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SL_2-Action on Hilbert Schemes and Calogero–Moser Spaces

Gwyn Bellamy & Victor Ginzburg

Abstract. We study the natural GL_2-action on the Hilbert scheme of points in the plane, resp. SL_2-action on the Calogero–Moser space. We describe the closure of the GL_2-orbit, resp. SL_2-orbit, of each point fixed by the corresponding diagonal torus. We also find the character of the representation of the group GL_2 in the fiber of the Procesi bundle and its Calogero–Moser analogue over the SL_2-fixed point.

1. Introduction

The natural action of the group GL_2 on $\mathbb{C}^2$ induces a GL_2-action on $\mathrm{Hilb}^n \mathbb{C}^2$, the Hilbert scheme of n points in the plane. There is also a similar action of the group SL_2 on $\mathsf{X}_{\mathbf{c}}$, the Calogero–Moser space. The fixed points of the corresponding maximal torus $\mathbb{C}^* \times \mathbb{C}^*$, resp. $\mathbb{C}^*$, of diagonal matrices, are labeled by partitions. Let $y_\lambda \in \mathrm{Hilb}^n \mathbb{C}^2$, resp. $x_\lambda \in \mathsf{X}_{\mathbf{c}}$, denote the point labeled by a partition λ. It turns out that such a point is fixed by the group SL_2 if and only if $\lambda = (m, m-1, \dots, 2, 1) =: \mathbf{m}$ is a *staircase* partition. In the Hilbert scheme case, this has been observed by Kumar and Thomsen [KT]. The case of the Calogero–Moser space can be deduced from the Hilbert scheme case using “hyper-Kähler rotation”. A different, purely algebraic proof is given in Section 3.

The theory of rational Cherednik algebras gives an $SL_2 \times \mathfrak{S}_n$-equivariant vector bundle $\mathcal{R}$ of rank $n!$ on the Calogero–Moser space. Thus, $\mathcal{R}|_{x_{\mathbf{m}}}$, the fiber of $\mathcal{R}$ over the SL_2-fixed point, acquires the structure of a $SL_2 \times \mathfrak{S}_n$-representation. We find the character formula of this representation in terms of Kostka–Macdonald polynomials. The vector bundle $\mathcal{R}$ is an analogue of the Procesi bundle $\mathcal{P}$, a $GL_2 \times \mathfrak{S}_n$-equivariant vector bundle of rank $n!$ on $\mathrm{Hilb}^n \mathbb{C}^2$. Our formula agrees with the character of the representation of $GL_2 \times \mathfrak{S}_n$ in $\mathcal{P}|_{y_{\mathbf{m}}}$, the fiber of $\mathcal{P}$ over the GL_2-fixed point, obtained by Haiman [H]. It is, in fact, possible to derive our character formula for $\mathcal{R}|_{x_{\mathbf{m}}}$ from the one for $\mathcal{P}|_{y_{\mathbf{m}}}$. However, the character formula for $\mathcal{P}|_{y_{\mathbf{m}}}$, as well as the construction of the Procesi bundle itself, involves the $n!$-theorem.

In Section 2, we review some general results about SL_2-actions. In Section 3, we apply these results to show that, for any λ, the SL_2-orbit of x_λ is closed in $\mathsf{X}_{\mathbf{c}}$. The GL_2-orbit of y_λ is not closed in $\mathrm{Hilb}^n \mathbb{C}^2$, in general, and we describe the closure in Section 4.

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2. $\mathfrak{sl}_2$-Actions

Let $T \subset SL_2$ be the maximal torus of diagonal matrices. The group T acts on the Lie algebra $\mathfrak{sl}_2$ by conjugation. Let (E, H, F) be the standard basis of $\mathfrak{sl}_2$.

Let X be an algebraic variety equipped with a T-action, and let $\mathrm{Vect}(X)$ be the Lie algebra of algebraic vector fields on X. The T-action on X induces a T-action on $\mathrm{Vect}(X)$ by Lie algebra automorphisms. An algebraic variety X equipped with a Lie algebra homomorphism $\mathfrak{sl}_2 \to \mathrm{Vect}(X)$ such that the action of $\mathrm{Lie}T \subset \mathfrak{sl}_2$ can be integrated to a T-action will be referred to as an $(\mathfrak{sl}_2, T)$-variety.

Given a group G and a G-variety X, we write X^G for the fixed point set of G. Given an $(\mathfrak{sl}_2, T)$-variety X, we write $X^{\mathfrak{sl}_2}$ for the closed subset with reduced scheme structure of X defined as the zero locus of all vector fields contained in the image of the map $\mathfrak{sl}_2 \to \mathrm{Vect}(X)$. Clearly, we have $X^{\mathfrak{sl}_2} \subset X^T$. Any variety with an SL_2-action has an obvious structure of an $(\mathfrak{sl}_2, T)$-variety. In such a case, we have $X^{SL_2} = X^{\mathfrak{sl}_2}$.

THEOREM 2.1. *Let X be smooth quasi-projective variety equipped with an $(\mathfrak{sl}_2, T)$-action. Then*:

(i) *If $x \in X^T$ is an isolated fixed point, then $x \in X^{\mathfrak{sl}_2}$ if and only if all the weights of T on $T_x X$ are odd.*

(ii) *If the $(\mathfrak{sl}_2, T)$-action on X comes from a nontrivial SL_2-action with dense orbit, then the set X^{SL_2} is finite.*

Proof. (i) Let $x \in X^{\mathfrak{sl}_2}$, and let $\mathfrak{m}$ be the maximal ideal in the local ring $\mathcal{O}_{X,x}$ defining this point. Then $\mathfrak{sl}_2$ acts on $\mathfrak{m}/\mathfrak{m}^2$. Since x is a isolated fixed point for the T-action, the degree zero weight space is 0, and so all $\mathfrak{sl}_2$-modules appearing in $\mathfrak{m}/\mathfrak{m}^2$ must have odd weight spaces only.

Conversely, assume that all nonzero weight spaces in $\mathfrak{m}/\mathfrak{m}^2$ have odd weight. We need to show that $\mathfrak{sl}_2$ acts in this case, that is, $\mathfrak{sl}_2(\mathfrak{m}) \subset \mathfrak{m}$. By Sumihiro's theorem [S] any T-orbit is contained in an affine T-stable Zariski open subset of X. Therefore, replacing $\mathcal{O}_{X,x}$ by some affine T-stable neighborhood, we may assume that X is an affine T-variety with $\mathfrak{sl}_2$-action and isolated fixed point defined by $\mathfrak{m} \lhd \mathbb{C}[X]$. Then $\mathbb{C}[X] = \mathbb{C}1 \oplus \mathfrak{m}$ as a T-module. In particular, every homogeneous element of nonzero degree belongs to $\mathfrak{m}$. If $z \in \mathfrak{m}$ is homogeneous of degree $\neq -2$, then $\deg E(z) = \deg z + 2 \neq 0$. Thus, $E(z) \in \mathfrak{m}$. On the other hand, if $\deg z = -2$, then our assumptions imply that $z \in \mathfrak{m}^2$, and hence $E(z) \in \mathfrak{m}$. A similar argument applies for F.

