Rational Singularities, ω-Multiplier Ideals, and Cores of Ideals

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Abstract. We define the ω-multiplier ideals on a normal variety. The main goal of this paper is to introduce an ω-multiplier ideal and prove its properties. We give characterizations of two-dimensional rational singularities by means of ω-multiplier ideals and cores of ideals.

1. Introduction

In this paper, we always assume that a ring is a domain essentially of finite type over $\mathbb{C}$ and a variety is an irreducible reduced separated scheme of finite type over $\mathbb{C}$.

Rees and Sally [27] introduced the cores of ideals. Okuma, Watanabe, and Yoshida [26] characterized a two-dimensional local ring with rational singularity via cores of ideals. However, in the higher-dimensional case, we have a counterexample to such a characterization. We give another characterization of a local ring with rational singularity of arbitrary dimension via cores of ideals. We, namely, will prove the following:

Theorem 1.1. *Let* $(A, \mathfrak{m})$ *be an n-dimensional Cohen–Macaulay local ring with an isolated singularity. Then A is a rational singularity if and only if* $\overline{I^n} \subset \mathrm{core}(I)$ *for any* $\mathfrak{m}$-*primary ideal I.*

By this theorem, we show that a Cohen–Macaulay local ring with an isolated singularity has a rational singularity if the Briançon–Skoda theorem holds for the ring. Lipman and Teissier [23] showed that the Briançon–Skoda theorem holds for a local ring with rational singularities. Therefore a Cohen–Macaulay local ring with an isolated singularity has a rational singularity if and only if the Briançon–Skoda theorem holds for the ring.

The multiplier ideals are fundamental tools in birational geometry. In this paper, we introduce a new notion of an "ω-multiplier ideal", which has similar properties and works in a slightly different way than a multiplier ideal. The main goal of this paper is to prove the properties of ω-multiplier ideals and show some applications.

For the definition of the multiplier ideals, we use the discrepancies. In order for the discrepancy to be well defined, we need to assume that the variety is normal

and $\mathbb{Q}$-Gorenstein. The advantage of ω-multiplier ideals is that they can be defined on any normal variety. If a variety X is normal Gorenstein, then the ω-multiplier ideal $\mathcal{J}^\omega(X, \mathfrak{a}^c)$ is equal to the usual multiplier ideal $\mathcal{J}(X, \mathfrak{a}^c)$ for any ideal $\mathfrak{a}$.

One of the most important theorems of the multiplier ideals is the Skoda theorem. We will prove that the Skoda theorem of ω-multiplier ideals of a local ring with a rational singularity.

PROPOSITION 1.2. *Let* $(A, \mathfrak{m})$ *be a two-dimensional local ring with a rational singularity,* $\mathfrak{a}$ *be an* $\mathfrak{m}$-*primary ideal, and* J *be a reduction of* $\mathfrak{a}$. *Then, for* $n \in \mathbb{Z}_{\geq 2}$,

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = \mathfrak{a}\mathcal{J}^\omega(A, \mathfrak{a}^{n-1}) = J\mathcal{J}^\omega(A, \mathfrak{a}^{n-1}).$$

Huneke and Swanson [11] proved many properties of cores of ideals of a two-dimensional regular local ring and the relationships between the core of an ideal and a multiplier ideal of a two-dimensional regular local ring. We generalize their results to rational singularities using ω-multiplier ideals. We will prove the following:

PROPOSITION 1.3. *Let* $(A, \mathfrak{m})$ *be a two-dimensional local ring with a rational singularity, and* $\mathfrak{a}$ *be an integrally closed* $\mathfrak{m}$-*primary ideal. Then*

(1) $\mathrm{core}(\mathfrak{a}) = \mathcal{J}^\omega(A, \mathfrak{a}^2) = \mathfrak{a}\mathcal{J}^\omega(A, \mathfrak{a})$.
(2) $\mathrm{e}(\mathfrak{a}) = \ell(A/\mathrm{core}(\mathfrak{a})) - 2\ell(A/\mathcal{J}^\omega(A, \mathfrak{a}))$.
(3) $\mathcal{J}^\omega(A, \mathrm{core}(\mathfrak{a})) = (\mathcal{J}^\omega(A, \mathfrak{a}))^2$.
(4) $\mathrm{core}(\mathfrak{a}^n) = \mathfrak{a}^{2n-1}\mathcal{J}^\omega(A, \mathfrak{a})$.
(5) $\mathrm{core}^n(\mathfrak{a}) = \mathfrak{a}(\mathcal{J}^\omega(A, \mathfrak{a}))^{2^n-1}$. *In particular,* $\mathrm{core}(\mathrm{core}(\mathfrak{a})) = \mathfrak{a}(\mathcal{J}^\omega(A, \mathfrak{a}))^3$.

Demailly, Ein, and Lazarsfeld [4] proved the subadditivity theorem for multiplier ideals on nonsingular varieties. This theorem gives many applications of commutative algebra and algebraic geometry. Takagi and Watanabe [30] proved that the subadditivity theorem holds for a two-dimensional log terminal local ring. Moreover, they showed the characterization of a two-dimensional log terminal local ring via the subadditivity of multiplier ideals. Hence it makes sense to consider the subadditivity of ω-multiplier ideals. We show the characterization of a two-dimensional local ring with a rational singularity via the subadditivity of ω-multiplier ideals.

THEOREM 1.4. *Let* $(A, \mathfrak{m})$ *be a two-dimensional normal local ring. Then* $X = \mathrm{Spec}\, A$ *has a rational singularity if and only if the subadditivity theorem of* ω-*multiplier ideals holds, that is, for any two ideals* $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$,

$$\mathcal{J}^\omega(X, \mathfrak{a}\mathfrak{b}) \subset \mathcal{J}^\omega(X, \mathfrak{a})\mathcal{J}^\omega(X, \mathfrak{b}).$$

To use the subadditivity of ω-multiplier ideals, we investigate the subadditivity of cores of ideals. We show the characterization of a two-dimensional local ring with a rational singularity via the subadditivity of cores of ideals.

COROLLARY 1.5. *Let* $(A, \mathfrak{m})$ *be a two-dimensional normal local ring. Then* $X = \mathrm{Spec}\, A$ *has a rational singularity if and only if the subadditivity theorem of cores*

of ideals holds, *that is*, *for any two* $\mathfrak{m}$-*primary integral closed ideals* $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$,

$$\operatorname{core}(\mathfrak{a}\mathfrak{b}) \subset \operatorname{core}(\mathfrak{a})\operatorname{core}(\mathfrak{b}).$$

Moreover, Takagi and Watanabe [30] showed that a two-dimensional normal ring is regular if the strong subadditivity theorem of multiplier ideals for the ring holds. We consider the problem of a version of ω-multiplier ideals. We will prove the following:

PROPOSITION 1.6. *Let* $(A, \mathfrak{m})$ *be a two-dimensional normal local ring essentially of finite type over* $\mathbb{C}$. *Then* $X = \operatorname{Spec} A$ *is regular if and only if the strong subadditivity theorem of* ω-*multiplier ideals holds*, *that is*, *for any two ideals* $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$ *and any rational number* $c, d > 0$,

$$\mathcal{J}^{\omega}(X, \mathfrak{a}^c \mathfrak{b}^d) \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^c)\mathcal{J}^{\omega}(X, \mathfrak{b}^d).$$

A multiplier ideal is an integrally closed ideal. It is natural to ask whether an integrally closed ideal is a multiplier ideal. In general, multiplier ideals are not integrally closed ideals (see [20; 21]). Favre and Jonsson [8] and Lipman and Watanabe [24] gave an answer to this question when a ring is two-dimensional regular local ring: they showed that all integrally closed ideals on a regular local ring are multiplier ideals. Moreover, Tucker [31] generalized the result to a log terminal local ring. On the other hand, we generalize this theorem to rational singularities by using ω-multiplier ideals. In other words, we will prove the following:

THEOREM 1.7. *Let* $(A, \mathfrak{m})$ *be a two-dimensional local normal ring. Suppose* $X = \operatorname{Spec} A$ *is a rational singularity. Then every integrally closed ideal is an* ω-*multiplier ideal*.

In Section 2, we define rational singularities, the Mather–Jacobian discrepancy and cores of ideals, and collect their results.

   In Section 3, we define ω-multiplier ideals and prove their properties. Further, we characterize a local ring with a rational singularity of arbitrary dimension via cores of ideals.

   In Section 4, we study ω-multiplier ideals of a two-dimensional local ring with a rational singularity. In Section 4.1, we discuss various relationships between the core of an ideal and an ω-multiplier ideal of a two-dimensional local ring with a rational singularity. In Section 4.2, we investigate when the subadditivity theorem of ω-multiplier ideals holds in the two-dimensional case. In Section 4.3, we show that all integrally closed ideals on surface with a rational singularity are ω-multiplier ideals.

2. Preliminaries

2.1. *Rational Singularities and Du Bois Singularities*

In this section, we define rational singularities and Du Bois singularities.

DEFINITION 2.1. We say that a local ring A has rational singularities if A is normal and there exists a desingularizaion $Y \to \operatorname{Spec} A$ with $H^i(Y, \mathcal{O}_Y) = 0$ for every $i > 0$.

The following is well known as a characterization of rational singularities in characteristic zero (see, e.g., [17]).

PROPOSITION 2.2 (Kempf's criterion for rational singularities). *Let A be a normal local Cohen–Macaulay ring essentially of finite type over a field of characteristic 0. The scheme $X = \operatorname{Spec} A$ has rational singularities if and only if there exists a desingularizaion $Y \to X$ with $f_* \omega_Y = \omega_X$, where ω_Y and ω_X are the canonical sheaves of Y and X, respectively.*

DEFINITION 2.3. Suppose that X is a reduced scheme embedded as a closed subscheme of a smooth scheme Y. Let $f : \widetilde{Y} \to Y$ be a log resolution of (Y, X) that is an isomorphism outside of X. Let E denote $(f^{-1}(X))_{\mathrm{red}}$. Then X is said to have Du Bois singularities if the natural map $\mathcal{O}_X \to \mathbf{R}f_* \mathcal{O}_E$ is a quasi-isomorphism.

First Du Bois singularities are introduced by Steenbrink with a different definition in [29], but Schwede [28] showed that it is equivalent to the condition in Definition 2.3.

Kovács, Schwede, and Smith characterized normal Cohen–Macaulay Du Bois singularities.

THEOREM 2.4 ([18, Thm. 3.1]). *Suppose that X is normal and Cohen–Macaulay. Let $\pi : Y \to X$ be any log resolution and denote the reduced exceptional divisor of π by G. Then X has Du Bois singularities if and only if $\pi_* \omega_Y(G) = \omega_X$.*

Using this theorem, we easily to see that Cohen–Macaulay log canonical singularities are Du Bois singularities and that Gorenstein Du Bois singularities are log canonical singularities.

REMARK 2.5. Kollár and Kovács [16] showed that log canonical singularities are Du Bois singularities even if the singularities are not Cohen–Macaulay.

2.2. Mather–Jacobian Minimal Log Discrepancy

We start by recalling the definition and basic properties of Mather–Jacobian log discrepancy defined in [5; 6]. We refer to [5] for further details. Let X be a variety of dimension $\dim X = n$. The sheaf Ω_X^n is invertible over the smooth locus X_{reg} of X, and hence the projection

$$\pi : \mathrm{P}(\Omega_X^n) \to X$$

is an isomorphism over X_{reg}. The Nash blow-up $\widehat{X} \to X$ is defined as the closure of $\pi^{-1}(X_{\mathrm{reg}})$ in $\mathrm{P}(\Omega_X^n)$.

If $V \supset X$ is an n-dimensional reduced, locally complete intersection scheme, then the Nash blow-up $\pi : \widehat{X} \to X$ is isomorphic to the blow-up of the ideal $j_V|_X$, where j_V is the Jacobian ideal of V (see Proposition 2.4 in [2]).

DEFINITION 2.6. Let $f : Y \to X$ be a resolution of singularities of X that factors through the Nash blow-up of X. The image of the canonical homomorphism

$$f^*(\Omega_X^n) \to \Omega_Y^n$$

is an invertible sheaf of the form $\mathrm{Jac}_f \, \Omega_Y^n$, where Jac_f is the relative Jacobian, which is an invertible ideal on Y and defines an effective divisor supported on the exceptional locus of f. The divisor is called the Mather discrepancy divisor and denoted by $\widehat{K}_{Y/X}$.

REMARK 2.7. Let X be an n-dimensional normal variety, and $V \supset X$ be an n-dimensional reduced, locally complete intersection scheme. If $f : Y \to X$ is a log resolution of $j_V|_X$ such that $j_V|_X \mathcal{O}_Y = \mathcal{O}_Y(-J_V)$, then we have $\widehat{K}_{Y/X} = K_Y + J_V - f^*(K_V|_X)$ (see [2]).

DEFINITION 2.8. Let $f : Y \to X$ be a log resolution of j_X, where j_X is the Jacobian ideal of a variety X. We denote by $J_{Y/X}$ the effective divisor on Y such that $j_X \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$. This divisor is called the Jacobian discrepancy divisor.

Here, we note that every log resolution of j_X factors through the Nash blow-up; see, for example, Remark 2.3 in [6].

DEFINITION 2.9. Let X be an n-dimensional normal variety, and V be a reduced locally complete intersection n-dimensional scheme containing X. The ideal $\mathfrak{d}_{X,V}$ is the ideal such that

$$\mathrm{Im}(\omega_X \to \omega_V|_X) = \mathfrak{d}_{X,V} \otimes \omega_V|_X.$$

REMARK 2.10. Let M be a smooth variety containing X and V. Consider the ideals I_X and I_V of X and V in M. Then, as $\mathcal{O}_V$-modules, we have

$$\omega_X \otimes \omega_V^{-1} = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_X, \mathcal{O}_V) = (I_V : I_X)/I_V,$$

and therefore

$$\mathfrak{d}_{X,V} = ((I_V : I_X) + I_X)/I_X.$$

In other words, if we write $V = X \cup X'$, where X' is the residual part of V with respect to X (given by the ideal $(I_V : I_X)$), then $\mathfrak{d}_{X,V}$ is the ideal defining the intersection $X \cap X'$ in X.

DEFINITION 2.11. Let X be a normal variety. The lci-defect ideal of X is defined to be

$$\mathfrak{d}_X = \sum_V \mathfrak{d}_{X,V},$$

where the sum is taken over all reduced, locally complete intersection schemes $V \supset X$ of the same dimension.

REMARK 2.12. The support of the lci-defect ideal of X is locally a noncomplete intersection locus of X. In particular, $\mathfrak{d}_X = \mathcal{O}_X$ if X is locally a complete intersection.

DEFINITION 2.13. A normal variety X is said to be $\mathbb{Q}$-Gorenstein if its canonical divisor K_X is $\mathbb{Q}$-Cartier.

DEFINITION 2.14. Let X be an n-dimensional normal $\mathbb{Q}$-Gorenstein variety, and V be a reduced locally complete intersection n-dimensional scheme containing X. Let r be a positive integer such that rK_X is Cartier. The ideal $\mathfrak{d}_{r,X,V}$ is the ideal such that

$$\mathrm{Im}(\mathcal{O}_X(rK_X) \to (\omega_V|_X)^{\otimes r}) = \mathfrak{d}_{r,X,V} \otimes (\omega_V|_X)^{\otimes r}.$$

DEFINITION 2.15. Let X be a normal $\mathbb{Q}$-Gorenstein variety. Let r be a positive integer such that rK_X is Cartier. The lci-defect ideal of level r of X is defined to be

$$\mathfrak{d}_{r,X} = \sum_V \mathfrak{d}_{r,X,V},$$

where the sum is taken over all reduced, locally complete intersection schemes $V \supset X$ of the same dimension.

PROPOSITION 2.16 ([2, Prop. 2.12]). *Let X be a normal $\mathbb{Q}$-Gorenstein variety. Let r be a positive integer such that rK_X is Cartier. Then $\mathfrak{d}_X^r \subset \overline{\mathfrak{d}_{r,X}}$.*

REMARK 2.17. If X is Gorenstein, then $\mathfrak{d}_X = \mathfrak{d}_{1,X}$. In general, however, $\overline{\mathfrak{d}_X^r} \neq \overline{\mathfrak{d}_{r,X}}$.

