

Relative Asymptotics for General Orthogonal Polynomials

BRIAN SIMANEK

ABSTRACT. In this paper, we study right limits of the Bergman shift matrix. Our results have applications to ratio asymptotics, weak asymptotic measures, relative asymptotics, and zero counting measures of orthogonal and orthonormal polynomials. Of particular interest are the applications to random orthogonal polynomials on the unit circle and real line.

1. Introduction

Let μ be a positive probability measure with compact and infinite support in the complex plane. Given such a measure, it is well known how to form the sequence of orthonormal polynomials for this measure, which we denote by $\{p_n(z; \mu)\}_{n \geq 0}$, satisfying

$$\int p_n(z; \mu) \overline{p_m(z; \mu)} d\mu(z) = \delta_{m,n}.$$

The polynomial $p_n(z; \mu)$ has a positive leading coefficient, which we denote by $\kappa_n(\mu)$. The monic polynomial $\kappa_n(\mu)^{-1} p_n(z; \mu)$ will be denoted by $P_n(z; \mu)$ and is of interest in its own right.

It is both interesting and informative to try to understand the relationship between the measure μ and the corresponding sequence of orthonormal polynomials. There are many ways that we can study the orthogonal polynomial asymptotics. One of the most interesting kinds of asymptotic behavior is known as *ratio asymptotics* and concerns the following limits:

$$\lim_{n \rightarrow \infty} \frac{p_{n-1}(z; \mu)}{p_n(z; \mu)}, \tag{1}$$

$$\lim_{n \rightarrow \infty} \frac{P_{n-1}(z; \mu)}{P_n(z; \mu)}, \tag{2}$$

provided that these limits exist. When studying ratio asymptotics, we are concerned with the existence of the limit, the domain of the limit, and a functional form of the limit.

A second kind of asymptotic behavior is known as *weak asymptotic measures* and concerns the weak limits of the measures $\{|p_n(z; \mu)|^2 d\mu(z)\}_{n \in \mathbb{N}}$ as $n \rightarrow \infty$. Since μ has compact support, weak limits always exist, and we can gain insight into properties of the measure μ by understanding these weak limits. In some

Received September 23, 2015. Revision received May 20, 2016.

cases, it is more practical to study the weak asymptotic moments, that is, to understand the moments of all the weak asymptotic measures. In general, the moments of a measure do not determine the measure, so this is a less precise object to study, but we will see that we can make precise statements about these moments even when we cannot identify the measure. If each of the following limits exists:

$$\lim_{n \rightarrow \infty} \int z^j |p_n(z; \mu)|^2 d\mu(z), \quad j \in \mathbb{N},$$

then we will say that the measure μ exhibits *weak asymptotic moments*.

A third way to study orthogonal polynomial asymptotics is to study the asymptotic distribution of the zeros of P_n as n becomes large. Let $\sigma_n(\mu)$ be the probability measure that assigns weight n^{-1} to each zero of P_n , where we count each zero a number of times equal to its multiplicity as a zero. If each of the following limits exists:

$$\lim_{n \rightarrow \infty} \int z^j d\sigma_n(z), \quad j \in \mathbb{N},$$

then we will say that the measure μ exhibits *Cesàro weak asymptotic moments*.

In order to understand precisely when each of these asymptotic behaviors can be observed, we need to consider the Bergman shift operator. For a compactly supported measure μ , the Bergman shift is a bounded operator on $L^2(\mu)$ that maps $f(z)$ to $zf(z)$. If we let $\mathcal{P}(\mu)$ denote the closure of the polynomials in $L^2(\mu)$, then it is clear that the Bergman shift maps $\mathcal{P}(\mu)$ to itself. If we use the orthonormal polynomials for the measure μ as a basis for $\mathcal{P}(\mu)$, then the orthogonality condition implies that the resulting matrix representation of the Bergman shift is a Hessenberg matrix

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} & \cdots \\ M_{21} & M_{22} & M_{23} & M_{24} & \cdots \\ 0 & M_{32} & M_{33} & M_{34} & \cdots \\ 0 & 0 & M_{43} & M_{44} & \cdots \\ 0 & 0 & 0 & M_{54} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In some cases, we will use the notation M_μ to emphasize the measure that gives rise to this matrix. If the support of μ is a compact subset of the real line, then the matrix M is often called a Jacobi matrix. In this case, the matrix M_μ is symmetric, its entries are zero away from the three main diagonals, and the matrix entries can be given explicitly in terms of the coefficients appearing in the three-term recurrence relation satisfied by the corresponding orthonormal polynomials (see [20]). If the support of μ is a subset of the unit circle, then M is sometimes called the GGT matrix for the measure μ (see [17, Sect. 4.1]), and its entries can be described explicitly in terms of the recursion coefficients for the corresponding monic orthogonal polynomials.

The structure of the matrix M for general μ is more difficult to describe. Some interesting properties appear in [13; 14; 15], all of which focus on the relationship between properties of the measure and the entries of M as we move along one

of its diagonals. We will continue this trend of using the limiting behavior of the entries of M along its diagonals to deduce properties of the measure μ and the corresponding orthonormal polynomials.

In [14], the author provides a precise description of when the monic orthogonal polynomials and orthonormal polynomials for μ exhibit ratio asymptotics. The result is stated as a condition on the entries of the Bergman shift matrix M_μ . Our first main goal in the present work is to prove a relative version of that result, so we must introduce several notions of relative asymptotics.

For any compactly supported measure μ , let us define $R_\mu := \sup\{|z| : z \in \text{supp}(\mu)\}$. We will say that two measures μ and ν exhibit *relative ratio asymptotics* if there is a nonnegative integer q so that

$$\lim_{n \rightarrow \infty} \left(\frac{P_{n-1}(z; \mu)}{P_n(z; \mu)} - \frac{P_{n-q-1}(z; \nu)}{P_{n-q}(z; \nu)} \right) = 0, \quad |z| > \max\{R_\mu, R_\nu\}. \tag{3}$$

We will say that two measures μ and ν exhibit *relative weak asymptotic moments* if there is a nonnegative integer q so that

$$\lim_{n \rightarrow \infty} \left(\int z^j |p_n(z; \mu)|^2 d\mu(z) - \int z^j |p_{n-q}(z; \nu)|^2 d\nu(z) \right) = 0, \quad j \in \mathbb{N}. \tag{4}$$

We will say that two measures μ and ν exhibit *relative Cesàro weak asymptotic moments* if

$$\lim_{n \rightarrow \infty} \left(\int z^j d\sigma_n(\mu)(z) - \int z^j d\sigma_n(\nu)(z) \right) = 0, \quad j \in \mathbb{N}. \tag{5}$$

A different but related notion of relative asymptotics is discussed in [8].

Results on the existence of relative ratio asymptotics can be found in [1, Thm. 3.7], [12, Thm. 6], and [5, Thm. 1.5]. Whereas our main result on relative ratio asymptotics is similar in both content and proof to [1, Thm. 3.7], we make the important generalization that we can take n to infinity along subsequences (which was observed in [12]) and even use different subsequences for each measure.

For a given measure μ and any $j, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let us define $h_{j,n}(\mu)$ by

$$h_{j,n}(\mu) := \frac{\kappa_{n-1-j}(\mu)}{\kappa_{n-1}(\mu)} (M_\mu)_{n-j,n}.$$

Now we can state our main result concerning relative ratio asymptotics.