Part (ii) is a result of Bialynicki-Birula, [BB, Theorem 1]. □

Let $N(T)$ be the normalizer of T in SL_2. The Borel subgroup of upper-triangular matrices in SL_2 is denoted B. Its opposite is B^-. The following two lemmas follow directly from the classification of closed subgroups of SL_2. We include proofs for the reader's convenience.

LEMMA 2.1. *Let $\mathcal{O}$ be a one-dimensional homogeneous SL_2-space. Then $\mathcal{O} \simeq SL_2/B$.*

Proof. Let $K = \mathrm{Stab}_{SL_2}(x)$ for some $x \in \mathcal{O}$, a closed subgroup of SL_2. Let $\mathfrak{k}$ be the Lie algebra of K. Since $\dim \mathfrak{k} = 2$, it is a solvable subalgebra of $\mathfrak{sl}_2$. Therefore it is conjugate to $\mathfrak{b}$. Without loss of generality, $\mathfrak{k} = \mathfrak{b}$. This means that $K^\circ = B \subset K \subset N_{SL_2}(B) = B$. □

LEMMA 2.2. *Let $\mathcal{O}$ be an SL_2-orbit in an affine variety X. Assume that the stabilizer of $x \in \mathcal{O}$ contains T. Then $\mathcal{O}$ is closed in X, and* $\mathrm{Stab}_{SL_2}(x)$ *is one of* $T, N(T)$, *or* SL_2.

Proof. Let X be an affine variety, and G a reductive group acting on X. If the stabilizer of a point x contains a maximal torus T of G, then $\mathcal{O}$ is closed. Indeed, since $B \cdot x = U \cdot x$ in this case and every U-orbit in X is closed, it follows that $B \cdot x$ is closed in X. This implies that $G \cdot x$ is closed since G/B is projective. The lemma follows since T, $N(T)$, and SL_2 are the only reductive subgroups of SL_2. □

LEMMA 2.3. *Let X be a complete SL_2-variety, and $\mathcal{O}$ an orbit such that the stabilizer of $x \in \mathcal{O}$ equals T, resp. $N(T)$.*

(1) *There is a finite (surjective) equivariant morphism $\mathbb{P}^1 \times \mathbb{P}^1 \twoheadrightarrow \overline{\mathcal{O}}$, resp. $\mathbb{P}^2 \twoheadrightarrow \overline{\mathcal{O}}$, which is the identity on $\mathcal{O}$.*
(2) *This morphism is an isomorphism if and only if $\overline{\mathcal{O}}$ is normal.*
(3) *In all cases, $\overline{\mathcal{O}} \smallsetminus \mathcal{O} \simeq \mathbb{P}^1$ and $\overline{\mathcal{O}}^{SL_2} = \emptyset$.*

Proof. We explain how the lemma can be deduced from the results of [M].

Matsuchima's theorem implies that $\mathcal{O}$ is affine. Therefore, by [EGA, Corollaire 21.12.7], the complement $Y = \overline{\mathcal{O}} \setminus \mathcal{O}$ has pure codimension one. By Theorem 2.1(ii) there are only finitely many zero-dimensional orbits in Y. Therefore Lemma 2.1 implies that each irreducible component Y_i of Y (being one-dimensional) must contain an orbit $\simeq SL_2/B$. Since this orbit is complete, it is closed in Y_i, that is, $Y_i \simeq SL_2/B$. Moreover, this implies that $Y_i \cap Y_j = \emptyset$ for $i \neq j$, and hence $\overline{\mathcal{O}}^{SL_2} = Y^{SL_2} = \emptyset$.

The space $\overline{\mathcal{O}}$ is an SL_2-equivariant completion of $\mathcal{O}$ in the sense of [M, Definition 1.1.1]. By [M, Theorem 5.1], $\mathbb{P}^1 \times \mathbb{P}^1$ is the unique (up to equivariant isomorphism) normal completion of $\mathcal{O} \simeq SL_2/T$ with $\mathcal{O}$ being equivariantly identified with the compliment $\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus \Delta$ of the diagonal. Similarly, loc. cit. implies that $\mathbb{P}^2$ is the unique (up to equivariant isomorphism) normal completion of $\mathcal{O} \simeq SL_2/N(T)$ with $\mathcal{O}$ being equivariantly identified with the complement $\mathbb{P}^2 \smallsetminus C$, where C is a nondegenerate quadric. In both cases, the complement is equivariantly identified with SL_2/B. □

3. Calogero–Moser Spaces

Let $(W, \mathfrak{h})$ be a finite Coxeter group with S the set of *all* reflections in W and $\mathbf{c}: S \to \mathbb{C}$ a conjugate invariant function. For each $s \in S$, we fix eigenvectors $\alpha_s \in \mathfrak{h}^*$ and $\alpha_s^\vee \in \mathfrak{h}$ with eigenvalue -1. Associated to this data is the rational Cherednik algebra $\mathsf{H}_{\mathbf{c}}(W)$ at $t=0$. It is the quotient of the skew group ring $T^*(\mathfrak{h}\oplus\mathfrak{h}^*)\rtimes W$ by the relations

$$[y,x] = -\sum_{s\in S}\mathbf{c}(s)\frac{\alpha_s(y)x(\alpha_s^\vee)}{\alpha_s(\alpha_s^\vee)}, \quad \forall x\in\mathfrak{h}^*, y\in\mathfrak{h},$$

and $[x,x']=[y,y']=0$ for $x,x'\in\mathfrak{h}^*$ and $y,y'\in\mathfrak{h}$. We choose a W-invariant inner product $(-,-)$ on $\mathfrak{h}$. The form defines a W-isomorphism $\mathfrak{h}^*\xrightarrow{\sim}\mathfrak{h}$, $x\mapsto\check{x}$.

