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Reduction of Local Uniformization to the Case of Rank One Valuations for Rings with Zero Divisors

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Abstract. This is a continuation of our previous paper, where it was proved that to obtain local uniformization for valuations centered on local domains, it suffices to prove it for rank one valuations. In this paper, we extend this result to the case of valuations centered on rings that are not necessarily integral domains and may even contain nilpotents.

1. Introduction

For an algebraic variety X over a field k, the problem of resolution of singularities is whether there exists a proper birational morphism $X' \longrightarrow X$ such that X' is regular. The problem of local uniformization can be seen as the local version of resolution of singularities for an algebraic variety. For a valuation ν of $k(X)$ having a center on X, the local uniformization problem asks whether there exists a proper birational morphism $X' \longrightarrow X$ such that the center of ν on X' is regular. This problem was introduced by Zariski in the 1940s as an important step to prove resolution of singularities. Zariski's approach consists in proving first that every valuation having a center on the given algebraic variety admits local uniformization. Then these local solutions have to be glued to obtain a global resolution of all singularities.

Zariski [10] succeeded in proving local uniformization for valuations centered on algebraic varieties over a field of characteristic zero. He used this to prove resolution of singularities for algebraic surfaces and threefolds over a field of characteristic zero (see [11]). Abhyankar [1] proved that local uniformization can be obtained for valuations centered on algebraic surfaces in any characteristic and used this fact to prove resolution of singularities for surfaces (see [2] and [3]). He also proved local uniformization and resolution of singularities for threefolds over fields of characteristic other than 2, 3, and 5 (see [4]). Very recently, Cossart and Piltant [5; 6] proved resolution of singularities (and, in particular, local uniformization) for threefolds over any field of positive characteristic and in the arithmetic case. They proved it using the approach of Zariski. However, the problem of local uniformization remains open for valuations centered on algebraic varieties of dimension greater than three over fields of positive characteristic.

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Since local uniformization is a local problem, we can work with local rings instead of algebraic varieties or schemes. A valuation ν centered on a local integral domain R is said to admit local uniformization if there exists local ring $R^{(1)}$ dominated by $\mathcal{O}_\nu$ and dominating R such that $R^{(1)}$ is regular. Let $\mathcal{N}$ be the category of all Noetherian local domains, and $\mathcal{M} \subseteq \mathcal{N}$ be a subcategory of $\mathcal{N}$ that is closed under taking homomorphic images and localizing any finitely generated birational extension at a prime ideal. We want to know for which subcategories $\mathcal{M}$ with these properties, all valuations centered on objects of $\mathcal{M}$ admit local uniformization. In Section 7.8 of [7], Grothendieck proved that any category of schemes, closed under passing to closed subschemes and finite radical extensions, in which resolution of singularities holds, is a subcategory of quasi-excellent schemes (it is known that the category of quasi-excellent schemes is closed under all the operations mentioned). He conjectured (see Remark 7.9.6 of [7]) that resolution of singularities holds in the most general possible context of quasi-excellent schemes. Translated into our local situation, this conjecture says that any valuation centered in a quasi-excellent local Noetherian domain admits local uniformization. For a discussion on quasi-excellent and excellent local rings, see Section 7.8 of [7]. However, this conjecture is widely open.

In most of the successful cases, including those mentioned before, local uniformization was first proved for rank one valuations. Then the general case was reduced to this a priori weaker one. In [9], we prove that this reduction works under very general assumptions. Namely, we consider a subcategory $\mathcal{M}$ of the category of all Noetherian local integral domains that are closed under taking homomorphic images and localizing any finitely generated birational extension at a prime ideal. The main result of [9] is that if every rank one valuation centered on an object of $\mathcal{M}$ admits local uniformization, then all the valuations centered on objects of $\mathcal{M}$ admit local uniformization. The main goal of this paper is to extend this result to rings that are not necessarily integral domains and, in particular, may contain nilpotent elements. The importance of nonintegral and nonreduced schemes in modern algebraic geometry is well known. Even if we were only interested in reduced schemes to start with, we are led to consider nonreduced ones since they are produced by natural constructions, for example, in deformation theory.

The motivation behind our work is the following: when trying to prove some result in local uniformization/resolution of singularities, it is sometimes convenient to expand the category that we are working on. For instance, if we are expecting to prove local uniformization by induction on the dimension of the varieties and if at some point we have to use deformation of the ring, then we have to prove local uniformization for every valuation of smaller rank in the category of "deformed" rings. Hence, it is desirable to have a reduction of local uniformization to rank one valuations to a category of rings as large as possible.

Even if deforming singularities does not intervene in the resolution process, it is desirable to prove simultaneous resolution in families of varieties. There are situations when flat families with nonreduced fibers appear by natural constructions, even when at the start the scheme we were deforming was reduced. It would be nice to be able to extend the notion of simultaneous resolution to this type of situation. It is therefore natural to try to state and prove resolution of singularities and local uniformization for nonreduced schemes.

Most of the recent results on resolution, such as the Cossart–Piltant theorem in dimension three, are stated and proved only for reduced schemes, so local uniformization in our sense does not formally follow from them. However, it should not be too hard to extend the existing results to the nonreduced schemes. We consider the present paper to be a step in this direction.

If R is not reduced, then we cannot expect, in general, to make $R^{(1)}$ regular by blowings up. The natural extension of our result to the case of rings with zero divisors is to require $(R^{(1)})_{\mathrm{red}}$ to be regular and $N_{(1)}^n/N_{(1)}^{n+1}$ to be an $(R^{(1)})_{\mathrm{red}}$-free module for every $n \in \mathbb{N}$ (here $N_{(1)}$ denotes the nilradical of $R^{(1)}$). For a more detailed motivation of our main theorem, see Section 2 (particularly, Remarks 2.4 and 2.5). Let $\mathcal{N}$ be the category of all Noetherian local rings, and $\mathcal{M} \subseteq \mathcal{N}$ a subcategory of $\mathcal{N}$ that is closed under taking homomorphic images and localizing any finitely generated birational extension at a prime ideal. Our main result is the following:

Theorem 1.1. *Assume that for every Noetherian local ring R in $Ob(\mathcal{M})$, every rank one valuation centered on R admits local uniformization. Then all the valuations centered on objects of $\mathcal{M}$ admit local uniformization.*

The proof of Theorem 1.1 consists of three main steps. The first step is to prove that given a local ring R and a valuation ν centered in R, there exists a local blowing up $\pi : R \longrightarrow R^{(1)}$ such that $R^{(1)}$ has only one associated prime ideal. The local blowing up π is constructed by first constructing a blowing up $X \longrightarrow \operatorname{Spec} R$ such that X has no embedded components and is locally irreducible as a topological space and then setting $R^{(1)} = \mathcal{O}_{X,\xi}$ where ξ is the center of ν on X. Then we consider a decomposition $\nu = \nu_1 \circ \nu_2$ of ν such that $\mathrm{rk}(\nu_1) < \mathrm{rk}(\nu)$ and $\mathrm{rk}(\nu_2) < \mathrm{rk}(\nu)$. Using induction, we can assume that both ν_1 and ν_2 admit local uniformization. The second main step consists in using this to prove that there exists a local blowing up $R^{(1)} \longrightarrow R^{(2)}$ such that $(R^{(2)})_{\mathrm{red}}$ is regular. The third and final step is to prove that there exists a further local blowing up $R^{(2)} \longrightarrow R^{(3)}$ such that $(R^{(3)})_{\mathrm{red}}$ is regular and $N_{(3)}^n/N_{(3)}^{n+1}$ is an $(R^{(3)})_{\mathrm{red}}$-free module for every $n \in \mathbb{N}$ (here $N_{(3)}$ denotes the nilradical of $R^{(3)}$).

This paper is organized as follows. In Section 2 we present the basic definitions and results that will be used in the sequel. Sections 3, 4, and 5 are dedicated to prove the results related to the first, second, and third steps, respectively. In the last section we present a proof of our main theorem.

2. Preliminaries

Let R be a Noetherian commutative ring with unity, and Γ an ordered Abelian group. Set $\Gamma_\infty := \Gamma \cup \{\infty\}$ and extend the addition and order from Γ to Γ_∞ as usual.