THEOREM 1.1. *Let μ and ν be two compactly supported and finite measures, each with infinitely many points in its support and let $q \in \mathbb{N}_0$ be fixed. We have*

$$\lim_{n \rightarrow \infty} \left(\frac{P_{n-1}(z; \mu)}{P_n(z; \mu)} - \frac{P_{n-q-1}(z; \nu)}{P_{n-q}(z; \nu)} \right) = 0, \quad |z| > \max\{R_\mu, R_\nu\} \tag{6}$$

if and only if for every $j \in \mathbb{N}_0$,

$$\lim_{n \rightarrow \infty} (h_{j,n}(\mu) - h_{j,n-q}(\nu)) = 0. \tag{7}$$

We might be able to obtain Theorem 1.1 as a direct consequence of [1, Thm. 3.7] or [12, Thm. 6], at least in the case $q = 0$. However, the setting we consider is

slightly different from both of those papers, so we will include a proof in Section 3 for completeness.

It is also meaningful to discuss a relative version of relation (1). Indeed, if $\{n_k\}_{k=1}^\infty = \mathcal{N}(\mu) \subseteq \mathbb{N}$ and $\{m_k\}_{k=1}^\infty = \mathcal{N}(\nu) \subseteq \mathbb{N}$ are two subsequences, then we will say that two measures μ and ν exhibit *normalized relative ratio asymptotics through $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$* precisely when

$$\lim_{k \rightarrow \infty} \left(\frac{p_{n_k+j-1}(z; \mu)}{p_{n_k+j}(z; \mu)} - \frac{p_{m_k+j-1}(z; \nu)}{p_{m_k+j}(z; \nu)} \right) = 0, \tag{8}$$

$$|z| > \max\{R_\mu, R_\nu\}, j \in \mathbb{Z}.$$

If $\mathcal{N}(\mu) = \mathcal{N}(\nu) = \mathbb{N}$, then we will say μ and ν exhibit *normalized relative ratio asymptotics*. A normal families argument implies that the convergence in (8) is always uniform on compact subsets of $\{z : |z| > \max\{R_\mu, R_\nu\}\}$.

Existence of the limit (8) is a stronger conclusion than the existence of the limit (3) because it assumes some convergence of the ratio of consecutive leading coefficients. Thus, normalized relative ratio asymptotics will be equivalent to a stronger statement than (7). To state this condition, we must recall the definition of a right limit of a matrix (see [15]).

For a matrix A whose rows and columns are indexed by natural numbers, let $A_n^{(m)}$ be the $2m + 1 \times 2m + 1$ submatrix of A centered at $A_{n,n}$. We will say that a matrix X is a right limit of the matrix A if there is a subsequence $\mathcal{N} \subseteq \mathbb{N}$ such that, for every $m \in \mathbb{N}$,

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} A_n^{(m)} = X_0^{(m)}.$$

In this case, we will also say that A approaches X through \mathcal{N} . If the matrix elements of A are bounded (which they are in the cases we are considering), then we can always find right limits. Also, notice that right limits are biinfinite, even though the matrix A is not. In the context of orthogonal polynomials on the real line, right limits have been used extensively for asymptotic calculations (see [20; 25] and references therein). For more on right limits, we refer the reader to [15] and [20, Chap. 7].

The recent publications [14; 15] shed some light on the relationship between right limits of the matrix M_μ and ratio asymptotics for general orthonormal polynomials under the added condition that

$$\liminf_{n \rightarrow \infty} \frac{\kappa_n}{\kappa_{n+1}} > 0. \tag{9}$$

Note that this is a very mild condition that is satisfied by many measures of interest including area measure on a smooth Jordan region [13] and measures on the unit circle with nonvanishing absolutely continuous component [17]. Using essentially the same techniques as in [14; 15], we will show the following result (recall the definition of R_μ given before equation (3)).

THEOREM 1.2. *Let μ be a finite measure with compact and infinite support such that (9) is satisfied. If $\mathcal{N} \subseteq \mathbb{N}$ is a subsequence, then the following statements are equivalent:*

- (I) *There is a sequence of functions $\{f_j\}_{j \in \mathbb{Z}}$, each of which is analytic in $\{z : R_\mu < |z| \leq \infty\}$, such that, for all $j \in \mathbb{Z}$,*

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} \frac{p_{n+j-1}(z; \mu)}{p_{n+j}(z; \mu)} = f_j(z), \quad |z| > R_\mu, \tag{10}$$

where the convergence is uniform on the exterior of $\{z : |z| > r\}$ for every $r > R_\mu$.

- (II) *The matrix M_μ approaches a right limit as $n \rightarrow \infty$ through \mathcal{N} .*

The implication (II) implies (I) is valid even if condition (9) is not satisfied.

REMARK. The proof of Theorem 1.2 will make it clear that if condition (II) of Theorem 1.2 applies to a measure μ , then the right limit of M_μ determines the sequence $\{f_j\}_{j \in \mathbb{Z}}$. If condition (9) is satisfied, then the sequence $\{f_j\}_{j \in \mathbb{Z}}$ in condition (I) also determines the right limit.

Theorem 1.2 and the remark immediately provide the following characterization of pairs of measures that exhibit normalized relative ratio asymptotics.

COROLLARY 1.3. *Let μ and ν be two positive and finite measures, each having compact and infinite support and satisfying (9). Furthermore, let $\{n_k\}_{k=1}^\infty = \mathcal{N}(\mu) \subseteq \mathbb{N}$ and $\{m_k\}_{k=1}^\infty = \mathcal{N}(\nu) \subseteq \mathbb{N}$ be two subsequences. The following statements are equivalent:*

- (i) *The measures μ and ν exhibit normalized relative ratio asymptotics through $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$.*
- (ii) *M_μ approaches a right limit X through $\{n_k\}_{k \in \mathcal{N}^* \subseteq \mathbb{N}}$ if and only if M_ν approaches X through $\{m_k\}_{k \in \mathcal{N}^*}$.*

In particular, the measures μ and ν exhibit normalized relative ratio asymptotics if and only if the unique right limit of $M_\mu - M_\nu$ is 0. The implication (ii) implies (i) is valid even if condition (9) is not satisfied.

REMARK. We will see by example in Section 5 that hypothesis (9) is essential for the implication (i) implies (ii) in the theorem.

Given a compactly supported finite measure with infinitely many points in its support, let \mathcal{A}_μ be the collection of analytic functions that occur as uniform limits of the sequence $\{p_{n-1}(z; \mu)/p_n(z; \mu)\}_{n \in \mathbb{N}}$ as $n \rightarrow \infty$ through a subsequence ([14, Prop. 2.2] implies that \mathcal{A}_μ is not empty). Let us also denote by \mathcal{B}_μ the collection of right limits of M_μ , and we have already remarked that \mathcal{B}_μ is not empty. Theorem 1.2 easily implies the following corollary.

COROLLARY 1.4. *Let μ and ν be two compactly supported and finite measures each having infinite support. If $\mathcal{B}_\mu = \mathcal{B}_\nu$, then $\mathcal{A}_\mu = \mathcal{A}_\nu$.*

In the next section, we will discuss some elementary consequences of the results in this section. In particular, we will be able to prove that relative ratio asymptotics

implies relative weak asymptotic moments. In Section 4, we discuss various universal properties of random orthogonal polynomials on the unit circle and real line by showing that such polynomials exhibit behavior that is dense in the appropriate spaces. The proofs of the results stated are contained in Section 3, whereas Section 5 contains some additional examples.

2. Asymptotic Relationships

The relationship between weak asymptotic moments and Cesàro weak asymptotic moments was proven in the context of orthogonal polynomials on the real line in [7, Thm. 2]. This was generalized in several ways by the author in [14], the results of which are summarized in the following theorem.

