

Slopes of Fibered Surfaces with a Finite Cyclic Automorphism

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ABSTRACT. We study slopes of finite cyclic covering fibrations of a fibered surface. We give the best possible lower bound of the slope of these fibrations. We also give the slope equality of finite cyclic covering fibrations of a ruled surface and observe the local concentration of the global signature of these surfaces on a finite number of fiber germs. We also give an upper bound of the slope of finite cyclic covering fibrations of a ruled surface.

Introduction

Let $f: S \rightarrow B$ be a surjective morphism from a complex smooth projective surface S to a smooth projective curve B with connected fibers. The datum (S, f, B) or simply f is called a *fibered surface* or a fibration. A fibered surface f is said to be *relatively minimal* if there exist no (-1) -curves contained in fibers of f , where a (-1) -curve is a nonsingular rational curve with self-intersection number -1 . The genus g of a fibered surface f is defined to be that of a general fiber of f . We put $K_f = K_S - f^*K_B$ and call it the relative canonical bundle.

Assume that $f: S \rightarrow B$ is a relatively minimal fibration of genus $g \geq 2$ and consider the following three relative invariants:

$$\begin{aligned} \chi_f &:= \chi(\mathcal{O}_S) - (g - 1)(b - 1), \\ K_f^2 &= K_S^2 - 8(g - 1)(b - 1), \\ e_f &:= e(S) - 4(g - 1)(b - 1), \end{aligned}$$

where b and $e(S)$ respectively denote the genus of the base curve B and the topological Euler characteristic of S . Then the following are well known:

- (Noether) $12\chi_f = K_f^2 + e_f$.
- (Arakelov) K_f is nef.
- (Ueno) $\chi_f \geq 0$, and $\chi_f = 0$ if and only if f is locally trivial (i.e., a holomorphic fiber bundle).
- (Segre) $e_f \geq 0$, and $e_f = 0$ if and only if f is smooth.

When f is not locally trivial, we put

$$\lambda_f = \frac{K_f^2}{\chi_f}$$

and call it the *slope* of f . Then we see $0 < \lambda_f \leq 12$ from the results mentioned. The slope of a fibration has proven to be sensible to a lot of geometric properties, both of the fibers of f and of the surface S itself (see [1]). A fibration of slope 12 is called a Kodaira fibration, first examples of which were constructed by Kodaira [8]. In particular, the upper bound is sharp. As to the lower bound, Xiao [15] showed the inequality

$$\lambda_f \geq 4 - \frac{4}{g},$$

which is nowadays called the *slope inequality*. Furthermore, fibrations with slope $4 - 4/g$ are turned out to be of hyperelliptic type ([9] and [15]). Hence, a sharper lower bound is expected for nonhyperelliptic fibrations.

It is generally believed that there is the lower bound of the slope depending on the gonality (or the Clifford index) of fibrations. Though there are several attempts, a general bound is still in fancy. To attack such a problem, one of the most hopeful strategies may be to extend a special kind of linear system (e.g., a gonality pencil) or an automorphism of a general fiber to the whole surface and to study the fibration through the covering structure on S thus obtained. However, it is usually impossible to have the desired extension, mainly because the object in question on the fiber is not unique and the monodromy forces it to change from one to another (see [3]). So, we are naturally led to consider the toy case that the fibration $f : S \rightarrow B$ is obtained from another fibration $W \rightarrow B$ via a covering map $S \rightarrow W$. Besides the hyperelliptic fibrations, one of the remarkable results in this direction is obtained by Cornalba and Stoppino [6]. They gave a lower bound for the slope of double covering fibrations and constructed examples showing its sharpness, extending a former result for bielliptic fibrations by Barja [2]. See also [12] for recent developments of double covering fibrations.