DEFINITION 2.1. A valuation ν on R is a mapping $\nu : R \longrightarrow \Gamma_\infty$ with the following properties:

(V1) $\nu(ab) = \nu(a) + \nu(b)$ for every $a, b \in R$;
(V2) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ for every $a, b \in R$;
(V3) $\nu(1) = 0$ and $\nu(0) = \infty$;
(V4) The **support of** ν, which is defined by $\mathrm{supp}(\nu) := \{a \in R \mid \nu(a) = \infty\}$, is a minimal prime ideal of R.

Take a multiplicative system S of R such that $\mathrm{supp}(\nu) \subseteq R \setminus S$. Then the extension (which we call again ν) of ν to R_S given by $\nu(a/s) := \nu(a) - \nu(s)$ is again a valuation. From now on, we will freely make such extensions of ν to R_S without mentioning it explicitly.

A valuation ν on R is said to have a center if $\nu(a) \geq 0$ for every $a \in R$. In this case the **center** of ν on R is defined by $\mathfrak{C}_\nu(R) := \{a \in R \mid \nu(a) > 0\}$. Moreover, if R is a local ring with unique maximal ideal $\mathfrak{m}$ (in which case we say "the local ring $(R, \mathfrak{m})$"), then a valuation ν on R is said to be **centered** at R if $\nu(a) \geq 0$ for every $a \in R$ and $\nu(a) > 0$ for every $a \in \mathfrak{m}$. We observe that if ν is a valuation having a center on R, then ν is centered on $R_{\mathfrak{C}_\nu(R)}$. The **value group of** ν, denoted by νR, is defined as the subgroup of Γ generated by $\{\nu(a) \mid a \in R\}$. The **rank of** ν is the number of proper convex subgroups of νR.

Let us denote the **nilradical** of R by N:

$$N = \mathrm{Nil}(R) := \{a \in R \mid a^l = 0 \text{ for some } l \in \mathbb{N}\}.$$

For $b \in R \setminus N$, we consider the canonical map $\Phi : R \longrightarrow R_b$ given by $\Phi(a) = a/1$. Let

$$J(b) := \ker \Phi = \bigcup_{i=1}^{\infty} \mathrm{ann}_R(b^i). \tag{1}$$

We have a natural embedding $R/J(b) \subseteq R_b$.

Assume, in addition, that $b \in R \setminus \mathrm{supp}(\nu)$. Take $a_1, \ldots, a_r \in R$ such that

$$\nu(a_i) \geq \nu(b) \quad \text{for each } i,\ 1 \leq i \leq r.$$

Consider the subring $R' := R/J(b)[a_1/b, \ldots, a_r/b]$ of R_b. Then the restriction of ν to R' has a center $\mathfrak{C}_\nu(R')$ in R'. We set $R^{(1)} := R'_{\mathfrak{C}_\nu(R')}$.

DEFINITION 2.2. The canonical map $R \longrightarrow R^{(1)}$ is called the **local blowing up of** R with respect to ν along the ideal $(b, a_1, \ldots, a_r)$. For a valuation μ having a center on R, we say that $R \longrightarrow R^{(1)}$ is **μ-compatible** if $b \notin \mathfrak{C}_\mu(R)$ and $a_i \in \mathfrak{C}_\mu(R)$ for every i, $1 \leq i \leq r$.

LEMMA 2.3. *The composition of finitely many local blowings up is again a local blowing up. Moreover, if each of these local blowings up is μ-compatible, then their composition is again μ-compatible.*

Proof. It suffices to prove that, for two local blowings up $\pi : R \longrightarrow R^{(1)}$ and $\pi' : R^{(1)} \longrightarrow R^{(2)}$ with respect to ν, there exists a local blowing up $R \longrightarrow R^{(3)}$

with respect to ν such that $R^{(3)} \simeq R^{(2)}$. We write

$$R^{(1)} = R'_{\mathfrak{C}_\nu(R')} \quad \text{for } R' = R/J(b)[a_1/b, \dots, a_r/b]$$

for some $a_1, \dots, a_r, b \in R$ and

$$R^{(2)} = R'^{(1)}_{\mathfrak{C}_\nu(R'^{(1)})} \quad \text{for } R'^{(1)} = R^{(1)}/J(\beta)[\alpha_1/\beta, \dots, \alpha_s/\beta]$$

for some $\alpha_1, \dots, \alpha_s, \beta \in R^{(1)}$. Then there exist $a_{r+1}, \dots, a_{r+s}, b' \in R$ such that $\alpha_i/\beta = \pi(a_{r+i})/\pi(b')$ for each i, $1 \le i \le s$. Consider the local blowing up

$$R \longrightarrow R^{(3)}$$

given by

$$R^{(3)} = R''_{\mathfrak{C}_\nu(R'')}$$
$$\text{for } R'' = R/J(bb')[a_1b'/bb', \dots, a_rb'/bb', a_{r+1}b/bb', \dots, a_{r+s}b/bb'].$$

It is straightforward to prove that $R^{(2)} \simeq R^{(3)}$. □

In view of Lemma 2.3, we will freely use the fact that the composition of finitely many local blowings up is itself a local blowing up without mentioning it explicitly.

REMARK 2.4. If $b \in N$, then R_b is the zero ring (i.e., the one-element ring in which $0 = 1$). If $b \in \operatorname{supp}(\nu) \setminus N$, then the ring R_b and the homomorphism $\Phi : R \to R_b$ are well defined, but there do not exist a localization $R^{(1)}$ of R_b and a valuation $\nu^{(1)}$ centered in $R^{(1)}$ whose restriction to R is ν. This is why in the definition of local blowing up we limit ourselves to the case $b \notin \operatorname{supp}(\nu)$.

REMARK 2.5. The ring $R_{\operatorname{supp}(\nu)}$ has only one associated prime ideal. If $R_{\operatorname{supp}(\nu)}$ contains nonzero nilpotent elements, then so does every local blowing up $R^{(1)}$ of R. Therefore in this case there is no hope of making $R^{(1)}$ regular; the best we can ask is for $(R^{(1)})_{\text{red}}$ to be regular. Furthermore, it is both natural and possible to look for some form of "constant" or "uniform" behavior of the nilradical of $R^{(1)}$ along $\operatorname{Spec}(R^{(1)})_{\text{red}}$. The intuitive idea of uniform behavior of a module along a scheme in algebraic geometry is often embodied in the concept of flatness. Therefore, in order to define local uniformization for rings with nilpotents, it is natural to ask that the nilradical of $R^{(1)}$ become a flat $(R^{(1)})_{\text{red}}$-module. In fact, we can do slightly better and require not only the nilradical itself but all of the successive quotients of its powers to be flat. A finitely generated module over a local ring is flat if and only if it is free. These considerations motivate the following definitions.

DEFINITION 2.6. Assume that R_{red} is regular. We say that Spec R is **normally flat along** Spec R_{red} if N^n/N^{n+1} is an R_{red}-free module for every $n \in \mathbb{N}$.

Since R is a Noetherian ring, there exists $n_0 \in \mathbb{N}$ such that $N^n = (0)$ for every $n > n_0$. Hence, the condition in Definition 2.6 is equivalent to the freeness of each of the finite collection modules $N/N^2, \dots, N^{n_0}/N^{n_0+1} = N^{n_0}$.

DEFINITION 2.7. For a local ring R, a valuation ν centered on R is said to admit **local uniformization** if there exists a local blowing up $R \longrightarrow R^{(1)}$ with respect to ν such that $R^{(1)}_{\text{red}}$ is regular and Spec $R^{(1)}$ is normally flat along Spec $R^{(1)}_{\text{red}}$.