DEFINITION 2.18. Let X be a normal $\mathbb{Q}$-Gorenstein variety. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \ldots, t_r \in \mathbb{R}$. Given a log resolution $f : Y \to X$ of $\mathfrak{a}_1 \cdots \mathfrak{a}_r$, we denote by $Z_1, \ldots, Z_r$ the effective divisors on Y such that $\mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$ for $1 \leq i \leq r$. For a prime divisor E over X such that E appears on Y, we define the log discrepancy at E as

$$a(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) := \mathrm{ord}_E(K_{Y/X}) - \mathrm{ord}_E(t_1 Z_1 + \cdots + t_r Z_r) + 1.$$

DEFINITION 2.19. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \ldots, t_r \in \mathbb{R}$. Given a log resolution $f : Y \to X$ of $j_X \mathfrak{a}_1 \cdots \mathfrak{a}_r$, we denote by $Z_1, \ldots, Z_r$ the effective divisors on Y such that $\mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$ for $1 \leq i \leq r$. For a prime divisor E over X such that E appears on Y, we define the Mather–Jacobian-log discrepancy at E as

$$a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) := \mathrm{ord}_E(\widehat{K}_{Y/X} - J_{Y/X} - t_1 Z_1 - \cdots - t_r Z_r) + 1.$$

REMARK 2.20. If X is normal and locally a complete intersection, then $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = a(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$. Indeed, in this case the image of the canonical map $\Omega_X^n \to \omega_X$ is $j_X \omega_X$, hence, $\widehat{K}_{Y/X} - J_{Y/X} = K_{Y/X}$. In particular, we see that $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = a(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$ if X is smooth.

Note that the Mather–Jacobian log discrepancy at a prime divisor E does not depend on the choice of f. We denote $\mathrm{ord}_E \widehat{K}_{Y/X}$ by $\widehat{k}_E$.

DEFINITION 2.21. Let X be a normal $\mathbb{Q}$-Gorenstein variety, $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \ldots, t_r \in \mathbb{R}$. Then $(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$ is canonical (resp. log canonical) if for every exceptional prime divisor E over X, the inequality $a(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) \geq 1$ (resp. ≥ 0) holds.

DEFINITION 2.22. Let X be a variety, $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \ldots, t_r \in \mathbb{R}$. Then $(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$ is MJ-canonical (resp. MJ-log canonical) if for every exceptional prime divisor E over X, the inequality $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) \geq 1$ (resp. ≥ 0) holds.

REMARK 2.23. Fix a log resolution $Y \to X$ of $\mathfrak{j}_X \mathfrak{a}_1, \ldots, \mathfrak{a}_r$. Then $(X, \mathfrak{a}_1 \cdots \mathfrak{a}_r)$ is MJ-canonical (resp. MJ-log canonical) if and only if $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) \geq 1$ (resp. ≥ 0) for all exceptional prime divisors E on Y. This is proved by using the fact that

$$\widehat{K}_{Y'/X} - J_{Y'/X} = K_{Y'/Y} + g^*(\widehat{K}_{Y/X} - J_{Y/X})$$

for a sequence $Y' \xrightarrow{g} Y \xrightarrow{f} X$ of such log resolution of $\mathfrak{j}_X \mathfrak{a}_1, \ldots, \mathfrak{a}_r$.

DEFINITION 2.24. Let X be a normal $\mathbb{Q}$-Gorenstein variety, $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \ldots, t_r \in \mathbb{R}$. Let $f : Y \to X$ be a log resolution of $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$. Define $Z_1, \ldots, Z_r$ by $\mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$ for $1 \leq i \leq r$. Then we can define the multiplier ideal as follows:

$$\mathcal{J}(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = f_* \mathcal{O}_Y(\lceil K_{Y/X} - t_1 Z_1 - \cdots - t_r Z_r \rceil).$$

DEFINITION 2.25. Let X be a normal $\mathbb{Q}$-Gorenstein variety. X is said to be a log terminal singularities if $\mathcal{J}(X, \mathcal{O}_X) = \mathcal{O}_X$.

REMARK 2.26. Log terminal singularities are rational singularities.

DEFINITION 2.27. Let X be a variety, $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be nonzero ideals on X, and $t_1, \ldots, t_r \in \mathbb{R}$. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \mathfrak{a}_1, \ldots, \mathfrak{a}_r$. Define $Z_1, \ldots, Z_r$ by $\mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$ for $1 \leq i \leq r$. Then we can define the Mather–Jacobian multiplier ideal (or MJ-multiplier ideal for short) as follows:

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - [t_1 Z_1 - \cdots - t_r Z_r]).$$

REMARK 2.28. Multiplier ideals and Mather–Jacobian multiplier ideals are independent of the choice of a log resolution.

PROPOSITION 2.29 ([2; 6]). *If X is MJ-canonical, then it is normal and has rational singularities.*

PROPOSITION 2.30 ([2]). *If X is MJ-log canonical, then it has Du Bois singularities.*

There are the relations between jet scheme and Mather–Jacobian minimal log discrepancy (see [2; 5; 15]). For the theory on jet schemes and arc space, see, for example, [7].

2.3. Cores of Ideals

In this section, we define cores of ideals and collect their results.

DEFINITION 2.31. Let A be a ring, and I be an ideal of A. An ideal $J \subset I$ is called a reduction of I if there is a positive number r such that $JI^r = I^{r+1}$. An ideal $J \subset I$ is called a minimal reduction of I if J is minimal among the reductions of I.

DEFINITION 2.32. Let A be a ring, and I be an ideal of A. Let $f : Y \to X =$ Spec A be the normalized blowing up of I such that $I\mathcal{O}_Y = \mathcal{O}_Y(-F)$. The integral closure of I is defined to be $f_*\mathcal{O}_Y(-F)$. We denote it by $\overline{I}$.

DEFINITION 2.33. Let X be an n-dimensional scheme. Suppose that a dualizing complex for X exists. A canonical sheaf ω_X for X is defined to be the coherent sheaf given by $(-n)$th cohomology of a normalized dualizing complex for X.

REMARK 2.34. Dualizing complexes exist for any equidimensional scheme essentially of finite type over an affine Gorenstein scheme (see [10]). If X is a normal algebraic variety, then the usual notion of the canonical sheaf provides the canonical sheaf of X. In the case, $X = $ Spec A where A is a local ring, and ω_X coincides with the sheafification of the canonical module ω_A.

Let $f : Y \to X$ be a birational morphism of integral schemes. Then the trace map $\mathrm{Tr}_f : f_*\omega_Y \to \omega_X$ is injective, and it is important to observe that in this case we can consider Tr_f as an inclusion $f_*\omega_Y \subset \omega_X$.

Hyry and Villamayor [14] proved the following lemma.

LEMMA 2.35 ([14, Lemma 2.2]). *Let $(A, \mathfrak{m})$ be a local ring. Let $f : Y \to X =$ Spec A be a proper birational morphism such that Y has rational singularities. Then $H^0(Y, \omega_Y) \subset H^0(Z, \omega_Z)$ for any proper birational morphism $g : Z \to X$. It follows, in particular, that $H^0(Y, \omega_Y) = H^0(Z, \omega_Z)$ if Z has rational singularities.*

DEFINITION 2.36. Let A be a Noetherian local ring, and I an ideal. The core of I, denoted core(I), is the intersection of all its reductions.

DEFINITION 2.37. Let $(A, \mathfrak{m})$ be a local ring. An ideal I of A is equimultiple if a minimal reduction of I is generated by h elements, where $h = \mathrm{ht}(I)$.

EXAMPLE 2.38. Every $\mathfrak{m}$-primary ideal in a local ring is equimultiple.

By the following theorem, we are able to compute the core of ideals for equimultiple ideals in Cohen–Macaulay local rings whose residue fields have characteristic 0.

THEOREM 2.39 ([12, Thm. 3.7]). *Let A be a Cohen–Macaulay local ring. Let I be an equimultiple ideal of A with $h = \mathrm{ht}(I) \geq 1$, let J be a minimal reduction of I, and let r be a positive number such that $J I^r = I^{r+1}$. Then*

$$\mathrm{core}(I) = J^{r+1} : I^r.$$

LEMMA 2.40 ([13, Lemma 3.1.5]). *Let $(A, \mathfrak{m})$ be a local ring, and let I be a proper ideal of A of height greater than one. Let $Y = \mathrm{Proj}\, A[I]$. Then $H^0(Y, I^{n+p}\omega_Y) :_{\omega_A} I^p = H^0(Y, I^n\omega_Y)$ for all $n \geq 0$ and all $p \geq 1$.*

LEMMA 2.41 ([13, Lemma 5.1.6]). *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring, and I be an equimultiple ideal of height h. Then*

$$H^0(Y, I^h\omega_Y) :_A \omega_A = J^{r+1} :_A I^r,$$

where $Y = \mathrm{Proj}\, A[I]$, $H^0(Y, I^h\omega_Y)$ is considered as a submodule of ω_A via the trace map, and J is any minimal reduction of I with $J I^r = I^{r+1}$.

THEOREM 2.42 ([13, Cor. 5.3.1]). *Let $(A, \mathfrak{m})$ be a Gorenstein local ring with rational singularities, and I be an equimultiple ideal of height h such that the Rees ring $A[It]$ is normal and Cohen–Macaulay. Let $Y = \mathrm{Proj}\, A[It]$. Then the following conditions are equivalent:*

(1) *$A[It]$ has rational singularities;*
(2) *$H^0(Y, I^n\omega_Y) = \mathcal{J}(I^n)$ for all $n \geq 0$;*
(3) *$\mathrm{core}(I) = \mathcal{J}(I^h)$.*

If this is the case, then

$$\mathrm{core}(I) = I\mathcal{J}(I^{h-1}),$$
$$\mathcal{J}(I^{h-1}) = \mathrm{core}(I) : I.$$

3. ω-Multiplier Ideals and Cores of Ideals

3.1. ω-Multiplier Ideals

In this section, we define ω-multiplier ideals and prove some properties of ω-multiplier ideals.

DEFINITION 3.1. Let X be a normal variety, $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_X$, $c \in \mathbb{Q}_{>0}$, and let $f : Y \to X$ be a log resolution of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. The ω-multiplier ideal of a pair $(X, \mathfrak{a}^c)$ is defined to be $f_*\omega_Y(-[cF]) : \omega_X$. We denote it by $\mathcal{J}^\omega(X, \mathfrak{a}^c)$.

DEFINITION 3.2. Let X be a variety with rational singularities, and $\mathfrak{a} \subsetneq \mathcal{O}_X$ be a nonzero ideal of $\mathcal{O}_X$. The rational threshold of a pair $(X, \mathfrak{a})$ is defined to be $\sup\{c > 0 | \mathcal{J}^\omega(X, \mathfrak{a}^c) = \mathcal{O}_X\}$. We denote it by $\mathrm{rt}(X, \mathfrak{a})$.

THEOREM 3.3 ([2, Thm. 6.15]). *Let X be a normal variety, $\mathfrak{a}$ be a nonzero ideal, and $c \in \mathbb{Q}_{>0}$. Then we have*

$$\mathcal{J}^{\omega}(X, \mathfrak{a}^c) = \mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^c \mathfrak{d}_X^{-1}).$$

De Fernex and Docampo [2] proved the following:

THEOREM 3.4 (See the proof of Theorem 6.15 in [2]). *Let X be a normal variety, and $\mathfrak{a}$ be a nonzero ideal sheaf of $\mathcal{O}_X$. Let V be a reduced locally complete intersection scheme containing X of the same dimension. Let $\mathfrak{d}_{V,X}$ be the ideal determined by the image of $\omega_X \to \omega_V|_X$. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_V|_X \cdot \mathfrak{d}_{V,X} \cdot \mathfrak{a}$ such that $\mathfrak{j}_V|_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_V)$, $\mathfrak{d}_{X,V} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_V)$, and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors J_V, D_V, and F on Y. Then*

$$\mathcal{J}^{\omega}(X, \mathfrak{a}^c) = f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_V + D_V - [cF]).$$

COROLLARY 3.5. *Let X be a normal variety, $\mathfrak{a}$ be a nonzero ideal, and $c \in \mathbb{Q}_{>0}$. Then we have*

$$\mathcal{J}^{\omega}(X, \mathfrak{a}^c) \supset \mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^c).$$

In particular, if X is locally a complete intersection, then

$$\mathcal{J}^{\omega}(X, \mathfrak{a}^c) = \mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^c).$$

THEOREM 3.6 ([2, Thm. 7.1]). *Let X be a normal variety, and let $\mathfrak{d}_X \subset \mathcal{O}_X$ be the lci-defect ideal of X. Let $f : Y \to X$ be a log resolution of $\mathfrak{d}_X$ and denote by E the reduced exceptional divisor. Then the following properties hold*:

(i) *The pair $(X, \mathfrak{d}_X^{-1})$ is MJ-canonical if and only if $\mathcal{J}^{\omega}(X, \mathcal{O}_X) = \mathcal{O}_X$.*
(ii) *The pair $(X, \mathfrak{d}_X^{-1})$ is MJ-log canonical if and only if $f_* \omega_Y(E) = \omega_X$.*

COROLLARY 3.7 ([2, Cor. 7.2]). *Let X be a normal variety, and let $\mathfrak{d}_X \subset \mathcal{O}_X$ be the lci-defect ideal of X. Then the following properties hold*:

(i) *If X has rational singularities, then $(X, \mathfrak{d}_X^{-1})$ is MJ-canonical.*
(ii) *If X has Du Bois singularities, then $(X, \mathfrak{d}_X^{-1})$ is MJ-log canonical.*

Moreover, the converse holds in both cases whenever X is Cohen–Macaulay.

Kempf's criterion for rational singularities implies the following proposition.

PROPOSITION 3.8. *Let X be a Cohen–Macaulay normal variety. Then X has rational singularities if and only if $\mathcal{J}^{\omega}(X, \mathcal{O}_X) = \mathcal{O}_X$.*

The following proposition gives the relation of Mather Jacobian discrepancies and usual multiplier discrepancies.

PROPOSITION 3.9 ([2, Prop. 3.4]). *Let X be a $\mathbb{Q}$-Gorenstein normal variety. Let r be a positive integer such that rK_X is Cartier. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X \cdot \mathfrak{d}_{r,X}$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$, and*

$\mathfrak{d}_{r,X} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{r,Y/X})$ *for some effective divisors* $J_{Y/X}$, $D_{Y/X}$, *and* $D_{r,Y/X}$ *on* Y. *Then*

$$\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} \geq \widehat{K}_{Y/X} - J_{Y/X} + D_{r,Y/X} = K_{Y/X}.$$

In particular, if X *is Gorenstein, then*

$$\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} = K_{Y/X}.$$

The following proposition is an immediate consequence of the previous proposition and gives a relation between ω-multiplier ideals and usual multiplier ideals.

PROPOSITION 3.10. *Let* X *be a* $\mathbb{Q}$-*Gorenstein normal variety,* $\mathfrak{a}$ *be a nonzero ideal of* $\mathcal{O}_X$, *and* $c \in \mathbb{Q}_{>0}$. *Then* $\mathcal{J}^{\omega}(X, \mathfrak{a}^c) \supset \mathcal{J}(X, \mathfrak{a}^c)$. *In particular, if* X *is Gorenstein, then* $\mathcal{J}^{\omega}(X, \mathfrak{a}^c) = \mathcal{J}(X, \mathfrak{a}^c)$.

The next proposition is an immediate consequence of the definition.

PROPOSITION 3.11. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be nonzero ideals on a normal variety* X, *and* $c > 0$.

(1) *If* $\mathfrak{a} \subset \mathfrak{b}$, *then* $\mathcal{J}^{\omega}(X, \mathfrak{a}^c) \subset \mathcal{J}^{\omega}(X, \mathfrak{b}^c)$.
(2) *If* $c \geq d$ *are in* $\mathbb{Q}_{>0}$, *then* $\mathcal{J}^{\omega}(X, \mathfrak{a}^c) \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^d)$.
(3) $\mathcal{J}^{\omega}(X, \mathfrak{a}^c) = \mathcal{J}^{\omega}(X, \overline{\mathfrak{a}}^c)$, *where* $\overline{\mathfrak{a}}$ *is the integral closure of* $\mathfrak{a}$.

PROPOSITION 3.12. *Let* $\mathfrak{a}$ *be a nonzero ideal on a normal variety* X, *and* $c > 0$.

(1) *The* ω-*multiplier ideal* $\mathcal{J}^{\omega}(X, \mathfrak{a}^c)$ *is an integrally closed ideal of* $\mathcal{O}_X$.
(2) *Suppose that* X *has rational singularities. Then* $\mathfrak{a} \subset \mathcal{J}^{\omega}(X, \mathfrak{a})$.
(3) $\mathcal{J}^{\omega}(X, \mathfrak{a}^c) = \mathcal{J}^{\omega}(X, \mathfrak{a}^{c+\varepsilon})$ *for* $0 < \varepsilon \ll 1$.