In this paper, we consider fibered surfaces induced from a particular type of cyclic coverings and try to generalize results for double coverings. More precisely, we shall work in the following situation. A relatively minimal fibration $f : S \rightarrow B$ of genus $g \geq 2$ is called a *primitive cyclic covering fibration* of type (g, h, n) if there exist a (not necessarily relatively minimal) fibration $\tilde{\varphi} : \tilde{W} \rightarrow B$ of genus $h \geq 0$ and a finite cyclic covering

$$\tilde{\theta} : \tilde{S} = \text{Spec}_{\tilde{W}} \left(\bigoplus_{j=0}^{n-1} \mathcal{O}_{\tilde{W}}(-j\tilde{d}) \right) \rightarrow \tilde{W}$$

of order n branched along a smooth curve $\tilde{R} \in |n\tilde{d}|$ for some $n \geq 2$ and $\tilde{d} \in \text{Pic}(\tilde{W})$ such that f is the relatively minimal model of $\tilde{f} := \tilde{\varphi} \circ \tilde{\theta}$. Here, we put the adjective “primitive” because a finite cyclic covering between nonsingular surfaces is not necessarily obtained in this way. Nevertheless, the assumption would be acceptable because the situation in [6] is exactly the case $n = 2$ and $h > 0$, and a hyperelliptic fibration is nothing more than a primitive cyclic covering fibration of type $(g, 0, 2)$. Furthermore, Kodaira fibrations in [8] are among primitive cyclic covering fibrations with \tilde{W} being a product of two curves.

As the first main result, we shall show the following:

THEOREM 0.1. *Let $f: S \rightarrow B$ be a primitive cyclic covering fibration of type (g, h, n) . If $h \geq 1$ and $g \geq (2n - 1)(2hn + n - 1)/(n + 1)$, then we have*

$$K_f^2 \geq \frac{24(g - 1)(n - 1)}{2(2n - 1)(g - 1) - n(n + 1)(h - 1)} \chi_f.$$

Moreover, we will construct an example showing that the inequality is sharp (Example 2.5). Putting $n = 2$, we recover Cornalba–Stoppino’s inequality shown in [6]. The essential idea of the proof is analogous to the double covering case. We take a relatively minimal model $\varphi: W \rightarrow B$ of $\tilde{\varphi}: \tilde{W} \rightarrow B$ (unique when $h > 0$) and consider the effects to the invariants of the “canonical resolution” of singular points of the branch curve $R \subset W$ (obtained as the direct image of \tilde{R}). One of the striking facts in our setting is that the multiplicities of singular points must be either 0 or 1 modulo n (Lemma 1.5).

When $h = 0$, we can move from one relatively minimal model $\varphi: W \rightarrow B$ to another via elementary transformations among \mathbb{P}^1 -bundles, in order to standardize the branch locus. This enables us to prove a more accurate result. Namely, we shall show the *slope equality* for them.

THEOREM 0.2. *There exists a function $\text{Ind}: \mathcal{A}_{g,0,n} \rightarrow \mathbb{Q}_{\geq 0}$ from the set $\mathcal{A}_{g,0,n}$ of all fiber germs of primitive cyclic covering fibrations of type $(g, 0, n)$ such that $\text{Ind}(F_p) = 0$ for a general $p \in B$ and*

$$K_f^2 = \frac{24(g - 1)(n - 1)}{2(2n - 1)(g - 1) + n(n + 1)} \chi_f + \sum_{p \in B} \text{Ind}(F_p)$$

for any primitive cyclic covering fibration $f: S \rightarrow B$ of type $(g, 0, n)$.

This is a generalization of the hyperelliptic case (see [16]). The number $\text{Ind}(F_p)$ is called the *Horikawa index* of the fiber F_p and is defined in terms of *singularity indices* (Definition 4.1) over $p \in B$ of the branch locus (see Theorem 4.3 for details). This enables us to observe that the signature of S is concentrated on singular fibers by introducing the *local signature* of a fiber (Corollary 4.5). For a general discussion on the slope equalities, Horikawa index and the local signature, see [1].