THEOREM 2.1. *Let μ be a finite measure with compact and infinite support.*

- (i) *If μ exhibits weak asymptotic moments, then it exhibits Cesàro weak asymptotic moments.*
- (ii) *If the monic orthogonal polynomials for μ exhibit ratio asymptotics, then μ exhibits weak asymptotic moments.*

The results in Section 1 allow us to prove the following relative version of that result.

THEOREM 2.2. *Let μ and ν be two finite measures with compact and infinite support.*

- (i) *If μ and ν exhibit relative weak asymptotic moments, then they also exhibit relative Cesàro weak asymptotic moments.*
- (ii) *If μ and ν exhibit relative ratio asymptotics, then they also exhibit relative weak asymptotic moments.*

To prove Theorem 2.2, we will require the following lemma, the proof of which is contained in the proof of [14, Thm. 1.2].

LEMMA 2.3. *Let π_n be the projection onto the span of $\{1, \dots, z^{n-1}\}$ inside $\mathcal{P}(\mu)$.*

- (a) *For any natural number $m \geq 1$, we can write*

$$((\pi_n M_\mu \pi_n)^m)_{n,n} = h_{m-1,n}(\mu) + \beta_{n,m},$$

where $\beta_{n,m}$ can be expressed as a sum of products of factors of the form $h_{j,n-k}(\mu)$ for appropriate values of $j, k \in \{0, 1, \dots, m-2\}$.

- (b) *For any natural number $m \geq 1$, $(M_\mu^m)_{n,n}$ can be expressed as a sum of products of factors of the form $h_{j,n-k}(\mu)$ where $j \in \{0, 1, \dots, m-1\}$ and $k \in \{-m+1, -m+2, \dots, m-2, m-1\}$.*

REMARK. Lemma 2.3(a) is stated in the proof of [1, Thm. 3.7]. A result similar to Lemma 2.3(b) for tridiagonal matrices is used in the proof of [9, Lemma 3] (see also [3, eq. (4.3)] for a related result for banded matrices).

Proof of Theorem 2.2. (i) We proceed exactly as in the proof of Theorem 2.1(i) in [14]. Indeed, suppose that μ and ν exhibit relative weak asymptotic moments. We know from [19, Prop. 2.3] that

$$\begin{aligned} & \left| \int z^j d\sigma_n(\mu)(z) - \int z^j d\sigma_n(\nu)(z) \right. \\ & \quad \left. - \frac{1}{n} \sum_{k=0}^{n-1} \left(\int z^j |p_n(z; \mu)|^2 d\mu(z) - \int z^j |p_n(z; \nu)|^2 d\nu(z) \right) \right| \\ & \leq \frac{2j(R_\mu + R_\nu)}{n}, \quad j \in \mathbb{N}. \end{aligned}$$

As $n \rightarrow \infty$, the sum over k tends to zero by hypothesis, and the right-hand side tends to zero, so the difference between the first two integrals must also tend to zero.

(ii) Suppose that μ and ν exhibit relative ratio asymptotics. Theorem 1.1 implies that, for some $q \in \mathbb{N}_0$,

$$\lim_{n \rightarrow \infty} (h_{j,n}(\mu) - h_{j,n-q}(\nu)) = 0, \quad j \in \mathbb{N}_0.$$

Combining this with the second part of Lemma 2.3 and noting that

$$\int z^m |p_n(z; \mu)|^2 d\mu(z) = (M_\mu^m)_{n+1,n+1}$$

proves the desired result. □

In the next section, we will prove the results stated in Section 1.

3. Proof of the Main Theorems

Our main objective in this section is to prove the results stated in Section 1. Our methods are similar to those used in [14]; indeed, we rely on results from [14] in some of our proofs. We begin by recalling the following fact, which is contained in Proposition 2.2 in [19].

PROPOSITION 3.1. *Let π_n be the projection onto the n -dimensional subspace given by the span of $\{1, z, \dots, z^{n-1}\}$ inside $\mathcal{P}(\mu)$. The polynomial $P_n(z; \mu)$ and the matrix M_μ are related by*

$$P_n(z; \mu) = \det(z - \pi_n M_\mu \pi_n), \tag{11}$$

where the determinant is the usual determinant of a finite matrix.

We will also make repeated use of the fact that $\{P_{n-1}(z; \mu)/P_n(z; \mu)\}_{n \in \mathbb{N}}$ (and hence also $\{p_{n-1}(z; \mu)/p_n(z; \mu)\}_{n \in \mathbb{N}}$) is a normal family (see [14, Prop. 2.2]).

Our first task is to prove Theorem 1.1, which we will do by appealing to Lemma 2.3.

Proof of Theorem 1.1. Suppose (7) holds for all $j \in \mathbb{N}_0$. Let $\rho_{M_\mu, n}(z) = (z - \pi_n M_\mu \pi_n)^{-1}$ be the resolvent of the truncated M_μ matrix. By (11) and Cramer’s rule we can write (when $|z|$ is sufficiently large)

$$\frac{P_{n-1}(z; \mu)}{P_n(z; \mu)} = \rho_{M_\mu, n}(z)_{n,n} = \sum_{m=0}^{\infty} \frac{((\pi_n M_\mu \pi_n)^m)_{n,n}}{z^{m+1}} \tag{12}$$

(see also [16, eq. (2.21)]). Therefore, when $|z|$ is sufficiently large, we have

$$\begin{aligned} & \frac{P_{n-1}(z; \mu)}{P_n(z; \mu)} - \frac{P_{n-q-1}(z; \nu)}{P_{n-q}(z; \nu)} \\ &= \sum_{m=0}^{\infty} \frac{((\pi_n M_\mu \pi_n)^m)_{n,n} - ((\pi_{n-q} M_\nu \pi_{n-q})^m)_{n-q, n-q}}{z^{m+1}}. \end{aligned} \tag{13}$$

We will show that all the Laurent coefficients of this expansion converge to 0 as $n \rightarrow \infty$. This and [14, Prop. 2.2] will prove the desired result.

To this end, recall that we assume that $h_{j,n}(\mu) = h_{j,n-q}(\nu) + o(1)$ as $n \rightarrow \infty$. Therefore, Lemma 2.3 implies that, for each $m \in \mathbb{N}_0$, $((\pi_n M_\mu \pi_n)^m)_{n,n} = ((\pi_{n-q} M_\nu \pi_{n-q})^m)_{n-q, n-q} + o(1)$ as $k \rightarrow \infty$, which is our desired conclusion.

For the converse, suppose that the monic orthogonal polynomials for μ and ν satisfy the relation (6). In this case, we know that each Laurent coefficient in (13) converges to 0 as $n \rightarrow \infty$. We will proceed by induction to show that (7) holds.

For the base case of the induction, we consider the $m = 1$ term in (13). The numerator is $h_{0,n}(\mu) - h_{0,n-q}(\nu)$, so we have proven (7) for $j = 0$. This will serve as the base case of our induction. As our induction hypothesis, assume that (7) holds for all $j < k$ for some fixed $k \in \mathbb{N}$. If we look at the $m = k + 1$ term in (13), then we see that the coefficient is $((\pi_n M_\mu \pi_n)^{k+1})_{n,n} - ((\pi_{n-q} M_\nu \pi_{n-q})^{k+1})_{n-q, n-q}$, which therefore must tend to zero as $n \rightarrow \infty$. Lemma 2.3 and the induction hypothesis imply that

$$\begin{aligned} & \lim_{n \rightarrow \infty} (((\pi_n M_\mu \pi_n)^{k+1})_{n,n} - h_{k,n}(\mu) - (((\pi_{n-q} M_\nu \pi_{n-q})^{k+1})_{n-q, n-q} \\ & \quad - h_{k,n-q}(\nu))) = 0. \end{aligned}$$

This implies (7) for $j = k$ and completes the induction. □

Now we turn our attention to right limits and normalized relative ratio asymptotics. Our first task is to prove Theorem 1.2, which concerns a necessary and sufficient condition for the matrix M_μ to converge to a right limit through a subsequence \mathcal{N} .