Proof. Let j_X be the Jacobian ideal of X, and $\mathfrak{d}_X$ be the lci-defect ideal of X. Let $f : Y \to X$ be a log resolution of $j_X \cdot \mathfrak{d}_X$ such that $j_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$, and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors $J_{Y/X}$, $D_{Y/X}$, and F on Y. Then we have $\mathcal{J}^{\omega}(X, \mathfrak{a}^c) = f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - [cF])$ by Theorem 3.3. Therefore $\mathcal{J}^{\omega}(X, \mathfrak{a}^c)$ is an integrally closed ideal of $\mathcal{O}_X$.

If X has rational singularities, then $\mathcal{J}^{\omega}(X, \mathcal{O}_X) = \mathcal{O}_X$ by Proposition 3.8. Therefore $\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}$ is effective. Thus $\mathcal{J}^{\omega}(X, \mathfrak{a}) \supset f_*\mathcal{O}_Y(-F) = \overline{\mathfrak{a}} \supset \mathfrak{a}$.

Since $[cF] = [(c + \varepsilon)F]$ for $0 < \varepsilon \ll 1$, we have $\mathcal{J}^{\omega}(X, \mathfrak{a}^c) = \mathcal{J}^{\omega}(X, \mathfrak{a}^{c+\varepsilon})$. $\qquad \square$

Blickle [1] defined the multiplier module.

DEFINITION 3.13. *Let* X *be a normal variety, and let* $\mathfrak{a}$ *be a nonzero ideal on* X. *Let* $f : Y \to X$ *be a log resolution of* $\mathfrak{a}$ *such that* $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. *Then the multiplier module is defined as*

$$\mathcal{J}_{\omega}(\mathfrak{a}^c) = f_*\mathcal{O}_Y(K_Y - [cF]) \subset \omega_X$$

for $c > 0$.

PROPOSITION 3.14. *Let X be a normal variety, and let $\mathfrak{a}$ be a nonzero ideal on X. Then $\mathcal{J}^{\omega}(\mathfrak{a}^c) = \mathcal{J}_{\omega}(\mathfrak{a}^c) : \omega_X$ for all $c > 0$.*

Proof. This follows immediately from the definition of ω-multiplier ideals. $\square$

Blickle [1] gave a formula computing the multiplier module of a monomial ideal on an arbitrary affine toric variety.

THEOREM 3.15 ([1, Thm. 1]). *Let X_σ be an affine toric variety, and $\mathfrak{a}$ a monomial ideal. Then*

$$\mathcal{J}_{\omega}(X_\sigma, \mathfrak{a}^c) = \langle x^m | m \in \text{ interior of } c \operatorname{Newt}(\mathfrak{a}) \rangle \subset \omega_{X_\sigma}.$$

PROPOSITION 3.16. *Let X_σ be an n-dimensional affine toric variety, and $\mathfrak{m}$ be the maximal ideal. Then $\operatorname{rt}(\mathfrak{m}) \geq 1$.*

Proof. Note that $\omega_{X_\sigma} = \langle x^m | m \in \operatorname{int}(\sigma) \cap \mathbb{Z}^n \rangle \subset \mathcal{O}_{X_\sigma}$. By Theorem 3.15, we have

$$\mathcal{J}_{\omega}(X_\sigma, \mathfrak{m}^c) = \langle x^m | m \in \text{ interior of } c \operatorname{Newt}(\mathfrak{m}) \rangle \subset \omega_{X_\sigma}.$$

Therefore, if $c < 1$, then we have $x^m \in \mathcal{J}_{\omega}(X_\sigma, \mathfrak{m}^c)$ for any $x^m \in \omega_{X_\sigma}$. This implies that $\operatorname{rt}(\mathfrak{m}) \geq 1$. $\square$

In general, $\operatorname{rt}(\mathfrak{m})$ is not necessarily greater than or equal to 1.

EXAMPLE 3.17. Let $A = (\mathbb{C}[x, y, z]/(x^2 + y^2 z + z^3))_{(x,y,z)}$, $\mathfrak{m} = (x, y, z)$. Then A is a Du Val singularity of type D_4. Let Y be the minimal resolution of $X = \operatorname{Spec} A$. The dual graph of the exceptional divisors on the minimal resolution of A is as follows;
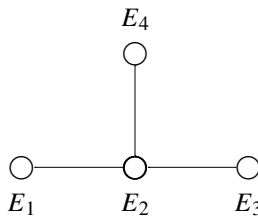
Therefore the fundamental cycle of the minimal resolution of $\operatorname{Spec} A$ is $Z = E_1 + 2E_2 + E_3 + E_4$, where $E_1, \ldots, E_4$ are exceptional divisors on the minimal resolution of $\operatorname{Spec} A$. Since A is a Gorenstein rational singularity, we have $K_{Y/X} = 0$, $\mathfrak{m}\mathcal{O}_Y = \mathcal{O}_Y(-Z)$. This implies that $\operatorname{lct}(\mathfrak{m}) = \frac{1}{2}$. Since A is Gorenstein, $\operatorname{rt}(\mathfrak{m})$ is equal to $\operatorname{lct}(\mathfrak{m})$. Thus we have $\operatorname{rt}(\mathfrak{m}) = \frac{1}{2}$.

LEMMA 3.18. *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen–Macaulay normal local ring, and $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of A. Then $\mathcal{J}^{\omega}(A, \mathfrak{a}^n) \subset \operatorname{core}(\mathfrak{a})$. In particular, if $\operatorname{Proj} A[\mathfrak{a}]$ has rational singularities, then $\mathcal{J}^{\omega}(A, \mathfrak{a}^n) = \operatorname{core}(\mathfrak{a})$.*

Proof. Let $f : Y \to X$ be the blowing-up along $\mathfrak{a}$, and $g : Z \to X$ be a log resolution of $\mathfrak{a}$. By Theorem 2.39 and Lemma 2.41, we have

$$\text{core}(\mathfrak{a}) = H^0(Y, \mathfrak{a}^n \omega_Y) :_A \omega_A.$$

Let $h : Z \to Y$ be a morphism with $g = f \circ h$. Then $h_*(\mathfrak{a}^n \omega_Z) \subset \mathfrak{a}^n \omega_Y$. Hence we have $H^0(Z, \mathfrak{a}^n \omega_Z) \subset H^0(Y, \mathfrak{a}^n \omega_Y)$. Therefore we have

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = H^0(Z, \mathfrak{a}^n \omega_Z) :_A \omega_A \subset H^0(Y, \mathfrak{a}^n \omega_Y) :_A \omega_A = \text{core}(\mathfrak{a}).$$

We assume that $Y = \text{Proj } A[\mathfrak{a}]$ has rational singularities. Then $h_*(\mathfrak{a}^n \omega_Z) = \mathfrak{a}^n \omega_Y$ by the projection formula. Therefore we have $\mathcal{J}^\omega(A, \mathfrak{a}^n) = \text{core}(\mathfrak{a})$.   □

LEMMA 3.19. *Let $(A, \mathfrak{m})$ be an n-dimensional local ring with rational singularities, and I be a minimal reduction of $\mathfrak{m}$. Then $\mathfrak{m}^{n+1-\lceil \text{rt}(\mathfrak{m}) \rceil} \subset I$.*

Proof. Let $X = \text{Spec } A$. Let j_X be the Jacobian ideal of X, and $\mathfrak{d}_X$ be the lci-defect ideal of X. Let $f : Y \to X$ be a log resolution of $j_X \cdot \mathfrak{d}_X \cdot \mathfrak{m}$ such that $j_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$, and $\mathfrak{m} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors $J_{Y/X}$, $D_{Y/X}$, and F on Y. Since $\lceil \text{rt}(\mathfrak{m}) \rceil - 1 < \text{rt}(\mathfrak{m})$, we have $\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - (\lceil \text{rt}(\mathfrak{m}) \rceil - 1)F \geq 0$. Therefore

$$\begin{aligned}
I \supset \text{core}(\mathfrak{m}) \supset \mathcal{J}^\omega(A, \mathfrak{m}^n) &= f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - nF) \\
&\supset f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - (\lceil \text{rt}(\mathfrak{m}) \rceil - 1)F - (n + 1 - \lceil \text{rt}(\mathfrak{m}) \rceil)F) \\
&\supset f_* \mathcal{O}_Y(-(n + 1 - \lceil \text{rt}(\mathfrak{m}) \rceil)F) \supset \mathfrak{m}^{n+1-\lceil \text{rt}(\mathfrak{m}) \rceil}.
\end{aligned}$$
□

PROPOSITION 3.20. *Let X be an n-dimensional variety with rational singularities. For a closed point $x \in X$,*

(1) $\text{rt}(\mathfrak{m}_x) \leq n$,
(2) $\text{rt}(\mathfrak{m}_x) = n$ *if and only if x is a nonsingular point,*
(3) *If $\text{rt}(\mathfrak{m}_x) > n - 1$, then x is a nonsingular point.*

Proof. For part (1), let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$. Let I be a minimal reduction of $\mathfrak{m}$, then $I \supset \mathfrak{m}^{n+1-\lceil \text{rt}(\mathfrak{m}_x) \rceil}$ by Lemma 3.19. Here, if $\text{rt}(\mathfrak{m}_x) > n$, then we obtain $I \supset \mathcal{O}_{X,x}$, a contradiction.

For part (2), suppose x is a nonsingular point. Replacing X by a small neighborhood of x, we may assume that X is nonsingular. Let $f : Y \to X$ be the blow-up of $\mathfrak{m}_x$, and E the exceptional divisor. Then f is a log resolution of $\mathfrak{m}_x$, and the equalities $K_Y - f^* K_X = (n - 1)E$ and $\text{val}_E(\mathfrak{m}_x) = 1$ hold. Hence $\text{rt}(\mathfrak{m}_x) = n$. Conversely, suppose $\text{rt}(\mathfrak{m}_x) = n$. Then, by Lemma 3.19, we have $\mathfrak{m} = I$. Therefore $\mathfrak{m}$ is generated by n elements. This implies that x is a nonsingular point.

For part (3), suppose $\text{rt}(\mathfrak{m}_x) > n - 1$. By the same way as before, x is a nonsingular point.   □

PROPOSITION 3.21. *Let X be a variety with rational singularities, and $\mathfrak{a}$ a nonzero ideal of $\mathcal{O}_X$. Then $\text{rt}(\mathfrak{a}) > 1$ if and only if for every nonzero ideal $\mathfrak{b} \subset \mathcal{O}_X$, we have $\mathcal{J}^\omega(X, \mathfrak{b}) \supset \mathfrak{b} : \mathfrak{a}$.*

Proof. First, suppose that $\mathcal{J}^\omega(X, \mathfrak{b}) \supset (\mathfrak{b} : \mathfrak{a})$ for every ideal $\mathfrak{b} \subset \mathcal{O}_X$. Considering the case where $\mathfrak{a} = \mathfrak{b}$, we have $\mathcal{J}^\omega(X, \mathfrak{a}) = \mathcal{O}_X$. Since $\mathcal{J}^\omega(X, \mathfrak{a}) = \mathcal{J}^\omega(X, \mathfrak{a}^{1+\varepsilon}) = \mathcal{O}_X$ for $0 < \varepsilon \ll 1$ by Proposition 3.12, we have $\mathrm{rt}(\mathfrak{a}) \geq 1 + \varepsilon > 1$.

Conversely, assume that $\mathrm{rt}(\mathfrak{a}) > 1$. Let $f : Y \to X$ be a log resolution of $j_X \cdot \mathfrak{d}_X \cdot \mathfrak{a} \cdot \mathfrak{b}$ such that $j_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$, $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_\mathfrak{a})$, and $\mathfrak{b} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_\mathfrak{b})$ for some effective divisors $J_{Y/X}$, $D_{Y/X}$, $F_\mathfrak{a}$, and $F_\mathfrak{b}$ on Y. Since $\mathrm{rt}(\mathfrak{a}) > 1$, we have $\mathcal{J}^\omega(X, \mathfrak{a}) = \mathcal{O}_X$. This implies that

$$\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - F_\mathfrak{a} \geq 0.$$

We may assume that $\mathfrak{b}$ is an integrally closed ideal, that is, $\mathfrak{b} = f_*\mathcal{O}_Y(-F_\mathfrak{b})$. Then $x \in \mathfrak{b} : \mathfrak{a} \Leftrightarrow x\mathfrak{a} \subset \mathfrak{b} \Leftrightarrow f^*x \cdot \mathcal{O}_Y(-F_\mathfrak{a}) \subset \mathcal{O}_Y(-F_\mathfrak{b}) \Leftrightarrow f^*x \in \mathcal{O}_Y(F_\mathfrak{a} - F_\mathfrak{b})$. Therefore we have $\mathrm{div}\, f^*x + F_\mathfrak{a} - F_\mathfrak{b} \geq 0$. Hence we have

$$\mathrm{div}\, f^*x + \widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - F_\mathfrak{b} \geq \widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} - F_\mathfrak{a} \geq 0.$$

Thus $x \in \mathcal{J}^\omega(X, \mathfrak{b})$.   □

COROLLARY 3.22. *Let X be a variety with rational singularities. Then $\mathrm{rt}(\mathfrak{m}_x) > 1$ for a closed point $x \in X$ if and only if for every $\mathfrak{m}_x$-primary ideal $\mathfrak{a} \subset \mathcal{O}_X$, we have a strict containment $\mathcal{J}^\omega(X, \mathfrak{a}) \supsetneq \mathfrak{a}$.*

Proof. First, suppose $\mathcal{J}^\omega(X, \mathfrak{a}) \supsetneq \mathfrak{a}$ for every $\mathfrak{m}_x$-primary ideal $\mathfrak{a} \subset \mathcal{O}_X$. Considering the case where $\mathfrak{a} = \mathfrak{m}_x$, we have $\mathcal{J}^\omega(X, \mathfrak{m}_x) = \mathcal{O}_X$. Since $\mathcal{J}^\omega(X, \mathfrak{m}_x) = \mathcal{J}^\omega(X, \mathfrak{m}_x^{1+\varepsilon}) = \mathcal{O}_X$ for $0 < \varepsilon \ll 1$ by Proposition 3.12, we have $\mathrm{rt}(\mathfrak{m}_x) \geq 1 + \varepsilon > 1$.

Conversely, assume that $\mathrm{rt}(\mathfrak{m}_x) > 1$. By Proposition 3.21, we have $\mathcal{J}^\omega(X, \mathfrak{a}) \supset (\mathfrak{a} : \mathfrak{m}_x)$ for every $\mathfrak{m}_x$-primary ideal $\mathfrak{a} \subset \mathcal{O}_X$. If $\mathfrak{m}_x^l \subset \mathfrak{a}$, then $\mathfrak{m}_x^{l-1} \subset (\mathfrak{a} : \mathfrak{m}_x)$. Therefore we have $(\mathfrak{a} : \mathfrak{m}_x) \supsetneq \mathfrak{a}$. This implies that $\mathcal{J}^\omega(X, \mathfrak{a}) \supsetneq \mathfrak{a}$.   □

De Fernex and Hacon [3] defined the log canonical, log terminal singularities on an arbitrary normal variety. These singularities are generalizations of log canonical, log terminal singularities for a $\mathbb{Q}$-Gorenstein variety. Moreover, in [3], they defined the $\natural$-pull back of an arbitrary divisor on a normal variety. In a local situation, as we can take an effective divisor $-K_X$, let $f : Y \to X$ be a log resolution of $\mathcal{O}_X(K_X)$. Define the divisor $f^\natural(-K_X)$ on Y by $\mathcal{O}_X(K_X)\mathcal{O}_Y = \mathcal{O}_Y(-f^\natural(-K_X))$.

