical differential equation results from their difference counterparts in Section 13. A reformulation of logarithmic derivative lemma and its proof are given in Section 14. We shall use Nevanlinna's notation freely throughout this paper. See [11; 19] for their meanings.

2. Main Results for Vanishing Period

In this section, our main results are for fixed $r := |z|$. In this sense, $m(r, \frac{f(z+\eta)}{f(z)})$, $N(r, f(z+\eta))$, and $T(r, f(z+\eta))$ are functions of η. We sometimes write $m_{\eta}(r, \frac{f(z+\eta)}{f(z)})$, $N_{\eta}(r, f(z+\eta))$, and $T_{\eta}(r, f(z+\eta))$ when we want to emphasize the dependence on η.

THEOREM 2.1. *Let $f(z)$ be a meromorphic function in $\mathbb{C}$, and $r = |z|$ be fixed. Then we have*

$$\lim_{\eta\to 0} m_{\eta}\left(r, \frac{f(z+\eta)}{f(z)}\right) + \lim_{\eta\to 0} m_{\eta}\left(r, \frac{f(z)}{f(z+\eta)}\right) = 0. \tag{2.1}$$

Moreover, if $0 < |\eta| < \alpha_1(r)$, where

$$\alpha_1(r) = \min\{\log^{-1/2} r, 1/(n(r+1))^2\}, \qquad n(r) = n(r, f) + n(r, 1/f), \tag{2.2}$$

then

$$\lim_{r\to\infty} m_{\eta}\left(r, \frac{f(z+\eta)}{f(z)}\right) + \lim_{r\to\infty} m_{\eta}\left(r, \frac{f(z)}{f(z+\eta)}\right) = 0. \tag{2.3}$$

From the theorem we deduce the following corollary.

COROLLARY 2.2. *Let $f(z)$ be a meromorphic function. Then, for arbitrary fixed $0 < r = |z|$,*

$$\lim_{\eta \to 0} m_\eta(r, f(z+\eta)) = m(r, f(z)). \tag{2.4}$$

Our next result relates the counting function and its varying steps.

THEOREM 2.3. *Let $f(z)$ be a meromorphic function in $\mathbb{C}$, let $r = |z|$ be such that $0 < |\eta| < \alpha_2(r)$, where*

$$\alpha_2(r) = \min\left\{ r, \log^{-1/2} r, \frac{h}{2}, \frac{1}{\sum_{0<|b_\mu|<r+\frac{1}{2}} 1/|b_\mu|} \right\}, \tag{2.5}$$

where $(b_\mu)_{\mu \in N}$ is the sequence of poles of $f(z)$, and let $h \in (0,1)$ be such that $f(z)$ has no poles in $\overline{D}(0,h) \setminus \{0\}$. Then

$$N_\eta(r, f(z+\eta)) = N(r, f(z)) + \varepsilon_1(r), \tag{2.6}$$

where $|\varepsilon_1(r)| \le n(0, f(z)) \log r + 3$.

Combining these asymptotic relations, we obtain the following estimate for the Nevanlinna characteristic function.

THEOREM 2.4. *Let $f(z)$ be a meromorphic function in $\mathbb{C}$. Then, for each fixed $r = |z|$, there exists $\beta(r) > 0$ with $\lim_{r\to\infty} \beta(r) = 0$ such that*

$$T_\eta(r, f(z+\eta)) = T(r, f(z)) + \varepsilon(r) \tag{2.7}$$

whenever $0 < |\eta| < \beta(r)$, where $|\varepsilon(r)| \le n(0, f(z)) \log r + 4$.

REMARK 2.5. We may choose $\beta(r) = \min\{\alpha_1(r), \alpha_2(r)\}$.

To beter understand these asymptotic relations, we give the following remark.

REMARK 2.6. Miles showed in [15] that the deficiency of a meromorphic function may change when choosing different origins. But, for a fixed η, Chiang and Feng [3] showed the asymptotic relations

$$N(r, f(z+\eta)) = N(r, f) + O(r^{\lambda-1+\varepsilon}) + O(\log r) \tag{2.8}$$

and

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r) \tag{2.9}$$

for finite-order meromorphic function $f(z)$, where λ denotes the exponent of convergence of poles of $f(z)$. This implies that the deficiency does not change after shifting the origin if the difference between the order and lower order is less than unity. By applying estimates (2.8) and (2.9) we can easily obtain an alternative proof of an earlier result of Valiron [18] that if a finite-order meromorphic function has the difference between its order and lower order less than unity, then the deficiency at the origin, that is, $\delta(0)$, is invariant against any finite shift. This result of Valiron no longer holds in general; see Miles [15]. However, Theorems 2.3 and 2.4 indicate that the deficiency remains the same by allowing the period to tend to zero without any restriction of order.

3. Main Results for Infinite Period

In this section, we distinguish two cases of meromorphic functions, those with finite positive order and those with zero order of growth. In the former, the varying steps ω are restricted by $0 < |\omega| < r^{\beta}$, where the constant β depends on the growth order of f. In the latter, we have $0 < |\omega| < \log^{1/2} r$. The ω is otherwise free to vary within the given upper bounds. For example, $|\omega|$ can tend to zero or to infinity as $r \to \infty$. In case we choose ω to be constant, the results for finite-order meromorphic functions essentially agree with the results in Chiang and Feng [3]. We shall stick to the standard notations $m(r, \frac{f(z+\omega)}{f(z)})$, $N(r, f(z+\omega))$, and $T(r, f(z+\omega))$ with the understanding that ω is free to vary with respect to an upper bound, which may depend on f in this section.

THEOREM 3.1. *Let $f(z)$ be a meromorphic function of finite order σ, $0 < \beta < 1$, and $0 < |\omega| < r^{\beta}$. Then, given $0 < \varepsilon < (1-\beta)/(2-\beta)$, we have*

$$m\left(r, \frac{f(z+\omega)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\omega)}\right) = O(r^{\sigma-(1-\beta)(1-\varepsilon)+\varepsilon}). \tag{3.1}$$

We note that the upper bound and the latter consideration in this section remains valid even as $\omega \to 0$ or remains constant, say, $\omega = 1$.

Similarly, we have asymptotic relations for the Nevanlinna counting function and characteristic function of infinite period.

THEOREM 3.2. *Let $f(z)$ be a meromorphic function of finite order $\sigma = \sigma(f)$.*

(i) *If $\sigma \geq 1$, $0 < \beta < 1$, and $0 < |\omega| < r^{\beta}$, then there exists $0 < \varepsilon < \beta'$, where $\beta' = \min\{(\sigma-1)(1-\beta)/\beta, 1-\beta\}$, such that*

$$N(r, f(z+\omega)) = N(r, f) + O(r^{\sigma-(1-\beta)+\varepsilon}) \tag{3.2}$$

outside a set of finite logarithmic measure.

(ii) *If $0 < \sigma < 1$, $0 < \beta < \sigma$, and $0 < |\omega| < r^{\beta}$, then we have*

$$N(r, f(z+\omega)) = N(r, f) + O(r^{\beta}) \tag{3.3}$$

outside a set of finite logarithmic measure.

(iii) *If $\sigma = 0$, $0 < |\omega| < \log^{1/2} r$ for $r > 1$, and $0 < |\omega| < 1$ for $r \leq 1$, then we have*

$$N(r, f(z+\omega)) = N(r, f) + O(\log r) \tag{3.4}$$

outside a set of finite logarithmic measure.

COROLLARY 3.3. *Let $f(z)$ be a meromorphic function of finite logarithmic order $\sigma_{\log} = \limsup_{r\to\infty} \log^{+} T(r, f)/\log\log r > 1$, and $0 < |\omega| < \log^{\beta} r$, where $1 < \beta < \sigma_{\log}$. Then we have*

$$N(r, f(z+\omega)) = N(r, f) + O(\log^{\beta} r) \tag{3.5}$$

outside a set of finite logarithmic measure.

From Theorems 3.1 and 3.2 and Corollary 3.3 we deduce the following theorem.

THEOREM 3.4. *Let $f(z)$ be a meromorphic function of finite order $\sigma = \sigma(f)$, and let $\varepsilon > 0$ be a positive constant.*

(i) *If $\sigma \geq 1$, $0 < \beta < 1$, $0 < |\omega| < r^{\beta}$, and $0 < \varepsilon < \beta''$, where $\beta'' = \min\{(\sigma - 1)(1-\beta)/\beta, (1-\beta)/(2-\beta)\}$, then we have*

$$T(r, f(z+\omega)) = T(r, f) + O(r^{\sigma-(1-\beta)(1-\varepsilon)+\varepsilon}) \tag{3.6}$$

outside a set of finite logarithmic measure.

(ii) *If $0 < \sigma < 1$, $0 < \beta < \sigma$, and $0 < |\omega| < r^{\beta}$, then we have*

$$T(r, f(z+\omega)) = T(r, f) + O(r^{\beta}) \tag{3.7}$$

outside a set of finite logarithmic measure.

(iii) *Moreover, if $\sigma = 0$, $0 < |\omega| < \log^{1/2} r$ for $r > 1$, and $0 < |\omega| < 1$ for $r \leq 1$, then we have*

$$T(r, f(z+\omega)) = T(r, f) + O(\log r) \tag{3.8}$$

outside a set of finite logarithmic measure.

Then we immediately have the following corollary.

COROLLARY 3.5. *Let $f(z)$ be a meromorphic function of finite logarithmic order $\sigma_{\log} > 1$. Suppose $0 < |\omega| < \log^{\beta} r$ with $1 < \beta < \sigma_{\log}$. Then we have*

$$T(r, f(z+\omega)) = T(r, f) + O(\log^{\beta} r) \tag{3.9}$$

outside a set of finite logarithmic measure.

4. Nevanlinna Theory for Difference Operator with Vanishing Period

In this section, we assume that the step size $c = \eta$ in (1.1) is nonzero and its upper bound tends to zero as $z \to \infty$.