3.1. The centre of $\mathsf{H}_{\mathbf{c}}(W)$

The center $Z(\mathsf{H}_{\mathbf{c}}(W))$ of $\mathsf{H}_{\mathbf{c}}(W)$ has a natural Poisson structure, making $\mathsf{H}_{\mathbf{c}}(W)$ into a Poisson module. Let $x_1,\dots,x_n$ be a basis of $\mathfrak{h}^*$, and $y_1,\dots,y_n$ the dual basis. Then the elements

$$E=-\frac{1}{2}\sum_i x_i^2, \qquad F=\frac{1}{2}\sum_i y_i^2, \qquad H=\frac{1}{2}\sum_i x_iy_i+y_ix_i \tag{3.1}$$

are central and form an $\mathfrak{sl}_2$-triple under the Poisson bracket. Their action on $\mathsf{H}_{\mathbf{c}}(W)$ is given by

$$\{E,x\}=\{F,\check{x}\}=0, \qquad \{E,\check{x}\}=x, \qquad \{F,x\}=\check{x},$$
$$\{H,x\}=x, \qquad \{H,\check{x}\}=-\check{x},$$

and $\{\mathfrak{sl}_2,w\}=0$ for all $w\in W$. Their action on $\mathsf{H}_{\mathbf{c}}(W)$ is locally finite. Therefore this action can be integrated to get a locally finite action of $SL_2(\mathbb{C})$ on $\mathsf{H}_{\mathbf{c}}(W)$ by algebra automorphisms. Explicitly, this action is given on generators by

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot x = ax+c\check{x}, \qquad \begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot\check{x}=bx+d\check{x},$$
$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot w=w, \quad \forall x\in\mathfrak{h}^*, w\in W.$$

The Calogero–Moser space $\mathsf{X}_{\mathbf{c}}(W)$ is an affine variety defined as $\operatorname{Spec} Z(\mathsf{H}_{\mathbf{c}}(W))$. The action of $SL_2(\mathbb{C})$ restricts to $Z(\mathsf{H}_{\mathbf{c}}(W))$ and induces a Hamiltonian action on $\mathsf{X}_{\mathbf{c}}(W)$ such that its differential is the action of $\mathfrak{sl}_2$ given by the vector fields $\{E,-\}$, $\{F,-\}$, and $\{H,-\}$.

There are only finitely many T-fixed points on $\mathsf{X}_{\mathbf{c}}(W)$. When the Calogero–Moser space is smooth, the T-fixed points are naturally labeled x_λ with $\lambda\in\operatorname{Irr}(W)$. These fixed points are uniquely specified by the fact that the simple head $L(\lambda)$ of the baby Verma module $\Delta(\lambda)$ is supported at x_λ; see [G] for details.

Consider the element $w_0=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ in SL_2. It normalizes T.

LEMMA 3.1. *Assume that* $\mathsf{X}_{\mathbf{c}}(W)$ *is smooth. Let* $x_\lambda\in\mathsf{X}_{\mathbf{c}}(W)$ *be the* T*-fixed point labeled by the representation* $\lambda\in\operatorname{Irr}(W)$. *Then* $w_0\cdot x_\lambda$ *is the fixed point labeled by* $\lambda\otimes\operatorname{sgn}$, *where* sgn *is the sign representation.*

Proof. The automorphism of $\mathsf{H}_{\mathbf{c}}(W)$ defined by w_0 is the Fourier transform $\mathbb{F}$ of order 4; it is defined by

$$\mathbb{F}: \quad x\mapsto\check{x}, \qquad y\mapsto-\check{y}, \qquad w\mapsto w, \quad \forall x\in\mathfrak{h}^*, y\in\mathfrak{h}, w\in W;$$

see [EG, p. 283]. The fixed point $w_0 \cdot x$ is the support of ${}^{w_0}L(\lambda)$. Thus, it suffices to show that ${}^{w_0}L(\lambda) \simeq L(\lambda \otimes \mathrm{sgn})$. This is a standard result. □

DEFINITION 3.1. An $(\mathsf{H}_{\mathbf{c}}, \mathfrak{sl}_2)$-module M is both a left $\mathsf{H}_{\mathbf{c}}(W)$-module and left $\mathfrak{sl}_2$-module such that the morphism $\mathsf{H}_{\mathbf{c}}(W) \otimes M \to M$ is a morphism of $\mathfrak{sl}_2$-modules.

Every finite-dimensional $(\mathsf{H}_{\mathbf{c}}(W), \mathfrak{sl}_2)$-module is set-theoretically supported at an SL_2-fixed point. However, not every finite-dimensional $\mathsf{H}_{\mathbf{c}}(W)$-module set-theoretically supported at an SL_2-fixed point has a compatible $\mathfrak{sl}_2$-action.

Let e denote the trivial idempotent in $\mathbb{C}W$. Then e is SL_2-invariant, and hence $\mathsf{H}_{\mathbf{c}}(W)e$ is an $(\mathsf{H}_{\mathbf{c}}, \mathfrak{sl}_2)$-module. Thinking of $\mathsf{H}_{\mathbf{c}}(W)e$ as a finitely generated $Z(\mathsf{H}_{\mathbf{c}}(W))$-module, we get an $SL_2 \times W$-equivariant coherent sheaf $\mathcal{R}$ on $\mathsf{X}_{\mathbf{c}}(W)$. When the latter space is smooth, $\mathcal{R}$ is a vector bundle of rank $|W|$.

3.2. *Type A*

Let $\mathsf{H}_{\mathbf{c}}$ be the rational Cherednik algebra for the symmetric group $\mathfrak{S}_n$ at $t = 0$ and $\mathbf{c} \neq 0$. In this case, both the set of T-fixed points in the CM-space $\mathsf{X}_{\mathbf{c}} := \mathsf{X}_{\mathbf{c}}(\mathfrak{S}_n)$ and the set of (isomorphism classes of) simple irreducible representations of $\mathfrak{S}_n$ are labeled by partitions of n. We write $\mathfrak{m}_\lambda$ for the maximal ideal of the T-fixed point corresponding to a partition λ.

NOTATION 3.1. From now on, the staircase partition $(m, m-1, \dots, 1)$ will be denoted $\mathbf{m}$. Given a partition λ, the corresponding representation of the symmetric group will be denoted π_λ. The finite-dimensional irreducible SL_2-module with highest weight $m \geq 0$ will be denoted $V(m)$.

7	5	3	1
5	3	1	
3	1		
1			

(3.2)

Let x be a box of the partition λ. The *hook length* $h(x)$ of x is the number of boxes strictly to the right of x plus the number of boxes strictly below plus one. In the staircase partition (3.2), the entry of the box is the corresponding hook length. The *hook polynomial* of λ is defined to be

$$H_\lambda(q) = \prod_{x \in \lambda} (1 - q^{h(x)}).$$

Let $(q)_n = \prod_{i=1}^{n} (1 - q^i)$ and denote by $n(\lambda)$ the partition statistic $\sum_{i \geq 1} (i-1)\lambda_i$. We write $\chi_T(U)$ for the character of a finite-dimensional T-representation U.