We assume that mK_X is effective. Let $f : Y \to X$ be a log resolution of $\mathcal{O}_X(-mK_X)$. Define the divisor D_m on Y by $\mathcal{O}_X(-mK_X)\mathcal{O}_Y = \mathcal{O}_Y(-D_m)$. Under this notation, we define the divisor

$$K_{m,Y/X} = K_Y - \frac{1}{m}D_m$$

with the support on the exceptional divisor. De Fernex and Hacon [3] showed that for $m, q \geq 1$,

$$K_{m,Y/X} \leq K_{qm,Y/X} \leq K_Y + f^\natural(-K_X).$$

PROPOSITION 3.23. *Let $X \subset \mathbb{A}^N$ be an n-dimensional affine normal variety. Then there is a log resolution $f : Y \to X$ of $j_X \mathfrak{d}_X \mathcal{O}_X(K_X)$ such that*

$$\widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X} = K_Y + f^\natural(-K_X).$$

Proof. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \mathfrak{d}_X \mathcal{O}_X(K_X)$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$ and $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$. Take a reduced complete intersection scheme $M \subset \mathbb{A}^N$ of codimension $c = N - n$ such that M contains X as an irreducible component. Then we have the sequence

$$\bigwedge^n \Omega_X \xrightarrow{\eta} \omega_X \xrightarrow{u} \omega_M|_X.$$

By Proposition 9.1 of [7], $\mathrm{Im}(u \circ \eta) = \mathfrak{j}_M|_X \omega_M|_X$. Note that $\mathcal{O}_X(K_X)\mathcal{O}_Y = \mathcal{O}_Y(-f^\natural(-K_X))$. We have the sequence

$$f^*\left(\bigwedge^n \Omega_X\right) \xrightarrow{\eta'} \mathcal{O}_Y(-f^\natural(-K_X)) \xrightarrow{u'} f^*(\omega_M|_X).$$

Since $\mathcal{O}_Y(-f^\natural(-K_X))$ and $f^*(\omega_M|_X)$ are invertible, we can write

$$\mathrm{Im}\,\eta' = I\mathcal{O}_Y(-f^\natural(-K_X)),$$
$$\mathrm{Im}\,u' = J_M f^*(\omega_M|_X),$$

with the ideal $I, J_M \subset \mathcal{O}_Y$. Then we obtain $IJ_M = \mathfrak{j}_M|_X \mathcal{O}_Y$. Consider all M and define $J = \sum_M J_M$; then we have $IJ = \mathfrak{j}_X \mathcal{O}_Y$. Let $g : Z \to Y$ be a log resolution of IJ such that $I\mathcal{O}_Z = \mathcal{O}_Z(-B)$, and $h : Z \to X$ be the composition of f and g. Then $B + D_{Z/X} = J_{Z/X}$ since $\mathfrak{d}_X \mathcal{O}_Y = J$.

Since h factors through the Nash blow-up, the torsion-free sheaf $h^*(\bigwedge^n \Omega_X)/\mathrm{Tor}$ is invertible, it is written as $\mathcal{O}_Z(C)$ by a divisor C on Z. Then, by the definition of $\widehat{K}_{Z/X}$, we have $\widehat{K}_{Z/X} = K_Z - C$. On the other hand, we have $C = g^*(-f^\natural(-K_X)) - B = -h^\natural(-K_X) - B$ by Lemma 2.7 in [3]. Therefore we have

$$\widehat{K}_{Z/X} - J_{Z/X} + D_{Z/X} = K_Z - C - B = K_Z + h^\natural(-K_X),$$

which completes the proof of the lemma.   □

De Fernex and Hacon [3] introduced a multiplier ideal for a pair $(X, \mathfrak{a}^t)$ with normal variety X and an ideal $\mathfrak{a}$ on X. For $m \in \mathbb{N}$, they defined mth multiplier ideal as follows:

$$\mathcal{J}_m(X, \mathfrak{a}^t) = f_* \mathcal{O}_Y(\lceil K_{m,Y/X} - tZ \rceil),$$

where $f : Y \to X$ is log resolution of $\mathfrak{a}\mathcal{O}_X(-K_X)$ and $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z)$. They proved that the family of ideals $\{\mathcal{J}_m(X, \mathfrak{a}^t)\}_m$ has the unique maximal element and call it the multiplier ideal of $(X, \mathfrak{a}^t)$ and denote it by $\mathcal{J}(X, \mathfrak{a}^t)$.

COROLLARY 3.24. *Let X be a normal variety, and $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_X$. Then, for $c \in \mathbb{Q}_{>0}$,*

$$\mathcal{J}_m(X, \mathfrak{a}^c) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c).$$

Proof. Since $K_{m,Y/X} \le K_Y + f^\natural(-K_X)$, we have $\mathcal{J}_m(X, \mathfrak{a}^c) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c)$.   □

COROLLARY 3.25. *Let X be a normal variety, and $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_X$. Then for $c \in \mathbb{Q}_{>0}$,*

$$\mathcal{J}(X, \mathfrak{a}^c) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c).$$

Proof. Since $\mathcal{J}_m(X, \mathfrak{a}^c) \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^c)$ for any m, we have $\mathcal{J}(X, \mathfrak{a}^c) \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^c)$. $\square$

3.2. Characterization Rational Singularities via Cores of Ideals

In this section, we characterize rational singularities via cores of ideals.

THEOREM 3.26 ([23, Thm. 2.1], Briançon–Skoda theorem). *Let $(A, \mathfrak{m})$ be an n-dimensional local ring with rational singularities, and I be an ideal of A. Then we have*

$$\overline{I^n} \subset I$$

where $^{-}$ denotes integral closure.

COROLLARY 3.27. *Let $(A, \mathfrak{m})$ be an n-dimensional local ring with rational singularities, I be an ideal of A, and J be a reduction of I. Then we have*

$$\overline{I^n} \subset J.$$

Proof. By the Briançon–Skoda theorem, we have $\overline{J^n} \subset J$. Since J is a reduction of I, we have $\overline{I^n} = \overline{J^n}$. Therefore we have $\overline{I^n} \subset J$. $\square$

LEMMA 3.28. *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen–Macaulay isolated singularity local ring. Suppose that A is not a rational singularity. Then there exists an $\mathfrak{m}$-primary ideal I of A such that $I^n \not\subset \text{core}(I)$.*

Proof. Let I be an $\mathfrak{m}$-primary ideal such that $f : Y = \text{Proj } A[I] \to \text{Spec } A$ is a desingularization. Since A is not a rational singularity, we have $H^0(Y, \omega_Y) \not\supset \omega_A$. By Theorem 2.39 and Lemma 2.41, we have

$$\text{core}(I) = H^0(Y, I^n \omega_Y) :_A \omega_A.$$

By Lemma 2.40, we have $H^0(Y, I^n \omega_Y) :_{\omega_A} I^n = H^0(Y, \omega_Y)$. This implies that $I^n \omega_A \not\subset H^0(Y, I^n \omega_Y)$ since $H^0(Y, \omega_Y) \not\supset \omega_A$. Therefore we have $I^n \not\subset H^0(Y, I^n \omega_Y) :_A \omega_A = \text{core}(I)$. $\square$

THEOREM 3.29. *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen–Macaulay isolated singularity local ring. Then A is a rational singularity if and only if $\overline{I^n} \subset \text{core}(I)$ for any $\mathfrak{m}$-primary ideal I.*

Proof. If A is a rational singularity, then $\overline{I^n} \subset \text{core}(I)$ for any $\mathfrak{m}$-primary ideal I by Corollary 3.27. For the converse proof, we assume that A is not a rational singularity. By Lemma 3.28, there is an $\mathfrak{m}$-primary ideal I of A such that $I^n \not\subset \text{core}(I)$. Thus we have $\overline{I^n} \not\subset \text{core}(I)$. $\square$

The following corollary implies that a Cohen–Macaulay isolated singularity local ring is a rational singularity if the Briançon–Skoda theorem holds for the ring.

COROLLARY 3.30. *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen–Macaulay isolated singularity local ring. A is a rational singularity if and only if $\overline{I^n} \subset I$ for any $\mathfrak{m}$-primary ideal I.*

Proof. If A is a rational singularity, then $\overline{I^n} \subset I$ for any $\mathfrak{m}$-primary ideal I by the Briançon–Skoda theorem. Hence we will show the converse implication. We assume that A is not a rational singularity. By Theorem 3.29, there are an $\mathfrak{m}$-primary ideal I and a reduction J of I such that $\overline{I^n} \not\subset J$. Therefore we have $\overline{J^n} \not\subset J$ since $\overline{I^n} = \overline{J^n}$. $\qquad\square$

COROLLARY 3.31. *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen–Macaulay isolated singularity local ring. Then A is a rational singularity if and only if $\overline{I} \subset \mathcal{J}^\omega(I)$ for any $\mathfrak{m}$-primary ideal I.*

Proof. We assume that A is a rational singularity. Let $f : Y \to X = \operatorname{Spec} A$ be a log resolution of $\jmath_X \eth_X I$ such that $\jmath_X \mathcal{O}_Y = \mathcal{O}_Y(-J)$, $\eth_X \mathcal{O}_Y = \mathcal{O}_Y(-D)$, and $I\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Then, by Theorem 3.3 and Proposition 3.8,

$$\widehat{K}_{Y/X} - J + D \geq 0, \qquad \mathcal{J}^\omega(X, I) = f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J + D - F).$$

Therefore we have

$$\mathcal{J}^\omega(I) = f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J + D - F) \supset f_* \mathcal{O}_Y(-F) = \overline{I}.$$

We assume that A is not a rational singularity. Then, by Theorem 3.29, there exists an $\mathfrak{m}$-primary I such that $\overline{I^n} \not\subset \operatorname{core}(I)$. Since by Lemma 3.18,

$$\mathcal{J}^\omega(I^n) \subset \operatorname{core}(I),$$

we have

$$\overline{I^n} \not\subset \mathcal{J}^\omega(I^n). \qquad\square$$

DEFINITION 3.32. Let $(A, \mathfrak{m})$ be a local domain that is a homomorphic image of a Gorenstein local ring. Suppose that $\operatorname{Spec} A \setminus \mathfrak{m}$ has rational singularities and that there exists a proper birational morphism $f : Y \to \operatorname{Spec} A$ such that Y has rational singularities. We define the number $r(A)$ as the smallest integer r such that $\mathfrak{m}^r \omega_A \subset \Gamma(Y, \omega_Y)$.

Hyry and Villamayor [14] gave an extension of the Briançon–Skoda theorem to normal Cohen–Macaulay local rings that have rational singularities in the punctured spectrum.

THEOREM 3.33 ([14, Thm. 2.6]). *Let $(A, \mathfrak{m})$ be an n-dimensional normal Cohen–Macaulay local domain which is a homomorphic image of a Gorenstein local ring. Suppose that $\operatorname{Spec} A \setminus \mathfrak{m}$ has rational singularities and that there exists a proper birational morphism $f : Y \to \operatorname{Spec} A$ such that Y has rational singularities. Set $r = r(A)$. Then $\overline{I^{n+r}} \subset I$ for all ideals $I \subset A$.*

PROPOSITION 3.34. *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen–Macaulay isolated singularity local ring. If A is a Du Bois singularity, then $\overline{I^{n+1}} \subset \operatorname{core}(I)$ for all ideals $I \subset A$.*

Proof. Let $f : Y \to \operatorname{Spec} A$ be a resolution of $\operatorname{Spec} A$ such that f is an isomorphism over $\operatorname{Spec} A \setminus \mathfrak{m}$, $f^{-1}(\mathfrak{m})$ is a simple normal crossing divisor, and

$\mathfrak{m}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ for a divisor F on Y. Let G be the reduced exceptional divisor of f. Since A is a Du Bois singularity, we have $\Gamma(Y, \omega_Y(G)) = \omega_A$ by Theorem 2.4. Therefore $\mathfrak{m}\omega_A = \mathfrak{m}\Gamma(Y, \omega_Y(G)) \subset \Gamma(Y, \omega_Y(G - F)) \subset \Gamma(Y, \omega_Y)$. Thus $r(A) = 1$. By Theorem 3.33, we have $\overline{I^{n+1}} \subset \mathrm{core}(I)$. $\qquad \square$

This proposition does not give a characterization of a Cohen–Macaulay Du Bois singularity. We have an example of an n-dimensional Cohen–Macaulay local ring A with a non-Du Bois isolated singularity such that $\overline{I^{n+1}} \subset \mathrm{core}(I)$ for all ideals $I \subset A$.

EXAMPLE 3.35. Let $A = (\mathbb{C}[x, y, z]/(x^3 + y^3 + z^4))_{(x,y,z)}$. Note that Gorenstein Du Bois singularities are log canonical singularities. Then Spec A is Gorenstein but not log canonical. Therefore A is not a Du Bois singularity. Let $f : Y \to$ Spec A be the blow-up at $\mathfrak{m}$. Then f is a resolution of Spec A. Therefore we have $r(A) = 1$. By Theorem 3.33, $\overline{I^3} \subset \mathrm{core}(I)$ for any ideal I.

4. Cores of Ideals and ω-Multiplier Ideals of Two-Dimensional Local Rings with a Rational Singularity

4.1. The Arithmetic of Cores of Ideals and ω-Multiplier Ideals

In this section, we discuss various relationships between the core of an ideal and the ω-multiplier ideal of a two-dimensional local ring with a rational singularity.

DEFINITION 4.1. Let $(A, \mathfrak{m})$ be a two-dimensional rational singularity and fix a resolution of singularities $f : Y \to$ Spec A. For any integral divisor D on Y, the f-anti-nef closure of D is defined to be the unique smallest integral f-anti-nef divisor that is bigger than or equal to D. We will denote it by $\mathrm{an}_f(D)$.

The followings are quite useful.

THEOREM 4.2 ([9, Prop. 1.10], [22]). *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity and fix a resolution of singularities $f : X \to$ Spec A. Then there is a one-to-one correspondence between the set of integrally closed ideals I in A such that $I\mathcal{O}_X$ is invertible and the set of effective f-anti-nef cycles Z on X. The correspondence is given by $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ and $I = H^0(X, \mathcal{O}_X(-Z))$.*

LEMMA 4.3 ([24, Lemma 1.2], [31, Lemma 2.1]). *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity and fix a resolution of singularities $f : Y \to$ Spec A. For any divisor D on Y, we have $f_*\mathcal{O}_Y(-D) = f_*\mathcal{O}_Y(-\mathrm{an}_f(D))$.*

PROPOSITION 4.4 ([23, Cor. 5.4]). *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an integrally closed ideal of A, and I be a reduction of $\mathfrak{a}$. Then $I\mathfrak{a} = \mathfrak{a}^2$.*

The following is a generalization of Lemma 5.6 in [25].

LEMMA 4.5. *Let $(A, \mathfrak{m})$ be a two-dimensional normal local ring, I be an $\mathfrak{m}$-primary ideal, and J be a minimal reduction of I with $JI = I^2$. Then, for $n \in \mathbb{Z}_{\geq 0}$,*

$$J^{n+1} : I = J^n(J : I) = I^n(J : I).$$

Proof. We will show that $J^{n+1} : I = J^n(J : I) = I^n(J : I)$ by induction on n. When $n = 0$, the assertion is trivial. If $n = 1$, then the equalities hold by Lemma 5.6 in [25]. Thus we may assume that $n \geq 2$. It is clear that $J^n(J : I) \subset I^n(J : I)$. Let $x \in I^n$ and $y \in (J : I)$. Then $xyI \subset yI^{n+1} = yIJ^n \subset J^{n+1}$. Therefore we have $I^n(J : I) \subset J^{n+1} : I$. Hence we will show the inclusion $J^{n+1} : I \subset J^n(J : I)$. Let $J = (x_1, x_2)$. Assume that $x \in J^{n+1} : I$. Since $J^{n+1} : I \subset J^{n+1} : J \subset J^n$, there exist $a_{i_1, i_2} \in A$ such that $x = \sum_{i_1+i_2=n} a_{i_1,i_2} x_1^{i_1} x_2^{i_2}$. Since $x \in J^{n+1} : I$, for any $f \in I$, there exist $b_{i_1,i_2} \in A$ such that $xf = \sum_{j_1+j_2=n+1} b_{j_1,j_2} x_1^{j_1} x_2^{j_2}$. Then we have $a_{n,0} x_1^n f - b_{n+1,0} x_1^{n+1} \in (x_2)$, $a_{0,n} x_2^n f - b_{0,n+1} x_2^{n+1} \in (x_1)$. Since x_1, x_2 is a regular sequence, we have $a_{n,0} f - b_{n+1,0} x_1 \in (x_2)$ and $a_{0,n} f - b_{0,n+1} x_2 \in (x_1)$. Thus $a_{n,0} f, a_{0,n} f \in J$. This shows that $a_{n,0}, a_{0,n} \in J : I$. We can write

$$x - a_{n,0} x_1^n - a_{0,n} x_2^n = x_1 x_2 \sum_{i_1+i_2=n, i_1, i_2 \neq 0, n} a_{i_1,i_2} x_1^{i_1-1} x_2^{i_2-1}.$$

Let $y = \sum_{i_1+i_2=n, i_1,i_2 \neq 0,n} a_{i_1,i_2} x_1^{i_1-1} x_2^{i_2-1}$. Since $x \in J^{n+1} : I$ and $a_{n,0} x_1^n$, $a_{0,n} x_2^n \in J^n(J : I) \subset J^{n+1} : I$, we have $x_1 x_2 y \in J^{n+1} : I$. For any $f \in I$, we have

$$x_1 x_2 y f \in J^{n+1}.$$

Hence we have

$$yf \in J^{n-1}.$$

Therefore we have

$$y \in J^{n-1} : I.$$

By induction hypothesis, we have $y \in J^{n-2}(J : I)$. Thus we have $x = a_{n,0} x_1^n + a_{0,n} x_2^n + x_1 x_2 y \in J^n(J : I)$. $\qquad\square$

PROPOSITION 4.6. *Let $(A, \mathfrak{m})$ be a two-dimensional normal local ring, I be an $\mathfrak{m}$-primary ideal, and J be a minimal reduction of I with $JI = I^2$. Then, for $n \in \mathbb{Z}_{\geq 1}$,*

$$J^{n-1} \operatorname{core}(I) = I^{n-1} \operatorname{core}(I) = J^{n+1} : I = J^n(J : I) = I^n(J : I).$$

Proof. By Theorem 2.39 and Lemma 4.5, we have

$$\operatorname{core}(I) = J^2 : I = J(J : I) = I(J : I).$$

Thus, by Lemma 4.5, we have

$$J^{n-1} \operatorname{core}(I) = I^{n-1} \operatorname{core}(I) = J^{n+1} : I = J^n(J : I) = I^n(J : I). \qquad\square$$

We need the following theorem to prove the properties of ω-multiplier ideals of a two-dimensional local ring with a rational singularity.