Let $\nu = \nu_1 \circ \nu_2$ be a fixed decomposition of ν. For simplicity of notation, we set $\mathfrak{p} := \mathfrak{C}_{\nu_1}(R)$, and for a local blowing up $R \longrightarrow R^{(1)}$, we set

$$\mathfrak{p}^{(1)} := \mathfrak{C}_{\nu_1}(R^{(1)}). \tag{2}$$

We need to guarantee that the main properties of $R_\mathfrak{p}$ and $R/\mathfrak{p}$ are preserved under ν_1-compatible local blowings up. More precisely, we have to prove the following:

PROPOSITION 2.8. *Let $\pi : R \longrightarrow R^{(1)}$ be a ν_1-compatible local blowing up. Then the canonical maps $R_\mathfrak{p} \longrightarrow R^{(1)}_{\mathfrak{p}^{(1)}}$ and $R/\mathfrak{p} \longrightarrow R^{(1)}/\mathfrak{p}^{(1)}$ induced by π are isomorphisms.*

To prove Proposition 2.8, we need the following basic lemma.

LEMMA 2.9. *Let S be a multiplicative system of R contained in $R \setminus \mathfrak{C}_\nu(R)$. Then the canonical map $\Phi : R_{\mathfrak{C}_\nu(R)} \longrightarrow (R_S)_{\mathfrak{C}_\nu(R_S)}$ given by $\Phi(a/b) = (a/1)/(b/1)$ is an isomorphism.*

Proof. For an element $(a/b)/(c/d) \in (R_S)_{\mathfrak{C}_\nu(R_S)}$, we have

$$\nu(b) = \nu(c) = \nu(d) = 0.$$

Consequently, $\nu(bc) = 0$ and $ad/bc \in R_{\mathfrak{C}_\nu(R)}$. Then

$$(a/b)/(c/d) = (ad/1)/(bc/1) = \Phi(ad/bc).$$

Suppose that $\Phi(a/b) = 0$. This means that there exists $c/d \in R_S \setminus \mathfrak{C}_\nu(R_S)$ such that $ac/d = 0$ in R_S. Thus, there exists $s \in S$ such that $sac = 0$. Moreover, since $c/d \notin \mathfrak{C}_\nu(R_S)$, we also have $c \notin \mathfrak{C}_\nu(R)$. This and the fact that $s \in S \subseteq R \setminus \mathfrak{C}_\nu(R)$ imply that $sc \notin \mathfrak{C}_\nu(R)$. Hence, $a/b = 0$ in $R_{\mathfrak{C}_\nu(R)}$, which is what we wanted to prove. □

Proof of Proposition 2.8. Applying Lemma 2.9 to R (with $S = \{1, b, b^2, \ldots\}$) and R' (with $S' = R' \setminus \mathfrak{C}_\nu(R')$) and the valuation ν_1, we obtain that the canonical maps $R_\mathfrak{p} \longrightarrow (R_b)_{\mathfrak{C}_{\nu_1}(R_b)}$ and $R'_{\mathfrak{C}_{\nu_1}(R')} \longrightarrow R^{(1)}_{\mathfrak{p}^{(1)}}$ are isomorphisms. Hence, to prove the first assertion, it suffices to show that the canonical map $(R_b)_{\mathfrak{C}_{\nu_1}(R_b)} \longleftarrow R'_{\mathfrak{C}_{\nu_1}(R')}$ is an isomorphism.

Since $R' \subseteq R_b$ and $\mathfrak{C}_{\nu_1}(R') = R' \cap \mathfrak{C}_{\nu_1}(R_b)$, we have that $R'_{\mathfrak{C}_{\nu_1}(R')} \longrightarrow (R_b)_{\mathfrak{C}_{\nu_1}(R_b)}$ is injective. On the other hand, any element $(a/b^n)/(c/b^m)$ in $(R_b)_{\mathfrak{C}_{\nu_1}(R_b)}$ can be written as $(ab^m/1)/(cb^n/1)$, which is the image of ab^m/cb^n. Hence the map

$$R'_{\mathfrak{C}_{\nu_1}(R')} \longrightarrow (R_b)_{\mathfrak{C}_{\nu_1}(R_b)}$$

is surjective and consequently an isomorphism.

Set $R_0 = R/J(b)$ and consider the induced map $R_0 \longrightarrow R^{(1)}$. Since the canonical map $R \longrightarrow R_0$ is surjective, to prove the surjectivity of $R \longrightarrow R^{(1)}/\mathfrak{p}^{(1)}$, it suffices to show that $R_0 \longrightarrow R^{(1)}/\mathfrak{p}^{(1)}$ is surjective. For an element $\alpha \in R^{(1)}$, we write $\alpha = p/q$ where $p = P(a_1/b, \ldots, a_r/b)$ and $q = Q(a_1/b, \ldots, a_r/b)$ for some

$$P(X_1, \ldots, X_r), Q(X_1, \ldots, X_r) \in R_0[X_1, \ldots, X_r].$$

Set $p_0 = P(0, \ldots, 0)$ and $q_0 = Q(0, \ldots, 0)$. Then

$$p_1 := p - p_0 = \sum_{i=1}^{r} a_i/b \cdot P_i(a_1/b, \ldots, a_r/b)$$

and

$$q_1 := q - q_0 = \sum_{i=1}^{r} a_i/b \cdot Q_i(a_1/b, \ldots, a_r/b)$$

for some $P_i, Q_i \in R_0[X_1, \ldots, X_r]$, $1 \le i \le r$. Since $\nu_1(a_i/b) > 0$, we obtain that $\nu_1(p_1) > 0$ and $\nu_1(q_1) > 0$. This implies that

$$\nu_1(q_0) = 0, \tag{3}$$

$$\nu_1(q_0 q) = 0, \tag{4}$$

and

$$\nu_1(q_0 p_1 - p_0 q_1) > 0. \tag{5}$$

Therefore,

$$p/q - p_0/q_0 = (q_0 p_1 - p_0 q_1)/q_0 q \in \mathfrak{p}^{(1)}.$$

It remains to prove that $p_0/q_0 \in R_0$. Since $\nu_1(q_1) > 0$, also $\nu(q_1) > 0$. Hence $\nu(q_0) = \nu(q - q_1) = 0$, and consequently q_0 is a unit in R_0. Therefore $p_0/q_0 \in R_0$.

To finish our proof, it suffices to show that the kernel of $R \longrightarrow R^{(1)}/\mathfrak{p}^{(1)}$ is $\mathfrak{p}$. This follows immediately from the definition of $\mathfrak{p}$ and $\mathfrak{p}^{(1)}$ as the centers of ν_1 on R and $R^{(1)}$. □

Lemmas 2.10 and 2.11 are generalizations of Lemma 2.18 and Corollary 2.20 of [9], respectively. The proofs presented there can be adapted to our more general case. We include the proofs here for convenience of the reader.

LEMMA 2.10. *For each local blowing up $R_\mathfrak{p} \longrightarrow \widetilde{R}^{(1)}$ with respect to ν_1, there exists a local blowing up $R \longrightarrow R^{(1)}$ with respect to ν such that $\widetilde{R}^{(1)} \simeq R^{(1)}_{\mathfrak{p}^{(1)}}$.*

Proof. We consider the local blowing up $R_\mathfrak{p} \longrightarrow \widetilde{R}^{(1)}$ given by

$$\widetilde{R}^{(1)} = \widetilde{R}'_{\mathfrak{C}_{\nu_1}(\widetilde{R}')} \quad \text{for } \widetilde{R}' = R_\mathfrak{p}/J(\beta)[\alpha_1/\beta, \ldots, \alpha_r/\beta].$$

Choose $a_1, \ldots, a_r, b \in R$ such that for each i, $1 \le i \le r$, we have $\Phi(a_i)/\Phi(b) = \alpha_i/\beta$ where $\Phi : R \longrightarrow R_\mathfrak{p}$ is the canonical map. If $\nu(a_i) < \nu(b)$ for some i, $1 \le i \le r$, then we have $\nu_1(\alpha_i) = \nu_1(\beta)$. Choose i so as to minimize the value $\nu(a_i)$ or, in other words, so that $\nu(a_i) \le \nu(a_j)$ for all $j \in \{1, \ldots, r\}$. Set

$$\widetilde{R}'' := R_\mathfrak{p}/J(\alpha_i)\left[\frac{\alpha_1}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{\beta}{\alpha_i}, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_r}{\alpha_i}\right].$$