LEMMA 3.2. *Let x_λ be the T-fixed point of $\mathsf{X}_{\mathbf{c}}$ labeled by the partition λ. Then*

$$\chi_T(T_{x_\lambda}\mathsf{X}_{\mathbf{c}}) = \sum_{x \in \lambda} q^{h(x)} + q^{-h(x)}.$$

Proof. It is known that the graded multiplicity of π_λ in the coinvariant ring $\mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}]^W_+ \rangle$ is given by $(q)_{(n)} q^{n(\lambda)} H_\lambda(q)^{-1}$, the so called "fake polynomial". If we decompose $T_{x_\lambda}\mathsf{X}_{\mathbf{c}} = (T_{x_\lambda}\mathsf{X}_{\mathbf{c}})^+ \oplus (T_{x_\lambda}\mathsf{X}_{\mathbf{c}})^-$ into its positive and negative weight parts, then Theorem 4.1 and Corollary 4.4 of [B2] imply that

$$\chi_T((T_{x_\lambda}\mathsf{X}_{\mathbf{c}})^+) = \sum_{x\in\lambda} q^{h(x)}, \quad \text{since } \chi_T(\mathbb{C}[(T_{x_\lambda}\mathsf{X}_{\mathbf{c}})^+]) = \frac{1}{H_\lambda(q)}.$$

The fact that T preserves the symplectic form on $\mathsf{X}_{\mathbf{c}}$ implies that $\chi_T((T_{x_\lambda}\mathsf{X}_{\mathbf{c}})^-) = \sum_{x\in\lambda} q^{-h(x)}$. □

The following observation is elementary.

LEMMA 3.3. *Let λ be a partition such that every hook length in λ is odd. Then λ is a staircase partition.*

Lemma 3.3, together with Lemma 3.2 and Theorem 2.1, implies that SL_2-fixed points in $\mathsf{X}_{\mathbf{c}}$ are very rare:

THEOREM 3.1. *If $n = \frac{m(m+1)}{2}$ for some integer m, then $\mathsf{X}_{\mathbf{c}}^{\mathfrak{sl}_2} = \{x_{\mathbf{m}}\}$. Otherwise, $\mathsf{X}_{\mathbf{c}}^{\mathfrak{sl}_2} = \emptyset$.*

The lemma, together with Theorem 2.1, implies the following:

PROPOSITION 3.1. *There exists a finite-dimensional $(\mathsf{H}_c, \mathfrak{sl}_2)$-module if and only if $n = \frac{m(m+1)}{2}$ for some m. In this case, any such module M is set-theoretically supported at the fixed point $x_{\mathbf{m}}$ labeled by the staircase partition.*

Proof. If M is an $(\mathsf{H}_{\mathbf{c}}, \mathfrak{sl}_2)$-module, then its set-theoretic support must be SL_2-stable. If M is also finite dimensional, then this support is a finite collection of points. These points must be SL_2-fixed since the group is connected. The result follows from Theorem 3.1.

Finally, we must show that there exists at least one $(\mathsf{H}_{\mathbf{c}}, \mathfrak{sl}_2)$-module supported at $x_{\mathbf{m}}$. Let $\mathfrak{m} \lhd Z(\mathsf{H}_{\mathbf{c}})$ be the maximal ideal of $x_{\mathbf{m}}$. Then $\{\mathfrak{sl}_2, \mathfrak{m}\} \subset \mathfrak{m}$. Recall that the $\mathsf{H}_{\mathbf{c}}$-module $\mathsf{H}_{\mathbf{c}}e$ is an $(\mathsf{H}_{\mathbf{c}}, \mathfrak{sl}_2)$-module. Thus, $\mathsf{H}_{\mathbf{c}}e/\mathfrak{m}\mathsf{H}_c e$ is a (simple) $(\mathsf{H}_{\mathbf{c}}, \mathfrak{sl}_2)$-module supported at $x_{\mathbf{m}}$. □

Recall that there is a unique simple $\mathsf{H}_{\mathbf{c}}$-module $L(\lambda)$ supported at each of the T-fixed points x_λ. Notice that we have shown the following:

COROLLARY 3.1. *The simple module $L(\mathbf{m}) \simeq \mathsf{H}_{\mathbf{c}}e/\mathfrak{m}_{\mathbf{m}}\mathsf{H}_{\mathbf{c}}e$ is an $(\mathsf{H}_{\mathbf{c}}, \mathfrak{sl}_2)$-module.*

Equivalently, the above arguments show that $\mathfrak{sl}_2$ acts on the fiber $\mathcal{R}_{\mathbf{m}}$ of $\mathcal{R}$ at $x_{\mathbf{m}}$. The formula for the character of the tangent space of $\mathsf{X}_{\mathbf{c}}(\mathfrak{S}_n)$ at $x_{\mathbf{m}}$ given by Lemma 3.2 shows that

$$T_{x_{\mathbf{m}}}\mathsf{X}_{\mathbf{c}} \simeq V(m) \otimes V(m-1) \tag{3.3}$$

as SL_2-modules.

Next, we describe the SL_2-orbits $\mathcal{O}_\lambda := SL_2 \cdot x_\lambda$ of the T-fixed points x_λ. First, we note that Lemma 2.2 implies the following:

LEMMA 3.4. *The orbit $\mathcal{O}_\lambda$ is closed, and* $\mathrm{Stab}_{SL_2}(x_\lambda)$ *is reductive.*

Lemma 3.1, Theorem 3.1, and Lemma 3.4 imply that

PROPOSITION 3.2. *Let λ be a partition of n. Then, we have the following three alternatives*:

1. $\lambda \neq \lambda^t$ *and* $\mathcal{O}_\lambda = \mathcal{O}_{\lambda^t} \simeq SL_2/T$;
2. $\lambda = \lambda^t \neq \mathbf{m}$ *and* $\mathcal{O}_\lambda \simeq SL_2/N(T)$;
3. $\lambda = \mathbf{m}$ *and* $\mathcal{O}_\lambda = \{x_\mathbf{m}\}$.

3.3. The SL_2-Structure of $\mathcal{R}_\mathbf{m}$

We define the SL_2-module

$$U_m := (V(m-1) \oplus V(m-2)) \otimes \bigotimes_{i=1}^{m-2} (V(i) \oplus V(i-1))^{\otimes 2}.$$

PROPOSITION 3.3. *There is an isomorphism of SL_2-modules*:

$$\mathcal{R}_\mathbf{m} \simeq [U_m \otimes U_{m-2} \otimes \cdots \otimes U_{2,1}]^{\oplus \dim \pi_\mathbf{m}}, \tag{3.4}$$

where the final term $U_{2,1}$ is either U_2 or U_1 depending on whether m is even or odd.