THEOREM 4.1. *Let $f(z)$ be a meromorphic function such that $\Delta_\eta f \not\equiv 0$ for all z, let $p \geq 2$ be a positive integer, and let $a_1, \dots, a_p$ be p distinct points in $\mathbb{C}$. Then there exists $\delta(r) > 0$ such that*

$$m(r, f) + \sum_{k=1}^{p} m(r, 1/(f - a_k)) \leq 2T(r, f) - N_{\Delta_\eta}(r, f) + \gamma \tag{4.1}$$

whenever $0 < |\eta| < \delta(r)$, where γ is a constant that depends on $a_1, \dots, a_p$ and r but is independent of z, and where

$$N_{\Delta_\eta}(r, f) := 2N(r, f) - N(r, \Delta_\eta f) + N(r, 1/\Delta_\eta f). \tag{4.2}$$

Proof. In the proof, we denote by $\gamma_1, \gamma_2, \dots$ some definite constants that depend on $a_1, \dots, a_p$ and r but are independent of z. Setting

$$P(f) = \prod_{k=1}^{p} (f - a_k),$$

we have

$$\begin{aligned}\sum_{k=1}^{p} m(r, 1/(f-a_k)) &= \sum_{k=1}^{p} T(r, 1/(f-a_k)) - \sum_{k=1}^{p} N(r, 1/(f-a_k))\\ &= pT(r, f) - N(r, 1/P(f)) + \gamma_1\\ &= T(r, P(f)) - N(r, 1/P(f)) + \gamma_2\\ &= m(r, 1/P(f)) + \gamma_2.\end{aligned}$$

We deduce from (2.1) of Theorem 2.1 that, for each fixed $r > 0$, there is $\delta(r) > 0$ such that

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) \le 1$$

whenever $0 < |\eta| < \delta(r)$. Then

$$m\left(r, \frac{\Delta_\eta f}{f-a_k}\right) \le \gamma_3,$$

which implies that

$$m\left(r, \frac{\Delta_\eta f}{P(f)}\right) \le \gamma_4.$$

We deduce

$$\begin{aligned}\sum_{k=1}^{p} m\left(r, \frac{1}{f-a_k}\right) &\le m\left(r, \frac{1}{\Delta_\eta f}\right) + \gamma = T(r, \Delta_\eta f) - N\left(r, \frac{1}{\Delta_\eta f}\right) + \gamma_5\\ &= m\left(r, f \cdot \frac{\Delta_\eta f}{f}\right) + N(r, \Delta_\eta f) - N\left(r, \frac{1}{\Delta_\eta f}\right) + \gamma_5\\ &\le m(r, f) + N(r, \Delta_\eta f) - N\left(r, \frac{1}{\Delta_\eta f}\right) + \gamma_6.\end{aligned}$$

Hence, there exist $\delta(r) > 0$ and a constant γ such that

$$m(r, f) + \sum_{j=1}^{p} m\left(r, \frac{1}{f-a_j}\right) \le 2T(r, f) - N_{\Delta_{\eta z}}(r, f) + \gamma$$

whenever $0 < |\eta| < \delta(r)$, where

$$N_{\Delta_\eta}(r, f) := 2N(r, f) - N(r, \Delta_\eta f) + N\left(r, \frac{1}{\Delta_\eta f}\right). \qquad \square$$

DEFINITION 4.2. Let $f(z)$ be a meromorphic function, and a be a finite complex number. Then

(i) $n_{\Delta_\eta}(r, 1/(f-a))$ is the number of common zeros of $f-a$ and $\Delta_\eta f$ in $\overline{D(0,r)} := \{z : |z| \le r\}$ (counting multiplicity), and we define the multiplicity to be the minimum of those of $f-a$ and $\Delta_\eta f$ for such points;

(ii) $n_{\Delta_\eta}(r, f) := n_{\Delta_\eta}(r; 0; 1/f)$, which stands for the number of common zeros of $1/f$ and $\Delta_\eta 1/f$ in $\overline{D(0,r)}$ (counting multiplicity), and the multiplicity is defined to be the minimum of those of $1/f$ and $\Delta_\eta 1/f$ for such points.

DEFINITION 4.3. We define the *varying-step difference integrated counting function* of $f(z)$ as

$$N_{\Delta_\eta}\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{n_{\Delta_\eta}(t, \frac{1}{f-a}) - n_{\Delta_\eta}(0, \frac{1}{f-a})}{t}\, dt + n_{\Delta_\eta}\left(0, \frac{1}{f-a}\right) \log r, \tag{4.3}$$

$$N_{\Delta_\eta}(r, f) := \int_0^r \frac{n_{\Delta_\eta}(t, f) - n_{\Delta_\eta}(0, f)}{t}\, dt + n_{\Delta_\eta}(0, f) \log r. \tag{4.4}$$

Besides,

$$\widetilde{N}_{\Delta_\eta}(r, 1/(f-a)) := N(r, 1/(f-a)) - N_{\Delta_\eta}(r, 1/(f-a)), \tag{4.5}$$

$$\widetilde{N}_{\Delta_\eta}(r, f) := N(r, f) - N_{\Delta_\eta}(r, f). \tag{4.6}$$

We have the following second main theorem for varying-step difference operator with vanishing period.

THEOREM 4.4. *Let $f(z)$ be a meromorphic function such that $\Delta_\eta f \not\equiv 0$ in $D(0, r) := \{z : |z| \le r\}$. Let $a_1, \ldots, a_p$ be $p \ge 2$ distinct points in $\mathbb{C}$. Then, there exists $\delta'(r) > 0$ such that*

$$(p-1)T(r, f) \le \widetilde{N}_{\Delta_\eta}(r, f) + \sum_{k=1}^{p} \widetilde{N}_{\Delta_\eta}(r, 1/(f-a_k)) + \hat{\varepsilon}(r) \tag{4.7}$$

whenever $0 < |\eta| < \delta'(r)$, where $|\hat{\varepsilon}(r)| \le n(0, f(z)) \log r + \gamma$ with constant γ depending only on $a_1, \ldots, a_p$ and r but independent of z, and $\lim_{r\to\infty} \delta'(r) = 0$.

Proof. We deduce from Theorem 4.1, after adding $\sum_{i=1}^{p} N(r, 1/(f-a_i))$ and applying Nevanlinna's first fundamental theorem, that

$$(p-1)T(r, f) \le \sum_{k=1}^{p} N(r, 1/(f-a_k)) + N(r, \Delta_\eta f) - N(r, 1/\Delta_\eta f) - N(r, f) + \gamma \tag{4.8}$$

whenever $0 < |\eta| < \delta(r)$.

According to Definition 4.3, we have

$$\sum_{k=1}^{p} N(r, 1/(f-a_k)) - \sum_{k=1}^{p} \widetilde{N}_{\Delta_\eta}(r, 1/(f-a_k)) = \sum_{k=1}^{p} N_{\Delta_\eta}(r, 1/(f-a_k)) \le N(r, 1/\Delta_\eta f),$$

and hence

$$\sum_{k=1}^{p} N(r, 1/(f-a_k)) - N(r, 1/\Delta_\eta f) \le \sum_{k=1}^{p} \widetilde{N}_{\Delta_\eta}(r, 1/(f-a_k)). \tag{4.9}$$

Moreover, if $z = z_0$ is a common pole of $f(z)$ and $f(z+\eta)$ with multiplicity m_1 and m_2, respectively, then we can write

$$f(z) = \frac{g(z)}{(z-z_0)^{m_1}}, \qquad f(z+\eta) = \frac{h(z)}{(z-z_0)^{m_2}},$$

where both $g(z)$ and $h(z)$ are analytic at $z = z_0$ and $g(z_0) \neq 0$ and $h(z_0) \neq 0$. Without loss of generality, we may assume that $m_1 \geq m_2$. Thus,

$$\Delta_\eta f := f(z+\eta) - f(z) = \frac{(z-z_0)^{m_1-m_2}h(z) - g(z)}{(z-z_0)^{m_1}}$$

and

$$\Delta_\eta \frac{1}{f} := \frac{1}{f(z+\eta)} - \frac{1}{f(z)} = \frac{(z-z_0)^{m_2}[g(z) - (z-z_0)^{m_1-m_2}h(z)]}{h(z)g(z)}.$$

If $m_1 > m_2$, then the multiplicity for the pole of $\Delta_\eta f$ and the zero of $\Delta_\eta \frac{1}{f}$ at $z = z_0$ are m_1 and m_2, respectively, from which it follows that the minimum multiplicity of the zero of $\frac{1}{f}$ and $\Delta_\eta \frac{1}{f}$ at $z = z_0$ is m_2. If $m_1 = m_2$, then we can write $h(z) - g(z) = (z-z_0)^m \cdot d(z)$, where m is a nonnegative integer, $d(z)$ is analytic at $z = z_0$, and $d(z_0) \neq 0$. Thus, the multiplicity for the pole of $\Delta_\eta f$ at $z = z_0$ is $m_1 - m$ if $m_1 \geq m$ and is 0 if $m_1 < m$, whereas similar consideration for the zero of $\Delta_\eta \frac{1}{f}$ at $z = z_0$ is $m_1 + m$, which implies that the minimum multiplicity of $\frac{1}{f}$ and $\Delta_\eta \frac{1}{f}$ at $z = z_0$ is m_1. Hence, we deduce

$$N(r, \Delta_\eta f) + N_{\Delta_\eta}(r, f) \leq N(r, f) + N(r, f(z+\eta)),$$

and Theorem 2.3 guarantees that we can find $0 < \delta'(r) < \delta(r)$ such that

$$N(r, \Delta_\eta f) - N(r, f) \leq \widetilde{N}_{\Delta_\eta}(r, f) + \varepsilon_1(r) \tag{4.10}$$

whenever $0 < |\eta| < \delta'(r)$. Combining (4.8), (4.9), and (4.10), we deduce

$$(p-1)T(r, f) \leq \widetilde{N}_{\Delta_\eta}(r, f) + \sum_{k=1}^{p} \widetilde{N}_{\Delta_\eta}(r, 1/(f - a_k)) + \hat{\varepsilon}(r)$$

whenever $0 < |\eta| < \delta'(r)$, where $|\hat{\varepsilon}(r)| \leq n(0, f(z)) \log r + \gamma$. □

4.1. *Defect Relation and Little Picard's Theorem for Varying-Step Difference Operator*

We define the *multiplicity index* and *ramification index* for a varying-step difference operator with vanishing period to be

$$\vartheta_{\Delta_\eta}(a, f) := \lim \inf_{r\to\infty} \frac{N_{\Delta_\eta}(r, 1/(f-a))}{T(r, f)}$$

and

$$\Theta_{\Delta_\eta}(a, f) := 1 - \lim \sup_{r\to\infty} \frac{\widetilde{N}_{\Delta_\eta}(r, 1/(f-a))}{T(r, f)}.$$

Then Theorem 4.4 immediately implies the following corollary.