Then $R^{(1)} \simeq \widetilde{R}''_{\mathfrak{C}_{\nu_1}(\widetilde{R}'')}$. Hence, after a suitable permutation of the set $\{a_1, \ldots, a_r, b\}$, we may assume that $\nu(a_i) \geq \nu(b)$ for every i, $1 \leq i \leq r$. Consider the local blowing up

$$R^{(1)} = R'_{\mathfrak{C}_\nu(R')} \quad \text{for } R' = R/J(b)[a_1/b, \ldots, a_r/b]$$

with respect to ν. It is straightforward to prove that $R^{(1)}_{\mathfrak{p}^{(1)}} \simeq \widetilde{R}^{(1)}$. □

LEMMA 2.11. *For each local blowing up $R/\mathfrak{p} \longrightarrow \overline{R}^{(1)}$ with respect to ν_2, there exists a local blowing up $R \longrightarrow R^{(1)}$ with respect to ν such that $R^{(1)}/\mathfrak{p}^{(1)} \simeq \overline{R}^{(1)}$ and $R_\mathfrak{p} \simeq R^{(1)}_{\mathfrak{p}^{(1)}}$.*

Proof. For an element $a \in R$, we denote its image under the canonical map

$$R \longrightarrow R/\mathfrak{p}$$

by $\overline{a}$. Then

$$\overline{R}^{(1)} = \overline{R}'_{\mathfrak{C}_{\nu_2}(\overline{R}')} \quad \text{with } \overline{R}' = (R/\mathfrak{p})/J(\overline{b})[\overline{a}_1/\overline{b}, \ldots, \overline{a}_r/\overline{b}]$$

for some $a_1, \ldots, a_r, b \in R \setminus \mathfrak{p}$. Since $\nu_2(\overline{a}_i) \geq \nu_2(\overline{b})$, we have $\nu(a_i) \geq \nu(b)$ for every i, $1 \leq i \leq r$. Then we can consider the local blowing up

$$R^{(1)} = R'_{\mathfrak{C}_\nu(R')} \quad \text{with } R' = R/J(b)[a_1/b, \ldots, a_r/b]$$

with respect to ν. It is again straightforward to prove that $R^{(1)}/\mathfrak{p}^{(1)} \simeq \overline{R}^{(1)}$ and $R_\mathfrak{p} \simeq R^{(1)}_{\mathfrak{p}^{(1)}}$. □

3. Associated Prime Ideals of R

Let R be a Noetherian ring. The main results of this section are the following.

PROPOSITION 3.1. *There exists a blowing up $X \longrightarrow \operatorname{Spec} R$ such that X has no embedded components and is locally irreducible as a topological space.*

Let R be a local Noetherian ring, and ν a valuation centered in R.

COROLLARY 3.2. *There exists a local blowing up $R \longrightarrow R^{(1)}$ with respect to ν such that* $\operatorname{Nil}(R^{(1)})$ *is the only associated prime of $R^{(1)}$.*

We start with the following lemma.

LEMMA 3.3. *Let R be a Noetherian ring, not necessarily local. Let $N = \operatorname{Nil}(R)$. Fix a finite collection of elements $a_1, \ldots, a_r \in R$, $b \in R \setminus N$, and $J(b)$ as in (1). Let*

$$R' = R/J(b)[a_1/b, \ldots, a_r/b] \subseteq R_b.$$

Then, for every $c' \in R'$, the ideal $\operatorname{ann}_{R'}(c')$ *can be written as* $\operatorname{ann}_{R'}(c/1)$ *for some $c \in R$. Moreover, if* $\operatorname{ann}_{R'}(c')$ *is prime, then* $\operatorname{ann}_R(b^{n_0}c)$ *is a prime ideal of R for some $n_0 \in \mathbb{N}$.*

Proof. Choose $c \in R$ such that $c' = c/b^l$ for some $l \in \mathbb{N}$. Fix $a' \in R'$ and write $a' = a/b^m$ for some $m \in \mathbb{N}$ and $a \in R$. Then we have

$$a' \in \operatorname{ann}_{R'}(c') \Longleftrightarrow acb^n = 0 \quad \text{for some } n \in \mathbb{N} \Longleftrightarrow a' \in \operatorname{ann}_{R'}(c/1).$$

Now assume that $\operatorname{ann}_{R'}(c')$ is prime and set $R_0 := R/J(b)$. Then

$$\operatorname{ann}_{R_0}(c/1) = \operatorname{ann}_{R'}(c') \cap R_0$$

is also prime. Moreover,

$$\pi^{-1}(\operatorname{ann}_{R_0}(c/1)) = \bigcup_{n=1}^{\infty} \operatorname{ann}_R(b^n c), \tag{6}$$

where $\pi : R \longrightarrow R/J(b)$ is the canonical epimorphism. Indeed,

$$\begin{aligned} a \in \pi^{-1}(\operatorname{ann}_{R_0}(c/1)) &\Longleftrightarrow ac/1 = 0 \text{ in } R_b \\ &\Longleftrightarrow b^n ac = 0 \text{ in } R \text{ for some } n \in \mathbb{N} \\ &\Longleftrightarrow a \in \textstyle\bigcup_{n=1}^{\infty} \operatorname{ann}_R(b^n c). \end{aligned}$$

Since R is Noetherian and

$$\operatorname{ann}_R(bc) \subseteq \operatorname{ann}_R(b^2 c) \subseteq \cdots \subseteq \operatorname{ann}_R(b^n c) \subseteq \cdots,$$

we have

$$\operatorname{ann}_R(b^{n_0} c) = \bigcup_{n=1}^{\infty} \operatorname{ann}_R(b^n c) \quad \text{for some } n_0 \in \mathbb{N}. \tag{7}$$

By (6) and (7) we conclude that $\operatorname{ann}_R(b^{n_0} c)$ is a prime ideal of R. □

COROLLARY 3.4. *Keep the notation of Lemma 3.3. The natural map*

$$\operatorname{Spec} R' \longrightarrow \operatorname{Spec} R$$

induces a bijection between $\operatorname{Ass}(R')$ *and the set of associated primes of* $\operatorname{Ass}(R)$ *not containing* b.

COROLLARY 3.5. *Keep the notation of Lemma 3.3. Let $S \subset R'$ be a multiplicative set. Put $R^{(1)} = R'_S$. If N is the only associated prime ideal of R, then* $\operatorname{Nil}(R^{(1)})$ *is the only associated prime ideal of* $R^{(1)}$.

Proof. By Theorem 6.2 of [8] we have $\operatorname{Ass}(R^{(1)}) = \operatorname{Ass}(R') \cap \operatorname{Spec}(R^{(1)})$. This and Lemma 3.3 guarantee that $|\operatorname{Ass}(R^{(1)})| \leq |\operatorname{Ass}(R)| = 1$. Consequently, $R^{(1)}$ has only one associated prime ideal, say $\mathfrak{q}$. The primary decomposition theorem now gives us that $\mathfrak{q} = \operatorname{Nil}(R^{(1)})$, which is what we wanted to prove. □

COROLLARY 3.6. *Assume, in addition, that R is local and ν is a valuation centered in R. For a local blowing up $R \longrightarrow R^{(1)}$, if N is the only associated prime ideal of R, then* $\operatorname{Nil}(R^{(1)})$ *is the only associated prime ideal of* $R^{(1)}$.

We will use Corollary 3.6 throughout this paper without always mentioning it explicitly.

Proof of Proposition 3.1. Let $(0)=\bigcap_{j=1}^{n} Q_j$ be a primary decomposition of (0) in R such that the ideals $P_j := \sqrt{Q_j}$ are pairwise distinct. Recalling the Q_j, if necessary, we may assume that there is $s \le n$ such that $P_1, \dots, P_s$ are the minimal primes of R, whereas $P_{s+1}, \dots, P_n$ are the embedded components of Spec R.

Let

$$J := \left(\prod_{j=s+1}^{n} P_j\right)\sum_{j=1}^{s}\left(\bigcap_{\substack{1\le i\le s\\ i\neq j}} Q_i\right).$$

Note that if $s=1$, then $\sum_{j=1}^{s}(\bigcap_{\substack{1\le i\le s\\ i\neq j}} Q_i)=R$ and $J=\prod_{j=2}^{n} P_j$.