Proof. As an $(\mathsf{H}_\mathbf{c}, \mathfrak{sl}_2)$-module, $\mathcal{R}_\mathbf{m}$ equals $\mathsf{H}_\mathbf{c}e/\mathfrak{m}\mathsf{H}_\mathbf{c}e$. As an $\mathsf{H}_\mathbf{c}$-module, $\mathsf{H}_\mathbf{c}e/\mathfrak{m}\mathsf{H}_\mathbf{c}e$ is isomorphic to $L(\mathbf{m})$. Thus, it suffices to show that the character of $L(\mathbf{m})$ as an SL_2-module equals the character of the right-hand side of equation (3.4). The character of $L(\mathbf{m})$ is given in [B1, Lemma 3.3]. However, we must shift the grading on $L(\mathbf{m})$ from that given in loc. cit., so that the isomorphism $\mathsf{H}_\mathbf{c}e/\mathfrak{m}\mathsf{H}_\mathbf{c}e \to L(\mathbf{m})$ is graded, that is, we require that the one-dimensional space $eL(\mathbf{m})$ lies in degree zero. Then,

$$\chi_T(L(\mathbf{m})) = q^{-n(\mathbf{m})} \frac{H_\mathbf{m}(q)}{(1-q)^n} \dim \pi_\mathbf{m}.$$

Note that $n(\mathbf{m}) = \frac{1}{6}(m-1)m(m+1)$. For the staircase partition, the character of $L(\mathbf{m})$ has a natural factorization. The largest hook in $\mathbf{m}$ is $(m, 1^{m-1})$, and $\mathbf{m} = (m, 1^{m-1}) + [m-2]$; therefore peeling away the hooks gives $q^{-n(\mathbf{m})}/q^{-n([m-2])} = q^{-(m-1)^2}$ and

$$\frac{H_\mathbf{m}(q)}{(1-q)^{2m-1} H_{[m-2]}(q)} = \frac{1}{(1-q)^{2m-1}} \left((1-q^{2m-1}) \prod_{i=1}^{m-1} (1-q^{2i-1})^2 \right)$$
$$= \frac{1-q^{2m-1}}{1-q} \prod_{i=1}^{m-1} \left(\frac{1-q^{2i-1}}{1-q} \right)^2.$$

Thus,

$$\frac{H_{\mathbf{m}}(q)q^{-(m-1)^2}}{(1-q)^{2m-1}H_{[m-2]}(q)} = (q^{m-1}+q^{m-2}+\cdots+q^{-(m-1)}) \cdot \prod_{i=1}^{m-2}(q^i+q^{i-1}+\cdots+q^{-i})^2.$$

This is precisely the character of U_m. □

We would like to refine this character by taking into account the action of W too. We decompose $L(\mathbf{m})$ as a $W \times SL_2$-module,

$$L(\mathbf{m}) = \bigoplus_{\lambda \vdash n} \pi_\lambda \otimes V_\lambda. \tag{3.5}$$

Then the *exponents* of λ are defined to be the positive integers $0 \le e_1 \le e_2 \le \cdots$ such that $V_\lambda = \bigoplus_i V(e_i)$. The fact that $L(\mathbf{m})$ is the regular representation as a W-module implies that

$$\dim \pi_\lambda = \sum_i (e_i+1) = \dim V_\lambda.$$

EXAMPLE 3.1. For $m=3$, we have $n=6$ and

λ	$e_1, e_2, \dots$
(6)	0
(5, 1)	1, 2
(4, 2)	1, 2, 3
(4, 1, 1)	0, 1, 2, 3
(3, 3)	0, 3
(3, 2, 1)	$0, 1^2, 2^2, 4$
(3, 1, 1, 1)	0, 1, 2, 3
(2, 2, 2)	0, 3
(2, 2, 1, 1)	1, 2, 3
(2, 1, 1, 1, 1)	1, 2
(1, 1, 1, 1, 1, 1)	0

LEMMA 3.5. *The exponents of λ equal the exponents of λ^t.*

Proof. There is an algebra isomorphism $\mathrm{sgn} : \mathsf{H}_{\mathbf{c}} \xrightarrow{\sim} \mathsf{H}_{-\mathbf{c}}$ defined by $\mathrm{sgn}(x) = x$, $\mathrm{sgn}(y) = y$, and $\mathrm{sgn}(w) = (-1)^{\ell(w)} w$, where $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $w \in \mathfrak{S}_n$, and ℓ is the length function. It is clear from (3.1) that sgn is SL_2-equivariant. Moreover, ${}^{\mathrm{sgn}}L(\lambda) \simeq L(\lambda^t)$. In particular, ${}^{\mathrm{sgn}}L(\mathbf{m}) \simeq L(\mathbf{m})$. This isomorphism maps V_λ to V_{λ^t} since ${}^{\mathrm{sgn}}\pi_\lambda \simeq \pi_\lambda \otimes \mathrm{sgn} \simeq \pi_{\lambda^t}$. □

Using the deeper combinatorics of Macdonald polynomials, we prove the following:

PROPOSITION 3.4. $\chi_T(V_\lambda) = \widetilde{K}_{\lambda,\mathbf{m}}(q, q^{-1})$.

Proof. Let s_λ denote the Schur polynomial associated to the partition λ so that $s_\lambda[\frac{Z}{1-q}]$ is a particular plethystic substitution of s_λ; we refer the reader to [H] for details.

The module $L(\mathbf{m})$ is a graded quotient of the Verma module $\Delta(\mathbf{m}) = \mathsf{H}_{\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}^*]\rtimes W} \pi_{\mathbf{m}}$. The graded W-character of $\Delta(\mathbf{m})$ is given by $s_{\mathbf{m}}[\frac{Z}{1-q}]$. As shown in [G], the graded multiplicity of $L(\mathbf{m})$ in $\Delta(\mathbf{m})$ is given by

$$(q)_n^{-1} q^{-n(\mathbf{m})} f_{\mathbf{m}}(q) = H_{\mathbf{m}}(q)^{-1} = \prod_{i=1}^{m} (1-q^{2i-1})^{-(m-i)}.$$

Therefore, the graded W-character, shifted by $q^{-n(\mathbf{m})}$ so that $eL(\mathbf{m})$ is in degree zero, of $L(\mathbf{m})$ equals $q^{-n(\mathbf{m})} H_{\mathbf{m}}(q) s_{\mathbf{m}}[\frac{Z}{1-q}]$. This implies that

$$\chi_T(V_\lambda) = \left\langle s_\mu, q^{-n(\mathbf{m})} \prod_{i=1}^{m} (1-q^{2i-1})^{m-i} s_{\mathbf{m}}\left[\frac{Z}{1-q}\right]\right\rangle. \tag{3.6}$$