COROLLARY 4.5. *Let $f(z)$ be a transcendental meromorphic function such that $\Delta_\eta f \not\equiv 0$. Then*

$$\sum_{a\in\widehat{\mathbb{C}}}(\delta(a, f) + \vartheta_{\Delta_\eta}(a, f)) \le \sum_{a\in\widehat{\mathbb{C}}} \Theta_{\Delta_\eta}(a, f) \le 2.$$

Next, we shall define Picard exceptional values for a varying-step difference operator with vanishing period.

DEFINITION 4.6. We call $a \in \widehat{\mathbb{C}}$ is a *Picard exceptional value for a varying-step difference operator with vanishing period* of $f(z)$ if there is a sequence $\eta_n \to 0$ as $n \to \infty$ such that $\widetilde{N}_{\Delta_{\eta_n}}(r, 1/(f-a)) = O(1)$.

We have the following Picard theorem for a varying-step difference operator with vanishing period.

THEOREM 4.7. *Let $f(z)$ be a meromorphic function having three Picard exceptional values for a varying-step difference operator with vanishing period. Then $f(z)$ is a constant.*

Proof. Without loss of generality, we may assume that the three exceptional values are 0, 1, and ∞. According to Theorem 4.4, we can find $\delta'(r) > 0$ such that

$$T(r, f) \le \widetilde{N}_{\Delta_\eta}(r, f) + \widetilde{N}_{\Delta_\eta}(r, 1/f) + \widetilde{N}_{\Delta_\eta}(r, 1/(f-1)) + \hat{\varepsilon}(r)$$

whenever $0 < |\eta| < \delta'(r)$, where $|\hat{\varepsilon}(r)| \le n(0, f(z)) \log r + \gamma$ with bounded constant γ.

If $f(z)$ is a transcendental meromorphic function, then

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} = \infty.$$

Thus, there exists $r_0 > 0$ such that

$$T(r, f) > 2\hat{\varepsilon}(r)$$

whenever $r \ge r_0$. For each $r \ge r_0$, since $\lim_{n\to\infty} \eta_n = 0$ ($\eta_n \ne 0$), there exists $N(r) > 0$ such that $0 < |\eta_n| < \delta'(r)$ whenever $n > N(r)$. Note that

$$\widetilde{N}_{\Delta_{\eta_n}}(r, f) = \widetilde{N}_{\Delta_{\eta_n}}(r, 1/f) = \widetilde{N}_{\Delta_{\eta_n}}(r, 1/(f-1)) = 0$$

whenever $n > N(r)$. Thus, $T(r, f) \le \hat{\varepsilon}(r)$, which is a contradiction.

Hence, $\Delta_{\eta_n} f \equiv 0$ on $\{z : |z| \le r\}$ whenever $n > N(r)$.

We claim that $f(z)$ is an entire function. Otherwise, there would exist z_1 such that $f(z_1) = \infty$, which implies that $1/f(z_1 + \eta_n) = 1/f(z_1) = 0$ whenever $n > N(r_1)$, where $r_1 \ge \max\{r_0, |z_1|\}$. Note that if $\lim_{n\to\infty} \eta_n = 0$ ($\eta_n \ne 0$), then z_1 is a nonisolated zero, which is a contradiction. So $f(z)$ must be an entire function.

Moreover, we have $f(\eta_n) = f(0)$ whenever $n > N(r_0)$. By the identity theorem we deduce that $f(z) \equiv f(0)$ on $\mathbb{C}$, which is impossible because of the assumption that $f(z)$ is transcendental.

Therefore, $f(z)$ is a rational function, which must reduce to a constant. $\square$

5. Nevanlinna Theory for Difference Operator with Infinite Period

When considering analogous Picard exceptional values for varying-step operators with infinite periods, the second main theorem that we state next allows ω to vary within the upper bound r^{β}. However, we need to further restrict ω to $r^{\beta/4} < |\omega| < r^{\beta}$ if $f(z)$ is of finite positive order and to $\log^{1/8} r < |\omega| < \log^{1/2} r$ if $f(z)$ is of order zero.

DEFINITION 5.1. Let $f(z)$ be a meromorphic function of finite order σ, and $r = |z|$. If $\sigma > 0$, then we define a *varying-step difference operator* by $\Delta_{\omega} f := f(z+\omega) - f(z)$, where $0 < |\omega| < r^{\beta}, 0 < \beta < \min\{1, \sigma\}$. If $\sigma = 0$, then we define $\Delta_{\omega} f := f(z+\omega) - f(z)$, where $0 < |\omega| < \log^{1/2} r$ for $r > 1$, and $0 < |\omega| < 1$ for $r \le 1$.

Based on this definition, we have the following versions of the second main theorem.

THEOREM 5.2. *Let $f(z)$ be a meromorphic function of finite order σ such that $\Delta_{\omega} f \not\equiv 0$. Let $a_1, \dots, a_p$ be $p \ge 2$ distinct points in $\mathbb{C}$. Then*

$$m(r, f) + \sum_{j=1}^{p} m(r, 1/(f - a_j)) \\ \le 2T(r, f) - N_{\Delta_{\omega}}(r, f) + o(T(r, f)) + O(\log r) \tag{5.1}$$

outside a set of finite logarithmic measure, *where*

$$N_{\Delta_{\omega}}(r, f) := 2N(r, f) - N(r, \Delta_{\omega} f) + N(r, 1/\Delta_{\omega} f). \tag{5.2}$$

THEOREM 5.3. *Let $f(z)$ be a meromorphic function of finite order σ such that $\Delta_{\omega} f \not\equiv 0$. Let $a_1, \dots, a_p$ be $p \ge 2$ distinct points in $\mathbb{C}$. Then*

$$(p-1)T(r, f) \le \widetilde{N}_{\Delta_{\omega}}(r, f) + \sum_{j=1}^{p} \widetilde{N}_{\Delta_{\omega}}(r, 1/(f - a_j)) \\ + o(T(r, f)) + O(\log r) \tag{5.3}$$

outside a set of finite logarithmic measure.

An analogue of Picard exceptional values is defined as follows.

DEFINITION 5.4. Let $f(z)$ be a meromorphic function of finite order σ. We call $a \in \widehat{\mathbb{C}}$ a *Picard exceptional value for a varying-step difference operator with infinite period* if $\widetilde{N}_{\Delta_{\omega}}(r, 1/(f - a)) = O(1)$:

(i) if $\sigma > 0$, then
  (a) $|\omega| = |z|^{\beta/2}, 0 < \beta < \min\{1, \sigma\}$, for $|z| > 1$, and $|z + \omega| < |z| - |z|^{\beta/4}$,
  (b) $|\omega| = |z|/2$ for $|z| \le 1$, and $|z + \omega| < \frac{3}{4}|z|$;

(ii) if $\sigma = 0$, then
  (a) $|\omega| = \log^{1/4} |z|$ for $|z| > 1$, and $|z + \omega| < |z| - \log^{1/8} |z|$,
  (b) $|\omega| = |z|/2$ for $|z| \le 1$, and $|z + \omega| < \frac{3}{4}|z|$.

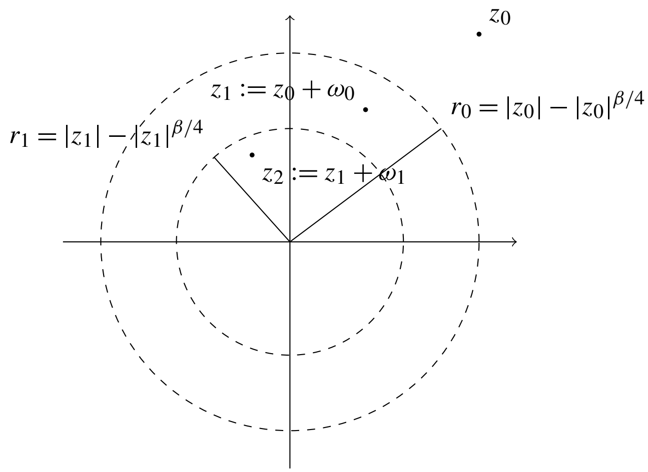

Figure 1 This figure shows the locations of three successive points that lie on the preimage of a *Picard exceptional value for a varying-step difference operator with infinite period,* where $z_n \to 0$ as $n \to \infty$

Here is an illustration of an example of the sequence (Figure 1).

Then we have the following Picard theorem for difference operator with infinite period.

THEOREM 5.5. *Let $f(z)$ be a meromorphic function of finite order σ. Suppose that f has three Picard exceptional values with respect to a varying-step difference operator with infinite period. Then $f(z)$ is a constant.*

Proof. Without loss of generality, we may assume that the three exceptional values are 0, 1, and ∞. We deduce from Theorem 5.3 that if $\Delta_\omega f \not\equiv 0$, then

$$T(r, f) \le o(T(r, f)) + O(\log r).$$

This is a contradiction unless either $f(z+\omega) \equiv f(z)$ and $f(z)$ is a transcendental meromorphic function or $f(z)$ is a rational function. We first show that in the latter case, f must must reduce to a constant. Otherwise, the definition of Picard exceptional values for varying-steps difference operator with infinite period would imply that the f has an infinite sequence of zeros/poles/a-points, which is a contradiction.

Next, we consider the case of $f(z)$ being a transcendental meromorphic function with $f(z+\omega) \equiv f(z)$. We claim that $f(z)$ must be an entire function. Without loss of generality, we may assume that $\arg\omega = -\arg z$, which is guaranteed under the assumption $|z+\omega| < |z| - |z|^{\beta/4}$ and $|z+\omega| < \frac{3}{4}|z|$ for $|z| > 1$ and $|z| \le 1$, respectively, when $\sigma > 0$, and $|z+\omega| < |z| - \log^{1/8}|z|$ and $|z+\omega| < \frac{3}{4}|z|$ for $|z| > 1$ and $|z| \le 1$, respectively, when $\sigma = 0$.