As the third result, we will give an upper bound of the slope.

THEOREM 0.3. *Let $f: S \rightarrow B$ be a primitive cyclic covering fibration of type $(g, 0, n)$ and assume that $n \geq 4$. Put*

$$r := \frac{2g}{n - 1} + 2, \quad \delta := \begin{cases} 0 & \text{if } r \in 2n\mathbb{Z}, \\ 1 & \text{if } r \notin 2n\mathbb{Z}. \end{cases}$$

Then, the following hold:

(1) *If $n \leq r < n(n - 1)$, then*

$$K_f^2 \leq \left(12 - \frac{48n^2(r - 1)}{(n - 1)(n + 1)(r^2 - \delta n^2)} \right) \chi_f.$$

(2) If $r \geq n(n-1)$, then

$$K_f^2 \leq \left(12 - \frac{48n(n-1)(r-1)}{n(n+1)r^2 - 8(2n-1)r + 24n - \delta n^3(n+1)} \right) \chi_f.$$

This in particular implies that we cannot have a Kodaira fibration when $h = 0$ and $n \geq 4$, even if we consider not only Kodaira's original constructions but also such an extended category of primitive cyclic covering fibrations. Unfortunately, it seems that our method does not work well when $n = 3$, and we fail to obtain an upper bound. We leave it as a future study. An upper bound for the slope of hyperelliptic fibrations was obtained by Matsusaka [13], which was improved by Xiao [17].

1. Primitive Cyclic Covering Fibrations

Throughout the paper, n denotes an integer greater than 1. We begin with the following elementary lemma. Note that it does not hold for $n = 3$, as the case $a = b = 1$ shows.

LEMMA 1.1. *Let n be a positive integer, and a, b integers such that $\gcd(a, b, n) = 1$. If $n \geq 4$, then either $a + 2b \notin n\mathbb{Z}$ or $2a + b \notin n\mathbb{Z}$.*

Proof. Suppose on the contrary that $a + 2b \in n\mathbb{Z}$ and $2a + b \in n\mathbb{Z}$. Then we have $3(a + b) \in n\mathbb{Z}$ and $a - b \in n\mathbb{Z}$ by adding and substituting them. If $n \notin 3\mathbb{Z}$, then it follows from $3(a + b) \in n\mathbb{Z}$ that $a + b \in n\mathbb{Z}$, and we conclude $a, b \in n\mathbb{Z}$, which contradicts $\gcd(a, b, n) = 1$. So we can assume that $n \in 3\mathbb{Z}$. Put $n = 3k$ with an integer $k \geq 2$. Since $a - b \in n\mathbb{Z}$, we may write $a = b + 3kl$ with an integer l . Then $2a + b = 3(b + 2kl)$. Since $2a + b \in n\mathbb{Z} = 3k\mathbb{Z}$, we have $b \in k\mathbb{Z}$. Then, it follows that $a \in k\mathbb{Z}$, contradicting $\gcd(a, b, n) = 1$. \square

Let X be a smooth projective surface, and $\sigma \in \text{Aut}(X)$ a holomorphic automorphism of X of order n . We denote by $\text{Fix}(\sigma)$ the set of all fixed points of σ .

Take a point $x \in \text{Fix}(\sigma)$ and an open neighborhood of U of x such that $\sigma(U) = U$. Let (z_1, z_2) be a system of local coordinates on U with $x = (0, 0)$ and write $\sigma(z) = (\sigma_1(z_1, z_2), \sigma_2(z_1, z_2))$. If $\sigma_i(z_1, z_2) = a_{i,1}z_1 + a_{i,2}z_2 + \cdots$ is the expansion around x , then the Jacobian matrix at x is given by

$$(J\sigma)_x = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

Since $\sigma^n = \text{Id}$, we may assume that

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} \zeta^{k_1} & 0 \\ 0 & \zeta^{k_2} \end{pmatrix},$$