THEOREM 4.7 ([26, Thms. 4.4 and 4.6]). *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal, and I be a minimal reduction of $\mathfrak{a}$. Let $f : Y \to X = \operatorname{Spec} A$ be a log resolution of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, and $f : Y_0 \to X$ be the minimal resolution of singularities. Then*

$$I : \mathfrak{a} = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - F)),$$
$$\operatorname{core}(\mathfrak{a}) = \mathfrak{a} H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - F))$$
$$= I H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - F)) = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - 2F)).$$

Hyry and Smith proved the following in the proof of Lemma 5.1.6 in [13]. We need the following lemma to prove Proposition 4.9.

LEMMA 4.8 (See the proof of Lemma 5.1.6 in [13]). *Let $(A, \mathfrak{m})$ be an n-dimensional Cohen–Macaulay local ring, $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal, and J be a minimal reduction of $\mathfrak{a}$ with $J\mathfrak{a}^r = \mathfrak{a}^{r+1}$. Let Y be the blow-up of $\mathfrak{a}$. Then, for $m \in \mathbb{Z}_{\geq 1}$,*

$$H^0(Y, \mathfrak{a}^m \omega_Y) = J^{m+r+1-n} \omega_A :_{\omega_A} \mathfrak{a}^r$$

and

$$J^m \omega_A : \omega_A = J^m.$$

PROPOSITION 4.9. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an $\mathfrak{m}$-primary integrally closed ideal, and J be a minimal reduction of $\mathfrak{a}$. Then, for $n \in \mathbb{N}$,*

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = J^n : \mathfrak{a} = J^{n-1}(J : \mathfrak{a}) = \mathfrak{a}^{n-1}(J : \mathfrak{a}).$$

Proof. Let $f : Y \to X$ be the blow-up along $\mathfrak{a}$, and $g : Z \to X$ be a log resolution of $\mathfrak{a}$. Then Y is normal because $\mathfrak{a}^m$ is an integrally closed ideal for any $m \in \mathbb{N}$ (see Theorem 7.1 in [22]). By Proposition 1.2 in [22], Y has a rational singularity. Therefore, by the projection formula, we have

$$H^0(Z, \mathfrak{a}^n \omega_Z) = H^0(Y, \mathfrak{a}^n \omega_Y).$$

Thus, by Proposition 4.4 and Lemma 4.8, we have

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = H^0(Y, \mathfrak{a}^n \omega_Y) : \omega_A$$
$$= (J^n \omega_A :_{\omega_A} \mathfrak{a}) : \omega_A$$
$$= (J^n \omega_A : \omega_A) : \mathfrak{a} = J^n : \mathfrak{a}.$$

Thus, by Lemma 4.5, we have

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = J^n : \mathfrak{a} = J^{n-1}(J : \mathfrak{a}) = \mathfrak{a}^{n-1}(J : \mathfrak{a}). \qquad \square$$

The following proposition implies that the Skoda theorem of ω-multiplier ideals holds for a two-dimensional local ring with a rational singularity.

PROPOSITION 4.10. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal, and J be a reduction of $\mathfrak{a}$. Then, for $n \in \mathbb{Z}_{\geq 2}$,*

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = \mathfrak{a}\mathcal{J}^\omega(A, \mathfrak{a}^{n-1}) = J\mathcal{J}^\omega(A, \mathfrak{a}^{n-1}).$$

Proof. We may assume that $\mathfrak{a}$ is an integrally closed ideal and J is a minimal reduction of $\mathfrak{a}$. By Proposition 4.9, we have

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = J^{n-1}(J : \mathfrak{a}) = \mathfrak{a}^{n-1}(J : \mathfrak{a}),$$
$$\mathcal{J}^\omega(A, \mathfrak{a}^{n-1}) = J^{n-2}(J : \mathfrak{a}) = \mathfrak{a}^{n-2}(J : \mathfrak{a}).$$

Therefore we have

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = \mathfrak{a}\mathcal{J}^\omega(A, \mathfrak{a}^{n-1}) = J\mathcal{J}^\omega(A, \mathfrak{a}^{n-1}). \qquad \square$$

THEOREM 4.11. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Let $f : Y \to X$ be a log resolution of singularities of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, and $f_0 : Y_0 \to X$ be the minimal resolution of singularities. Then, for $n \in \mathbb{N}$,*

$$\mathcal{J}^\omega(A, \mathfrak{a}^n) = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - nF)).$$

Proof. We may assume that $\mathfrak{a}$ is an integrally closed ideal. Let I be a minimal reduction of $\mathfrak{a}^n$. By Theorem 4.7, we have $I : \mathfrak{a}^n = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - nF))$. By Proposition 4.9, we have $I : \mathfrak{a}^n = \mathcal{J}^\omega(A, \mathfrak{a}^n)$. Therefore $\mathcal{J}^\omega(A, \mathfrak{a}^n) = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - nF))$. $\qquad \square$

COROLLARY 4.12. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, $\mathfrak{a}$ be an $\mathfrak{m}$-primary integrally closed ideal, and J be a minimal reduction of $\mathfrak{a}$. Then*

$$\operatorname{core}(\mathfrak{a}) = \mathcal{J}^\omega(A, \mathfrak{a}^2) = \mathfrak{a}\mathcal{J}^\omega(A, \mathfrak{a}) = J\mathcal{J}^\omega(A, \mathfrak{a}).$$

Proof. By Theorem 4.7, Proposition 4.10, and Theorem 4.11, we have

$$\operatorname{core}(\mathfrak{a}) = \mathcal{J}^\omega(A, \mathfrak{a}^2) = \mathfrak{a}\mathcal{J}^\omega(A, \mathfrak{a}) = J\mathcal{J}^\omega(A, \mathfrak{a}). \qquad \square$$

PROPOSITION 4.13. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, and $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal. Then*

$$e(\mathfrak{a}) = \ell(A/\operatorname{core}(\mathfrak{a})) - 2\ell(A/\mathcal{J}^\omega(A, \mathfrak{a})).$$

Proof. Let $I = (x_1, x_2)$ be a minimal reduction of $\mathfrak{a}$. We have

$$e(\mathfrak{a}) = \ell(A/I) = \ell(A/I\mathcal{J}^\omega(A, \mathfrak{a})) - \ell(I/I\mathcal{J}^\omega(A, \mathfrak{a})).$$

By Corollary 4.12, $\ell(A/I\mathcal{J}^\omega(A, \mathfrak{a})) = \ell(A/\operatorname{core}(\mathfrak{a}))$.

We will show that $I/I\mathcal{J}^\omega(A, \mathfrak{a})$ is isomorphic to $A/\mathcal{J}^\omega(A, \mathfrak{a}) \oplus A/\mathcal{J}^\omega(A, \mathfrak{a})$. Let $\phi : A/\mathcal{J}^\omega(A, \mathfrak{a}) \oplus A/\mathcal{J}^\omega(A, \mathfrak{a}) \to I/I\mathcal{J}^\omega(A, \mathfrak{a})$ be a map defined by $\phi(a + \mathcal{J}^\omega(A, \mathfrak{a}), b + \mathcal{J}^\omega(A, \mathfrak{a})) = x_1 a + x_2 b + I\mathcal{J}^\omega(A, \mathfrak{a})$. It is clear that ϕ is surjective. Let $(a + \mathcal{J}^\omega(A, \mathfrak{a}), b + \mathcal{J}^\omega(A, \mathfrak{a})) \in \ker \phi$. Then, by Proposition 4.9,

$$x_1 a + x_2 b \in I\mathcal{J}^\omega(A, \mathfrak{a}) = I(I : \mathfrak{a}) = I^2 : \mathfrak{a}.$$

Then for any element $h \in \mathfrak{a}$, $(x_1 a + x_2 b) h \in I^2$. Therefore there are $c_1, c_2, c_3 \in A$ such that $(x_1 a + x_2 b) h = c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2$. Since $x_1 a h - c_1 x_1^2 \in (x_2)$, $x_2 b h - c_3 x_2^2 \in (x_1)$ and x_1, x_2 is a regular sequence, we have $a h - c_1 x_1 \in (x_2)$, $b h - c_3 x_2 \in (x_1)$. Therefore we have $a h, b h \in (x_1, x_2)$. Thus we have $a, b \in I : \mathfrak{a}$. Since $\mathcal{J}^\omega(A, \mathfrak{a}) = I : \mathfrak{a}$, ϕ is injective. Hence ϕ is an isomorphism. This implies that $\ell(I / I \mathcal{J}^\omega(A, \mathfrak{a})) = 2\ell(A / \mathcal{J}^\omega(A, \mathfrak{a}))$. Thus we have

$$e(\mathfrak{a}) = \ell(A / \mathrm{core}(\mathfrak{a})) - 2\ell(A / \mathcal{J}^\omega(A, \mathfrak{a})). \qquad \square$$

LEMMA 4.14. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity. Let $f : Y \to X = \mathrm{Spec}\, A$ be a resolution of singularities of $\mathrm{Spec}\, A$. We assume that the morphism f is factorized as*

$$Y := Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X,$$

where $f_i : Y_i \to Y_{i-1}$ is a contraction of a (-1)-curve E_i on Y_i for every $i = 1, \ldots, n$, and $f_0 : Y_0 \to X$ is the minimal resolution of X. We denote by $\pi_i : Y \to Y_i$ the composition of $f_{i+1}, \ldots, f_n$ for $i = 0, 1, \ldots, n-1$ and by $\pi_n : Y \to Y$ the identity morphism on Y. Let Z be an f-anti-nef cycle on Y, and $K = K_{Y/Y_0} = \sum_{i=1}^{n} \pi_i^* E_i$. Let

$$C = \{j \in \mathbb{N} \mid 1 \le j \le n,\ Z \cdot \pi_j^* E_j < 0\}.$$

Then

$$\mathrm{an}_f(Z - K) = Z - \sum_{i \in C} \pi_i^* E_i.$$

Proof. First, we will show that $Z - \sum_{i \in C} \pi_i^* E_i$ is f-anti-nef. For each f_0-exceptional curve F, we have

$$\left(Z - \sum_{i \in C} \pi_i^* E_i \right) \cdot \pi_{0*}^{-1} F \le Z \cdot \pi_{0*}^{-1} F \le 0.$$

We assume that for $i \in C$ and $j \notin C$, $\pi_i^* E_i \cdot \pi_{j*}^{-1} E_j = 1$. Then $f_i : Y_i \to Y_{i-1}$ is the blow-up at a closed point of the strict transform of E_j on Y_{i-1}. This implies that $\pi_i^* E_i \le \pi_j^* E_j$. Therefore $Z \cdot \pi_j^* E_j \le Z \cdot \pi_i^* E_i < 0$ since Z is f-anti-nef. This implies that $j \in C$, which is a contradiction. Hence we have $\pi_i^* E_i \cdot \pi_{j*}^{-1} E_j = 0$ for $i \in C$ and $j \notin C$. Thus, for $j \notin C$, we have

$$\left(Z - \sum_{i \in C} \pi_i^* E_i \right) \cdot \pi_{j*}^{-1} E_j = Z \cdot \pi_{j*}^{-1} E_j = 0.$$

We assume that $Z \cdot \pi_{j*}^{-1} E_j < 0$ for $j \in C$. Then we have

$$\left(Z - \sum_{i \in C} \pi_i^* E_i \right) \cdot \pi_{j*}^{-1} E_j \le Z \cdot \pi_{j*}^{-1} E_j - \pi_j^* E_j \cdot \pi_{j*}^{-1} E_j = Z \cdot \pi_{j*}^{-1} E_j + 1 \le 0.$$

We assume that $Z \cdot \pi_{j*}^{-1} E_j = 0$ for $j \in C$. Then there exists $k \in C$ such that $Z \cdot \pi_k^* E_k < 0$, $\pi_k^* E_k \le \pi_j^* E_j$, and $\pi_k^* E_k \cdot \pi_{j*}^{-1} E_j = 1$. Therefore

$$\left(Z - \sum_{i \in C} \pi_i^* E_i \right) \cdot \pi_{j*}^{-1} E_j = - \sum_{i \in C} \pi_i^* E_i \cdot \pi_{j*}^{-1} E_j$$

$$\le -\pi_j^* E_j \cdot \pi_{j*}^{-1} E_j - \pi_k^* E_k \cdot \pi_{j*}^{-1} E_j = 0.$$

By the above discussion, $Z - \sum_{i \in C} \pi_i^* E_i$ is f-anti-nef. This implies that

$$\mathrm{an}_f(Z - K) \le Z - \sum_{i \in C} \pi_i^* E_i.$$

Let Z' be a cycle such that $Z - K \le Z' < Z - \sum_{i \in C} \pi_i^* E_i$. Next, we will show that Z' is not f-anti-nef. Let $F = Z - \sum_{i \in C} \pi_i^* E_i - Z'$ and $\pi_{j*}^{-1} E_j \le F$. Then there exists $k \notin C$ such that $\pi_{j*}^{-1} E_j \le \pi_k^* E_k$. Thus we have $j \notin C$. Since $Z \cdot \pi_{j*}^{-1} E_j = 0$ and $\pi_i^* E_i \cdot \pi_{j*}^{-1} E_j = 0$ for $i \in C$ and $j \notin C$,

$$Z' \cdot F = \left(Z - \sum_{i \in C} \pi_i^* E_i - F \right) \cdot F = -F \cdot F > 0.$$

Thus Z' is not f-anti-nef. Therefore the minimal f-anti-nef cycle that is bigger than or equal to $Z - K$ is $Z - \sum_{i \in C} \pi_i^* E_i$. $\qquad\square$

LEMMA 4.15. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity. Let $f : Y \to X = \mathrm{Spec}\, A$ be a log resolution of $\jmath_X \partial_X$ such that $\jmath_X \mathcal{O}_Y = \mathcal{O}_Y(-J)$ and $\partial_X \mathcal{O}_Y = \mathcal{O}_Y(-D)$. Let Z be an exceptional f-anti-nef divisor on Y. Let $K^\omega = \widehat{K}_{Y/X} - J + D$ and $K = K_{Y/Y_0}$, where Y_0 is the minimal resolution of X. Then*

$$\mathrm{ord}_F K^\omega = \mathrm{ord}_F K$$

for any exceptional prime divisor F with $Z \cdot F < 0$.