The fact that the right-hand side of (3.6) equals $\widetilde{K}_{\lambda,\mathbf{m}}(q, q^{-1})$ follows from the property of transformed Macdonald polynomials [H, Proposition 3.5.10]. □

3.4. Other Coxeter Groups

In this section we sketch how we can perform a similar analysis for other Coxeter groups W. First, $\mathsf{X}_{\mathbf{c}}(W)$ might be singular. In this case the torus fixed points x_Ω are labeled by Calogero–Moser families $\Omega \subset \operatorname{Irr} W$. Lemma 3.1 still holds, except now $w_0 \cdot x_\Omega = x_{\Omega \otimes \mathrm{sgn}}$, where $\Omega \otimes \mathrm{sgn} := \{\lambda \otimes \mathrm{sgn} \mid \lambda \in \Omega\}$ is another Calogero–Moser family. Thus, if x_Ω is fixed by SL_2, then necessarily $\Omega = \Omega \otimes \mathrm{sgn}$. Next, provided that the fixed point $x = x_\lambda$ is smooth, the analogue of Lemma 3.2 still holds. Using Theorem 4.1 and Corollary 4.4 of [B2], we can compute the character $\chi_T(T_{x_\lambda}\mathsf{X}_{\mathbf{c}})$, though it is hard to give a formula in general. For instance, when W is a Weyl group of type B/C and $\mathbf{c}$ generic, then $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ is a bipartition of n, and

$$\chi_T(T_{x_\lambda}\mathsf{X}_{\mathbf{c}}) = \sum_{x \in \lambda^{(1)} \cup \lambda^{(2)}} q^{2h(x)} + q^{-2h(x)}. \tag{3.7}$$

These two observations give partial information on $\mathsf{X}_{\mathbf{c}}(W)^{\mathfrak{sl}_2}$, which is sufficient in some cases to determine all SL_2-fixed points. Again, if W is a Weyl group of type B/C and $\mathbf{c}$ generic, then (3.7) implies that all weights of T on the tangent space $T_{x_\lambda}\mathsf{X}_{\mathbf{c}}$ are even. Thus, it cannot be an $\mathfrak{sl}_2$-module. This implies that $\mathsf{X}_{\mathbf{c}}^{\mathfrak{sl}_2} = \emptyset$.

Similarly, if W is of type G_2 and $\mathbf{c}$ is generic, then there are five T-fixed points, four of which are smooth and one is singular. This is the unique isolated singularity. Since the singular locus is SL_2-stable, this singular point is fixed by SL_2. The other four T-fixed points are not SL_2-fixed (already w_0 as in Lemma 3.1 does not fix any of these points).

More generally, SL_2 preserves the symplectic leaves in $\mathsf{X}_{\mathbf{c}}(W)$. In particular, the zero-dimensional leaves give SL_2-fixed points. These zero-dimensional leaves are labeled by *cuspidal* Calogero–Moser families; see [BT]. Therefore each cuspidal Calogero–Moser family gives rise to an SL_2-fixed point. The cuspidal families for Coxeter groups of type A, B, D and $I_2(m)$ are classified in loc. cit.

4. The Hilbert Scheme of Points in the Plane

The group SL_2 also acts naturally on the Hilbert scheme $\mathrm{Hilb}^n \mathbb{C}^2$ of n points in the plane. This is the restriction of a GL_2-action induced by the natural action of GL_2 on $\mathbb{C}^2$.

4.1. Fixed points

The T-fixed points y_λ in $\mathrm{Hilb}^n \mathbb{C}^2$ are also labeled by partitions λ of n. If I is the T-fixed codimension n ideal labeled by λ, then it is uniquely defined by the fact that the corresponding quotient $\mathbb{C}[x, y]/I_\lambda$ has basis given by $x^i y^j$ with

$$(i, j) \in Y_\lambda := \{(i, j) \in \mathbb{Z}^2 \mid 0 \le j \le \ell(\lambda) - 1, 0 \le i \le \lambda_j - 1\},$$

the *Young tableau* of λ. The orbit $GL_2 \cdot y_\lambda$ is denoted $\mathcal{O}_\lambda$. We identify $\mathbb{C}^\times$ with the scalar matrices in GL_2. Then $(\mathrm{Hilb}^n \mathbb{C}^2)^{\mathbb{C}^\times}$ is the moduli space of homogeneous ideals of codimension n in $\mathbb{C}[x, y]$, as studied in [I]. It is a smooth projective GL_2-stable subvariety of $\mathrm{Hilb}^n \mathbb{C}^2$ containing the points y_λ. Notice that the GL_2-orbits and SL_2-orbits in $(\mathrm{Hilb}^n \mathbb{C}^2)^{\mathbb{C}^\times}$ agree since the action factors through PGL_2.

LEMMA 4.1. *If* $n = \frac{m(m+1)}{2}$ *for some integer* m, *then* $(\mathrm{Hilb}^n \mathbb{C}^2)^{GL_2} = \{y_{\mathbf{m}}\}$. *Otherwise*, $(\mathrm{Hilb}^n \mathbb{C}^2)^{GL_2} = \emptyset$.

Proof. This follows from [KT, Lemma 12]. Alternatively, notice that if y_λ is fixed by GL_2, then $\mathbb{C}[x, y]/I_\lambda$ is a GL_2-module. Since each graded piece of $\mathbb{C}[x, y]$ is an irreducible GL_2-module, this implies that there is some m such that $I_\lambda = \mathbb{C}[x, y]_{\ge m}$ and hence $\lambda = \mathbf{m}$. □

We say that a partition λ is *steep* if $\lambda_1 > \cdots > \lambda_\ell > 0$.

PROPOSITION 4.1. *Let* $\lambda \neq \mathbf{m}$ *be a partition of* n *and set* $K = \mathrm{Stab}_{SL_2}(y_\lambda)$.

(1) *If* λ *is steep, then* $K = B$, *and if* λ^t *is steep, then* $K = B_-$. *In both cases,* $\mathcal{O}_\lambda \simeq \mathbb{P}^1$.
(2) *If neither* λ *nor* λ^t *is steep, then* $K = T$ *if* $\lambda \neq \lambda^t$ *and* $K = N(T)$ *if* $\lambda = \lambda^t$. *In both cases the complement to* $\mathcal{O}_\lambda$ *in* $\overline{\mathcal{O}_\lambda}$ *equals* $\mathbb{P}^1$.
(3) *The orbit* $\mathcal{O}_\lambda$ *is closed if and only if* λ *or* λ^t *is steep.*

Proof. If λ is steep, then [KT, Lemma 12] shows that $B \subset K$. If $\dim K > \dim B$, then $\dim K = 3$, that is, $K = SL_2$ and $\lambda = \mathbf{m}$ (notice that $\mathbf{m}$ is the only partition such that both λ and λ^t are steep). Therefore $\dim B = \dim K$, and hence $K^\circ = B$. But then $N_{SL_2}(B) = B$ implies that $K = B$. Since $y_{\lambda^t} = w_0 \cdot y_\lambda$, if λ^t is steep, then $K = w_0 B w_0^{-1} = B_-$. This proves part (1).