where $\zeta = \exp(2\pi\sqrt{-1}/n)$, and k_1, k_2 are integers satisfying $0 \leq k_1 \leq k_2 \leq n-1$. Since $\sigma \neq \text{Id}$, we have $k_2 > 0$. It is clear that x is a smooth point on a one-dimensional fixed locus when $k_1 = 0$ and that it is an isolated fixed point when $k_1 > 0$. Since σ is of order n , we have $\gcd(k_1, k_2, n) = 1$. If the Jacobian matrix at x has the canonical form described before, then we call x a fixed point of type (k_1, k_2) . Let $\rho_1: X_1 \rightarrow X$ be the blow-up at a fixed point x of type (k_1, k_2) .

Put $E := \rho_1^{-1}(x)$ and let σ_1 be the automorphism of X_1 of order n induced by σ . Then we easily see that $E \subset \text{Fix}(\sigma_1)$ when $k_1 = k_2$ and that there are exactly two isolated fixed points on E of respective types $(k_1, k_2 - k_1)$ and $(k_1 - k_2, k_2)$ when $k_1 \neq k_2$. With this remark, the following can be shown by using Lemma 1.1 repeatedly.

LEMMA 1.2. *Let X be a smooth projective surface with an automorphism σ of order n . If there exists a birational morphism $\rho: \tilde{X} \rightarrow X$ such that the automorphism $\tilde{\sigma}$ on \tilde{X} induced by σ has no isolated fixed points, then either $n \leq 3$, or any isolated fixed point of σ is of type (k, k) for some k ($0 < k < n, \text{gcd}(k, n) = 1$).*

We apply the previous observation to the situation we are interested in. Before going further, we need some preparations.

Let Y be a smooth projective surface, and R an effective divisor on Y divisible by n in the Picard group $\text{Pic}(Y)$, that is, R is linearly equivalent to $n\mathfrak{d}$ for some divisor $\mathfrak{d} \in \text{Pic}(Y)$. Then we can construct a finite n -sheeted covering of Y with branch locus R as follows. Put $\mathcal{A} = \bigoplus_{j=0}^{n-1} \mathcal{O}_Y(-j\mathfrak{d})$ and introduce a graded \mathcal{O}_Y -algebra structure on \mathcal{A} by multiplying the section of $\mathcal{O}_Y(n\mathfrak{d})$ defining R . According to [4], we call $Z := \text{Spec}_Y(\mathcal{A})$ equipped with the natural surjective morphism $\varphi: Z \rightarrow Y$ a classical n -cyclic covering of Y branched over R . Locally, Z is defined by $z^n = r(x, y)$, where $r(x, y)$ denotes the local analytic equation of R . From this we see that Z is normal if and only if R is reduced and is smooth if and only if so is R . When Z is smooth, we have

$$\varphi^*R = nR_0, \quad K_Z = \varphi^*K_Y + (n - 1)R_0, \quad \text{Aut}(Z/Y) \simeq \mathbb{Z}/n\mathbb{Z}, \quad (1.1)$$

where R_0 is the effective divisor (usually called the ramification divisor) on Z defined locally by $z = 0$, and $\text{Aut}(Z/Y)$ is the covering transformation group for φ .

DEFINITION 1.3. A relatively minimal fibration $f: S \rightarrow B$ of genus $g \geq 2$ is called a primitive cyclic covering fibration of type (g, h, n) if there exist a (not necessarily relatively minimal) fibration $\tilde{\varphi}: \tilde{W} \rightarrow B$ of genus $h \geq 0$ and a classical n -cyclic covering

$$\tilde{\theta}: \tilde{S} = \text{Spec}_{\tilde{W}} \left(\bigoplus_{j=0}^{n-1} \mathcal{O}_{\tilde{W}}(-j\tilde{\mathfrak{d}}) \right) \rightarrow \tilde{W}$$

branched over a smooth curve $\tilde{R} \in |n\tilde{\mathfrak{d}}|$ for some $n \geq 2$ and $\tilde{\mathfrak{d}} \in \text{Pic}(\tilde{W})$ such that f is the relatively minimal model of $\tilde{f} := \tilde{\varphi} \circ \tilde{\theta}$.