Proof. The morphism f can be factorized as

$$Y := Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X,$$

where $f_i : Y_i \to Y_{i-1}$ is a contraction of a (-1)-curve E_i on Y_i for every $i = 1, \ldots, n$, and $f_0 : Y_0 \to X$ is the minimal resolution of X. We denote by $\pi_i : Y \to Y_i$ the composition of $f_{i+1}, \ldots, f_n$ for $i = 0, 1, \ldots, n-1$ and by $\pi_n : Y \to Y$ the identity morphism on Y. Let

$$C = \{ j \in \mathbb{N} \mid 1 \le j \le n, Z \cdot \pi_j^* E_j < 0 \}.$$

Then

$$\mathrm{an}_f(nZ - K) = nZ - \sum_{i \in C} \pi_i^* E_i$$

for any positive integer n by Lemma 4.14. Let $\mathfrak{a} = f_* \mathcal{O}_Y(-Z)$. Then $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, and we have $\mathfrak{a} \mathcal{O}_Y = \mathcal{O}_Y(-Z)$ by Theorem 4.2. Therefore $\mathcal{J}^\omega(X, \mathfrak{a}^n) = f_* \mathcal{O}_Y(K^\omega - nZ)$ by Theorem 3.3. By Theorem 4.11 and

Lemma 4.14, we have

$$nZ - K^\omega \le \mathrm{an}_f(nZ - K^\omega) = \mathrm{an}_f(nZ - K) = nZ - \sum_{i \in C} \pi_i^* E_i.$$

This implies that $\sum_{i \in C} \pi_i^* E_i \le K^\omega$. Since $\mathrm{ord}_F \pi_j^* E_j = 0$ for $j \notin C$, we have

$$\mathrm{ord}_F K = \mathrm{ord}_F \sum_{i=1}^{n} \pi_i^* E_i = \mathrm{ord}_F \sum_{i \in C} \pi_i^* E_i.$$

Therefore we have $\mathrm{ord}_F K^\omega \ge \mathrm{ord}_F K$.

We assume that $\mathrm{ord}_F K^\omega > \mathrm{ord}_F K$. Then we have

$$K^\omega \ge \sum_{i \in C} \pi_i^* E_i + F.$$

Since $Z \cdot F < 0$, there exists $n \in \mathbb{N}$ such that $nZ - \sum_{i \in C} \pi_i^* E_i - F$ is f-anti-nef. Then

$$\begin{aligned}
\mathrm{an}_f(nZ - K^\omega) &\le \mathrm{an}_f\left(nZ - \left(\sum_{i \in C} \pi_i^* E_i + F\right)\right) \\
&\le nZ - \sum_{i \in C} \pi_i^* E_i - F < nZ - \sum_{i \in C} \pi_i^* E_i \\
&= \mathrm{an}_f(nZ - K^\omega),
\end{aligned}$$

which is a contradiction. Therefore we have $\mathrm{ord}_F K^\omega = \mathrm{ord}_F K$. $\square$

LEMMA 4.16. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity. Let $f : Y \to X = \mathrm{Spec}\, A$ be a resolution of singularities of X, and F be a prime exceptional divisor on Y. Then there exists an exceptional f-anti-nef divisor Z on Y with $Z \cdot F < 0$.*

Proof. Let Z_f be the fundamental cycle of f. Then there exists a prime exceptional divisor F_1 with $Z \cdot F_1 < 0$. Since $f^{-1}(\mathfrak{m})$ is connected, there exists a sequence $\{F_1, \ldots, F_n\}$ such that F_i is an exceptional prime divisor, $F_i \cdot F_{i+1} = 1$ for $1 \le i \le n - 1$, and $F_n = F$.

We will make an exceptional f-anti-nef divisor Z_i such that $Z_i \cdot F_i < 0$ for i by induction on i. When $i = 1$, we can take Z_f as Z_1. By the induction hypothesis, there exists an exceptional f-anti-nef divisor Z_i such that $Z_i \cdot F_i < 0$. Since $Z_i \cdot F_i < 0$, there exists a positive integer n such that $nZ_i - F_i$ is an f-anti-nef divisor. Then $(nZ_i - F_i) \cdot F_{i+1} \le -F_i \cdot F_{i+1} < 0$. Therefore we can take $nZ_i - F_i$ as Z_{i+1}.

Thus there exists an exceptional f-anti-nef divisor Z on Y with $Z \cdot F < 0$. $\square$

PROPOSITION 4.17. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity. Let $f : Y \to X = \mathrm{Spec}\, A$ be a log resolution of $\mathfrak{j}_X \mathfrak{d}_X$ such*

that $j_X \mathcal{O}_Y = \mathcal{O}_Y(-J)$ *and* $\mathfrak{d}_X \mathcal{O}_Y = \mathcal{O}_Y(-D)$. *Let* $K^\omega = \widehat{K}_{Y/X} - J + D$ *and* $K = K_{Y/Y_0}$, *where* Y_0 *is the minimal resolution of* X. *Then*

$$K^\omega = K.$$

Proof. By Lemma 4.16, for any prime exceptional divisor F on Y, there exists an exceptional f-anti-nef divisor Z on Y with $Z \cdot F < 0$. By Lemma 4.15, we have $\mathrm{ord}_F K^\omega = \mathrm{ord}_F K$. Therefore we have $K^\omega = K$. $\qquad\square$

We need the following lemma to prove Lemma 4.19.

LEMMA 4.18 ([19, Lemma 9.2.19]). *Let X be a smooth variety of dimension n, and D any $\mathbb{Q}$-divisor on X with simple normal crossing support. Suppose that $f : Y \to X$ is a log resolution of D. Then*

$$f_* \mathcal{O}_Y(K_{Y/X} - [f^*D]) = \mathcal{O}_X(-[D]).$$

LEMMA 4.19. *Let $(A, \mathfrak{m})$ be a two-dimensional normal local ring, and $\mathfrak{a}$ be a nonzero ideal of A. Let $f : Y \to X = \mathrm{Spec}\, A$ be a log resolution of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, and $f_0 : Y_0 \to X$ be the minimal resolution of singularities. Then, for $c > 0$, $f_* \mathcal{O}_Y(K_{Y/Y_0} - [cF])$ is independent of the choice of log resolutions.*

Proof. Since any two log resolutions can be dominated by a third, we consider the case of two log resolutions of $\mathfrak{a}$, $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$, with a map between them:

$$
\begin{array}{ccc}
Y_2 & \xrightarrow{\ g\ } & Y_1 \\
& \searrow{\scriptstyle f_2} & \downarrow{\scriptstyle f_1} \\
& & X
\end{array}
$$

Let $\mathfrak{a}\mathcal{O}_{Y_1} = \mathcal{O}_{Y_1}(-F_1)$, $\mathfrak{a}\mathcal{O}_{Y_2} = \mathcal{O}_{Y_2}(-F_2)$, and $g : Y_2 \to Y_1$ be the morphism with $f_2 = f_1 \circ g$. Then we have $K_{Y_2/Y_0} = K_{Y_2/Y_1} + g^*(K_{Y_1/Y_0})$ and $F_2 = g^*(F_1)$. By the projection formula and Lemma 4.18,

$$
\begin{aligned}
f_{2*}\mathcal{O}_{Y_2}(K_{Y_2/Y_0} - [cF_2]) &= f_{1*}g_*\mathcal{O}_{Y_2}(K_{Y_2/Y_1} + g^*K_{Y_1/Y_0} - [cg^*F_1]) \\
&= f_{1*}(g_*\mathcal{O}_{Y_2}(K_{Y_2/Y_1} - [cg^*F_1]) \otimes \mathcal{O}_{Y_1}(K_{Y_1/Y_0})) \\
&= f_{1*}\mathcal{O}_{Y_1}(K_{Y_1/Y_0} - [cF_1]).
\end{aligned}
$$

Therefore $f_*\mathcal{O}_Y(K_{Y/Y_0} - [cF])$ is independent of the choice of log resolutions. $\qquad\square$

THEOREM 4.20. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, and $\mathfrak{a}$ be a nonzero ideal of A. Let $f : Y \to X = \mathrm{Spec}\, A$ be a log resolution of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z)$, and $f_0 : Y_0 \to X$ be the minimal resolution of singularities. Then, for $c > 0$,*

$$\mathcal{J}^\omega(A, \mathfrak{a}^c) = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - [cZ])).$$

Proof. By Lemma 4.19, we may assume that f is a log resolution of $\mathfrak{j}_X \mathfrak{d}_X \mathfrak{a}$. Let J and D be divisors on Y such that $\mathfrak{j}_X \mathcal{O}_Y = \mathcal{O}_Y(-J)$ and $\mathfrak{d}_X \mathcal{O}_Y = \mathcal{O}_Y(-D)$. Let $K^\omega = \widehat{K}_{Y/X} - J + D$ and $K = K_{Y/Y_0}$. By Proposition 4.17, we have $K^\omega = K$. Hence we have

$$\mathcal{J}^\omega(A, \mathfrak{a}^c) = H^0(Y, \mathcal{O}_Y(K_{Y/Y_0} - [cZ]))$$

by Theorem 3.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

PROPOSITION 4.21. *Let* $(A, \mathfrak{m})$ *be a two-dimensional local ring with a rational singularity, and* $\mathfrak{a}$ *be an integrally closed* $\mathfrak{m}$*-primary ideal. Then*

$$\mathcal{J}^\omega(A, \mathrm{core}(\mathfrak{a})) = (\mathcal{J}^\omega(A, \mathfrak{a}))^2.$$

Proof. Let $f : Y \to X$ be a log resolution of $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z)$ for some effective divisor Z on Y. The morphism f can be factorized as

$$Y := Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X,$$

where $f_i : Y_i \to Y_{i-1}$ is a contraction of a (-1)-curve E_i on Y_i for every $i = 1, \ldots, n$, and $f_0 : Y_0 \to X$ is the minimal resolution of X. We denote by $\pi_i : Y \to Y_i$ the composition of $f_{i+1}, \ldots, f_n$ for $i = 0, 1, \ldots, n-1$ and by $\pi_n : Y \to Y$ the identity morphism on Y. Let $K = K_{Y/Y_0}$ and

$$C = \{ j \in \mathbb{N} | 1 \le j \le n, Z \cdot \pi_j^* E_j < 0 \}.$$

By Lemma 4.14, we have

$$\mathrm{an}_f(Z - K) = Z - \sum_{i \in C} \pi_i^* E_i.$$

By Theorem 4.7, we have

$$\mathrm{core}(\mathfrak{a}) = f_* \mathcal{O}_Y \left(\sum_{i \in C} \pi_i^* E_i - 2Z \right).$$

Let

$$C' = \left\{ j \in \mathbb{N} | 1 \le j \le n, \left(2Z - \sum_{i \in C} \pi_i^* E_i \right) \cdot \pi_j^* E_j < 0 \right\}.$$

Then, by Lemma 4.14, we have

$$\mathrm{an}_f \left(2Z - \sum_{i \in C} \pi_i^* E_i - K \right) = 2Z - \sum_{i \in C} \pi_i^* E_i - \sum_{i \in C'} \pi_i^* E_i.$$

We will show that $C = C'$. Let $j \in C$. Since $Z \cdot \pi_j^* E_j < 0$ and $\sum_{i \in C} \pi_i^* E_i \cdot \pi_j^* E_j = -1$, we have $(2Z - \sum_{i \in C} \pi_i^* E_i) \cdot \pi_j^* E_j < 0$. Therefore $C \subset C'$.

Hence we will show the opposite inclusion. We assume that we can take $j \in C' \setminus C$. Then $Z \cdot \pi_j^* E_j = 0$ and $\sum_{i \in C} \pi_i^* E_i \cdot \pi_j^* E_j > 0$ since $(2Z - \sum_{i \in C} \pi_i^* E_i) \cdot \pi_j^* E_j < 0$. On the other hand, since $\pi_i^* E_i \cdot \pi_j^* E_j = 0$ for $i \ne j$,

we have $\sum_{i \in C} \pi_i^* E_i \cdot \pi_j^* E_j$ is 0, which is a contradiction. Thus we have $C = C'$. This implies that

$$\operatorname{an}_f \left(2Z - \sum_{i \in C} \pi_i^* E_i - K \right) = 2 \left(Z - \sum_{i \in C} \pi_i^* E_i \right).$$

Thus we have

$$\mathcal{J}^\omega(A, \operatorname{core}(\mathfrak{a})) = f_* \mathcal{O}_Y \left(K - \left(2Z - \sum_{i \in C} \pi_i^* E_i \right) \right)$$

$$= f_* \mathcal{O}_Y \left(-2 \left(Z - \sum_{i \in C} \pi_i^* E_i \right) \right)$$

$$= (\mathcal{J}^\omega(A, \mathfrak{a}))^2. \qquad \square$$

PROPOSITION 4.22. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, and $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal. Then, for $n \in \mathbb{N}$,*

$$\operatorname{core}(\mathfrak{a}^n) = \mathfrak{a}^{2n-1} \mathcal{J}^\omega(A, \mathfrak{a}).$$

Proof. We have $\operatorname{core}(\mathfrak{a}^n) = \mathfrak{a}^n \mathcal{J}^\omega(A, \mathfrak{a}^n) = \mathfrak{a}^{2n-1} \mathcal{J}^\omega(A, \mathfrak{a})$ by Proposition 4.10 and Corollary 4.12. $\qquad \square$

Now we introduce some notation: $\operatorname{core}^1(\mathfrak{a}) = \operatorname{core}(\mathfrak{a})$ and, for $n > 1$, $\operatorname{core}^n(\mathfrak{a}) = \operatorname{core}^{n-1}(\operatorname{core}(\mathfrak{a}))$.

PROPOSITION 4.23. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring with a rational singularity, and $\mathfrak{a}$ be an integrally closed $\mathfrak{m}$-primary ideal. Then, for $n \in \mathbb{N}$,*

$$\operatorname{core}^n(\mathfrak{a}) = \mathfrak{a}(\mathcal{J}^\omega(A, \mathfrak{a}))^{2^n - 1}.$$

In particular, $\operatorname{core}(\operatorname{core}(\mathfrak{a})) = \mathfrak{a}(\mathcal{J}^\omega(A, \mathfrak{a}))^3.$

Proof. We have $\operatorname{core}(\mathfrak{a}) = \mathfrak{a}(\mathcal{J}^\omega(A, \mathfrak{a}))$ by Corollary 4.12. Now let $n > 1$ and assume that the proposition holds for $n - 1$. Then, by Proposition 4.21,

$$\operatorname{core}^n(\mathfrak{a}) = \operatorname{core}^{n-1}(\operatorname{core}(\mathfrak{a}))$$

$$= \operatorname{core}(\mathfrak{a})(\mathcal{J}^\omega(A, \operatorname{core}(\mathfrak{a})))^{2^{n-1}-1}$$

$$= \mathfrak{a}\mathcal{J}^\omega(A, \mathfrak{a})((\mathcal{J}^\omega(A, \mathfrak{a}))^2)^{2^{n-1}-1}$$

$$= \mathfrak{a}(\mathcal{J}^\omega(A, \mathfrak{a}))^{2^n - 1}. \qquad \square$$

4.2. Subadditivity Theorem for ω-Multiplier Ideals of a Two-Dimensional Singularity

In this section, we investigate when the subadditivity theorem of ω-multiplier ideals holds in the two-dimensional case.

Demailly, Ein, and Lazarsfeld proved the following theorem, which is called the subadditivity theorem.

THEOREM 4.24 ([4]). *Let $(A, \mathfrak{m})$ be a regular local ring. Then, for any two nonzero ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$ and any rational numbers $c, d > 0$,*

$$\mathcal{J}(X, \mathfrak{a}^c \mathfrak{b}^d) \subset \mathcal{J}(X, \mathfrak{a}^c) \mathcal{J}(X, \mathfrak{b}^d).$$

In this paper, we say that the subadditivity theorem holds if $\mathcal{J}^\omega(X, \mathfrak{a}\mathfrak{b}) \subset \mathcal{J}^\omega(X, \mathfrak{a}) \mathcal{J}^\omega(X, \mathfrak{b})$ for any two nonzero ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$ and that the strong subadditivity theorem holds if $\mathcal{J}^\omega(X, \mathfrak{a}^c \mathfrak{b}^d) \subset \mathcal{J}^\omega(X, \mathfrak{a}^c) \mathcal{J}^\omega(X, \mathfrak{b}^d)$ for any two nonzero ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$ and any rational numbers $c, d > 0$.

The following lemma seems to be well known to the specialists, but because of lack of an explicit reference, we give its proof.