Proof of Theorem 1.2. We use (12) again to write

$$\begin{aligned} \frac{P_{n+j-1}(z; \mu)}{P_{n+j}(z; \mu)} &= \rho_{M_\mu, n+j}(z)_{n+j, n+j} \\ &= \sum_{m=0}^{\infty} \frac{((\pi_{n+j} M_\mu \pi_{n+j})^m)_{n+j, n+j}}{z^{m+1}}. \end{aligned} \tag{14}$$

Therefore, we have convergence uniformly on $\{z : |z| \geq r\}$ for every $r > R_\mu$ if and only if each of the Laurent coefficients in (14) converges.

Let us suppose that $\mathcal{N} \subseteq \mathbb{N}$ is a subsequence through which M_μ converges to a right limit. Then $\mathcal{N} + j := \{n + j : n \in \mathcal{N}, n + j > 0\}$ is also a subsequence through which M_μ converges to a right limit. Therefore, for every $j, k \in \mathbb{Z}$, the following limit exists:

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} (M_\mu)_{n-j, n-j-k}.$$

Lemma 2.3 implies that the coefficient $((\pi_{n+j} M_\mu \pi_{n+j})^m)_{n+j, n+j}$ appearing in (14) can be written as a finite sum of products of elements of this form. Therefore, the Laurent coefficients in (14) converge as $n \rightarrow \infty$ through \mathcal{N} , and hence we have the desired ratio asymptotics for the monic orthogonal polynomials. However, convergence to a right limit also implies convergence of the ratio $\kappa_{n+j-1} \kappa_{n+j}^{-1}$ to a (j -dependent) limit as $n \rightarrow \infty$ through \mathcal{N} . Therefore, we have shown that condition (I) of the theorem holds. Notice that this half of the proof did not make use of condition (9).

Now let us suppose that there are functions $\{f_j\}_{j \in \mathbb{Z}}$ as in the statement of condition (I) of the theorem. In this case, we know that $\kappa_{n+j-1} \kappa_{n+j}^{-1}$ converges to a (j -dependent) limit as $n \rightarrow \infty$ through \mathcal{N} . In other words, we know that the following limits exist:

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} M_{n+j+1, n+j}, \quad j \in \mathbb{Z}.$$

Since we are assuming that each of these limits is nonzero, we know that each of the Laurent coefficients appearing in (14) converges as $n \rightarrow \infty$ through \mathcal{N} . If we examine the $m = 1$ term in (14), then we also conclude that the following limits exist:

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} M_{n+j, n+j}, \quad j \in \mathbb{Z}.$$

This will serve as the base case of our induction. Suppose as our induction hypothesis that the following limits exist for all integers $k < q$:

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} M_{n+j-k, n+j}, \quad j \in \mathbb{Z}.$$

We will show that the same limits exist for $k = q$, which will prove that M_μ converges to a right limit as $n \rightarrow \infty$ through \mathcal{N} .

Consider then the coefficient of z^{-q-2} in (14). Lemma 2.3 implies that we can write

$$((\pi_{n+j} M_\mu \pi_{n+j})^{q+1})_{n+j, n+j} = h_{q, n+j}(\mu) + \beta_{n+j, q+1}, \tag{15}$$

where the induction hypothesis implies that $\beta_{n+j, q+1}$ converges as $n \rightarrow \infty$ through \mathcal{N} . Since the left-hand side of (15) also converges as $n \rightarrow \infty$ through \mathcal{N} ,

we conclude that $h_{q,n+j}(\mu)$ converges as $n \rightarrow \infty$ through \mathcal{N} . To complete the proof of this half of the theorem, we need only recall that

$$\frac{\kappa_{n-2}(\mu)}{\kappa_{n-1}(\mu)} = (M_\mu)_{n,n-1}. \tag{16}$$

Our assumptions tell us that each ratio $\{\kappa_{n+\ell-1}\kappa_{n+\ell}^{-1}\}_{\ell \in \mathbb{Z}}$ converges as $n \rightarrow \infty$ through \mathcal{N} and (9) implies that the limit is not zero. Therefore, it must be the case that $\lim_{n \rightarrow \infty} (M_\mu)_{n+j-q,n+j}$ exists for all $j \in \mathbb{Z}$ as $n \rightarrow \infty$ through \mathcal{N} . This completes the induction. \square

As we remarked earlier, the proof of Theorem 1.2 shows that the right limit of M_μ determines the functions $\{f_j\}_{j \in \mathbb{Z}}$ appearing in (10) and also shows that if (9) is satisfied, then the right limit of M_μ is determined by the functions $\{f_j\}_{j \in \mathbb{Z}}$ appearing in (10). This observation will enable us to prove Corollary 1.3, which concerns a necessary and sufficient condition for normalized relative ratio asymptotics through subsequences.

Proof of Corollary 1.3. Suppose that μ and ν exhibit normalized relative ratio asymptotics through $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$. Let \mathcal{N}^* be a subsequence such that M_μ converges to a right limit X as $n \rightarrow \infty$ through $\{n_k\}_{k \in \mathcal{N}^*}$. By Theorem 1.2 we know that, for each $j \in \mathbb{Z}$, there is a function f_j such that

$$\lim_{\substack{k \rightarrow \infty \\ k \in \mathcal{N}^*}} \frac{p_{n_k+j-1}(z; \mu)}{p_{n_k+j}(z; \mu)} = f_j(z), \quad |z| > R_\mu.$$

From the definition of normalized relative ratio asymptotics we see that these same equalities hold and with the same functions $\{f_j\}_{j \in \mathbb{Z}}$ if we replace n_k by m_k and $p_n(z; \mu)$ by $p_n(z; \nu)$. Invoking Theorem 1.2 again shows that M_ν converges to X as $n \rightarrow \infty$ through $\{m_k\}_{k \in \mathcal{N}^*}$.

For the converse, suppose that condition (ii) in the statement of the corollary is true for the sequences $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$. Suppose for contradiction that there are $z_0 \in \{z : |z| > \max\{R_\mu, R_\nu\}\}$, $j \in \mathbb{Z}$, and a subsequence \mathcal{N}^* such that

$$\lim_{\substack{k \rightarrow \infty \\ k \in \mathcal{N}^*}} \left(\frac{p_{n_k+j-1}(z_0; \mu)}{p_{n_k+j}(z_0; \mu)} - \frac{p_{m_k+j-1}(z_0; \nu)}{p_{m_k+j}(z_0; \nu)} \right) = \mathcal{Z} \neq 0.$$

By taking a subsequence of \mathcal{N}^* if necessary we may assume that both M_μ and M_ν converge to a right limit as $n \rightarrow \infty$ through $\{n_k\}_{k \in \mathcal{N}^*}$ and $\{m_k\}_{k \in \mathcal{N}^*}$, respectively. By assumption these must be the same right limit, and hence the sequences $\{f_j\}_{j \in \mathbb{Z}}$ appearing in (10) must be the same. However, this would imply that $\mathcal{Z} = 0$, which is a contradiction.

Notice that the proof of the implication (ii) implies (i) only relies upon the fact that condition (II) of Theorem 1.2 implies condition (I) of Theorem 1.2. Since that implication does not require assumption (9), this proves the last statement of the corollary. \square

In the next section, we will consider applications of our results to random orthogonal polynomials on the unit circle and the real line.