REMARK 1.4. Note that a finite cyclic covering between complex manifolds is not necessarily obtained in this way. For example, the most typical cyclic n -sheeted covering between complex lines given by $z \mapsto z^n$ branches over only two points (0 and ∞). This is why we put the adjective ‘‘primitive’’ to our cyclic covering fibrations.

Let $f: S \rightarrow B$ be a primitive cyclic covering fibration of type (g, h, n) . We freely use the notation in Definition 1.3. Let \tilde{F} and $\tilde{\Gamma}$ be general fibers of \tilde{f} and $\tilde{\varphi}$,

respectively. Then the restriction map $\tilde{\theta}|_{\tilde{F}}: \tilde{F} \rightarrow \tilde{\Gamma}$ is a classical n -cyclic covering branched over $\tilde{R} \cap \tilde{\Gamma}$. Since the genera of \tilde{F} and $\tilde{\Gamma}$ are g and h , respectively, the Hurwitz formula gives us

$$r := \tilde{R}\tilde{\Gamma} = \frac{2(g-1-n(h-1))}{n-1}. \quad (1.2)$$

Note that r is a multiple of n . Let $\tilde{\sigma}$ be a generator of $\text{Aut}(\tilde{S}/\tilde{W}) \simeq \mathbb{Z}/n\mathbb{Z}$, and $\rho: \tilde{S} \rightarrow S$ the natural birational morphism. By assumption, $\text{Fix}(\tilde{\sigma})$ is a disjoint union of smooth curves and $\tilde{\theta}(\text{Fix}(\tilde{\sigma})) = \tilde{R}$. Let $\varphi: W \rightarrow B$ be a relatively minimal model of $\tilde{\varphi}$, and $\psi: \tilde{W} \rightarrow W$ the natural birational morphism. Since $\tilde{\psi}$ is a succession of blow-ups, we can write $\tilde{\psi} = \psi_1 \circ \cdots \circ \psi_N$, where $\psi_i: W_i \rightarrow W_{i-1}$ denotes the blow-up at $x_i \in W_{i-1}$ ($i = 1, \dots, N$) with $W_0 = W$ and $W_N = \tilde{W}$. We define reduced curves R_i on W_i inductively as $R_{i-1} = (\psi_i)_* R_i$ starting from $R_N = \tilde{R}$ down to $R_0 =: R$. We also put $E_i = \psi_i^{-1}(x_i)$ and $m_i = \text{mult}_{x_i}(R_{i-1})$ for $i = 1, 2, \dots, N$.

LEMMA 1.5. *With this notation, the following hold for any $i = 1, \dots, N$.*

- (1) *Either $m_i \in n\mathbb{Z}$ or $m_i \in n\mathbb{Z} + 1$. Moreover, $m_i \in n\mathbb{Z}$ if and only if E_i is not contained in R_i .*
- (2) *$R_i = \psi_i^* R_{i-1} - n[\frac{m_i}{n}]E_i$, where $[t]$ is the greatest integer not exceeding t .*
- (3) *There exists $\mathfrak{d}_i \in \text{Pic}(W_i)$ such that $\mathfrak{d}_i = \psi_i^* \mathfrak{d}_{i-1} - [\frac{m_i}{n}]E_i$ and $R_i \sim n\mathfrak{d}_i$, $\mathfrak{d}_N = \tilde{\mathfrak{d}}$.*