LEMMA 4.25. *Let $(A, \mathfrak{m})$ be a two-dimensional rational singularity and fix a resolution of singularities $f : Y \to \operatorname{Spec} A$. Let Z_1, Z_2 be two effective f-anti-nef divisors on Y. Then $f_* \mathcal{O}_Y(-Z_1) \subset f_* \mathcal{O}_Y(-Z_2)$ if and only if $Z_1 \geq Z_2$.*

Proof. If $Z_1 \geq Z_2$, then $f_* \mathcal{O}_Y(-Z_1) \subset f_* \mathcal{O}_Y(-Z_2)$. Hence we will show the converse implication. Suppose, by way of contradiction, $f_* \mathcal{O}_Y(-Z_1) \subset f_* \mathcal{O}_Y(-Z_2)$ and $Z_1 \not\geq Z_2$. Note that $f_* \mathcal{O}_Y(-Z_1) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z_1)$ by Theorem 4.2. Then

$$
\begin{aligned}
x \in f_* \mathcal{O}_Y(-Z_2) : f_* \mathcal{O}_Y(-Z_1) &\Leftrightarrow x f_* \mathcal{O}_Y(-Z_1) \subset f_* \mathcal{O}_Y(-Z_2) \\
&\Leftrightarrow f^* x \cdot \mathcal{O}_Y(-Z_1) \subset \mathcal{O}_Y(-Z_2) \\
&\Leftrightarrow f^* x \in \mathcal{O}_Y(Z_1 - Z_2) \\
&\Leftrightarrow x \in f_* \mathcal{O}_Y(Z_1 - Z_2).
\end{aligned}
$$

Therefore we have $f_* \mathcal{O}_Y(-Z_2) : f_* \mathcal{O}_Y(-Z_1) = f_* \mathcal{O}_Y(Z_1 - Z_2)$. Since

$$f_* \mathcal{O}_Y(-Z_1) \subset f_* \mathcal{O}_Y(-Z_2),$$

we have $f_* \mathcal{O}_Y(-Z_2) : f_* \mathcal{O}_Y(-Z_1) = A$. On the other hand, we have $f_* \mathcal{O}_Y(Z_1 - Z_2) \neq A$ since $Z_1 \not\geq Z_2$. Indeed, there exists a prime divisor E on Y such that $\operatorname{ord}_E Z_1 < \operatorname{ord}_E Z_2$. Therefore we have $f_* \mathcal{O}_Y(Z_1 - Z_2) \subset f_* \mathcal{O}_Y(-E) \neq A$. Thus if $f_* \mathcal{O}_Y(-Z_1) \subset f_* \mathcal{O}_Y(-Z_2)$, then $Z_1 \geq Z_2$. □

THEOREM 4.26. *Let $(A, \mathfrak{m})$ be a two-dimensional normal local ring. Then $X = \operatorname{Spec} A$ has a rational singularity if and only if the subadditivity theorem of ω-multiplier ideals holds, that is, for any two nonzero ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$,*

$$\mathcal{J}^\omega(X, \mathfrak{a}\mathfrak{b}) \subset \mathcal{J}^\omega(X, \mathfrak{a}) \mathcal{J}^\omega(X, \mathfrak{b}).$$

Proof. If the subadditivity theorem holds, then $\mathcal{J}^\omega(X, \mathcal{O}_X) \subset \mathcal{J}^\omega(X, \mathcal{O}_X)^2$. Thus $\mathcal{J}^\omega(X, \mathcal{O}_X) = \mathcal{O}_X$, namely, X has a rational singularity. Hence we will show the converse implication, that is, we will prove that for any two ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$, $\mathcal{J}^\omega(X, \mathfrak{a}\mathfrak{b}) \subset \mathcal{J}^\omega(X, \mathfrak{a}) \mathcal{J}^\omega(X, \mathfrak{b})$ when X has a rational singularity. Let $f : Y \to X$ be a resolution of singularities such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F_\mathfrak{a})$ and $\mathfrak{b}\mathcal{O}_Y = \mathcal{O}_Y(-F_\mathfrak{b})$ are invertible and $\operatorname{Exc}(f) \cup \operatorname{Supp} F_\mathfrak{a} \cup \operatorname{Supp} F_\mathfrak{b}$ is a simple normal

crossing divisor. Denote by K the relative canonical divisor K_{Y/Y_0}, where Y_0 is the minimal resolution of X. By Theorem 4.20, we have

$$\mathcal{J}^\omega(X, \mathfrak{a})\mathcal{J}^\omega(X, \mathfrak{b}) = H^0(Y, \mathcal{O}_Y(K - F_\mathfrak{a}))H^0(Y, \mathcal{O}_Y(K - F_\mathfrak{b})),$$
$$\mathcal{J}^\omega(X, \mathfrak{ab}) = H^0(Y, \mathcal{O}_Y(K - F_\mathfrak{a} - F_\mathfrak{b})).$$

Since X has a rational singularity, the product of integrally closed ideals of X is also integrally closed (see [22]). Hence $\mathcal{J}^\omega(X, \mathfrak{a})\mathcal{J}^\omega(X, \mathfrak{b})$ and $\mathcal{J}^\omega(X, \mathfrak{ab})$ are integrally closed, and $\mathcal{J}^\omega(X, \mathfrak{a})\mathcal{J}^\omega(X, \mathfrak{b})$ and $\mathcal{J}^\omega(X, \mathfrak{ab})$ correspond to the cycles $\mathrm{an}_f(F_\mathfrak{a} - K) + \mathrm{an}_f(F_\mathfrak{b} - K)$ and $\mathrm{an}_f(F_\mathfrak{a} + F_\mathfrak{b} - K)$, respectively. Therefore, it suffices to show that

$$\mathrm{an}_f(F_\mathfrak{a} - K) + \mathrm{an}_f(F_\mathfrak{b} - K) \leq \mathrm{an}_f(F_\mathfrak{a} + F_\mathfrak{b} - K).$$

In order to prove this, we prepare some notation. The morphism f can be factorized as

$$Y := Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X,$$

where $f_i : Y_i \to Y_{i-1}$ is a contraction of a (-1)-curve E_i on Y_i for every $i = 1, \ldots, n$, and $f_0 : Y_0 \to X$ is the minimal resolution of X. We denote by $\pi_i : Y \to Y_i$ the composition of $f_{i+1}, \ldots, f_n$ for $i = 0, 1, \ldots, n - 1$ and by $\pi_n : Y \to Y$ the identity morphism on Y. Using Lemma 4.14, we will prove that

$$\mathrm{an}_f(F_\mathfrak{a} - K) + \mathrm{an}_f(F_\mathfrak{b} - K) \leq \mathrm{an}_f(F_\mathfrak{a} + F_\mathfrak{b} - K).$$

Let

$$C_\mathfrak{a} = \{j \in \mathbb{N} | 1 \leq j \leq n, F_\mathfrak{a} \cdot \pi_j^* E_j < 0\},$$
$$C_\mathfrak{b} = \{j \in \mathbb{N} | 1 \leq j \leq n, F_\mathfrak{b} \cdot \pi_j^* E_j < 0\},$$

and

$$C_\mathfrak{ab} = \{j \in \mathbb{N} | 1 \leq j \leq n, (F_\mathfrak{a} + F_\mathfrak{b}) \cdot \pi_j^* E_j < 0\}.$$

Then we have $C_\mathfrak{ab} \subset C_\mathfrak{a} \cup C_\mathfrak{b}$. Therefore, by Lemma 4.14,

$$\mathrm{an}_f(F_\mathfrak{a} - K) + \mathrm{an}_f(F_\mathfrak{b} - K) = F_\mathfrak{a} - \sum_{i \in C_\mathfrak{a}} \pi_i^* E_i + F_\mathfrak{b} - \sum_{i \in C_\mathfrak{b}} \pi_i^* E_i$$
$$\leq F_\mathfrak{a} + F_\mathfrak{b} - \sum_{i \in C_\mathfrak{ab}} \pi_i^* E_i$$
$$= \mathrm{an}_f(F_\mathfrak{a} + F_\mathfrak{b} - K). \qquad \square$$

LEMMA 4.27. *Let $(A, \mathfrak{m})$ be an n-dimensional local ring, and I be a nonzero ideal of A. Let $f : Y \to X = \mathrm{Spec}\, A$ be a log resolution of I such that $I\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Then, for any divisor K on Y,*

$$f_*\mathcal{O}_Y(K) : I = f_*\mathcal{O}_Y(K + F).$$

Proof. We have

$$x \in f_*\mathcal{O}_Y(K) : I \Leftrightarrow xI \subset f_*\mathcal{O}_Y(K)$$
$$\Leftrightarrow f^*x \cdot \mathcal{O}_Y(-F) \subset \mathcal{O}_Y(K)$$

$$\Leftrightarrow f^* x \in \mathcal{O}_Y(K + F)$$
$$\Leftrightarrow x \in f_* \mathcal{O}_Y(K + F).$$

Therefore we have $f_* \mathcal{O}_Y(K) : I = f_* \mathcal{O}_Y(K + F)$. □

COROLLARY 4.28. *Let $(A, \mathfrak{m})$ be a two-dimensional normal local ring. Then $X = \operatorname{Spec} A$ has a rational singularity if and only if the subadditivity theorem of cores of ideals holds, that is, for any two $\mathfrak{m}$-primary integral closed ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$,*

$$\operatorname{core}(\mathfrak{a}\mathfrak{b}) \subset \operatorname{core}(\mathfrak{a}) \operatorname{core}(\mathfrak{b}).$$

Proof. If A has a rational singularity, then

$$\operatorname{core}(\mathfrak{a}\mathfrak{b}) = \mathcal{J}^\omega(X, \mathfrak{a}^2\mathfrak{b}^2) \subset \mathcal{J}^\omega(X, \mathfrak{a}^2)\mathcal{J}^\omega(X, \mathfrak{b}^2) = \operatorname{core}(\mathfrak{a}) \operatorname{core}(\mathfrak{b})$$

by Corollary 4.12 and Theorem 4.26. Hence we will show the converse implication. Let I be an $\mathfrak{m}$-primary integral closed ideal such that $g : Z = \operatorname{Proj} A[I] \to X = \operatorname{Spec} A$ is a resolution of singularities. Let F' be an effective divisor on Z such that $I\mathcal{O}_Z = \mathcal{O}_Z(-F')$. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X \cdot I$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$, and $I \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisors $J_{Y/X}$, $D_{Y/X}$, and F on Y. Let $K = \widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}$. Then

$$\operatorname{core}(I) = g_* \mathcal{O}_Z(K_Z - 2F') : \omega_X$$
$$= f_* \mathcal{O}_Y(K_Y - 2F) : \omega_X$$
$$= \mathcal{J}^\omega(X, I^2) = f_* \mathcal{O}_Y(K - 2F)$$

by Lemma 2.39, Lemma 2.41, and Theorem 3.3. In the same manner, we have

$$\operatorname{core}(I^2) = f_* \mathcal{O}_Y(K - 4F).$$

Next we will show that

$$f_* \mathcal{O}_Y(K - 2F) \subset f_* \mathcal{O}_Y(2^{n-1}K - 2F)$$

for any $n \in \mathbb{N}$ by induction on n. When $n = 1$, the assertion is trivial. By the induction hypothesis and subadditivity of cores of ideals, we have

$$f_* \mathcal{O}_Y(K - 4F) = \operatorname{core}(I^2) \subset (\operatorname{core}(I))^2 = (f_* \mathcal{O}_Y(K - 2F))^2$$
$$\subset (f_* \mathcal{O}_Y(2^{n-1}K - 2F))^2 \subset f_* \mathcal{O}_Y(2^n K - 4F).$$

Therefore we have

$$f_* \mathcal{O}_Y(K - 2F) = f_* \mathcal{O}_Y(K - 4F) : I^2$$
$$\subset f_* \mathcal{O}_Y(2^n K - 4F) : I^2 = f_* \mathcal{O}_Y(2^n K - 2F)$$

by Lemma 4.27. By the previous discussion, we have

$$f_* \mathcal{O}_Y(K - 2F) \subset f_* \mathcal{O}_Y(2^{n-1}K - 2F)$$

for any $n \in \mathbb{N}$. By Lemma 4.27, we have that for any $n \in \mathbb{N}$,

$$f_* \mathcal{O}_Y(K) = f_* \mathcal{O}_Y(K - 2F) : I^2 \subset f_* \mathcal{O}_Y(2^{n-1}K - 2F) : I^2 = f_* \mathcal{O}_Y(2^{n-1}K).$$

This implies that K is effective. Since $\mathcal{J}^{\omega}(A) = f_*\mathcal{O}_Y(K) = A$, A has a rational singularity. $\qquad\square$

In order that the strong subadditivity theorem of ω-multiplier ideal holds, nonsingularness is necessary.

PROPOSITION 4.29. *Let $(A, \mathfrak{m})$ be a two-dimensional normal local ring. Then $X = \operatorname{Spec} A$ is regular if and only if the strong subadditivity theorem of ω-multiplier ideals holds, that is, for any two nonzero ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$ and any rational numbers $c, d > 0$,*

$$\mathcal{J}^{\omega}(X, \mathfrak{a}^c \mathfrak{b}^d) \subset \mathcal{J}^{\omega}(X, \mathfrak{a}^c) \mathcal{J}^{\omega}(X, \mathfrak{b}^d).$$

Proof. If A is regular, then the strong subadditivity theorem holds (see [4]). Hence we will show the converse implication. In order for the strong subadditivity theorem to hold, by Theorem 4.26, it is necessary that A is a rational singularity. Assume that A is not regular. Let $f : Y \to X$ be the minimal resolution, and F be the fundamental cycle of f.

We assume that the exceptional locus of f is irreducible. Then F is the f-exceptional prime divisor. Let $g : Z \to Y$ be the blow-up at a closed point of F, and $h : Z \to X$ be the composite morphism of f and g. We denote by E_1 the strict transform of F, and by E_2 the exceptional divisor of g. Let $n = -E_1 \cdot E_1$, $C = (n-1)E_1 + 2nE_2$, and $K = K_{Z/Y} = E_2$. Then C and $(n-1)E_1 + (2n-1)E_2$ are h-anti-nef since $n = -E_1 \cdot E_1 = -F \cdot F + 1 \geq 3$. Since $E_1 + E_2$ is the fundamental cycle of h, we have

$$\operatorname{an}_h\left(\left\lfloor \frac{1}{n}C - K \right\rfloor\right) = E_1 + E_2,$$
$$\operatorname{an}_h(C - K) = (n-1)E_1 + (2n-1)E_2.$$

These imply that

$$h_*\mathcal{O}_Z(-\operatorname{an}_h(C - K)) \not\subset \left(h_*\mathcal{O}_Z\left(-\operatorname{an}_h\left(\left\lfloor \frac{1}{n}C - K \right\rfloor\right)\right)\right)^n$$

by Lemma 4.25. Therefore, denoting the ideal $I = h_*\mathcal{O}_Z(-C)$, we have $\mathcal{J}^{\omega}(X, I) \not\subset \mathcal{J}^{\omega}(X, I^{1/n})^n$ by Theorem 4.20. Thus the strong subadditivity theorem does not hold on A.

We assume that the exceptional locus of f is reducible. Let E be an f-exceptional prime divisor such that $F \cdot E < 0$. Then there exists $n \in \mathbb{N}$ such that $nF - E$ is f-anti-nef. Since F is the fundamental cycle of f, we have

$$\operatorname{an}_f\left(\left\lfloor \frac{1}{n}(nF - E) \right\rfloor\right) = F,$$
$$\operatorname{an}_f(nF - E) = nF - E.$$

These imply that

$$f_*\mathcal{O}_Y(-\operatorname{an}_f(nF - E)) \not\subset \left(f_*\mathcal{O}_Y\left(-\operatorname{an}_f\left(\left\lfloor \frac{1}{n}(nF - E) \right\rfloor\right)\right)\right)^n$$

by Lemma 4.25. Therefore, denoting the ideal $I = f_* \mathcal{O}_Y(-nF + E)$, we have $\mathcal{J}^\omega(X, I) \not\subset \mathcal{J}^\omega(X, I^{1/n})^n$ by Theorem 4.20. Thus the strong subadditivity theorem does not hold on A.

According to the previous discussion, if A is not regular, then the strong subadditivity theorem does not hold on A. □

REMARK 4.30. In the higher-dimensional case, we have a counterexample to Theorem 4.26.

Takagi and Watanabe [30] gave the following counterexample to the subadditivity of multiplier ideals in a three-dimensional hypersurface local ring. Since the ring is Gorenstein, the multiplier ideals are ω-multiplier ideals by Proposition 3.10.