Clearly $J \not\subset N$. Let X be the blowing up of Spec R along the ideal J. We claim that X satisfies the conclusion of the proposition. The scheme X is a union of finitely many affine charts of the form Spec R', where R' is as in Lemma 3.3. It suffices to check the conclusion of the proposition with X replaced by Spec R'. Moreover, the ideal $\sum_{j=1}^{s}(\bigcap_{\substack{1\le i\le s\\ i\neq j}} Q_i)$ has a set of generators each of which is contained in $(\bigcap_{\substack{1\le i\le s\\ i\neq j}} Q_i)$ for some $j \in \{1, \dots, s\}$, say, in $\bigcap_{2\le i\le s} Q_i$. Therefore, we may assume, in addition, that $b \in (\prod_{j=s+1}^{n} P_j)(\bigcap_{2\le i\le s} Q_i)$.

By Corollary 3.4, R' has a unique associated prime ideal Q', and the natural preimage of Q' is Q_1. This completes the proof. □

REMARK 3.7. If I is the only associated prime ideal of R, then for every $b \notin N$, we have $J(b)=(0)$. In this case we do not need to mention the ideal $J(b)$ in the definition of a local blowing up. We will use this throughout this paper without mentioning it explicitly.

4. Making R_{red} Regular

Let R be a local ring, and ν a valuation centered on R. Assume that $\nu=\nu_1\circ\nu_2$ and denote by $\mathfrak{p}$ the center of ν_1 on R. As usual, we denote by N the nilradical of R, and for a local blowing up $R \longrightarrow R^{(1)}$ (as in Definition 2.2), we denote the nilradical of $R^{(1)}$ by $N_{(1)}$. Also, as usual, $\mathfrak{p}^{(1)}$ is the center of ν_1 in $R^{(1)}$ as in (2). Assume that N is the only associated prime ideal of R. The main goal of this section is to prove the following proposition.

PROPOSITION 4.1. *Assume that $(R_{\mathfrak{p}})_{\text{red}}$ and $R/\mathfrak{p}$ are regular. Then there exists a ν_1-compatible local blowing up $R \longrightarrow R^{(1)}$ such that $(R^{(1)})_{\text{red}}$ is regular. Moreover, for every local blowing up $R^{(1)} \longrightarrow R^{(2)}$ along an ideal $(b, a_1, \dots, a_r)$ with $b \notin \mathfrak{p}^{(1)}$ and $a_1, \dots, a_r \in N_{(1)}$, we have that $(R^{(2)})_{\text{red}}$ is regular.*

To prove Proposition 4.1, we need a few lemmas.

LEMMA 4.2. *Assume that $(R_{\mathfrak{p}})_{\text{red}}$ is regular. Then there exists a ν_1-compatible local blowing up $R \longrightarrow R^{(1)}$ such that the $R^{(1)}/\mathfrak{p}^{(1)}$-module $\mathfrak{p}^{(1)}/((\mathfrak{p}^{(1)})^2+N_{(1)})$ is free. Moreover, if $y_1^{(1)}, \dots, y_r^{(1)}$ are elements of $\mathfrak{p}^{(1)}$ whose images in*

$\mathfrak{p}^{(1)}/((\mathfrak{p}^{(1)})^2 + N_{(1)})$ *form a basis of* $\mathfrak{p}^{(1)}/((\mathfrak{p}^{(1)})^2 + N_{(1)})$, *then their images in* $(R^{(1)}_{\mathfrak{p}^{(1)}})_{\mathrm{red}}$ *form a regular system of parameters of* $(R^{(1)}_{\mathfrak{p}^{(1)}})_{\mathrm{red}}$.

LEMMA 4.3. *Let* $\pi : R \longrightarrow R^{(1)}$ *be a local blowing up along an ideal* $(b, a_1, \ldots, a_r)$ *with* $b \notin \mathfrak{p}$ *and* $a_1, \ldots, a_r \in N$. *If* $\mathfrak{p}/(\mathfrak{p}^2 + N)$ *is a free* $R/\mathfrak{p}$*-module, then*

$$\mathfrak{p}^{(1)}/((\mathfrak{p}^{(1)})^2 + N_{(1)})$$

is a free $R^{(1)}/\mathfrak{p}^{(1)}$*-module.*

LEMMA 4.4. *Take* $y_1, \ldots, y_r \in \mathfrak{p}$ *and* $x_1, \ldots, x_t \in \mathfrak{m} \setminus \mathfrak{p}$ *whose images form a regular system of parameters of* $(R_{\mathfrak{p}})_{\mathrm{red}}$ *and* $R/\mathfrak{p}$, *respectively. If* $\mathfrak{p}/(\mathfrak{p}^2 + N)$ *is an* $R/\mathfrak{p}$*-free module with basis* $y_1 + (\mathfrak{p}^2 + N), \ldots, y_r + (\mathfrak{p}^2 + N)$, *then* R_{red} *is regular.*

Proof of Proposition 4.1, assuming Lemmas 4.2, 4.3, and 4.4. We apply Lemma 4.2 to obtain a ν_1-compatible local blowing up $R \longrightarrow R^{(1)}$ and $y_1^{(1)}, \ldots, y_r^{(1)} \in R^{(1)}$ such that their images in $\mathfrak{p}^{(1)}/((\mathfrak{p}^{(1)})^2 + N_{(1)})$ form an $(R^{(1)})_{\mathrm{red}}$- basis and their images in $(R^{(1)}_{\mathfrak{p}^{(1)}})_{\mathrm{red}}$ form a regular system of parameters. Moreover, by Proposition 2.8, $R^{(1)}/\mathfrak{p}^{(1)}$ is regular. Also, by Lemma 4.3 and Proposition 2.8, for every local blowing up $R^{(1)} \longrightarrow R^{(2)}$ along an ideal $(b, a_1, \ldots, a_r)$ with $b \notin \mathfrak{p}^{(1)}$ and $a_1, \ldots, a_r \in N_{(1)}$, the hypotheses of Lemma 4.4 are satisfied for $R^{(2)}$. Hence, we obtain that $(R^{(1)})_{\mathrm{red}}$ and $(R^{(2)})_{\mathrm{red}}$ are regular. □

We now proceed with the proofs of Lemmas 4.2, 4.3, and 4.4.

LEMMA 4.5. *Take generators* $y_1, \ldots, y_r, y_{r+1}, \ldots, y_{r+s}$ *of* $\mathfrak{p}$ *and* $b \notin \mathfrak{p}$. *Let*

$$\pi : R \longrightarrow R^{(1)}$$

be the local blowing up along the ideal $(b, y_1, \ldots, y_r)$. *Set*

$$y_i^{(1)} = \pi(y_i)/b \quad \text{for } 1 \le i \le r \quad \text{and} \quad y_{r+k}^{(1)} = \pi(y_{r+k}) \quad \text{for } 1 \le k \le s.$$

Then $\mathfrak{p}^{(1)}$ *is generated by* $y_1^{(1)}, \ldots, y_{r+s}^{(1)}$.

Proof. Obviously $y_i^{(1)} \in \mathfrak{p}^{(1)}$ for every i, $1 \le i \le r + s$. Take an element $p/q \in \mathfrak{p}^{(1)}$. This implies that $p = p(y_1/b, \ldots, y_r/b)$ for some $p(X_1, \ldots, X_r) \in R[X_1, \ldots, X_r]$ (see Remark 3.7). If we set $p_0 = p(0, \ldots, 0)$, then

$$p = p_0 + \frac{y_1}{b} p_1 + \cdots + \frac{y_r}{b} p_r \quad \text{for some } p_1, \ldots, p_r \in R'.$$

This implies that $p_0 \in \mathfrak{p}$. Hence, there exist $a_1, \ldots, a_{r+s} \in R$ such that $p_0 = a_1 y_1 + \cdots + a_{r+s} y_{r+s}$. Thus

$$\frac{p}{q} = \sum_{i=1}^{r} \frac{\pi(ba_i) + p_i}{q} y_i^{(1)} + \sum_{k=1}^{s} \frac{\pi(a_{r+k})}{q} y_{r+k}^{(1)} \in (y_1^{(1)}, \ldots, y_{r+s}^{(1)}) R^{(1)}.$$

This concludes our proof. □

Proof of Lemma 4.2. Since $(R_{\mathfrak{p}})_{\rm red}$ is regular, there are elements $y_1,\dots,y_r\in\mathfrak{p}$ such that their images in $(R_{\mathfrak{p}})_{\rm red}$ form a regular system of parameters. The first step is to reduce to the case where $y_1,\dots,y_r$ generate $\mathfrak{p}$.