Proof. Since $\tilde{R} = R_N$ is reduced, every R_i is reduced. Set $\mathfrak{d}_N = \tilde{\mathfrak{d}}$. Since $\text{Pic}(W_N) = \psi_N^* \text{Pic}(W_{N-1}) \oplus \mathbb{Z}[E_N]$, there exist $\mathfrak{d}_{N-1} \in \text{Pic}(W_{N-1})$ and $d_N \in \mathbb{Z}$ such that $\mathfrak{d}_N = \psi_N^* \mathfrak{d}_{N-1} - d_N E_N$. Then, inductively, we can take $\mathfrak{d}_{i-1} \in \text{Pic}(W_{i-1})$ and $d_i \in \mathbb{Z}$ such that $\mathfrak{d}_i = \psi_i^* \mathfrak{d}_{i-1} - d_i E_i$ for $i = N, N-1, \dots, 1$. Since $\tilde{R} = R_N \sim n\tilde{\mathfrak{d}} = \psi_N^*(n\mathfrak{d}_{N-1}) - nd_N E_N$ and $R_{N-1} = (\psi_N)_* R_N$, we get $R_{N-1} \sim n\mathfrak{d}_{N-1}$, where the symbol \sim means linear equivalence of divisors. Then, by induction, we have $R_i \sim n\mathfrak{d}_i$ for any i .

Now, if E_i is not contained in R_i , then R_i is the proper transform of R_{i-1} by ψ_i , and hence we have $m_i = nd_i$. Similarly, if E_i is contained in R_i , then, since R_i is reduced, $R_i - E_i$ is the proper transform of R_{i-1} by ψ_i , and hence we have $m_i = d_i n + 1$. In either case, $d_i = [m_i/n]$. \square

A smooth rational curve with self-intersection number $-k$ is called a $(-k)$ -curve. An irreducible curve on \tilde{S} is called \tilde{f} -vertical if it is contained in a fiber of \tilde{f} ; otherwise, it is called \tilde{f} -horizontal.

LEMMA 1.6. *Let E be an \tilde{f} -vertical (-1) -curve on \tilde{S} and put $L = \tilde{\theta}(E)$.*

- (1) *If $E \subset \text{Fix}(\tilde{\sigma})$, then L is a $\tilde{\varphi}$ -vertical $(-n)$ -curve contained in \tilde{R} . Conversely, for any $\tilde{\varphi}$ -vertical $(-n)$ -curve $L' \subset \tilde{R}$, there exists an \tilde{f} -vertical (-1) -curve E' such that $E' \subset \text{Fix}(\tilde{\sigma})$ and $\tilde{\theta}^* L' = nE'$.*
- (2) *If $E \not\subset \text{Fix}(\tilde{\sigma})$, then L is a $\tilde{\varphi}$ -vertical (-1) -curve, and there exist \tilde{f} -vertical (-1) -curves E_2, \dots, E_n such that $\tilde{\theta}^* L = E_1 + E_2 + \cdots + E_n$ and*

E_1, E_2, \dots, E_n are disjoint, where $E_1 = E$. Moreover, if $\bar{\rho}: \tilde{S} \rightarrow \bar{S}$ and $\bar{\psi}: \tilde{W} \rightarrow \bar{W}$ denote the contractions of $\bigcup_{i=1}^n E_i$ and L , respectively, then there exists a natural classical n -cyclic covering $\bar{\theta}: \bar{S} \rightarrow \bar{W}$ branched over $\bar{R} = \bar{\psi}_* \tilde{R}$ such that $\bar{\theta} \circ \bar{\rho} = \bar{\psi} \circ \bar{\theta}$.

Proof. (1) Suppose that $E \subset \text{Fix}(\tilde{\sigma})$. Then L is $\tilde{\varphi}$ -vertical and $L \subset \tilde{R}$. From the last, it follows $\tilde{\theta}^* L = nE$. Since $nL^2 = (\tilde{\theta}^* L)^2 = n^2 E^2 = -n^2$, we get $L^2 = -n$. Since $\tilde{\theta}|_E: E \rightarrow L$ is an isomorphism, we have $L \simeq \mathbb{P}^1$, implying that L is a $(-n)$ -curve. The rest of (1) is clear.