Assume now that neither λ nor λ^t is steep. Let $\mathrm{Lie}\, K = \mathfrak{k}$. Since $\mathfrak{k} \supset \mathfrak{t}$, but $\mathfrak{k} \not\supset \mathfrak{b}, \mathfrak{sl}_2$, we have $\mathfrak{k} = \mathfrak{t}$, and hence $K = T$ or $N(T)$. Then part (2) follows from

Lemma 2.3. Notice that Lemma 2.3 is applicable here even though $\mathrm{Hilb}^n \mathbb{C}^2$ is not complete; this is because $\mathcal{O}_\lambda$ is contained in the punctual Hilbert scheme $\mathrm{Hilb}_0^n \mathbb{C}^2 \subset \mathrm{Hilb}^n \mathbb{C}^2$ of all ideals supported at $0 \in \mathbb{C}^2$. This SL_2-stable subvariety is complete.

Part (3) follows directly from parts (1) and (2). □

QUESTION 4.1. For which λ is $\overline{\mathcal{O}_\lambda}$ normal?

Associate with a partition λ the diagonals $d_k := |\{(i, j) \in Y_\lambda \mid i + j = k\}|$, where $k = 0, 1, \ldots$. That is, d_k is the number of boxes lying on the line $x + y = k$. For instance, if $\lambda = (4, 3, 3, 1, 1)$, then the diagonals $(d_0, d_1, \ldots)$ are $(1, 2, 3, 4, 2)$. Now construct a new partition $U(\lambda)$ from λ by setting $U(\lambda)_i = |\{d_k \mid d_k \geq i\}|$. It is again a partition of $|\lambda|$. Pictorially, if we visualize the Young tableau Y_λ in the English style, as in (3.2), then on the kth diagonal (where there are d_k boxes), we have simply moved all boxes as far to the top-right as possible. For example, $U(4, 3, 3, 1, 1) = (5, 4, 2, 1)$. If instead we move all boxes on the kth diagonal as far to the bottom left as possible, we get $U(\lambda)^t$.

LEMMA 4.2. *Let λ be a partition.*

(1) *The partition $U(\lambda)$ is steep, and $U(\lambda) = \lambda$ if and only if λ is steep.*
(2) *$U(\lambda) = \mathbf{m}$ if and only if $\lambda = \mathbf{m}$.*

Proof. It is clear from the construction that $U(\lambda)$ is steep; if $\lambda_{i-1} = \lambda_i$ for some i, then we can move the box at the end of ith row further up and to the right on the diagonal that it belongs to. Similarly, if λ is steep, then $\lambda_{i-1} > \lambda_i$ for all i such that $\lambda_i \neq 0$ implies that there is always a box "above and to the right" of a given box, that is, if $(i, j) \in Y_\lambda$ and $i \neq 0$, then $(i - 1, j + 1) \in Y_\lambda$ (this can be viewed as an alternative definition of steep).

Part (2) is also immediate from the construction. □

PROPOSITION 4.2. *Let λ be a partition such that neither λ nor λ^t is steep. Then $\overline{\mathcal{O}_\lambda} = \mathcal{O}_\lambda \sqcup \mathcal{O}_{U(\lambda)}$.*

Proof. Grade $\mathbb{C}[x, y]$ by putting x and y in degree one. Then every $I \in \mathcal{O}_\lambda$ is graded, $I = \bigoplus_{k \geq 0} I_k$, and $\dim I_k$ is independent of I. Since $\dim(I_\lambda)_k = k + 1 - d_k$, we deduce that $\dim I_k = k + 1 - d_k$ for all $I \in \mathcal{O}_\lambda$. By Proposition 4.1 (2) and Lemma 2.3 we know that $\overline{\mathcal{O}_\lambda} = \mathcal{O}_\lambda \sqcup \mathcal{O}'$, where $\mathcal{O}' \simeq SL_2/B$. Thus, there exists a steep partition $\mu \neq \mathbf{m}$ such that $\mathcal{O}' = \mathcal{O}_\mu$.

The Hilbert–Mumford criterion implies that there exists $I \in \mathcal{O}_\lambda$ such that $J = \lim_{t \to 0} t \cdot I$ is a T-fixed point in $\mathcal{O}_\mu$. Thus, either $J = I_\mu$ or $J = I_{\mu^t}$. Without loss of generality, $J = I_\mu$. This implies that $\dim(I_\mu)_k = k + 1 - d_k$. Since μ is steep, $(I_\mu)_k$ is a B-submodule of $\mathbb{C}[x, y]_k$; cf. Proposition 4.1 (1). Therefore, $\{x^k, x^{k-1}y, \ldots, x^{k+1-d_k} y^{d_k - 1}\}$ is a basis of $(\mathbb{C}[x, y]/I_\mu)_k$, that is, $\{(i, j) \in Y_\mu \mid i + j = k\}$ equals $\{(k, 0), (k-1, 1), \ldots, (k+1-d_k, d_k - 1)\}$. But $U(\lambda)$ is uniquely defined by this property. Hence $\mu = U(\lambda)$. □

REMARK 4.1. For any (homogeneous) ideal $I \in (\mathrm{Hilb}^n \mathbb{C}^2)^{\mathbb{C}^\times}$, I is fixed by B if and only if each I_k is a B-submodule of $\mathbb{C}[x, y]_k$. But the B-submodules of $\mathbb{C}[x, y]_k$ are the same as the U-submodules of $\mathbb{C}[x, y]_k$. This implies that I is B-fixed if and only if it is U-fixed.

It is known (see, e.g., [GS, Theorem 5.6]) that the Hilbert scheme fits into a flat family $p : \mathfrak{X} \to \mathbb{A}^1$ such that $p^{-1}(0) \simeq \mathrm{Hilb}^n \mathbb{C}^2$ and $p^{-1}(\mathbf{c}) \simeq \mathsf{X}_{\mathbf{c}}$ for $\mathbf{c} \neq 0$. Moreover, SL_2 acts on $\mathfrak{X}$ such that the map p is equivariant with SL_2 acting trivially on $\mathbb{C}$. The identification of the fibers is also equivariant. The set-theoretic fixed point set $\mathfrak{X}^T$ decomposes

$$\mathfrak{X}^T = \bigsqcup_{\lambda \vdash n} \mathbb{A}_\lambda,$$

into a union of connected components $\mathbb{A}_\lambda$, where $\mathbb{A}_\lambda \simeq \mathbb{A}^1$ with $p^{-1}(\mathbf{c}) \cap \mathbb{A}_\lambda = \{x_\lambda\}$ for $\mathbf{c} \neq 0$ and $p^{-1}(0) \cap \mathbb{A}_\lambda = \{y_\lambda\}$. The only thing that is not immediate here is that the parameterization of the fixed points in $\mathsf{X}_{\mathbf{c}}$ match those of $\mathrm{Hilb}^n \mathbb{C}^2$. But this can be seen from Lemma 3.2, [H, Lemma 5.4.5], and from the fact that a partition is uniquely defined by its hook polynomial.