Indeed, if $z=\mu_1\neq 0$ is a pole of $f(z)$, then, by $f(z+\omega)\equiv f(z)$, there is a sequence $\{\mu_n\}_{n=1}^{\infty}$ of poles of $f(z)$ with $\lim_{n\to\infty}\mu_n=0$. Thus, $z=0$ is a nonisolated singularity of $f(z)$, which contradicts with $f(z)$ being meromorphic. Hence, the only possible pole of $f(z)$ is $z=0$. Then we write $f(z)=g(z)/z^m$, where m is the multiplicity of the pole $z=0$, $g(z)$ is an entire function, and $g(0)\neq 0$. Since $f(z)$ is transcendental, we can find a finite complex number α such that the set $\{z : f(z)=\alpha\}$ is infinite. Thus, we can choose $0\neq\nu_1\in\{z : f(z)=\alpha\}$. By $f(z+\omega)\equiv f(z)$ we have a sequence of α-points $\{\nu_n\}_{n=1}^{\infty}$ of $f(z)$ with $\lim_{n\to\infty}\nu_n=0$. So $g(\nu_n)=\nu_n^m f(\nu_n)=\alpha\cdot\nu_n^m$. Note that $g(z)$ is entire, so we deduce that $g(0)=0$, which is a contradiction with $g(0)\neq 0$. Hence, $f(z)$ is an entire function.

Set $M=\{z : |z|\leq 10\}$. Then $f(M)$ is bounded. For each $z\in\mathbb{C}\setminus M$, we can find $z_0\in M$ such that $f(z)=f(z_0)$. Thus, $f(z)$ is bounded, which implies that $f(z)$ is a constant. This contradicts with $f(z)$ being transcendental.

Combining the two cases, we obtain that $f(z)$ is a constant. □

6. Preliminaries

LEMMA 6.1 (see [12, p. 60]). *Let* $0<\alpha<1$. *Then*

$$\left(\sum_{k=1}^{n} x_k\right)^{\alpha}\leq\sum_{k=1}^{n} x_k^{\alpha}, \tag{6.1}$$

where $x_k\geq 0$ $(k=1,2,\ldots,n)$.

LEMMA 6.2 (see [12, p. 60]). *Let* $\varphi(x)$ *be a positive function on* $[a,b]$. *Then* $\log\varphi(x)$ *is integrable, and*

$$\frac{1}{b-a}\int_a^b \log\varphi(x)\,dx\leq\log\left[\frac{1}{b-a}\int_a^b\varphi(x)\,dx\right]. \tag{6.2}$$

LEMMA 6.3 (see [12, p. 62]). *Let* $0<\alpha<1$. *Then, for every complex number* ω, *we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|re^{i\theta}-\omega|^{\alpha}}\,d\theta\leq\frac{1}{(1-\alpha)r^{\alpha}}. \tag{6.3}$$

LEMMA 6.4 (see [12, p. 62]). *Let* $f(z)$ *be a meromorphic function. Then*

$$\begin{aligned}\left|\frac{f'(z)}{f(z)}\right| &\leq \frac{8R}{(R-r)^2}\left(T(R,f)+T\left(R,\frac{1}{f}\right)\right)\\ &\quad+\sum_{|a_u|<R}\frac{2}{|z-a_u|}+\sum_{|b_v|<R}\frac{2}{|z-b_v|},\end{aligned} \tag{6.4}$$

where $\{a_u\}$ *and* $\{b_v\}$ *are the sets of zeros and poles of* $f(z)$ *in* $D(0,R)$, *respectively.*

LEMMA 6.5 (see [16]). *Let $f(z)$ be a nonconstant meromorphic function. Then, for all irreducible rational functions in $f(z)$,*

$$R(f) = \frac{P(f)}{Q(f)} = \frac{\sum_{i=0}^{p} a_i(z) f^i(z)}{\sum_{j=0}^{q} b_j(z) f^j(z)}, \tag{6.5}$$

where $\{a_i(z)\}$ and $\{b_j(z)\}$ are small functions of $f(z)$, and $a_p(z) \not\equiv 0$, $b_q(z) \not\equiv 0$. Then,

$$T(r, R(f)) = \max\{p, q\} T(r, f) + S(r, f). \tag{6.6}$$

Here, we say a meromorphic function $g(z)$ is a small function of $f(z)$ if $T(r, g) = o(T(r, f))$.

LEMMA 6.6 (see [3]). *Let $0 < \alpha \leq 1$. Then there exists a constant $C_\alpha > 0$ depending only on α such that*

$$\log(1 + x) \leq C_\alpha x^\alpha \tag{6.7}$$

for $x \geq 0$. In particular, $C_1 = 1$.

LEMMA 6.7 (see [3]). *Let $0 < \alpha \leq 1$, and let C_α be as in Lemma 6.6. Then, for any two complex numbers z_1 and z_2 not vanishing simultaneously, we have the inequality*

$$\left| \log \left| \frac{z_1}{z_2} \right| \right| \leq C_\alpha \left(\left| \frac{z_1 - z_2}{z_2} \right|^\alpha + \left| \frac{z_2 - z_1}{z_1} \right|^\alpha \right). \tag{6.8}$$

LEMMA 6.8 (see [7]). *Let $z_1, z_2, \ldots$ be an infinite sequence of complex numbers that has no finite limit point and is ordered by increasing moduli. Let $n(t)$ denote the number of the points $\{z_k\}$ that lie in $|z| \leq t$. Let $\alpha > 1$ be a given real constant. Then there exists a set $E \subset (1, \infty)$ of finite logarithmic measure such that if $|z| \notin E \cup [0, 1]$, then*

$$\sum_{|z_k| \leq \alpha r} \frac{1}{|z - z_k|} < \alpha^2 \frac{n(\alpha^2 r)}{r} \log^\alpha r \log n(\alpha^2 r), \tag{6.9}$$

where $r = |z|$.

7. Proof of Theorem 2.1

Proof. We distinguish two cases.

Case 1. Suppose that $f(z)$ has no poles and zeros in $D(0, r + 1) := \{z : |z| < r + 1\}$. Thus, we can choose $|\eta| \in (0, \frac{1}{2})$ such that $z + \eta \in D(0, r + 1)$ for all z on $\{z : |z| = r\}$. It follows that $f(z + \eta)/f(z)$ is analytic on an open set containing $\{z : |z| \leq r\}$. Thus f is uniformly continuous on the compact set $\{z : |z| \leq r\}$ and hence on $\{z : |z| = r\}$. Since $\lim_{\eta \to 0} f(z + \eta)/f(z) = 1$, for arbitrary $\varepsilon > 0$, there exists $h_1(r, \varepsilon) > 0$ such that

$$\left| \frac{f(z + \eta)}{f(z)} \right| < 1 + \varepsilon$$

whenever $|\eta| < h_1(r,\varepsilon)$, where $\lim_{r\to\infty} h_1(r,\varepsilon) = 0$. Hence,

$$m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) < \log(1+\varepsilon) \le \varepsilon$$

whenever $|\eta| < h_1(r,\varepsilon)$.

Therefore,

$$\lim_{\eta\to 0} m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) = 0. \tag{7.1}$$

Similarly, we have

$$\lim_{\eta\to 0} m_\eta\left(r, \frac{f(z)}{f(z+\eta)}\right) = 0.$$

We also deduce

$$\lim_{r\to\infty} m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) \le \varepsilon < 2\varepsilon$$

since $\eta \to 0$, and therefore we can apply (7.1). Hence,

$$\lim_{r\to\infty} m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) = 0 \quad \text{and} \quad \lim_{r\to\infty} m_\eta\left(r, \frac{f(z)}{f(z+\eta)}\right) = 0.$$

Case 2. Suppose that $f(z)$ has poles and zeros in $D(0, r+1)$. We define

$$F(z) = f(z)\frac{\prod_{v=1}^{N}(z-b_v)}{\prod_{u=1}^{M}(z-a_u)},$$

where a_u $(u = 1, 2, \dots, M)$ and b_v $(v = 1, 2, \dots, N)$ are the zeros and poles of $f(z)$ in $D(0, r+1)$, respectively. Thus, $F(z)$ has no poles and zeros in $D(0, r+1)$. For all z satisfying $|z| = r$, we can also choose $|\eta| \in (0, \frac{1}{2})$ such that $z + \eta \in D(0, r+1)$. Moreover,

$$F(z+\eta) = f(z+\eta)\frac{\prod_{v=1}^{N}(z+\eta-b_v)}{\prod_{u=1}^{M}(z+\eta-a_u)}.$$

Then,

$$\frac{f(z+\eta)}{f(z)} = \frac{F(z+\eta)}{F(z)} \prod_{u=1}^{M} \frac{z+\eta-a_u}{z-a_u} \prod_{v=1}^{N} \frac{z-b_v}{z+\eta-b_v}.$$

It follows from Lemma 6.3 and Lemma 6.7 with $0 < \alpha < 1$ that

$$\begin{aligned}
\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| &\le \left|\log\left|\frac{F(z+\eta)}{F(z)}\right|\right| + \sum_{u=1}^{M}\left|\log\left|\frac{z+\eta-a_u}{z-a_u}\right|\right| \\
&\quad + \sum_{v=1}^{N}\left|\log\left|\frac{z-b_v}{z+\eta-b_v}\right|\right| \\
&\le \left|\log\left|\frac{F(z+\eta)}{F(z)}\right|\right| + C_\alpha|\eta|^\alpha\left[\sum_{u=1}^{M}\left(\frac{1}{|z-a_u|^\alpha} + \frac{1}{|z+\eta-a_u|^\alpha}\right)\right. \\
&\quad \left. + \sum_{v=1}^{N}\left(\frac{1}{|z-b_v|^\alpha} + \frac{1}{|z+\eta-b_v|^\alpha}\right)\right].
\end{aligned}$$