Assume that $y_1,\dots,y_r$ do not generate $\mathfrak{p}$. Choose $y_{r+1},\dots,y_{r+s}\in\mathfrak{p}$ such that $y_1,\dots,y_r,y_{r+1},\dots,y_{r+s}$ generate $\mathfrak{p}$. For each k, $1\le k\le s$, we can find $b_k\in R\setminus\mathfrak{p}$, $b_{1k},\dots,b_{rk}\in R$, and $h_k\in(y_1,\dots,y_r)^2$ such that

$$b_ky_{r+k}+b_{1k}y_1+\cdots+b_{rk}y_r+h_k\in N.$$

Consider the local blowing up $\pi:R\longrightarrow R^{(1)}$ along $(b_1,y_1,\dots,y_r)$. It follows that

$$\pi(b_1)(y^{(1)}_{r+1}+\pi(b_{11})y^{(1)}_1+\cdots+\pi(b_{r1})y^{(1)}_r+h^{(1)}_1)\in N_{(1)}$$

and

$$\pi(b_k)y^{(1)}_{r+k}+\pi(b_1b_{1k})y^{(1)}_1+\cdots+\pi(b_1b_{rk})y^{(1)}_r+h^{(1)}_k\in N_{(1)}\quad\text{for } 2\le k\le s,$$

where $y^{(1)}_i=\pi(y_i)/b_1$ for $1\le i\le r$ and $y^{(1)}_{r+k}=\pi(y_{r+k})$ and some $h^{(1)}_k\in(y^{(1)}_1,\dots,y^{(1)}_r)^2$ for $1\le i\le s$. Since $N_{(1)}$ is prime and $\pi(b_1)\notin N_{(1)}$, we obtain that

$$y^{(1)}_{r+1}+\pi(b_{11})y^{(1)}_1+\cdots+\pi(b_{r1})y^{(1)}_r+h^{(1)}_1\in N_{(1)}.$$

Consequently,

$$(y^{(1)}_1,\dots,y^{(1)}_r,y^{(1)}_{r+1},\dots,y^{(1)}_{r+s})R^{(1)}=(y^{(1)}_1,\dots,y^{(1)}_r,y^{(1)}_{r+2},\dots,y^{(1)}_{r+s})R^{(1)}.$$

We proceed inductively to obtain a ν_1-compatible local blowing up $R\longrightarrow R^{(s)}$ such that

$$(y^{(s)}_1,\dots,y^{(s)}_r,y^{(s)}_{r+1},\dots,y^{(s)}_{r+s})R^{(s)}=(y^{(s)}_1,\dots,y^{(s)}_r)R^{(s)}.$$

By Lemma 4.5 we have $\mathfrak{p}^{(s)}=(y^{(s)}_1,\dots,y^{(s)}_r,y^{(s)}_{r+1},\dots,y^{(s)}_{r+s})R^{(s)}$, and by Lemma 2.8 the images of $y^{(s)}_1,\dots,y^{(s)}_r$ in $(R^{(s)}_{\mathfrak{p}^{(s)}})_{\rm red}$ form a regular system of parameters. This means that $y^{(s)}_1,\dots,y^{(s)}_r$ generate $\mathfrak{p}^{(s)}$. Thus we have reduced the problem to the case where $(y_1,\dots,y_r)$ generate $\mathfrak{p}$ and will make this assumption from now on.

Now, the only nontrivial fact that remains to be checked is that the images of $y_1,\dots,y_r$ in $\mathfrak{p}/\mathfrak{p}^2+N$ are $R/\mathfrak{p}$-linearly independent. Take $a_1,\dots,a_r\in R$ such that

$$a_1y_1+\cdots+a_ry_r\in\mathfrak{p}^2+N.$$

Since the images of $y_1,\dots,y_r$ in $(R_{\mathfrak{p}})_{\rm red}$ form a regular system of parameters, their images in $\mathfrak{p}R_{\mathfrak{p}}/(\mathfrak{p}^2+N)R_{\mathfrak{p}}$ form an $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$-basis of $\mathfrak{p}R_{\mathfrak{p}}/(\mathfrak{p}^2+N)R_{\mathfrak{p}}$. This implies that $a_1/1,\dots,a_r/1\in\mathfrak{p}R_{\mathfrak{p}}$ and consequently $a_1,\dots,a_r\in\mathfrak{p}$.

This completes the proof of the lemma. □

Proof of Lemma 4.3. Take $y_1,\dots,y_s\in\mathfrak{p}$ such that their images form an $R/\mathfrak{p}$-basis of $\mathfrak{p}/(\mathfrak{p}^2+N)$. We claim that the images of $\pi(y_1),\dots,\pi(y_s)$ form an $R^{(1)}/\mathfrak{p}^{(1)}$-basis of $\mathfrak{p}^{(1)}/((\mathfrak{p}^{(1)})^2+N_{(1)})$. Take an element $\alpha\in\mathfrak{p}^{(1)}$. Then $\alpha=p/q$

where $p, q \in R' := R[a_1/b, \ldots, a_r/b]$ with $\nu_1(p) > 0$ and $\nu(q) = 0$. Set $p_0 = p(0, \ldots, 0)$ and write

$$p = p_0 + \frac{a_1}{b} p_1 + \cdots + \frac{a_r}{b} p_r \quad \text{for some } p_1, \ldots, p_r \in R'.$$

This implies that $p_0 \in \mathfrak{p}$. By our assumption there exist $c_1, \ldots, c_s \in \mathfrak{p}$, $g \in \mathfrak{p}^2$, and $h \in N$ such that

$$p_0 = c_1 y_1 + \cdots + c_s y_r + g + h.$$

Consequently,

$$\alpha = \frac{\pi(c_1)}{q}\pi(y_1) + \cdots + \frac{\pi(c_s)}{q}\pi(y_s) + \frac{\pi(g)}{q} + \frac{\pi(h)}{q} + \frac{a_1}{b}\frac{p_1}{q} + \cdots + \frac{a_r}{b}\frac{p_r}{q}.$$

Since $a_1, \ldots, a_r, h \in N$, we have that

$$\frac{\pi(h)}{q} + \frac{a_1}{b}\frac{p_1}{q} + \cdots + \frac{a_r}{b}\frac{p_r}{q} \in N_{(1)}.$$

This and the fact that $\pi(g)/q \in (\mathfrak{p}^{(1)})^2$ imply that the images of $\pi(y_1), \ldots, \pi(y_r)$ generate $\mathfrak{p}^{(1)}/((\mathfrak{p}^{(1)})^2 + N_{(1)})$.

Now assume that there exists $\alpha_i = (a_i/b^{l_i})/(c_i/b^{m_i}) \in R^{(1)}$, $1 \le i \le r$, such that

$$\alpha_1\pi(y_1) + \cdots + \alpha_r\pi(y_r) \in (\mathfrak{p}^{(1)})^2 + N_{(1)}.$$

Then there exists $n \in \mathbb{N}$ such that

$$a_1 b^n y_1 + \cdots + a_r b^n y_r \in \mathfrak{p}^2 + N.$$

This implies that $a_i b^n \in \mathfrak{p}$ for every i, $1 \le i \le r$. Since $b \notin \mathfrak{p}$, this implies that $a_1, \ldots, a_r \in \mathfrak{p}$. Therefore, $\alpha_1, \ldots, \alpha_r \in \mathfrak{p}^{(1)}$, which concludes our proof. □

Proof of Lemma 4.4. Set $\mathfrak{p}' = \{a + N \in R_{\text{red}} \mid a \in \mathfrak{p}\}$. Since the images of the y_i in $\mathfrak{p}/(\mathfrak{p}^2 + N)$ form a basis of $\mathfrak{p}/(\mathfrak{p}^2 + N)$, we conclude that $(y_1, \ldots, y_r) + \mathfrak{p}^2 + N = \mathfrak{p}$. Applying Nakayama's lemma (corollary of Theorem 2.2 of [8]), we conclude that $(y_1, \ldots, y_r) + N = \mathfrak{p}$, and consequently $y_1 + N, \ldots, y_r + N$ generate $\mathfrak{p}'$.