Then the SL_2-varieties $SL_2 \cdot \mathbb{A}_\lambda$ are connected. Assume that neither λ nor λ^t is steep. Then there are equivariant trivializations

$$SL_2 \cdot \mathbb{A}_\lambda \simeq SL_2/N(T) \times \mathbb{A}^1 \quad \text{or} \quad SL_2 \cdot \mathbb{A}_\lambda \simeq SL_2/T \times \mathbb{A}^1,$$

depending on whether $\lambda = \lambda^t$ or not.

Let $\widetilde{\mathfrak{sl}_2} \to \mathfrak{sl}_2$ be Grothendieck's simultaneous resolution and write ϖ for the composition $\widetilde{\mathfrak{sl}_2} \to \mathfrak{sl}_2 \to \mathfrak{sl}_2/\!/SL_2 \cong \mathbb{A}^1$, where the second map is $a \mapsto \frac{1}{2}\mathrm{Tr}\, a$.

CONJECTURE 4.1. Let $\lambda \neq \mathbf{m}$ be a steep partition. There exists an SL_2-equivariant embedding $\widetilde{\mathfrak{sl}_2} \hookrightarrow \mathfrak{X}$ sending the B-fixed point $[1 : 0] \in \mathbb{P}^1 \subset \widetilde{\mathfrak{sl}_2}$ to y_λ and such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathfrak{sl}_2} & \hookrightarrow & \mathfrak{X} \\ & {\scriptstyle \varpi}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathbb{A}^1 & \end{array}$$

REMARK 4.2. Conjecture 4.1 has been confirmed by Li Yu in the case $n = 3$.

4.2. *The Procesi Bundle*

The Procesi bundle $\mathcal{P}$ on $\mathrm{Hilb}^n \mathbb{C}^2$ is a $GL_2 \times \mathfrak{S}_n$-equivariant vector bundle of rank $n!$. See [H] and references therein for details. The fiber $\mathcal{P}_{\mathbf{m}}$ is a $GL_2 \times \mathfrak{S}_n$-module, decomposing as

$$\mathcal{P}_{\mathbf{m}} = \bigoplus_{\mu \vdash n} V_\mu \otimes \pi_\mu.$$

As GL_2-modules, we have a decomposition $V_\mu = \bigoplus_i V(m_i, n_i)$ into a direct sum of irreducible GL_2-modules $V(m_i, n_i)$ with highest weight (m_i, n_i); here $m_i, n_i \in$

$\mathbb{Z}$ with $m_i \geq n_i$. We call $(m_1, n_1), (m_2, n_2), \ldots$ the *graded exponents* of μ. Let H denote the 2-torus of diagonal matrices in GL_2. The character of V_μ is given by the cocharge Kostka–Macdonald polynomial

$$\chi_H(V_\lambda) = \widetilde{K}_{\lambda,\mathbf{m}}(q,t). \tag{4.1}$$

Notice that this implies $\widetilde{K}_{\lambda,\mathbf{m}}(q,t) = \widetilde{K}_{\lambda,\mathbf{m}}(t,q)$. This can also be deduced directly from the definition of Macdonald polynomials (see e.g. [H, Proposition 3.5.10]). Similarly, equation (4.1), together with standard properties [H, Proposition 3.5.12] of Macdonald polynomials, implies that

$$V_{\lambda^t} \simeq V_\lambda^* \otimes \det{}^{\otimes n(\mathbf{m})}.$$

Thus, if the exponents of λ are $(m_1, n_1), \ldots$, then the exponents of λ^t are

$$(n(\mathbf{m}) - n_1, n(\mathbf{m}) - m_1), \ldots.$$

Question 4.2. Is there an explicit formula for the graded exponents of λ?

Next, we explain how Lemma 3.5 and Proposition 3.4 can be deduced from the statements of Section 4.2, *provided* that we use Haiman's $n!$ theorem.

Let u be a formal variable, and $\mathsf{H}_{u\mathbf{c}}$ the flat $\mathbb{C}[u]$-algebra such that $\mathsf{H}_{u\mathbf{c}}/\langle u \rangle \simeq \mathsf{H}_0$ and $\mathsf{H}_{u\mathbf{c}}/\langle u-1 \rangle \simeq \mathsf{H}_{\mathbf{c}}$. By [GS, Theorem 5.5], the space $\mathfrak{X}$ can be identified with a moduli space of λ-stable $\mathsf{H}_{u\mathbf{c}}$-modules L such that $L|_{\mathfrak{S}_n} \simeq \mathbb{C}\mathfrak{S}_n$. Here λ is a generic stability parameter; see loc. cit. for definitions. As such, $\mathfrak{X}$ comes equipped with a canonical bundle $\widetilde{\mathcal{P}}$ such that each fiber is an $\mathsf{H}_{u\mathbf{c}}$-module. The action of SL_2 on $\mathfrak{X}$ lifts to $\widetilde{\mathcal{P}}$.

Theorem 4.1. *For* $\mathbf{c} \neq 0$, $\widetilde{\mathcal{P}}|_{p^{-1}(\mathbf{c})} \simeq \mathcal{R}$ *and* $\widetilde{\mathcal{P}}|_{p^{-1}(0)} \simeq \mathcal{P}$.

Proof. The first claim follows from [EG, Section 3], and the second is a consequence of Haiman's proof of the $n!$-conjecture; see the proof of [GS, Theorem 5.3] and references therein. □

Corollary 4.1. *As* $\mathfrak{S}_n \times SL_2$*-modules,* $\mathcal{R}_{\mathbf{m}} \simeq \mathcal{P}_{\mathbf{m}}$*, and hence* $\chi_T(V_\lambda) = \chi_H(V_\lambda)|_{t=q^{-1}}$.

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G. Bellamy
School of Mathematics and Statistics
University of Glasgow
University Gardens
Glasgow, G12 8QW
UK

gwyn.bellamy@glasgow.ac.uk

V. Ginzburg
Department of Mathematics
University of Chicago
Chicago, IL 60637
USA

ginzburg@math.uchicago.edu