Thus,

$$
\begin{aligned}
& m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) + m_\eta\left(r, \frac{f(z)}{f(z+\eta)}\right) \\
&\quad \le m_\eta\left(r, \frac{F(z+\eta)}{F(z)}\right) + m_\eta\left(r, \frac{F(z)}{F(z+\eta)}\right) \\
&\qquad + C_\alpha |\eta|^\alpha \sum_{u=1}^{M}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|re^{i\theta}-a_u|^\alpha}\, d\theta\right. \\
&\qquad \left. + \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|re^{i\theta}+\eta-a_u|^\alpha}\, d\theta\right) \\
&\qquad + C_\alpha |\eta|^\alpha \sum_{v=1}^{N}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|re^{i\theta}-b_v|^\alpha}\, d\theta\right. \\
&\qquad \left. + \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|re^{i\theta}+\eta-b_v|^\alpha}\, d\theta\right) \\
&\quad \le m_\eta\left(r, \frac{F(z+\eta)}{F(z)}\right) + m_\eta\left(r, \frac{F(z)}{F(z+\eta)}\right) + \frac{2C_\alpha |\eta|^\alpha}{(1-\alpha) r^\alpha}(M+N) \\
&\quad \le m_\eta\left(r, \frac{F(z+\eta)}{F(z)}\right) + m_\eta\left(r, \frac{F(z)}{F(z+\eta)}\right) \\
&\qquad + \frac{2C_\alpha |\eta|^\alpha}{(1-\alpha) r^\alpha}\left[n(r+1, f) + n\left(r+1, \frac{1}{f}\right)\right].
\end{aligned} \tag{7.2}
$$

Since F is free of zeros and poles in $|z| < r$, we can apply Case 1 to F and combine this with the last inequality (7.2) to deduce

$$
\lim_{\eta\to 0} m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) = 0 \quad \text{and} \quad \lim_{\eta\to 0} m_\eta\left(r, \frac{f(z)}{f(z+\eta)}\right) = 0.
$$

On the other hand, we choose $\alpha = \frac{1}{2}$ in (7.2), and since $0 < |\eta| < \alpha_1(r)$, where $\alpha_1(r) = \min\{\log^{-1/2} r, 1/(n(r+1))^2\}$, $n(r) = n(r, f) + n(r, 1/f)$, we have

$$
\begin{aligned}
& m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) + m_\eta\left(r, \frac{f(z)}{f(z+\eta)}\right) \\
&\quad \le m_\eta\left(r, \frac{F(z+\eta)}{F(z)}\right) + m_\eta\left(r, \frac{F(z)}{F(z+\eta)}\right) + \frac{4C_{1/2}}{r^{1/2}},
\end{aligned}
$$

which clearly tends to zero as $r \to \infty$. This proves (2.3). □

8. Proof of Corollary 2.2

Proof. Since (2.1) holds for each positive real number r, for any given $\varepsilon > 0$, there exists $h(r, \varepsilon) > 0$ such that

$$
m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) < \varepsilon \quad \text{and} \quad m_\eta\left(r, \frac{f(z)}{f(z+\eta)}\right) < \varepsilon
$$

whenever $|\eta| < h(r,\varepsilon)$. Note that

$$\begin{aligned} m(r, f(z)) &\le m_\eta\left(r, \frac{f(z)}{f(z+\eta)}\right) + m_\eta(r, f(z+\eta)) \\ &\le m(r, f(z)) + m_\eta\left(r, \frac{f(z)}{f(z+\eta)}\right) + m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right), \end{aligned}$$

that is,

$$-m_\eta\left(r, \frac{f(z)}{f(z+\eta)}\right) \le m_\eta(r, f(z+\eta)) - m(r, f(z)) \le m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right).$$

This implies

$$\begin{aligned} &|m_\eta(r, f(z+\eta)) - m(r, f(z))| \\ &\quad \le m_\eta\left(r, \frac{f(z)}{f(z+\eta)}\right) + m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) < 2\varepsilon \end{aligned}$$

whenever $|\eta| < h(r,\varepsilon)$. Therefore

$$\lim_{\eta\to 0} m_\eta(r, f(z+\eta)) = m(r, f(z)).$$

□

9. Proof of Theorem 2.3

Proof. Let $\alpha_2(r)$ be defined in (2.5). Since $0 < |\eta| < \alpha_2(r)$, we have $n(0, f(z+\eta)) = 0$.

Applying the argument in [3, (5.1), (5.2), (5.3), and (5.4)], we deduce

$$\begin{aligned} &|N_\eta(r, f(z+\eta)) - N(r, f(z))| \\ &\quad \le |\eta| \Bigg(\sum_{\substack{0<|b_\mu-\eta|<r,\\ b_\mu\neq 0}} \frac{1}{|b_\mu-\eta|} + \sum_{\substack{0<|b_\mu|<r,\\ b_\mu-\eta\neq 0}} \frac{1}{|b_\mu|} \Bigg) + n(0, f(z)) \log r. \end{aligned} \tag{9.1}$$

Note that $f(z)$ has no poles in $\overline{D}(0,h)\setminus\{0\}$, which implies that

$$\sum_{\substack{0<|b_\mu-\eta|\le|\eta|,\\ b_\mu\neq 0}} \frac{1}{|b_\mu-\eta|} = 0$$

under the assumption $0 < |\eta| < \alpha_2(r)$. Thus,

$$\begin{aligned} \sum_{\substack{0<|b_\mu-\eta|<r,\\ b_\mu\neq 0}} \frac{1}{|b_\mu-\eta|} &= \sum_{\substack{0<|b_\mu-\eta|\le|\eta|,\\ b_\mu\neq 0}} \frac{1}{|b_\mu-\eta|} + \sum_{|\eta|<|b_\mu-\eta|<r} \frac{1}{|b_\mu-\eta|} \\ &\le \sum_{|\eta|<|b_\mu-\eta|<r} \frac{1}{|b_\mu|}\cdot\left(1 + \left|\frac{\eta}{b_\mu-\eta}\right|\right) \\ &\le 2\sum_{|\eta|<|b_\mu-\eta|<r} \frac{1}{|b_\mu|} \le 2\sum_{0<|b_\mu|<r+|\eta|} \frac{1}{|b_\mu|}. \end{aligned} \tag{9.2}$$

From (9.1) and (9.2) we deduce

$$\begin{aligned}&|N_\eta(r, f(z+\eta)) - N(r, f(z))|\\&\quad \leq 3|\eta|\left(\sum_{0<|b_\mu|<r+|\eta|} \frac{1}{|b_\mu|}\right) + n(0, f(z)) \log r.\end{aligned}$$

Since $0 < |\eta| < \alpha_2(r)$, we have

$$|N_\eta(r, f(z+\eta)) - N(r, f(z))| \leq n(0, f(z)) \log r + 3.$$

Hence,

$$N_\eta(r, f(z+\eta)) = N(r, f(z)) + \varepsilon_1(r),$$

where $|\varepsilon_1(r)| \leq n(0, f(z)) \log r + 3$. □

10. Proof of Theorem 2.4

Proof. It follows from the proofs of Theorem 2.1 and Theorem 2.3 that, for each $r > 0$, there exists $\beta(r) > 0$ such that whenever $0 < |\eta| < \beta(r) := \min\{\alpha_1(r), \alpha_2(r)\}$ (see the remark after Theorem 2.4),

$$\begin{aligned}T_\eta(r, f(z+\eta)) &= m_\eta(r, f(z+\eta)) + N_\eta(r, f(z+\eta))\\&\leq m(r, f(z)) + m_\eta\left(r, \frac{f(z+\eta)}{f(z)}\right) + N_\eta(r, f(z+\eta))\\&= T(r, f(z)) + \varepsilon_1(r) + 1,\end{aligned}$$

where $|\varepsilon_1(r)| \leq n(0, f(z)) \log r + 3$. Similarly, we have

$$T(r, f(z)) \leq T_\eta(r, f(z+\eta)) + \varepsilon_1(r) + 1.$$

This proves (2.7). □

11. Proof of Theorem 3.1

Proof. Since $f(z)$ is of finite order σ, we have

$$T(r, f) = O(r^{\sigma+\varepsilon}).$$

By choosing $R = 2r$, $R' = 3r$, and $\alpha = 1 - \varepsilon$ in [3, Thm. 2.4] we have

$$m\left(r, \frac{f(z+\omega)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\omega)}\right) = O(r^{\sigma-(1-\beta)(1-\varepsilon)+\varepsilon}).$$ □

12. Proof of Theorem 3.2

Proof. (1) If σ is nonzero, then we again apply [3, (5.1)–(5.4)] to obtain

$$\begin{aligned}&|N(r, f(z+\omega)) - N(r, f(z))|\\&\quad \leq |\omega|\left(\sum_{\substack{0<|b_\mu-\omega|<r,\\ b_\mu\neq 0}} \frac{1}{|b_\mu-\omega|} + \sum_{\substack{0<|b_\mu|<r,\\ b_\mu-\omega\neq 0}} \frac{1}{|b_\mu|}\right) + O(\log r).\end{aligned}$$

Note that if $|\omega| < r^\beta < r$ and $r > 1$, then

$$\sum_{\substack{0<|b_\mu-\omega|<r,\\ b_\mu\neq 0}} \frac{1}{|b_\mu-\omega|} = \sum_{\substack{0<|b_\mu-\omega|\le|\omega|,\\ b_\mu\neq 0}} \frac{1}{|b_\mu-\omega|} + \sum_{|\omega|<|b_\mu-\omega|<r} \frac{1}{|b_\mu-\omega|}. \tag{12.1}$$

Lemma 6.8 implies, with $\alpha = 2$, that

$$\begin{aligned}\sum_{\substack{0<|b_\mu-\omega|\le|\omega|,\\ b_\mu\neq 0}} \frac{1}{|b_\mu-\omega|} &\le \sum_{|b_\mu|\le 2|\omega|} \frac{1}{|\omega-b_\mu|}\\ &\le 4\cdot\frac{n(4|\omega|)}{|\omega|}\cdot\log^2|\omega|\cdot\log n(4|\omega|)\\ &= O(|\omega|^{\sigma-1+\varepsilon}\cdot\log^3|\omega|) \end{aligned}\tag{12.2}$$

when $|\omega|$ is sufficiently large and outside a set of finite logarithmic measure of $|\omega|$.