4. Random Orthogonal Polynomials

The results in Section 1 are most useful in settings where we have greater information about the structure of the Bergman shift matrix. In particular, if the measure μ is supported on the unit circle, then the Bergman shift matrix is unitary and has entries that can be written explicitly in terms of the coefficients in the recursion relation satisfied by the monic orthogonal polynomials. If μ is supported on a compact subset of the real line, then the Bergman shift matrix is self-adjoint and banded, and the entries are also expressed in terms of recursion coefficients. In this section, we explore some consequences of our results in cases where the measure is chosen randomly by choosing the recursion coefficients randomly.

4.1. The Unit Circle Case

Suppose μ satisfies $\text{supp}(\mu) \subseteq \partial\mathbb{D}$. In this case, it is well known that the entries of the Bergman shift matrix can be expressed in terms of the Verblunsky coefficients $\{\alpha_n\}_{n \geq 0}$, where $P_n(0; \mu) = -\bar{\alpha}_{n-1}$ (see [17, Sec. 4.1]). Verblunsky’s theorem (see [17, Chap. 1]) tells us that there is a one-to-one correspondence between infinite sequences in \mathbb{D} and nontrivial probability measures on the unit circle. Therefore, we can consider random measures on the unit circle by considering random sequences of Verblunsky coefficients (as in [2; 6; 22; 23]). Our goal is to show that the orthonormal polynomials for a random measure almost surely exhibit a universal ratio asymptotic behavior. The precise meaning of this statement depends on the random distribution from which we select our Verblunsky coefficients.

DEFINITION. If τ is a probability measure satisfying $\tau(\mathbb{D}) = 1$, then the class $K(\tau)$ is the set of all probability measures on the unit circle whose Verblunsky coefficients are all contained in $\text{supp}(\tau) \cap \mathbb{D}$.

Now we are ready to state our main result of this section.

THEOREM 4.1. *Suppose τ is a probability measure satisfying $\tau(\mathbb{D}) = 1$ and let μ be a probability measure on the unit circle chosen randomly by selecting the sequence of Verblunsky coefficients $\{\alpha_n\}_{n \geq 0}$ as i.i.d. random variables with distribution τ . Then almost surely, for every ν in the class $K(\tau)$, there exists a subsequence $\{m_n\}_{n=1}^\infty \subseteq \mathbb{N}$ such that*

$$\lim_{n \rightarrow \infty} \left(\frac{p_{m_n-1}(z; \mu)}{p_{m_n}(z; \mu)} - \frac{p_{m_n-1}(z; \nu)}{p_{m_n}(z; \nu)} \right) = 0, \quad |z| > 1.$$

The proof of this result is relatively straightforward given what we already know. Indeed, since the set $\{p_{n-1}(z; \nu)/p_n(z; \nu)\}_{n \in \mathbb{N}}$ is a normal family, it suffices to show that whenever we have a subsequence through which the orthonormal polynomials for ν exhibit ratio asymptotics, there is a subsequence through which the orthonormal polynomials for μ exhibit the same ratio asymptotics. We have seen that ratio asymptotic behavior is controlled by the right limit behavior of the Bergman shift matrix, so it will suffice to show that the Bergman shift matrix for

the measure μ almost surely exhibits every possible kind of right limit behavior within the class $K(\tau)$. This will follow from basic probability theory.

We begin with some useful lemmas. We have already mentioned our definition of the right limit of a matrix, but for the right limit of a sequence, we use the same definition as in [20]. The form of the Bergman shift matrix when $\text{supp}(\mu) \subseteq \partial\mathbb{D}$ makes it clear that if the sequence of Verblunsky coefficients converges to a right limit through a subsequence $\mathcal{N} \subseteq \mathbb{N}$, then so does the corresponding Bergman shift matrix and through the same subsequence \mathcal{N} .

LEMMA 4.2. *Let τ be a probability measure satisfying $\tau(\mathbb{D}) = 1$. There exists a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $(\mathbb{D} \cap \text{supp}(\tau))^{\mathbb{N}}$ that has all of $\text{supp}(\tau)^{\mathbb{Z}}$ as right limits.*

Proof. Let $S = \{s_1, s_2, \dots\}$ be a countable sequence of points in $\mathbb{D} \cap \text{supp}(\tau)$ that has all of $\text{supp}(\tau)$ as limits of subsequences. Let S^* be the sequence obtained by setting the first element in the sequence equal to s_1 , then following this with all permutations of $\{s_1, s_2\}$, then following this with all permutation of $\{s_1, s_2, s_3\}$, and so on. For example, the beginning of the sequence S^* is

$$S^* = \{s_1, s_1, s_2, s_2, s_1, s_1, s_2, s_3, s_1, s_3, s_2, s_2, \\ s_1, s_3, s_2, s_3, s_1, s_3, s_1, s_2, s_3, s_2, s_1, \dots\}.$$

We claim that S^* is our desired sequence.

To see this, let $\{a_j\}_{j \in \mathbb{Z}}$ be any element of $\text{supp}(\tau)^{\mathbb{Z}}$, and let us write $S^* = \{u_1, u_2, \dots\}$. Since S has all of $\text{supp}(\tau)$ as limits of subsequences, we may for each $j \in \mathbb{Z}$ find a subsequence $\{m_{n,j}\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} s_{m_{n,j}} = a_j.$$

It is straightforward to see that we can thin these subsequences so that each natural number belongs to the subsequence $\{m_{n,j}\}_{n \in \mathbb{N}}$ for at most one integer j . By construction, for every $k \in \mathbb{N}$, the $(2k + 1)$ -tuple $(s_{m_{k,-k}}, s_{m_{k,-k+1}}, \dots, s_{m_{k,k}})$ appears infinitely often as a contiguous block in S^* . So, we can build our desired subsequence $\mathcal{N} = \{n_1, n_2, \dots\}$ by first setting n_1 equal to any k_1 where $u_{k_1} = s_{m_{1,0}}$. For every $j \geq 2$, we set n_j equal to any $k_j > n_{j-1}$ such that

$$u_{k_j-j+1} = s_{m_{j,-j+1}}, \quad \dots, \quad u_{k_j} = s_{m_{j,0}}, \quad \dots, \quad u_{k_j+j-1} = s_{m_{j,j-1}}.$$

With this choice of n_j , we see that, for any $\ell \in \mathbb{Z}$, we have

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} u_{n+\ell} = \lim_{j \rightarrow \infty} u_{n_j+\ell} = \lim_{j \rightarrow \infty} s_{m_{j,\ell}} = a_\ell,$$

which is what we wanted to show. □

LEMMA 4.3. *Let τ be a probability measure satisfying $\tau(\mathbb{D}) = 1$ and suppose that $\{\alpha_n\}_{n \geq 0}$ is a sequence of i.i.d. random variables with distribution τ . Then with probability one, all of $\text{supp}(\tau)^{\mathbb{Z}}$ is a right limit of the sequence $\{\alpha_n\}_{n \geq 0}$.*

Proof. Let $S^* = \{u_1, u_2, \dots\}$ be the sequence whose existence is proven in Lemma 4.2. If $B(x, r)$ denotes the open ball with center x and radius r in the

complex plane, then let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence of positive real numbers converging to zero such that, for every $j \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \prod_{k=1}^j \tau(B(u_k, y_n)) = \infty$$

(it is straightforward to construct such a sequence). Define the events $\{A_n^{(j)}\}_{n \in \mathbb{N}}$ by

$$A_n^{(j)} = \{|\alpha_{n+k-1} - u_k| \leq y_n, k \in \{1, \dots, j\}\},$$

that is, $A_n^{(j)}$ is the event that $\{\alpha_n, \dots, \alpha_{n+j-1}\}$ very closely resembles the beginning of the sequence S^* . It is clear that, for each $j \in \mathbb{N}$, the events $\{A_{3jn}^{(j)}\}_{n \in \mathbb{N}}$ are independent, and

$$\sum_{n=1}^{\infty} \mathbb{P}(A_{3jn}^{(j)}) = \sum_{n=1}^{\infty} \prod_{k=1}^j \tau(B(u_k, y_n)) = \infty.$$