Since the images of $y_1, \ldots, y_r, x_1, \ldots, x_t$ in R_{red} generate $\mathfrak{m}' = \{a + N \in R_{\text{red}} \mid a \in \mathfrak{m}\}$, we conclude that $r + t \ge \dim R_{\text{red}}$. Also, since $r = \dim(R_\mathfrak{p})_{\text{red}} = \operatorname{ht}(\mathfrak{p}')$ and $t = \dim(R/\mathfrak{p}) = \operatorname{ht}(\mathfrak{m}/\mathfrak{p}) = \operatorname{ht}(\mathfrak{m}'/\mathfrak{p}')$, we have

$$\dim(R_{\text{red}}) = \operatorname{ht}(\mathfrak{m}') \ge \operatorname{ht}(\mathfrak{p}') + \operatorname{ht}(\mathfrak{m}'/\mathfrak{p}') = r + t \ge \dim(R_{\text{red}}).$$

Therefore, $r + t = \dim(R_{\text{red}})$, and hence R_{red} is regular. □

5. Making N^n/N^{n+1} Free

Let R be a local ring, and ν a valuation centered on R. Assume that

$$\nu = \nu_1 \circ \nu_2$$

and denote by $\mathfrak{p}$ the center of ν_1 on R. As usual, we set $N = \operatorname{Nil}(R)$ and $N_\mathfrak{p} := \operatorname{Nil}(R_\mathfrak{p})$. Also, for a local blowing up $R \longrightarrow R^{(k)}$, we set $N_{(k)} = \operatorname{Nil}(R^{(k)})$ and $N_{\mathfrak{p}^{(k)}} := \operatorname{Nil}(R^{(k)}_{\mathfrak{p}^{(k)}})$. Assume that N is the only associated prime ideal of R. The main goal of this section is to prove the following proposition.

PROPOSITION 5.1. *Assume that* $N_{\mathfrak{p}}^n/N_{\mathfrak{p}}^{n+1}$ *is an* $(R_{\mathfrak{p}})_{\mathrm{red}}$*-free module for every* $n \in \mathbb{N}$*. Then there exists a local blowing up* $R \longrightarrow R^{(1)}$ *with respect to* ν *along an ideal* $(b, a_1, \ldots, a_r)$ *with* $b \notin \mathfrak{p}$ *and* $a_1, \ldots, a_r \in N$ *such that the* $(R^{(1)})_{\mathrm{red}}$*-module* $N_{(1)}^n/N_{(1)}^{n+1}$ *is free for every* $n \in \mathbb{N}$*.*

To prove Proposition 5.1, we need some preliminary results.

LEMMA 5.2. *Take elements* $y_1, \ldots, y_{r+s} \in N^n$ *such that their images in* N^n/N^{n+1} *generate* N^n/N^{n+1} *as an* R_{red}*-module. Consider the local blowing up* $\pi : R \longrightarrow R^{(1)}$ *along the ideal* $(b, y_1, \ldots, y_r)$ *for some* $b \in R \setminus N$*. Set*

$$y_i^{(1)} = \pi(y_i)/b \quad \text{for } 1 \le i \le r \quad \text{and} \quad y_{r+k}^{(1)} = \pi(y_{r+k}) \quad \text{for } 1 \le k \le s.$$

Then the images of $y_1^{(1)}, \ldots, y_{r+s}^{(1)}$ *in* $N_{(1)}^n/N_{(1)}^{n+1}$ *form a set of generators of this module.*

Proof. Take an element $p/q \in N_{(1)}^n$. As in proof of the Lemma 4.5, we can write

$$p = p_0 + \frac{y_1}{b}p_1 + \cdots + \frac{y_r}{b}p_r \quad \text{for some } p_1, \ldots, p_r \in R'$$

with $p_0 \in N^n$. This means that there exist $a_1, \ldots, a_{r+s} \in R$ such that $p_0 - a_1 y_1 - \cdots - a_{r+s} y_{r+s} = y_0 \in N^{n+1}$. Consequently,

$$\frac{p}{q} - \sum_{i=1}^{r} \frac{\pi(ba_i) + p_i}{q} y_i^{(1)} - \sum_{i=r+1}^{r+s} \frac{\pi(a_i)}{q} y_i^{(1)} = \frac{\pi(y_0)}{q} \in N_{(1)}^{n+1}.$$

This concludes our proof. □

LEMMA 5.3. *Under the same assumptions as in the previous lemma, if the images of* $y_1, \ldots, y_r$ *in* N^n/N^{n+1} *are* R_{red}*-linearly independent, then the images of* $a_1^{(1)}, \ldots, a_r^{(1)}$ *in* $N_{(1)}^n/N_{(1)}^{n+1}$ *are* $(R^{(1)})_{\mathrm{red}}$*-linearly independent.*

Proof. Take elements $\alpha_1, \ldots, \alpha_r \in R^{(1)}$ such that

$$\alpha_1 y^{(1)} + \cdots + \alpha_r y^{(1)} \in N_{(1)}^{n+1}. \tag{8}$$

We have to show that $\alpha_1, \ldots, \alpha_r \in N_{(1)}$. For each i, $1 \le i \le r$, we write $\alpha_i = (a_i/b^{r_i})/(c_i/b^{s_i})$ for some $a_i, c_i \in R$ and $r_i, s_i \in \mathbb{N}$. Then equation (8) implies that there exist $l \in \mathbb{N}$ and $c \in R \setminus \mathfrak{p}$ such that

$$a_1 b^l c y_1 + \cdots + a_r b^l c y_r \in N^{n+1}.$$

Since $y_1 + N^{n+1}, \ldots, y_r + N^{n+1}$ are R_{red}-linearly independent, this implies that

$$a_i b^l c \in N \quad \text{for every } i, 1 \le i \le r.$$

Since N is prime (this is a consequence of the fact that it is the only associated prime ideal of R) and $b, c \in R \setminus N$, we obtain that $a_1, \ldots, a_r \in N$. Consequently, $\alpha_1, \ldots, \alpha_r \in N_{(1)}$, which concludes our proof. □

Proof of Proposition 5.1. By assumption we have that $N_{\mathfrak{p}}^n/N_{\mathfrak{p}}^{n+1}$ is $(R_{\mathfrak{p}})_{\mathrm{red}}$-free for every $n\in\mathbb{N}$. Hence, by Proposition 2.8, for every ν_1-compatible local blowing up $R\longrightarrow R^{(1)}$, we have that $N_{\mathfrak{p}^{(1)}}^n/N_{\mathfrak{p}^{(1)}}^{n+1}$ is $(R_{\mathfrak{p}^{(1)}}^{(1)})_{\mathrm{red}}$-free for every $n\in\mathbb{N}$. Therefore, it suffices to show that, for a fixed $n\in\mathbb{N}$, there exists a local blowing up $R\longrightarrow R^{(1)}$ along an ideal $(b,a_1,\ldots,a_r)$ with $b\notin\mathfrak{p}$ and $a_1,\ldots,a_r\in N$ such that $N_{(1)}^n/N_{(1)}^{n+1}$ is $(R^{(1)})_{\mathrm{red}}$-free.