Since

$$\begin{aligned}\sum_{|\omega|<|b_\mu-\omega|<r} \frac{1}{|b_\mu-\omega|} &\le \sum_{|\omega|<|b_\mu-\eta|<r} \frac{1}{|b_\mu|}\cdot\left(1+\left|\frac{\omega}{b_\mu-\omega}\right|\right)\\ &< \sum_{|\omega|<|b_\mu-\omega|<r} \frac{2}{|b_\mu|} \le \sum_{0<|b_\mu|<2r} \frac{2}{|b_\mu|}, \end{aligned}\tag{12.3}$$

we get

$$\begin{aligned}&|N(r, f(z+\omega)) - N(r, f(z))|\\ &\qquad\le 3|\omega|\left(\sum_{0<|b_\mu|<2r}\frac{1}{|b_\mu|}\right) + O(|\omega|^{\sigma+\varepsilon}\cdot\log^3|\omega|) + O(\log r). \end{aligned}\tag{12.4}$$

But a standard argument (see e.g. [3, (5.9)–(5.12)] implies that

$$\sum_{0<|b_\mu|<2r}\frac{1}{|b_\mu|} = O(r^{\sigma-1+\varepsilon})$$

when $\sigma \ge 1$ and

$$\sum_{0<|b_\mu|<2r}\frac{1}{|b_\mu|} = O(1) \tag{12.5}$$

when $\sigma < 1$.

Thus, when $\sigma \ge 1$, we choose $0 < \varepsilon < \min\{(\sigma-1)(1-\beta)/\beta, 1-\beta\}$. Hence,

$$\begin{aligned}N(r, f(z+\omega)) = N(r, f) &+ O(r^{\beta(\sigma+\varepsilon)}\cdot\log^3 r)\\ &+ O(r^{\sigma-(1-\beta)+\varepsilon}) + O(\log r)\end{aligned}\tag{12.6}$$

outside a set of finite logarithmic measure of $|\omega|$ and hence of $|r|$. Since $\varepsilon < 1-\beta$, this, together with (12.6), gives (3.2).

On the other hand, when $0 < \sigma < 1$, (12.4) becomes

$$N(r, f(z+\omega)) = N(r, f) + O(r^{\beta}) + O(r^{\beta(\sigma+\varepsilon)} \cdot \log^3 r) + O(\log r). \quad (12.7)$$

We choose $0 < \varepsilon < 1 - \sigma$ in (12.6). Since the term (12.5) becomes bounded, the assumption on ε means that the term $O(r^{\beta})$ is dominant over the term $r^{\beta(\sigma+\varepsilon)}$, and so (3.3) follows.

(2) If $\sigma = \sigma(f) = 0$, then we choose $|\omega| < \log^{1/2} r < r$ for $r > 1$. It follows similarly from (12.1) and Lemma 6.8 with a different ε, $0 < \varepsilon < 1$, that (12.2) holds with $\sigma = 0$.

It follows from (12.3) that (12.4) holds with $\sigma = 0$.

Note that $\sigma = \sigma(f) = 0$, so that (12.5) applies. Hence,

$$\begin{aligned} N(r, f(z+\omega)) &= N(r, f) + O(\log^{\varepsilon/2} r \cdot \log^3 \log r) + O(\log r) \\ &= N(r, f) + O(\log r) \end{aligned}$$

outside a set of finite logarithmic measure. □

13. Applications of Vanishing Period

It is known that we can recover the classical Painlevé equations from the corresponding discrete Painlevé equations [17] taking suitable limits of specifically designated change of variables. See for example, [10] and [6]. We further consider limits of different types between certain discrete equations and their continuous counterparts by making use of what we have established in this paper.

Example 13.1. If the difference equation

$$\begin{aligned} f(z+\eta) - f(z) &= R(f(z), z, \eta) \\ &= \frac{a_0(z,\eta) + a_1(z,\eta) f(z) + \cdots + a_p(z,\eta) f^p(z)}{b_0(z,\eta) + b_1(z,\eta) f(z) + \cdots + b_q(z,\eta) f^q(z)} \end{aligned} \quad (13.1)$$

with rational coefficients $a_i(z,\eta)$ $(i = 1, \ldots, p)$ and $b_j(z,\eta)$ $(j = 1, \ldots, q)$ admits a transcendental meromorphic solution $f(z)$ that is independent of η, where η is a nonzero parameter such that

$$\lim_{\eta \to 0} \frac{R(f(z), z, \eta)}{\eta} = \widehat{R}(f(z), z),$$

when taken as a formal limit, is a rational function of $f(z)$ with rational coefficients. Then, $q = 0$ and $p \le 2$. Moreover, (13.1) will be reduced into a Riccati differential equation of the form $f'(z) = a(z) + b(z) f(z) + c(z) f^2(z)$ with rational coefficients.

Proof. It follows from (13.1) after division of both sides by η and an application of Lemma 6.5 that

$$\begin{aligned} \max\{p,q\} T(r, f(z)) &= T(r, R(f(z), z, \eta)) + O(\log|\eta|) + S(r, f(z)) \\ &= T(r, f(z+\eta) - f(z)) + O(\log|\eta|) + S(r, f(z)) \\ &\le T(r, f(z+\eta)) + T(r, f(z)) + O(\log|\eta|) + S(r, f(z)). \end{aligned}$$

We deduce from Theorem 2.4 that

$$\max\{p,q\}T(r,f(z)) \le 2T(r,f(z)) + O(\log|\eta|) + \varepsilon(r) + S(r,f(z)). \quad (13.2)$$

We choose $|\eta| = \min\{\alpha_1(r), \alpha_2(r)\}/2$, where $\alpha_1(r)$ and $\alpha_2(r)$ were defined in (2.2) and (2.5), respectively. Note that if f has at most finitely many zeros, and we are done. If, however, f has infinitely many zeros, then we have, for a suitably chosen $\delta > 0$, that

$$\begin{aligned}\sum_{0<|b_\mu|<r+1/2} \frac{1}{|b_\mu|} &= \frac{n(r+1/2)}{r+1/2} + \int_\delta^{r+1/2} \frac{n(t)}{t^2}\,dt + O(1)\\ &\ge \frac{n(r+1/2)}{r+1/2} + O(1) + \int_{r/2+1/4}^{r+1/2} \frac{n(t)}{t^2}\,dt\\ &\ge \frac{n(r+1/2)}{r+1/2} + O(1) + \left(r+\frac{1}{2}-\frac{r}{2}-\frac{1}{4}\right)\frac{n(r/2+1/4)}{(r+1/2)^2}\\ &= \frac{n(r+1/2)}{r+1/2} + O(1) + \left(\frac{1}{2}+o(1)\right)\frac{n(r/2+1/4)}{r}, \quad (13.3)\end{aligned}$$

from which and from (13.2) we deduce that $\max\{p,q\} \le 2$.

On the other hand, note that (13.1) can be written as

$$\frac{f(z+\eta)-f(z)}{\eta} = \frac{R(f(z),z,\eta)}{\eta}.$$

Letting $\eta \to 0$ as a formal limit, we obtain

$$f'(z) = \widehat{R}(f(z),z), \quad (13.4)$$

which is an equation considered by Malmquist. Since this equation admits a meromorphic solution under our assumption, Malmquist's theorem (see [14, p. 193]) implies that equation (13.4) reduces to a Riccati differential equation of the form

$$f'(z) = a(z) + b(z)f(z) + c(z)f^2(z)$$

with rational coefficients. Therefore, $q = 0$ and $p \le 2$. □

EXAMPLE 13.2. Suppose that the difference equation

$$\begin{aligned}&f(z+\eta_1+\eta_2) - f(z+\eta_1) - f(z+\eta_2) + f(z)\\ &\quad = R(f(z),z,\eta_1,\eta_2)\\ &\quad = \frac{a_0(z,\eta_1,\eta_2) + a_1(z,\eta_1,\eta_2)f(z) + \cdots + a_p(z,\eta_1,\eta_2)f^p(z)}{b_0(z,\eta_1,\eta_2) + b_1(z,\eta_1,\eta_2)f(z) + \cdots + b_q(z,\eta_1,\eta_2)f^q(z)} \quad (13.5)\end{aligned}$$

with rational coefficients $a_i(z,\eta_1,\eta_2)$ $(i = 1,\dots,p)$ and $b_j(z,\eta_1,\eta_2)$ $(j = 1,\dots,q)$ admits a transcendental meromorphic solution $f(z)$ independent of η_1 and η_2 such that both

$$\lim_{\eta_1\to 0} \frac{R(f(z),z,\eta_1,\eta_2)}{\eta_1} = R_1(f(z),z,\eta_2)$$

and

$$\lim_{\eta_2\to 0}\lim_{\eta_1\to 0} = \frac{R(f(z), z, \eta_1, \eta_2)}{\eta_1\eta_2} = \lim_{\eta_2\to 0}\frac{R_1(f(z), z, \eta_2)}{\eta_2} = R_2(f(z), z)$$

are rational functions of $f(z)$ with rational coefficients. Then either (13.5) will be reduced to Painlevé equations (*I*) or (*II*) after taking limits, which implies that $q = 0$ and $p \le 3$, or (13.5) will be transformed into a reducible second-order differential equation or that without the Painlevé property.

Proof. We deduce from (13.5) and Lemma 6.5 that

$$\begin{aligned}\max\{p, q\}T(r, f(z)) &= T(r, R(f(z), z, \eta_1, \eta_2)) + O(\log|\eta_1|)\\ &\quad + O(\log|\eta_2|) + S(r, f(z))\\ &= T(r, f(z+\eta_1+\eta_2) - f(z+\eta_1) - f(z+\eta_2) + f(z))\\ &\quad + O(\log|\eta_1|) + O(\log|\eta_2|) + S(r, f(z))\\ &\le T_{\eta_1+\eta_2}(r, f(z+\eta_1+\eta_2)) + T_{\eta_1}(r, f(z+\eta_1))\\ &\quad + T_{\eta_2}(r, f(z+\eta_2)) + T(r, f(z))\\ &\quad + O(\log|\eta_1|) + O(\log|\eta_2|) + S(r, f(z)).\end{aligned}$$

We deduce from Theorem 2.4 that

$$\begin{aligned}\max\{p, q\}T(r, f(z)) \le 4T(r, f(z)) &+ O(\log|\eta_1|)\\ &+ O(\log|\eta_2|) + 3\varepsilon(r) + S(r, f(z)). \qquad (13.6)\end{aligned}$$

We choose $|\eta_1| = |\eta_2| = \min\{\alpha_1(r), \alpha_2(r)\}/2$, where $\alpha_1(r)$ and $\alpha_2(r)$ are defined in (2.2) and (2.5), respectively. According to (13.3), Theorem 2.4 applies, from which and from (13.6) we deduce that $\max\{p, q\} \le 4$.