EXAMPLE 4.31. Let $A = (\mathbb{C}[X, Y, Z, W]/(X^2 + Y^4 + Z^4 + W^5))_{(X,Y,Z,W)}$ and $\mathfrak{m} = (x, y, z, w)$, where x, y, z, w are the images of X, Y, Z, W in A. Then A is a Gorenstein canonical singularity but not a terminal singularity. Therefore A is a rational singularity, $\mathcal{J}^\omega(\mathfrak{m}) = \mathfrak{m}$, and $\overline{\mathfrak{m}^2} \subset \mathcal{J}^\omega(\mathfrak{m}^2)$. Since $x^2 \in \mathfrak{m}^4$, we have $x \in \overline{\mathfrak{m}^2}$. Hence $x \in \mathcal{J}^\omega(\mathfrak{m}^2)$ and $x \notin \mathcal{J}^\omega(\mathfrak{m})\mathcal{J}^\omega(\mathfrak{m})$. Thus $\mathcal{J}^\omega(\mathfrak{m}^2) \not\subset \mathcal{J}^\omega(\mathfrak{m})\mathcal{J}^\omega(\mathfrak{m})$.

4.3. Integrally Closed Ideals on Surface with a Rational Singularity

In this section, we show that all integrally closed ideals on surface with a rational singularity are ω-multiplier ideals.

THEOREM 4.32. *Let $(A, \mathfrak{m})$ be a two-dimensional normal local ring. Suppose $X = \operatorname{Spec} A$ has a rational singularity. Then every integrally closed ideal is an ω-multiplier ideal.*

Favre, Jonsson, Lipman, and Watanabe showed that all integrally closed ideals on regular surfaces are multiplier ideals (see [8] and [24]). Our result is a generalization of this theorem since ω-multiplier ideals of regular schemes are multiplier ideals.

DEFINITION 4.33. Let $(A, \mathfrak{m})$ be a two-dimensional normal local ring. Let $f : Y \to X = \operatorname{Spec} A$ be a resolution of singularities such that $f^{-1}(\mathfrak{m})$ is a simple normal crossing divisor. Let $E_1, \ldots, E_u$ be the irreducible components of $f^{-1}(\mathfrak{m})$. $\check{E}_i$ is defined to be an effective exceptional $\mathbb{Q}$-divisor such that

$$\check{E}_i \cdot E_j = \begin{cases} -1 & (i = j), \\ 0 & (i \neq j). \end{cases}$$

DEFINITION 4.34. Let Y be a two-dimensional regular scheme, and $x^{(i)}$ be a closed point of Y. A generic sequence of n-blow-ups over $x^{(i)}$ is

$$Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 = Y,$$

where f_1 is the blow-up of $Y_0 = Y$ at $x_1 := x^{(i)}$, and $f_k : Y_k \to Y_{k-1}$ is the blow-up of Y_{k-1} at a general closed point x_k of $(f_{k-1})^{-1}(x_{k-1})$ for $k = 2, \ldots, n$. Let $f : Y_n \to Y$ be the composition $f_1 \circ \cdots \circ f_n$. Let $E(x^{(i)}, k)$, $k = 1, \ldots, n$, be the strict transforms of the n new f-exceptional divisors created by blowing-ups $f_1, \ldots, f_n$, respectively.

Lipman and Watanabe [24] stated the following:

REMARK 4.35. $f^{-1}(x^{(i)})$ is a chain of n integral curves $E(x^{(i)}, 1), \ldots, E(x^{(i)}, n)$ such that for $0 < k < n$,

$$E(x^{(i)}, k) \cdot E(x^{(i)}, k+1) = 1,$$
$$E(x^{(i)}, k) \cdot E(x^{(i)}, k) = -2,$$

whereas

$$E(x^{(i)}, n) \cdot E(x^{(i)}, n) = -1;$$

and if $|k' - k| > 1$, then

$$E(x^{(i)}, k') \cdot E(x^{(i)}, k) = 0.$$

LEMMA 4.36. *Let Y be a two-dimensional regular scheme, and $x^{(i)}$ be a closed point of Y. Let $f : Y_n \to Y$ be a generic sequence of n-blow-ups over $x^{(i)}$. As in Definition 4.34, denote by $E(x^{(i)}, 1), \ldots, E(x^{(i)}, n)$ the strict transforms of the n exceptional divisors over $x^{(i)}$. Then*

$$K_f := K_{Y_n} - f^*(K_Y) = \sum_{k=1}^{n} k E(x^{(i)}, k).$$

Proof. We will show the lemma by induction of n. When $n = 1$, we have $K_f := K_{Y_1} - f^*(K_Y) = E(x^{(i)}, 1)$. By the induction hypothesis, $K_{Y_{n-1}/Y} = \sum_{k=1}^{n-1} k E(x^{(i)}, k)$. Therefore

$$K_{Y_n} - f^*(K_Y) = K_{Y_n/Y_{n-1}} + f_n^* K_{Y_{n-1}/Y}$$
$$= n E(x^{(i)}, n) + \sum_{k=1}^{n-1} k E(x^{(i)}, k)$$
$$= \sum_{k=1}^{n} k E(x^{(i)}, k). \qquad \square$$

LEMMA 4.37. *Let Y be a two-dimensional regular scheme, and $x^{(i)}$ be a closed point of Y. Let $f : Y_n \to Y$ be a generic sequence of n-blow-ups over $x^{(i)}$. As in Definition 4.34, denote by $E(x^{(i)}, 1), \ldots, E(x^{(i)}, n)$ the strict transforms of the n exceptional divisors over $x^{(i)}$. Let $K_f = K_{Y_n} - f^*(K_Y)$. Then*

$$K_f \cdot E(x^{(i)}, k) = \begin{cases} -1 & (k = n), \\ 0 & (k \neq n). \end{cases}$$

Proof. By Lemma 4.36, $K_f := K_{Y_n} - f^*(K_Y) = \sum_{k=1}^{n} k E(x^{(i)}, k)$. For $k \neq n$, by Remark 4.35,

$$
\begin{aligned}
&K_f \cdot E(x^{(i)}, k) \\
&= ((k-1)E(x^{(i)}, k-1) + k E(x^{(i)}, k) + (k+1)E(x^{(i)}, k+1)) \cdot E(x^{(i)}, k) \\
&= (k-1) - 2k + (k+1) = 0.
\end{aligned}
$$

By Remark 4.35,

$$
\begin{aligned}
K_f \cdot E(x^{(i)}, n) &= ((n-1)E(x^{(i)}, n-1) + n E(x^{(i)}, n)) \cdot E(x^{(i)}, n) \\
&= (n-1) - n = -1. \qquad \square
\end{aligned}
$$

Tucker [31] showed the following:

LEMMA 4.38 ([31, Lemma 2.2]). *Let $(A, \mathfrak{m})$ be a two-dimensional normal local ring. Let $f : Y \to X = \operatorname{Spec} A$ be a resolution of singularities such that $f^{-1}(\mathfrak{m})$ is a simple normal crossing divisor. Let $E_1, \ldots, E_u$ be the irreducible components of $f^{-1}(\mathfrak{m})$. Suppose $x^{(i)}$ is a closed point of E_i with $x^{(i)} \notin E_j$ for $j \neq i$. Let $g : Y_n \to Y$ be a generic sequence of n-blow-ups over $x^{(i)}$. As in Definition 4.34, denote by $E(x^{(i)}, 1), \ldots, E(x^{(i)}, n)$ the strict transforms of the n exceptional divisors over $x^{(i)}$ and by $E(i)$ the strict transforms of $E_1, \ldots, E_u$ on Y_n. Then*

(1) $\check{E}(i) \leq \check{E}(x^{(i)}, 1) \leq \cdots \leq \check{E}(x^{(i)}, n)$.

(2) *Suppose D is an integral $f \circ g$-anti-nef divisor on Y_n such that E_i is the unique component of $g_* D$ containing $x^{(i)}$. Then*

$$
\operatorname{ord}_{E(i)} D \leq \operatorname{ord}_{E(x^{(i)}, 1)} D \leq \cdots \leq \operatorname{ord}_{E(x^{(i)}, n)} D.
$$

Further, $\operatorname{ord}_{E(i)} D < \operatorname{ord}_{E(x^{(i)}, n)} D$ if and only if

$$
\sum_{k=1}^{n} (-D \cdot E(x^{(i)}, k)) \check{E}(x^{(i)}, k) \geq \check{E}(i).
$$

Tuker [31] showed that all integrally closed ideals on log terminal surfaces are multiplier ideals. Our proof is just an imitation of the proof of the Theorem 1.1 of [31].

We begin the proof of Theorem 4.32.

Proof. Let $I \subset \mathcal{O}_X$ be an integrally closed ideal. We will construct an ideal $\mathfrak{a}$ and $c \in \mathbb{Q}_{>0}$ such that $I = \mathcal{J}^{\omega}(X, \mathfrak{a}^c)$. Let $f : Y \to X$ be a log resolution of $\mathfrak{j}_X \cdot \mathfrak{d}_X \cdot I$ with exceptional divisors $E_1, \ldots, E_u$ such that $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, $\mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_{Y/X})$ and $I \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_0)$. Let $K = \widehat{K}_{Y/X} - J_{Y/X} + D_{Y/X}$. Write

$$
K = \sum_{i=1}^{u} b_i E_i,
$$

$$
F_0 = (f^{-1})_* f_*(F_0) + \sum_{i=1}^{u} a_i E_i.
$$

Note that $b_i \geq 0$ since X has a rational singularity. Let $0 < \varepsilon < 1/2$ such that $\lfloor \varepsilon (f^{-1})_* f_*(F_0) \rfloor = 0$ and $\varepsilon(a_i + 1) < 1 + b_i$ for $i = 1, \ldots, u$. Let $n_i := \lfloor (1 + b_i)/\varepsilon - (a_i + 1) \rfloor > 0$ and $e_i := (-F_0 \cdot E_i)$. Choose e_i distinct closed points $x_1^{(i)}, \ldots, x_{e_i}^{(i)}$ on E_i such that $x_j^{(i)} \notin \mathrm{Supp}((f^{-1})_* f_*(F_0))$ and $x_j^{(i)} \notin E_l$ for $l \neq i$. Denote by $g : Z \to Y$ the composition of a generic sequence of n_i-blow-ups over each of the points $x_j^{(i)}$ for $j = 1, \ldots, e_i$ and $i = 1, \ldots, u$. As in Definition 4.34, denote by $E(x_j^{(i)}, 1), \ldots, E(x_j^{(i)}, n_i)$ the strict transforms of the n_i exceptional divisors over $x_j^{(i)}$ and by $E(1), \ldots, E(u)$ the strict transforms of $E_1, \ldots, E_u$.

Let $h := f \circ g$ and $F = g^*(F_0)$. By Lemma 4.36 and Lemma 4.37,

$$K_g := K_Z - g^*(K_Y) = \sum_{i=1}^{u} \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} k E(x_j^{(i)}, k)$$

and

$$K_g \cdot E(i) = e_i,$$

$$K_g \cdot E(x_j^{(i)}, k) = \begin{cases} -1 & (k = n_i), \\ 0 & (k \neq n_i). \end{cases}$$

Then $F + K_g$ is h-anti-nef since

$$F \cdot E(i) = F_0 \cdot E_i = -e_i, \qquad F \cdot E(x_j^{(i)}, k) = 0.$$

Let $K' = K_g + g^*(K)$, $\mathfrak{a} = h_* \mathcal{O}_Z(-(F + K_g))$, and $c = 1 + \varepsilon$. Then, by Theorem 4.2, we have $\mathfrak{a} \mathcal{O}_Z = \mathcal{O}_Z(-(F + K_g))$.

We will show that $I = \mathcal{J}^\omega(X, \mathfrak{a}^c) = h_* \mathcal{O}_Z(-F)$. By Theorem 3.3,

$$\mathcal{J}^\omega(X, \mathfrak{a}^c) = h_* \mathcal{O}_Z(-\lfloor c(F + K_g) - K' \rfloor).$$

Therefore it suffices to show that

$$F' := \mathrm{an}_h(\lfloor c(F + K_g) - K' \rfloor) = F$$

by Lemma 4.3.

CLAIM 1. *We have $F' \leq F$ and $h_* F' = h_* F$. In addition, for $i = 1, \ldots, u$ and $j = 1, \ldots, e_i$,*

$$\mathrm{ord}_{E(x_j^{(i)}, n_i)}(F') = \mathrm{ord}_{E(x_j^{(i)}, n_i)}(F) = \mathrm{ord}_{E(i)}(F).$$

Proof. By the definition of a generic sequence of blow-up, we have

$$\mathrm{ord}_{E(x_j^{(i)}, n_i)}(F) = \mathrm{ord}_{E(i)}(F).$$

Since $F' = \mathrm{an}_h(\lfloor c(F + K_g) - K' \rfloor)$ and F are h-anti-nef, it suffices to show that

$$\lfloor c(F + K_g) - K' \rfloor \leq F,$$

$$h_* \lfloor c(F + K_g) - K' \rfloor = h_* F,$$

$$\mathrm{ord}_{E(x_j^{(i)}, n_i)}(\lfloor c(F + K_g) - K' \rfloor) = \mathrm{ord}_{E(x_j^{(i)}, n_i)}(F).$$

We have

$$\lfloor c(F + K_g) - K' \rfloor = F + \lfloor \varepsilon(F + K_g) - g^*K \rfloor.$$

Since $\lfloor \varepsilon(f^{-1})_* f_*(F_0) \rfloor = 0$, it follows that $h_* \lfloor c(F + K_g) - K' \rfloor = h_* F$. Consider the coefficients of $\varepsilon(F + K_g) - g^*K$. We have

$$\mathrm{ord}_{E(i)}(\varepsilon(F + K_g) - g^*K) = \varepsilon a_i - b_i < 1,$$
$$\mathrm{ord}_{E(x_j^{(i)}, k)}(\varepsilon(F + K_g) - g^*K) = \varepsilon(a_i + k) - b_i.$$

Since $0 < \varepsilon < 1/2$ and $(1 + b_i)/\varepsilon - (a_i + 1) - 1 < n_i \le (1 + b_i)/\varepsilon - (a_i + 1)$, we have

$$0 < 1 - 2\varepsilon < \varepsilon(a_i + n_i) - b_i \le 1 - \varepsilon < 1.$$

Therefore we have

$$\mathrm{ord}_{E(x_j^{(i)}, k)} \lfloor \varepsilon(F + K_g) - g^*K \rfloor \le 0,$$
$$\mathrm{ord}_{E(x_j^{(i)}, n_i)} \lfloor \varepsilon(F + K_g) - g^*K \rfloor = 0.$$

Thus we have $F' \le F$ and

$$\mathrm{ord}_{E(x_j^{(i)}, n_i)}(F') = \mathrm{ord}_{E(x_j^{(i)}, n_i)}(F). \qquad \square$$

CLAIM 2. *For each* $i = 1, \ldots, u$,

$$(-F' \cdot E(i))\check{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(x_j^{(i)}, k))\check{E}(x_j^{(i)}, k) \ge (-F \cdot E(i))\check{E}(i).$$

Proof. (1) We assume that $\mathrm{ord}_{E(i)} F' = \mathrm{ord}_{E(i)} F$.

We have $F' \cdot E(i) \le F \cdot E(i)$ since $F' \le F$ by Claim 1. Since $\check{E}(i)$ and $\check{E}(x_j^{(i)}, k)$ are effective and F' is h-anti-nef, we have

$$(-F' \cdot E(i))\check{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(x_j^{(i)}, k))\check{E}(x_j^{(i)}, k) \ge (-F \cdot E(i))\check{E}(i).$$

(2) We assume that $\mathrm{ord}_{E(i)} F' < \mathrm{ord}_{E(i)} F = \mathrm{ord}_{E(x_j^{(i)}, n_i)} F'$.

Then, for each $j = 1, \ldots, e_i$, we have

$$\sum_{k=1}^{n_i} (-F' \cdot E(x_j^{(i)}, k))\check{E}(x_j^{(i)}, k) \ge \check{E}(i)$$

by Lemma 4.38. Therefore we have

$$(-F' \cdot E(i))\check{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(x_j^{(i)}, k))\check{E}(x_j^{(i)}, k)$$

$$\ge \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(x_j^{(i)}, k))\check{E}(x_j^{(i)}, k)$$

$$\ge e_i \check{E}(i) = (-F \cdot E(i))\check{E}(i). \qquad \square$$

Next, we will prove that $F' \geq F$. By the two claims, we have

$$F' = h^* h_* F' + \sum_{i=1}^{u} \left((-F' \cdot E(i)) \check{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(x_j^{(i)}, k)) \check{E}(x_j^{(i)}, k) \right)$$

$$\geq h^* h_* F + \sum_{i=1}^{u} (-F \cdot E(i)) \check{E}(i) = F.$$

Therefore we have $F = F'$ by Claim 1. Thus $I = \mathcal{J}^{\omega}(X, \mathfrak{a}^c)$. $\qquad \square$

REMARK 4.39. In higher-dimensional case, we have counterexamples to Theorem 4.32 (see [20] and [21]).

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