Therefore, the second Borel–Cantelli lemma (see [21, Thm. 7.2.2]) tells us that, with probability 1, we can find a subsequence $\mathcal{N}_j \subseteq \mathbb{N}$ such that

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}_j}} \alpha_{n+k-1} = u_k, \quad k \in \{1, \dots, j\}.$$

It follows that we can construct a subsequence $\{n_k\}_{k=1}^{\infty}$ by choosing any $n_1 \in \mathcal{N}_1$ and then choosing $n_j \in \mathcal{N}_j$ sufficiently large compared to n_{j-1} for every $j \geq 2$ so that

$$\lim_{j \rightarrow \infty} \alpha_{n_j+k-1} = u_k, \quad k \in \mathbb{N}. \tag{17}$$

We conclude that, with probability one, there is a subsequence $\{n_k\}_{k=1}^{\infty}$ such that (17) holds. However, it is straightforward to see that (17) is sufficient to imply that every right limit of the sequence S^* is also a right limit of the sequence $\{\alpha_n\}_{n \in \mathbb{N}}$. By construction, the set of right limits of S^* is all of $\text{supp}(\tau)^{\mathbb{Z}}$, so the desired conclusion follows. \square

Proof of Theorem 4.1. Suppose μ and ν are as in the statement of the theorem. We begin with a simple observation: if $\mathcal{N} \subseteq \mathbb{N}$ is a subsequence such that M_ν converges to a right limit X as $n \rightarrow \infty$ through \mathcal{N} , then by Lemma 4.3 there is almost surely a subsequence $\mathcal{N}' \subseteq \mathbb{N}$ such that M_μ converges to the same right limit X as $n \rightarrow \infty$ through \mathcal{N}' .

Define

$$\mathcal{F}(n, k, r) := \inf_{m > k} \left(\sup_{|z| > r} \left| \frac{p_{m-1}(z; \mu)}{p_m(z; \mu)} - \frac{p_{n-1}(z; \nu)}{p_n(z; \nu)} \right| \right).$$

It suffices to show that, for every $r > 1$ and every subsequence $\{j_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \mathcal{F}(n, j_n, r) = 0. \tag{18}$$

Suppose for contradiction that there exist $r_0 > 1$ and subsequences $\{k_n\}_{n \in \mathbb{N}}$, $\{j_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} \mathcal{F}(k_n, j_{k_n}, r_0) = t > 0.$$

Take a subsequence $\{k'_n\}_{n \in \mathbb{N}}$ of $\{k_n\}_{n \in \mathbb{N}}$ so that M_ν converges to a right limit X_0 as $n \rightarrow \infty$ through $\{k'_n\}_{n \in \mathbb{N}}$. By our earlier observation there is almost surely a subsequence $\{h_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ such that M_μ converges to the right limit X_0 as $n \rightarrow \infty$ through $\{h_n\}_{n \in \mathbb{N}}$, and it is trivial to see that we may refine this subsequence so that $h_n > j_{k_n}$ for all $n \in \mathbb{N}$. Therefore, by Corollary 1.3, for all sufficiently large n , we have

$$\mathcal{F}(k_n, j_{k_n}, r_0) \leq \sup_{|z| > r_0} \left| \frac{p_{h_n-1}(z; \mu)}{p_{h_n}(z; \mu)} - \frac{p_{k_n-1}(z; \nu)}{p_{k_n}(z; \nu)} \right| < \frac{t}{2},$$

which gives us the desired contradiction. □

4.2. The Real Line Case

Suppose μ satisfies $\text{supp}(\mu) \subseteq \mathbb{R}$. In this case, it is well known that the entries of the Bergman shift matrix can be expressed in terms of the Jacobi parameters $\{a_n, b_n\}_{n \geq 1}$, where $a_n > 0$ and $b_n \in \mathbb{R}$. Indeed, the diagonal entries of the Bergman shift matrix are the sequence $\{b_n\}_{n \in \mathbb{N}}$, and the off-diagonal elements are the sequence $\{a_n\}_{n \in \mathbb{N}}$. Favard’s theorem (see [17, Chap. 1]) tells us that there is a one-to-one correspondence between bounded sequences of Jacobi parameters and nontrivial and compactly supported probability measures on the real line. Therefore, we can consider random measures on the real line by considering random sequences of Jacobi parameters. Our goal is to show that the orthonormal polynomials for a random measure almost surely exhibit a universal ratio asymptotic behavior. The precise meaning of this statement depends on the random distribution from which we select our Jacobi parameters.

DEFINITION. If τ_1 is a compactly supported probability measure on $(0, \infty)$ and τ_2 is a compactly supported probability measure on \mathbb{R} , then the class $K(\tau_1, \tau_2)$ is the set of all probability measures on the real line whose Jacobi parameters $\{a_n, b_n\}_{n \in \mathbb{N}}$ satisfy

$$a_n \in \text{supp}(\tau_1), \quad b_n \in \text{supp}(\tau_2), \quad n \in \mathbb{N}.$$

The class $K^*(\tau_1, \tau_2)$ is the set of all biinfinite tridiagonal symmetric matrices having diagonal entries in $\text{supp}(\tau_2)$ and off-diagonal entries in $\text{supp}(\tau_1)$.

Now we are ready to state the analog of Theorem 4.1 for measures on the real line.

THEOREM 4.4. *Suppose that τ_1 is a compactly supported probability measure on $(0, \infty)$ and τ_2 is a compactly supported probability measure on \mathbb{R} . Let μ be a probability measure on the real line chosen randomly by selecting the sequence of off-diagonal Jacobi parameters $\{a_n\}_{n \in \mathbb{N}}$ as i.i.d. random variables with distribution τ_1 and diagonal Jacobi parameters $\{b_n\}_{n \in \mathbb{N}}$ as i.i.d. random variables with*

distribution τ_2 . Then, almost surely, for every ν in the class $K(\tau_1, \tau_2)$, there exists a subsequence $\{m_n\}_{n=1}^\infty \subseteq \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} \left(\frac{p_{m_n-1}(z; \mu)}{p_{m_n}(z; \mu)} - \frac{p_{n-1}(z; \nu)}{p_n(z; \nu)} \right) = 0, \quad |z| > \max\{R_\mu, R_\nu\}.$$

The proof of Theorem 4.4 is very similar to that of Theorem 4.1. The only substantive modification is in the analog of Lemma 4.2, which we now provide.