On the other hand, note that (13.5) can be written as

$$\frac{f(z+\eta_2+\eta_1) - f(z+\eta_2)}{\eta_1} - \frac{f(z+\eta_1) - f(z)}{\eta_1} = \frac{R(f(z), z, \eta_1, \eta_2)}{\eta_1}.$$

Letting $\eta_1 \to 0$ as a formal limit, we get

$$\frac{f'(z+\eta_2) - f'(z)}{\eta_2} = \frac{1}{\eta_2}\cdot R_1(f(z), z, \eta_2).$$

Letting $\eta_2 \to 0$ as a formal limit, we have

$$f''(z) = R_2(f(z), z). \qquad (13.7)$$

Then (see, e.g., [13, Section 14.4]) either (13.7) is a reducible second-order differential equation, which can be solved by known special functions, or that without the Painlevé property, or it is $P_I : f''(z) = 6f^2(z) + z$ or $P_{II} : f''(z) = 2f^3(z) + zf(z) + \alpha$, where α is a constant.

In the second case, we deduce that $q = 0$ and $p \le 3$. □

The following theorem is a limiting analogue of the Clunie lemma. Although the basic idea goes back to Clunie [14], we apply our Theorem 2.1 instead of the logarithmic derivative estimate [14] and the logarithmic difference estimate [3].

THEOREM 13.3. *Let $f(z)$ be a nonconstant meromorphic solution of*

$$f^n(z)P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are difference polynomials in $f(z)$, and its steps of shifts are nonzero parameters. If the degree of $Q(z, f)$ is at most n, then

$$\lim_{\Gamma\to 0} m(r, P(z, f)) = o(T(r, f)),$$

where Γ is the set of all these steps in $P(z, f)$. Here, $\Gamma \to 0$ means that the maximum length of the steps tends to zero.

14. A Reformulation of Logarithmic Derivative Lemma

We give an alternative derivation of Nevanlinna's original logarithmic derivative lemma $m(r, f'/f) = O(\log r)$ via a formal limiting process of a new difference-type estimate of

$$m\left(r, \frac{1}{\eta}\left(\frac{f(z+\eta)}{f(z)} - 1\right)\right) \longrightarrow m\left(r, \frac{f'}{f}\right), \quad \eta \to 0.$$

THEOREM 14.1. *Let $f(z)$ be a meromorphic function, r, R, and R' be positive real numbers satisfying $0 < r < R < R'$, and $0 < \alpha < 1$ be a constant. Then*

$$\begin{aligned}
\lim_{\eta\to 0} m_\eta&\left(r, \frac{1}{\eta}\left(\frac{f(z+\eta)}{f(z)} - 1\right)\right) \\
&\le \frac{1}{\alpha}\log\left(1 + \frac{8R^\alpha}{(R-r)^{2\alpha}}\left(T^\alpha(R, f) + T^\alpha\left(R, \frac{1}{f}\right)\right)\right. \\
&\quad \left. + \frac{3(N(R', f) + N(R', \frac{1}{f}))}{(1-\alpha)r^\alpha \log \frac{R'}{R}}\right) \\
&\quad + \frac{1}{\alpha}\log\left(2^\alpha + \frac{N(R', f) + N(R', \frac{1}{f})}{(1-\alpha)r^\alpha \log\frac{R'}{R}}\right) + 2\log 2. \qquad (14.1)
\end{aligned}$$

Proof. We distinguish two cases.

Case 1. If $f(z)$ has no zeros and poles in $\overline{D}(0, R)$, then $f(z)$ is analytic on $D(0, R) := \{z : |z| < R\}$. Thus, for all z satisfying $|z| = r < R$, we can choose $|\eta|$ (> 0) sufficiently small such that $z + \eta \in D(0, R)$ and

$$f(z+\eta) - f(z) = \eta f'(z) + o(\eta)$$

as $\eta \to 0$. So

$$\frac{1}{\eta}\left(\frac{f(z+\eta)}{f(z)} - 1\right) = \frac{f'(z)}{f(z)} + \frac{o(\eta)}{\eta}\frac{1}{f(z)}.$$

Since $f(z)$ has no zeros in $\overline{D}(0, R)$, $1/f(z)$ is analytic on $D(0, R)$ and continuous on $\overline{D}(0, R)$. By the maximum modulus principle, there exists $M_1 > 0$ such that $|1/f(z)| < M_1$ for all $z \in D(0, R)$.

We further choose $|\eta|$ (> 0) sufficiently small such that $|\frac{o(\eta)}{\eta}\frac{1}{f(z)}| < \frac{1}{2}$ for all $z \in D(0, R)$. Hence, when $|\eta|$ (> 0) is sufficiently small, we have

$$\begin{aligned} m_\eta\left(r, \frac{1}{\eta}\left(\frac{f(z+\eta)}{f(z)} - 1\right)\right) &\le m\left(r, \frac{f'(z)}{f(z)}\right) + m\left(r, \frac{o(\eta)}{\eta}\frac{1}{f(z)}\right) + \log 2 \\ &\le m\left(r, \frac{f'(z)}{f(z)}\right) + \log 2. \end{aligned} \tag{14.2}$$

Case 2. If $f(z)$ has zeros and poles in $\overline{D}(0, R)$, then we define

$$F(z) = f(z)\frac{\prod_{v=1}^{N}(z - b_v)}{\prod_{u=1}^{M}(z - a_u)}, \tag{14.3}$$

where a_u ($u = 1, 2, \ldots, M$) and b_v ($v = 1, 2, \ldots, N$) are the zeros and poles of $f(z)$ on $\overline{D}(0, R)$, respectively. Thus, $F(z)$ is free of poles and zeros on $\overline{D}(0, R)$.

For all z satisfying $|z| = r < R$, we can choose $|\eta|$ (> 0) sufficiently small such that $z + \eta \in D(0, R)$. We deduce

$$\begin{aligned} \frac{1}{\eta}\left(\frac{f(z+\eta)}{f(z)} - 1\right) &= \frac{1}{\eta}\left(\frac{F(z+\eta)}{F(z)} - 1\right) + \frac{F(z+\eta)}{F(z)} \\ &\quad \cdot \frac{1}{\eta}\left(\prod_{u=1}^{M}\frac{z+\eta-a_u}{z-a_u}\prod_{v=1}^{N}\frac{z-b_v}{z+\eta-b_v} - 1\right). \end{aligned} \tag{14.4}$$

From (14.3) we have

$$\log F(z) = \log f(z) + \sum_{v=1}^{N}\log(z - b_v) - \sum_{u=1}^{M}\log(z - a_u) + 2k\pi i \tag{14.5}$$

for some $k \in \mathbb{Z}$. Taking logarithmic derivatives on both sides of (14.3) and an application of Lemma 6.4 allow us to deduce

$$\begin{aligned} \left|\frac{F'(z)}{F(z)}\right| &\le \left|\frac{f'(z)}{f(z)}\right| + \sum_{v=1}^{N}\frac{1}{|z-b_v|} + \sum_{u=1}^{M}\frac{1}{|z-a_u|} \\ &\le \frac{8R}{(R-r)^2}\left(T(R, f) + T\left(R, \frac{1}{f}\right)\right) + \sum_{|a_u|<R}\frac{2}{|z-a_u|} \\ &\quad + \sum_{|b_v|<R}\frac{2}{|z-b_v|} + \sum_{v=1}^{N}\frac{1}{|z-b_v|} + \sum_{u=1}^{M}\frac{1}{|z-a_u|} \\ &\le \frac{8R}{(R-r)^2}\left(T(R, f) + T\left(R, \frac{1}{f}\right)\right) \\ &\quad + \sum_{v=1}^{N}\frac{3}{|z-b_v|} + \sum_{u=1}^{M}\frac{3}{|z-a_u|}. \end{aligned} \tag{14.6}$$

Taking $\log^+$ of both sides of (14.6) and applying Lemma 6.1 with $0 < \alpha < 1$ yields

$$
\begin{aligned}
\log^+\left|\frac{F'(z)}{F(z)}\right| &\le \frac{1}{\alpha}\log\left(1+\left|\frac{F'(z)}{F(z)}\right|\right)^{\alpha} \\
&\le \frac{1}{\alpha}\log\left(1+\frac{8R}{(R-r)^2}\left(T(R,f)+T\left(R,\frac{1}{f}\right)\right)\right. \\
&\qquad \left.+\sum_{v=1}^{N}\frac{3}{|z-b_v|}+\sum_{u=1}^{M}\frac{3}{|z-a_u|}\right)^{\alpha} \\
&\le \frac{1}{\alpha}\log\left(1+\frac{8R^{\alpha}}{(R-r)^{2\alpha}}\left(T^{\alpha}(R,f)+T^{\alpha}\left(R,\frac{1}{f}\right)\right)\right. \\
&\qquad \left.+\sum_{v=1}^{N}\frac{3}{|z-b_v|^{\alpha}}+\sum_{u=1}^{M}\frac{3}{|z-a_u|^{\alpha}}\right).
\end{aligned}
\tag{14.7}
$$

A straightforward application of Lemma 6.2 and Lemma 6.3 to (14.7) gives the estimate

$$
\begin{aligned}
m\left(r,\frac{F'(z)}{F(z)}\right) &\le \frac{1}{\alpha}\log\left(1+\frac{8R^{\alpha}}{(R-r)^{2\alpha}}\left(T^{\alpha}(R,f)+T^{\alpha}\left(R,\frac{1}{f}\right)\right)\right. \\
&\qquad \left.+\frac{3}{(1-\alpha)r^{\alpha}}(M+N)\right).
\end{aligned}
\tag{14.8}
$$

Noting that $0 < R < R'$, we have

$$
n(R,f) \le \frac{N(R',f)}{\log\frac{R'}{R}} \quad \text{and} \quad n\left(R,\frac{1}{f}\right) \le \frac{N(R',\frac{1}{f})}{\log\frac{R'}{R}}.
\tag{14.9}
$$