Take elements $y_1/b_1,\ldots,y_r/b_r\in N_{\mathfrak{p}}^n$, $y_1,\ldots,y_r\in R$, and $b_1,\ldots,b_r\in R\setminus\mathfrak{p}$ such that

$$y_1/b_1+N_{\mathfrak{p}}^{n+1},\ldots,y_r/b_r+N_{\mathfrak{p}}^{n+1}$$

form a basis of $N_{\mathfrak{p}}^n/N_{\mathfrak{p}}^{n+1}$. We observe first that since N is prime and $y_i/b_i\in N_{\mathfrak{p}}^n$, we have $y_i\in N^n$ for each i, $1\le i\le r$. We claim that if

$$y_1+N^{n+1},\ldots,y_r+N^{n+1}$$

generate N^n/N^{n+1} as an R_{red}-module, then this module is free. Indeed, if there exist $a_i+N\in R_{\mathrm{red}}$ such that $a_1y_1+\cdots+a_ry_r\in N^{n+1}$, then

$$a_1b_1/1\cdot y_1/b_1+\cdots+a_rb_r/1\cdot y_r/b_r=(a_1y_1+\cdots+a_ry_r)/1\in N_{\mathfrak{p}}^{n+1}.$$

This implies that, for each i, $1\le i\le r$, $a_ib_i/1\in N_{\mathfrak{p}}$ and consequently $a_ib_ic_i\in N$ for some $c_i\in R\setminus\mathfrak{p}$. Since N is prime and $b_1c_1,\ldots,b_rc_r\in R\setminus N$, we conclude that $a_1,\ldots,a_r\in N$, which is what we wanted to prove.

If $y_1+N^{n+1},\ldots,y_r+N^{n+1}$ do not generate N^n/N^{n+1} (as an R_{red}-module), then we take $y_{r+1},\ldots,y_{r+s}\in N^n$ such that $y_1+N^{n+1},\ldots,y_{r+s}+N^{n+1}$ generate N^n/N^{n+1}. For each k, $1\le k\le s$, since $y_{r+k}\in N^n$, there exist $b_k\in R\setminus\mathfrak{p}$ such that

$$b_ky_{r+k}-b_{1k}y_1-\cdots-b_{rk}y_r\in N^{n+1} \tag{9}$$

for some $b_{1k},\ldots,b_{rk}\in R$. Consider now the local blowing up along the ideal $(b_1,y_1,\ldots,y_r)$. Set

$$y_i^{(1)}:=\pi(y_i)/b_1\in R^{(1)}\quad\text{for each } i,\ 1\le i\le r$$

and

$$y_{r+k}^{(1)}:=\pi(y_{r+k})\in R^{(1)}\quad\text{for each } k,\ 1\le k\le s.$$

From equation (9) we obtain that

$$y_{r+1}^{(1)}-\pi(b_{11})y_1^{(1)}-\cdots-\pi(b_{r1})y_r^{(1)}\in N_{(1)}^{n+1}$$

and

$$\pi(b_k)y_{r+k}^{(1)}-\pi(b_1b_{1k})y_1^{(1)}-\cdots-\pi(b_rb_{rk})y_r^{(1)}\in N_{(1)}^{n+1}$$

for every k, $2\le k\le s$. Consequently, $y_{r+1}^{(1)}+N_{(1)}^{n+1}$ is generated in the $(R^{(1)})_{\mathrm{red}}$-module $N_{(1)}^n/N_{(1)}^{n+1}$ by $y_1^{(1)}+N_{(1)}^{n+1},\ldots,y_r^{(1)}+N_{(1)}^{n+1}$. Moreover, using Lemma 5.2, we obtain that $N_{(1)}^n/N_{(1)}^{n+1}$ is generated as an $R_{\mathrm{red}}^{(1)}$-module by the images of

$$y_1^{(1)},\ldots,y_r^{(1)},y_{r+2}^{(1)},\ldots,y_{r+s}^{(1)}.$$

Also, by Lemma 5.3 the images of $y_1^{(1)},\ldots,y_r^{(1)}$ in $N_{(1)}^n/N_{(1)}^{n+1}$ are $(R^{(1)})_{\mathrm{red}}$-linearly independent.

We proceed inductively to obtain a local blowing up $R \longrightarrow R^{(s)}$ such that the $(R^{(s)})_{\mathrm{red}}$-module $N^n_{(s)}/N^{n+1}_{(s)}$ is generated by the images of $y_1^{(s)}, \ldots, y_r^{(s)}$ and the images of $y_1^{(s)}, \ldots, y_r^{(s)}$ in $N^n_{(s)}/N^{n+1}_{(s)}$ are $(R^{(s)})_{\mathrm{red}}$-linearly independent. □

6. Proof of the Main Theorem

In this section we present the proof of our main theorem.

Proof of Theorem 1.1. We will prove the assertion by induction on the rank. Since all rank one valuations admit local uniformization by assumption, we fix $n \in \mathbb{N}$ and will prove that if all valuations of rank smaller than n admit local uniformization, then also valuations of rank n admit local uniformization.

Let ν be a valuation centered in the local ring $R \in Ob(\mathcal{M})$ such that $\mathrm{rk}(\nu) = n$. By Lemma 3.2 there exists a local blowing up $R \longrightarrow R^{(1)}$ with respect to ν such that $\mathrm{Nil}(R^{(1)})$ is the only associated prime ideal of $R^{(1)}$. Hence, replacing R by $R^{(1)}$, we may assume that the only associated prime ideal of R is $\mathrm{Nil}(R)$.

Decompose ν as $\nu = \nu_1 \circ \nu_2$ for valuations ν_1 and ν_2 with rank smaller than n. By assumption we know that ν_1 and ν_2 admit local uniformization. Since ν_1 admits local uniformization, by Lemma 2.10 there exists a local blowing up $R \longrightarrow R^{(1)}$ with respect to ν such that $R^{(1)}_{\mathfrak{p}^{(1)}}$ is regular and $N^n_{\mathfrak{p}^{(1)}}/N^{n+1}_{\mathfrak{p}^{(1)}}$ is $(R^{(1)}_{\mathfrak{p}^{(1)}})_{\mathrm{red}}$-free for every $n \in \mathbb{N}$. Replacing R by $R^{(1)}$, we may assume that $(R_{\mathfrak{p}})_{\mathrm{red}}$ is regular and $N^n_{\mathfrak{p}}/N^{n+1}_{\mathfrak{p}}$ is $(R_{\mathfrak{p}})_{\mathrm{red}}$-free for every $n \in \mathbb{N}$.

Since ν_2 admits local uniformization, we can use Lemma 2.11 to obtain that there exists a local blowing up $R \longrightarrow R^{(1)}$ with respect to ν such that $(R^{(1)}_{\mathfrak{p}^{(1)}})_{\mathrm{red}}$ and $R^{(1)}/\mathfrak{p}^{(1)}$ are regular and $N^n_{\mathfrak{p}^{(1)}}/N^{n+1}_{\mathfrak{p}^{(1)}}$ is $(R^{(1)}_{\mathfrak{p}^{(1)}})_{\mathrm{red}}$-free for every $n \in \mathbb{N}$. Replacing R by $R^{(1)}$, we can assume that $(R_{\mathfrak{p}})_{\mathrm{red}}$ and $R/\mathfrak{p}$ are regular and that $N^n_{\mathfrak{p}}/N^{n+1}_{\mathfrak{p}}$ is $(R_{\mathfrak{p}})_{\mathrm{red}}$-free for every $n \in \mathbb{N}$.

Since $(R_{\mathfrak{p}})_{\mathrm{red}}$ and $R/\mathfrak{p}$ are regular, we apply Proposition 4.1 to obtain a ν_1-compatible local blowing up $R \longrightarrow R^{(1)}$ such that $(R^{(1)})_{\mathrm{red}}$ is regular. Using Proposition 2.8, we have that $N^n_{\mathfrak{p}^{(1)}}/N^{n+1}_{\mathfrak{p}^{(1)}}$ is a free $(R^{(1)}_{\mathfrak{p}^{(1)}})_{\mathrm{red}}$-module for every $n \in \mathbb{N}$. By Proposition 5.1 there exists a $\mathfrak{p}^{(1)}$-compatible local blowing up $R^{(1)} \longrightarrow R^{(2)}$ such that $N^n_{(2)}/N^{n+1}_{(2)}$ is an $(R^{(2)})_{\mathrm{red}}$-free module for every $n \in \mathbb{N}$. Moreover, since this local blowing up is along an ideal $(b, a_1, \ldots, a_r)$ with $b \notin \mathfrak{p}^{(1)}$ and $a_1, \ldots, a_r \in N_{(1)}$, we use Proposition 4.1 to obtain that $(R^{(2)})_{\mathrm{red}}$ is regular. This concludes our proof. □

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