LEMMA 4.5. *Suppose that τ_1 is a compactly supported probability measure on $(0, \infty)$ and τ_2 is a compactly supported probability measure on \mathbb{R} . There exists a Jacobi matrix J^* in the class $K(\tau_1, \tau_2)$ that has all of $K^*(\tau_1, \tau_2)$ as right limits.*

Proof. Let $S = \{s_1, s_2, \dots\}$ be a countable set in $\text{supp}(\tau_1)$ that has all of $\text{supp}(\tau_1)$ as limits of subsequences, and let $T = \{t_1, t_2, \dots\}$ be a countable set in $\text{supp}(\tau_2)$ that has all of $\text{supp}(\tau_2)$ as limits of subsequences. Let S^* be the sequence obtained by setting the first element in the sequence equal to s_1 , then following this with two consecutive appearances of each permutation of $\{s_1, s_2\}$. The next elements of S^* will be six consecutive appearances of each permutation of $\{s_1, s_2, s_3\}$. This pattern is repeated by including $n!$ appearances of each permutation of $\{s_1, \dots, s_n\}$. For example, the beginning of the sequence S^* is

$$S^* = \{s_1, s_1, s_2, s_1, s_2, s_2, s_1, s_2, s_1, s_1, s_2, s_3, s_1, s_2, s_3, s_1, s_2, s_3, s_1, s_2, s_3, s_1, s_2, s_3, \dots\}.$$

Let T^* be the sequence obtained by setting the first element in the sequence equal to t_1 , then following this with every possible permutation of $\{t_1, t_2\}$ repeated two times, then following this with every possible permutation of $\{t_1, t_2, t_3\}$ repeated six times, and so on. At each step of the construction, we will add on every possible permutation of $\{t_1, \dots, t_n\}$ repeated $n!$ times. For example, the beginning of the sequence T^* is

$$T^* = \{t_1, t_1, t_2, t_2, t_1, t_1, t_2, t_2, t_1, t_1, t_2, t_3, t_1, t_3, t_2, t_2, t_1, t_3, t_3, t_1, t_3, t_1, t_2, t_3, t_2, t_1, \dots\}.$$

We claim that we may set J^* equal to the matrix with the sequence T^* on the diagonal and S^* on the off-diagonals.

To see this, let $\{a_j\}_{j \in \mathbb{Z}}$ be any element of $\text{supp}(\tau_1)^\mathbb{Z}$, and let $\{b_j\}_{j \in \mathbb{Z}}$ be any element of $\text{supp}(\tau_2)^\mathbb{Z}$. Let us also write $S^* = \{u_n\}_{n \in \mathbb{N}}$ and $T^* = \{v_n\}_{n \in \mathbb{N}}$. By the properties of S and T , for each pair $j, k \in \mathbb{Z}$, we may find subsequences $\{m_{n,j}\}_{n \in \mathbb{N}}, \{q_{n,k}\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} s_{m_{n,j}} = a_j, \quad \lim_{n \rightarrow \infty} t_{q_{n,k}} = b_k.$$

It is straightforward to see that we can thin these subsequences so that each natural number belongs to the subsequence $\{m_{n,j}\}_{n \in \mathbb{N}}$ for at most one integer j and to the subsequence $\{q_{n,k}\}_{n \in \mathbb{N}}$ for at most one integer k .

By construction, for every $k \in \mathbb{N}$, the $(2k + 1)$ -tuple $(s_{m_{k,-k}}, s_{m_{k,-k+1}}, \dots, s_{m_{k,k}})$ appears infinitely often as a contiguous block in S^* and at the same position in S^*

as the block $(t_{qk,-k}, t_{qk,-k+1}, \dots, t_{qk,k})$ in T^* . So, we can build our desired subsequence $\mathcal{N} = \{n_1, n_2, \dots\}$ by first setting n_1 equal to any ℓ_1 where $u_{\ell_1} = s_{m_1,0}$ and $v_{\ell_1} = t_{q_1,0}$. For every $j \geq 2$, we set n_j equal to any $\ell_j > n_{j-1}$ such that

$$\begin{aligned} u_{\ell_j-j+1} &= s_{m_{j,-j+1}}, & \dots, & & u_{\ell_j} &= s_{m_{j,0}}, & \dots, & & u_{\ell_j+j-1} &= s_{m_{j,j-1}}; \\ v_{\ell_j-j+1} &= t_{q_{j,-j+1}}, & \dots, & & v_{\ell_j} &= t_{q_{j,0}}, & \dots, & & v_{\ell_j+j-1} &= t_{q_{j,j-1}}. \end{aligned}$$

With this choice of n_j , we see that, for any $r \in \mathbb{Z}$, we have

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} u_{n+r} &= \lim_{j \rightarrow \infty} u_{n_j+r} = \lim_{j \rightarrow \infty} s_{m_{j,r}} = a_r, \\ \lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} v_{n+r} &= \lim_{j \rightarrow \infty} v_{n_j+r} = \lim_{j \rightarrow \infty} t_{q_{j,r}} = b_r, \end{aligned}$$

as desired. □

With Lemma 4.5 in hand, it is now a simple matter to adapt the proof of Theorem 4.1 to complete the proof of Theorem 4.4, so we omit the details.

In the next section, we consider some additional examples that highlight some important applications of the results from Section 1.

5. Further Examples

5.1. Example: Alexandrov Measures on the Unit Circle

Suppose that μ is a probability measure supported on $\partial\mathbb{D}$ with Verblunsky coefficients $\{\alpha_n\}_{n=0}^\infty$. For every $\lambda \in \partial\mathbb{D}$, we can consider the measure μ_λ whose Verblunsky coefficients are related to those of μ by

$$\alpha_n(\mu_\lambda) = \lambda \alpha_n(\mu), \quad n \in \{0, 1, 2, 3, \dots\}.$$

The family of measures $\{\mu_\lambda\}_{\lambda \in \partial\mathbb{D}}$ is called the family of *Alexandrov measures* for the measure μ (see [17]).

If λ_1 and λ_2 are distinct complex numbers in the unit circle, then the relationship between μ_{λ_1} and μ_{λ_2} is highly nontrivial. Indeed, it is well known that these two measures need not be mutually absolutely continuous (see [18, Sec. 10.3]). However, the structure of the Bergman shift matrix reveals that $M_{\mu_{\lambda_1}}$ and $M_{\mu_{\lambda_2}}$ differ only in the first row, and thus the unique right limit of their difference is zero. Indeed, the formulas in [17, Sec. 4.1] or [4, eq. (28)] tell us that

$$(M_\mu)_{ij} = \begin{cases} -\alpha_{i-2} \bar{\alpha}_{j-1} \prod_{k=i-1}^{j-2} \sqrt{1 - |\alpha_k|^2}, & i \leq j, \\ \sqrt{1 - |\alpha_{j-1}|^2}, & i = j + 1, \end{cases}$$

where we set $\alpha_{-1} = -1$ and all unspecified entries equal to zero. It follows at once from Corollary 1.3 that the orthonormal polynomials for μ_{λ_1} and μ_{λ_2} exhibit normalized relative ratio asymptotics. We remark that this fact could also be deduced from [5, Thm. 1.5].

5.2. Example: Coefficient Stripping

If $\{\alpha_0, \alpha_1, \dots\}$ is a sequence of Verblunsky coefficients for a measure μ supported on $\partial\mathbb{D}$, then we can associate to it the measure μ^1 that has Verblunsky coefficient sequence $\{\alpha_1, \alpha_2, \dots\}$ and is called the *once stripped* measure (see [17, Sec. 3.4]). Similarly, for every $k \in \mathbb{N}$, the measure μ^k is defined as the measure corresponding to the Verblunsky coefficient sequence $\{\alpha_k, \alpha_{k+1}, \dots\}$. The orthonormal polynomials corresponding to the measure μ^k are often referred to as the *kth associated polynomials* (see [24]). It is easy to see that if M_μ approaches a right limit as $n \rightarrow \infty$ through $\mathcal{N} \subseteq \mathbb{N}$, then M_{μ^k} approaches that same right limit as $n \rightarrow \infty$ through $\mathcal{N} - k$. It follows from Corollary 1.3 that

$$\lim_{n \rightarrow \infty} \left[\frac{p_{n-k-1}(z; \mu^k)}{p_{n-k}(z; \mu^k)} - \frac{p_{n-1}(z; \mu)}{p_n(z; \mu)} \right] = 0, \quad |z| > 1.$$

A similar result holds for stripped orthogonal polynomials on the real line.