Combining (14.8) and (14.9), we deduce that

$$
\begin{aligned}
m\left(r,\frac{F'(z)}{F(z)}\right) &\le \frac{1}{\alpha}\log\left(1+\frac{8R^{\alpha}}{(R-r)^{2\alpha}}\left(T^{\alpha}(R,f)+T^{\alpha}\left(R,\frac{1}{f}\right)\right)\right. \\
&\qquad \left.+\frac{3(N(R',f)+N(R',\frac{1}{f}))}{(1-\alpha)r^{\alpha}\log\frac{R'}{R}}\right).
\end{aligned}
\tag{14.10}
$$

On the other hand, an application of L'Hospital's rule yields

$$
\lim_{\eta\to 0}\frac{1}{\eta}\left(\prod_{u=1}^{M}\frac{z+\eta-a_u}{z-a_u}\prod_{v=1}^{N}\frac{z-b_v}{z+\eta-b_v}-1\right)=\sum_{u=1}^{M}\frac{1}{z-a_u}-\sum_{v=1}^{N}\frac{1}{z-b_v},
$$

so that

$$
\lim_{\eta\to 0}\left|\frac{1}{\eta}\cdot\prod_{u=1}^{M}\frac{z+\eta-a_u}{z-a_u}\prod_{v=1}^{N}\frac{z-b_v}{z+\eta-b_v}-1\right| < \sum_{u=1}^{M}\frac{1}{|z-a_u|}+\sum_{v=1}^{N}\frac{1}{|z-b_v|}+1.
$$

Hence, there exists $h > 0$ such that

$$\left|\frac{1}{\eta}\cdot\prod_{u=1}^{M}\frac{z+\eta-a_u}{z-a_u}\prod_{v=1}^{N}\frac{z-b_v}{z+\eta-b_v}-1\right| < \sum_{u=1}^{M}\frac{1}{|z-a_u|}+\sum_{v=1}^{N}\frac{1}{|z-b_v|}+1 \tag{14.11}$$

whenever $0 < |\eta| < h$. We apply Lemma 6.1 to (14.11) to deduce

$$\begin{aligned}
&\log^+\left|\frac{1}{\eta}\cdot\prod_{u=1}^{M}\frac{z+\eta-a_u}{z-a_u}\prod_{v=1}^{N}\frac{z-b_v}{z+\eta-b_v}-1\right|\\
&\quad\le\frac{1}{\alpha}\log\left(1+\left|\frac{1}{\eta}\cdot\prod_{u=1}^{M}\frac{z+\eta-a_u}{z-a_u}\prod_{v=1}^{N}\frac{z-b_v}{z+\eta-b_v}-1\right|\right)^{\alpha}\\
&\quad\le\frac{1}{\alpha}\log\left(2+\sum_{u=1}^{M}\frac{1}{|z-a_u|}+\sum_{v=1}^{N}\frac{1}{|z-b_v|}\right)^{\alpha}\\
&\quad\le\frac{1}{\alpha}\log\left(2^{\alpha}+\sum_{u=1}^{M}\frac{1}{|z-a_u|^{\alpha}}+\sum_{v=1}^{N}\frac{1}{|z-b_v|^{\alpha}}\right).
\end{aligned}\tag{14.12}$$

Applying again Lemma 6.2, Lemma 6.3, and (14.9) to (14.12) enables us to deduce

$$\begin{aligned}
&m_\eta\left(r,\frac{1}{\eta}\cdot\prod_{u=1}^{M}\frac{z+\eta-a_u}{z-a_u}\prod_{v=1}^{N}\frac{z-b_v}{z+\eta-b_v}-1\right)\\
&\quad\le\frac{1}{\alpha}\log\left(2^{\alpha}+\sum_{u=1}^{M}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|re^{i\theta}-a_u|^{\alpha}}\,d\theta\right.\\
&\qquad\left.+\sum_{v=1}^{N}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|re^{i\theta}-b_v|^{\alpha}}\,d\theta\right)\\
&\quad\le\frac{1}{\alpha}\log\left(2^{\alpha}+\frac{1}{(1-\alpha)r^{\alpha}}(M+N)\right)\\
&\quad=\frac{1}{\alpha}\log\left(2^{\alpha}+\frac{1}{(1-\alpha)r^{\alpha}}\left(n(R,f)+n\left(R,\frac{1}{f}\right)\right)\right)\\
&\quad\le\frac{1}{\alpha}\log\left(2^{\alpha}+\frac{N(R',f)+N(R',\frac{1}{f})}{(1-\alpha)r^{\alpha}\log\frac{R'}{R}}\right).
\end{aligned}\tag{14.13}$$

Since $F(z)$ is free of zeros and poles in $D(0,R)$, we can apply case 1 of Theorem 2.1, which asserts that

$$\lim_{\eta\to 0} m_\eta\left(r,\frac{F(z+\eta)}{F(z)}\right)=0.$$

In particular, we can apply Case 1 with (14.2) to $F(z)$ so that

$$m\left(r, \frac{1}{\eta}\left(\frac{F(z+\eta)}{F(z)} - 1\right)\right) \le m\left(r, \frac{F'(z)}{F(z)}\right) + \log 2 \tag{14.14}$$

when $|\eta|$ is sufficiently small.

Then, inequality (14.1) follows from (14.4), (14.10), (14.13), and (14.14). $\square$

In particular, we have the following corollary for finite-order meromorphic functions.

COROLLARY 14.2. *Let $f(z)$ be a meromorphic function of finite order σ. Then*

$$\lim_{r\to\infty} \lim_{\eta\to 0} m_\eta\left(r, \frac{1}{\eta}\left(\frac{f(z+\eta)}{f(z)} - 1\right)\right) = O(\log r) \quad \textit{when } \sigma \ge 1; \tag{14.15}$$

$$\lim_{r\to\infty} \lim_{\eta\to 0} m_\eta\left(r, \frac{1}{\eta}\left(\frac{f(z+\eta)}{f(z)} - 1\right)\right) = O(1) \quad \textit{when } \sigma < 1. \tag{14.16}$$

Proof. Since $f(z)$ is a nonconstant meromorphic function of finite order σ, given $0 < \varepsilon < 2$, we have

$$T(r, f) \le r^{\sigma+\varepsilon/2}$$

when r is sufficiently large. We choose $\alpha = 1 - \frac{\varepsilon}{2}$, $R = 2r$, and $R' = 3r$ in Theorem 14.1. Then the stated limits follow. $\square$

15. Concluding Remarks

This paper has established a way that allows us to recover the classical little Picard theorem for meromorphic functions from the corresponding little Picard theorem for difference operators. One way to consider the original little Picard theorem is that it is a consequence of the meromorphic function belonging to the kernel of a differential operator. Our formulations of Nevanlinna theories enable us to see that the meromorphic functions belonging to the kernels of vanishing and infinite-period varying-step difference operators must reduce to constants. This allows us to treat the classical results as limiting cases of the discrete analogues. As the discrete–continuous interplay has always been a new source of inspiration (see e.g. [1; 17]), the current paper offers a concrete approach to achieve this interplay between discrete and continuous operators by Nevanlinna theory.

ACKNOWLEDGMENTS. We would like to acknowledge useful comments from our colleague Dr. T. K. Lam during the initial stage of this research. The authors would also like to thank the referees for their constructive comments that led to better presentation of the paper.

References

[1] M. J. Ablowitz, R. G. Halburd, and B. Herbst, *On the extension of the Painlevé property to difference equations,* Nonlinearity 13 (2000), 889–905.

[2] Z.-X. Chen, *Complex differences and difference equations,* Math. Monogr. Ser., 29, Science Press, Beijing, 2014.

[3] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane,* Ramanujan J. 16 (2008), 105–129.

[4] ______, *On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions,* Trans. Amer. Math. Soc. 361 (2009), no. 7, 3767–3791.

[5] Y. M. Chiang and S. N. M. Ruijsenaars, *On the Nevanlinna order of meromorphic solutions to linear analytic difference equations,* Stud. Appl. Math. 116 (2006), 257–287.

[6] B. Grammaticos and A. Ramani, *Discrete Painlevé equations: an integrability paradigm,* Phys. Scr. 89 (2014), 038002.

[7] G. G. Gundersen, *Estimates for the logarithmic derivative of meromorphic functions, plus similar estimates,* J. Lond. Math. Soc. (2) 37 (1988), 88–104.

[8] R. G. Halburd and R. J. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations,* J. Math. Anal. Appl. 314 (2006), 477–487.

[9] ______, *Nevanlinna theory for the difference operator,* Ann. Acad. Sci. Fenn. Math. 31 (2006), no. 2, 463–478.

[10] ______, *Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations,* J. Phys. A: Math. Theor. 40 (2007), R1–R38.

[11] W. K. Hayman, *Meromorphic functions,* Oxford Math. Monogr., The Clarendon Press, Oxford, 1964.

[12] Y. He and X. Xiao, *Algebroid functions and ordinary differential equations* (in Chinese), Science Press, Beijing, 1988.

[13] E. L. Ince, *Ordinary differential equations,* Dover Publications, New York, 1956.

[14] I. Laine, *Nevanlinna theory and complex differential equations,* de Gruyter, Berlin, 1993.

[15] J. Miles, *Some examples of the dependence of the Nevanlinna deficiency upon the choice of origin,* Proc. Lond. Math. Soc. (3) 47 (1983), 145–176.

[16] A. Z. Mohon'ko, *The Nevanlinna characteristics of certain meromorphic functions* (in Russian), Teor. Funkc. Funkc. Anal. Ih Prilož. 14 (1971), 83–87.

[17] A. Ramani and B. Grammaticos, *Discrete Painlevé equations: coalescences, limits and degeneracies,* Phys. A 228 (1996), 160–171.

[18] G. Valiron, *Valeurs exceptionnelles et valeurs déficientes des fonctions méromorphes,* C. R. Acad. Sci. Paris 225 (1947), 556–558.

[19] L. Yang, *Value distribution theory,* Science Press, Beijing, 1993.

Y.-M. Chiang
Department of Mathematics
Hong Kong University of Science
and Technology
Clear Water Bay
Kowloon
Hong Kong

machiang@ust.hk

X.-D. Luo
Department of Mathematics
Hong Kong University of Science
and Technology
Clear Water Bay
Kowloon
Hong Kong

Current address
Department of Applied Mathematics
University of Colorado at Boulder
Boulder, CO 80309
USA

lxdmathematics@gmail.com