5.3. Example: Measures on the Real Line

In this example, we revisit a result proven by Nevai and Van Assche [10]. Let us consider the case in which the measures of orthogonality μ and ν are each supported on (perhaps different) compact subsets of the real line. In this case, the Bergman shift matrix M_μ is symmetric and is zero away from its three main diagonals (and similarly for M_ν). If we label the diagonal elements of M_μ as $\{b_n(\mu)\}_{n \in \mathbb{N}}$ and the off diagonal elements as $\{a_n(\mu)\}_{n \in \mathbb{N}}$, then hypothesis (9) is equivalent to the condition $\inf_{n \in \mathbb{N}} a_n(\mu) > 0$. Corollary 1.3 then implies that if $\inf_{n \in \mathbb{N}} a_n(\mu)a_n(\nu) > 0$, then μ and ν exhibit normalized relative ratio asymptotics if and only if $M_\mu - M_\nu$ is compact, which in turn implies that the essential support of μ and ν is the same (this is by Weyl’s theorem; see [11, Thm. S.13]).

5.4. Example: Degenerate Cases

If hypothesis (9) is removed, then conclusion of Corollary 1.3 fails because it is possible that both μ and ν exhibit normalized ratio asymptotics with limit function 0, but the matrix $M_\mu - M_\nu$ has a nonzero right limit. To make this more concrete, we appeal to measures on the unit circle.

Define the measures μ and ν through their Verblunsky coefficients by

$$\alpha_n(\mu) = 1 - \frac{1}{n+2}, \quad \alpha_n(\nu) = \left(1 - \frac{1}{n+2}\right)e^{in}, \quad n \geq 0.$$

A similar pair of measures provided an illustrative example in [14]. It is well known that $\kappa_n(\mu)\kappa_{n+1}(\mu)^{-1} = \sqrt{1 - |\alpha_n(\mu)|^2}$ (see [17, eq. (1.5.22)]) and similarly for ν . Since $\{P_{n-1}(z; \mu)P_n(z; \mu)^{-1}\}_{n \in \mathbb{N}}$ is a normal family on $\{z : |z| > 1\}$, we have

$$\lim_{n \rightarrow \infty} \frac{P_{n-1}(z; \mu)}{P_n(z; \mu)} = 0 = \lim_{n \rightarrow \infty} \frac{P_{n-1}(z; \nu)}{P_n(z; \nu)}, \quad |z| > 1,$$

so μ and ν exhibit normalized relative ratio asymptotics. However, by appealing to the formulas in Example 5.1 we see that, for $n > 1$,

$$(M_\mu)_{n,n} = -\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{1}{n}\right),$$

$$(M_\nu)_{n,n} = -\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{1}{n}\right)e^{-i}.$$

From this it follows easily that 0 is not the unique right limit of $M_\mu - M_\nu$, and we see that condition (9) is necessary in the statement of Corollary 1.3. Furthermore, the measure ν shows that (9) is an essential assumption in Theorem 1.2.

ACKNOWLEDGMENTS. We thank Nikos Stylianopoulos for bringing the important paper [1] to our attention. We thank the anonymous referees for useful suggestions and especially for their suggestion to consult references [3; 4; 7; 9; 10; 24; 25]. We also thank P. Nevai for useful discussion of previous results and for helping us find the reference [7].

References

- [1] B. Beckermann, *Complex Jacobi matrices*, J. Comput. Appl. Math. 127 (2001), 17–65.
- [2] E. B. Davies and B. Simon, *Eigenvalue estimates for non-normal matrices and the zeros of random orthogonal polynomials on the unit circle*, J. Approx. Theory 141 (2006), no. 2, 189–213.
- [3] J. Geronimo, E. III Harrell, and W. Van Assche, *On the asymptotic distribution of eigenvalues of banded matrices*, Constr. Approx. 4 (1988), no. 4, 403–417.
- [4] L. Golinskii, P. Nevai, and W. Van Assche, *Perturbation of orthogonal polynomials on an arc of the unit circle*, J. Approx. Theory 83 (1995), no. 3, 392–422.
- [5] L. Golinskii and A. Zlatoš, *Coefficients of orthogonal polynomials on the unit circle and higher-order Szegő theorems*, Constr. Approx. 26 (2007), no. 3, 361–382.
- [6] R. Killip and M. Stoiciu, *Eigenvalue statistics for CMV matrices: from Poisson to clock via random matrix ensembles*, Duke Math. J. 146 (2009), no. 3, 361–399.
- [7] A. Máté, P. Nevai, and V. Totik, *Mean Cesáro summability of orthogonal polynomials*, Constructive theory of functions, pp. 588–599, Bulg. Acad. Sci, Sofia, 1984.
- [8] ———, *Extensions of Szegő's theory of orthogonal polynomials II*, Constr. Approx. 3 (1987), no. 1, 51–72.
- [9] P. Nevai and J. S. Dehesa, *On asymptotic average properties of zeros of orthogonal polynomials*, SIAM J. Math. Anal. 10 (1979), 1184–1192.
- [10] P. Nevai and W. Van Assche, *Compact perturbations of orthogonal polynomials*, Pacific J. Math. 153 (1992), no. 1, 163–184.
- [11] M. Reed and B. Simon, *Methods of modern mathematical physics I: functional analysis*, Academic Press, New York–London, 1972.
- [12] L. Robert and L. Santiago, *Finite sections method for Hessenberg matrices*, J. Approx. Theory 123 (2003), no. 1, 68–88.
- [13] E. B. Saff and N. Stylianopoulos, *Asymptotics for Hessenberg matrices for the Bergman shift operator on Jordan regions*, Complex Anal. Oper. Theory 8 (2014), no. 1, 1–24.

- [14] B. Simanek, *Ratio asymptotics, Hessenberg matrices, and weak asymptotic measures*, Int. Math. Res. Not. IMRN 24 (2014), 6798–6825.
- [15] ———, *The Bergman shift operator on polynomial lemniscates*, Constr. Approx. 41 (2015), no. 1, 113–131.
- [16] B. Simon, *Ratio asymptotics and weak asymptotic measures for orthogonal polynomials on the real line*, J. Approx. Theory 126 (2004), 198–217.
- [17] ———, *Orthogonal polynomials on the unit circle, part one: classical theory*, American Mathematical Society, Providence, RI, 2005.
- [18] ———, *Orthogonal polynomials on the unit circle, part two: spectral theory*, American Mathematical Society, Providence, RI, 2005.
- [19] ———, *Weak convergence of CD kernels and applications*, Duke Math. J. 146 (2009), 305–330.
- [20] ———, *Szegő's theorem and its descendants: spectral theory for L^2 perturbations of orthogonal polynomials*, Princeton University Press, Princeton, NJ, 2010.
- [21] ———, *A comprehensive course in analysis, part I: real analysis*, American Mathematical Society, Providence, RI, 2015.
- [22] M. Stoiciu, *The statistical distribution of the zeros of random paraorthogonal polynomials on the unit circle*, J. Approx. Theory 139 (2006), no. 1–2, 29–64.
- [23] A. V. Teplyaev, *The pure point spectrum of random orthogonal polynomials on the circle* (Russian), Dokl. Akad. Nauk SSSR 320 (1991), no. 1, 49–53, Translation in Sov. Math., Dokl. 44 (1992), no. 2, 407–411.
- [24] W. Van Assche, *Orthogonal polynomials, associated polynomials, and functions of the second kind*, J. Comput. Appl. Math. 37 (1991), 237–249.
- [25] ———, *Weak convergence of orthogonal polynomials*, Indag. Math. (N.S.) 6 (1995), 7–23.

Department of Mathematics
Baylor University
One Bear Place #97328
Waco, TX 76798
USA

Brian_Simanek@baylor.edu