(2) Suppose that $E \not\subset \text{Fix}(\tilde{\sigma})$. Then either $\tilde{\theta}^* L$ is irreducible, or it consists of n irreducible curves. The first alternative is impossible because if $\tilde{\theta}^* L = E$, then we would have $L^2 = -1/n$, which is absurd. Therefore, $\tilde{\theta}^* L$ consists of n irreducible curves E_1, \dots, E_n . We may assume that $E_1 = E$. Note that some power of $\tilde{\sigma}$ maps E isomorphically onto E_i for any i . Therefore, every E_i is an \tilde{f} -vertical (-1) -curve since so is E . All the E_i must be contracted by ρ since $g > 0$. It follows that E_i are mutually disjoint. In particular, we see that $L \cap \tilde{R} = \emptyset$. Let $\bar{\rho}: \tilde{S} \rightarrow \bar{S}$ and $\bar{\psi}: \tilde{W} \rightarrow \bar{W}$ be the contractions of $\bigcup_{i=1}^n E_i$ and L , respectively. Then $\bar{R} = \bar{\psi}_* \tilde{R}$ is isomorphic to \tilde{R} . In particular, it is smooth, and $\bar{\psi}^* \bar{R} = \tilde{R}$. There is a uniquely determined element $\bar{d} \in \text{Pic}(\bar{W})$ satisfying $\bar{d} = \bar{\psi}^* \bar{d}$. Then we have $\bar{R} \sim n\bar{d}$. Hence, we can construct a classical n -cyclic covering $\bar{\theta}: \text{Spec}(\bigoplus_{j=0}^{n-1} \mathcal{O}_{\bar{W}}(-j\bar{d})) \rightarrow \bar{W}$ branched over \bar{R} . Clearly, it is isomorphic to the natural map $\bar{S} \rightarrow \bar{W}$. □

By this lemma we can assume from the first time that all the \tilde{f} -vertical (-1) -curves are contained in $\text{Fix}(\tilde{\sigma})$. In what follows, we tacitly assume this. We decompose $\rho: \tilde{S} \rightarrow S$ into a succession of blow-ups as $\rho = \rho_1 \circ \dots \circ \rho_k$, where $\rho_i: \tilde{S}_i \rightarrow \tilde{S}_{i-1}$ is a blow-up ($i = 1, \dots, k$), $\tilde{S}_k = \tilde{S}$, and $\tilde{S}_0 = S$.

LEMMA 1.7. *$\tilde{\sigma}$ induces on \tilde{S}_i an automorphism σ_i of order n for each i , and the center of ρ_i is a fixed point of σ_{i-1} .*

Proof. Let E_i be the exceptional (-1) -curve of ρ_i . On $\tilde{S}_k = \tilde{S}$, we put $\sigma_k = \tilde{\sigma}$. Then we have $E_k \subset \text{Fix}(\sigma_k)$ by assumption. Then σ_k induces an automorphism σ_{k-1} of \tilde{S}_{k-1} of order n . If $\sigma_{k-1}(E_{k-1}) \neq E_{k-1}$, then, as in the proof of Lemma 1.6(2), we can show that $E_{k-1} \cap \text{Fix}(\sigma_{k-1}) = \emptyset$. This implies that E_{k-1} can be regarded as a (-1) -curve on \tilde{S} disjoint from $\text{Fix}(\tilde{\sigma})$, contradicting our assumption. Hence, $\sigma_{k-1}(E_{k-1}) = E_{k-1}$, and σ_{k-1} induces an automorphism σ_{k-2} of \tilde{S}_{k-2} of order n . Now, an easy inductive argument shows the assertion. □

We put $\sigma := \sigma_0$. Since $\tilde{\sigma}$ has no isolated fixed points, it follows from Lemma 1.2 applied to $\rho: \tilde{S} \rightarrow S$ that either $n \leq 3$ or every isolated fixed point of σ is of type (l, l) for some l . When $n = 2$, it is clear that every isolated fixed point is of type $(1, 1)$. For $n = 3$, we can show the following:

LEMMA 1.8. *If $n = 3$, then any isolated fixed point x of σ is of type either $(1, 1)$ or $(2, 2)$.*

Proof. Assume that x is an isolated fixed point of type $(1, 2)$. Let $\rho_1: \bar{S}_1 \rightarrow S$ be the blow-up at x . Then the automorphism σ_1 induced by σ has two isolated fixed points of respective types $(1, 1)$ and $(2, 2)$ on $\rho_1^{-1}(x)$. We need to blow \bar{S}_1 up at two such fixed points to get \tilde{S} . We denote by E the proper transform of $\rho_1^{-1}(x)$ and by E_i the new exceptional (-1) -curves coming from the fixed point of type (i, i) ($i = 1, 2$). Then E is a (-3) -curve not contained in $\text{Fix}(\tilde{\sigma})$, whereas $E_1, E_2 \subset \text{Fix}(\tilde{\sigma})$. Let L, L_1, L_2 be the images of E, E_1, E_2 under $\tilde{\theta}$, respectively. We easily see that L is a (-1) -curve not contained in \tilde{R} and that L_1, L_2 are (-3) -curves contained in \tilde{R} . Furthermore, $L + L_1 + L_2$ is contained in a fiber of $\tilde{\varphi}$.

On the other hand, $\tilde{\theta}|_E: E \rightarrow L$ is a classical 3-cyclic covering branched over $\tilde{R} \cap L$. Recall that $\tilde{R} \cap L$ contains at least two points $L \cap L_1$ and $L \cap L_2$. Since E and L are smooth rational curves, we apply the Hurwitz formula to see that it branches over exactly two points. Therefore, L meets \tilde{R} exactly at $L \cap (L_1 \cup L_2)$. Since L is a $\tilde{\varphi}$ -vertical (-1) -curve, we can assume that $\tilde{\psi}$ contracts L . By what we have just seen, after the contraction, the image of L will be a double point on the branch locus. This contradicts Lemma 1.5(1). \square

Therefore, we get the following lemma.

LEMMA 1.9. *Any irreducible curve contracted by ρ is an \tilde{f} -vertical (-1) -curve contained in $\text{Fix}(\tilde{\sigma})$.*

Proof. We write $\rho = \rho_1 \circ \dots \circ \rho_k$ as before. It suffices to show that the center of ρ_i is an isolated fixed point of σ_{i-1} for any i . Suppose that the center of ρ_i is a smooth point of a one-dimensional fixed locus of σ_{i-1} . Let its type be $(0, l)$, $\text{gcd}(l, n) = 1$. Then, on the exceptional (-1) -curve of ρ_i , we can find an isolated fixed point of type $(n - l, l)$. This is impossible by Lemmas 1.2 and 1.8. \square

From this lemma we can reconstruct $(\tilde{S}, \tilde{\sigma})$ from (S, σ) by blowing up isolated fixed points of σ .

2. Lower Bound of the Slope

The purpose of this section is to show the following theorem.

THEOREM 2.1. *Let $f: S \rightarrow B$ be a primitive cyclic covering fibration of type (g, h, n) . If $h \geq 1$ and $g \geq (2n - 1)(2hn + n - 1)/(n + 1)$, then $K_f^2 \geq \lambda_{g,h,n} \chi_f$, where*

$$\lambda_{g,h,n} = \frac{24(n - 1)(g - 1)}{2(2n - 1)(g - 1) - n(n + 1)(h - 1)}. \tag{2.1}$$

REMARK 2.2. (1) By (1.2) the condition $g \geq (2n - 1)(2hn + n - 1)/(n + 1)$ is equivalent to $r \geq 3g/(2n - 1) + 3$, and

$$\lambda_{g,h,n} = \frac{8}{1 + r(n + 1)/6(g - 1)}.$$

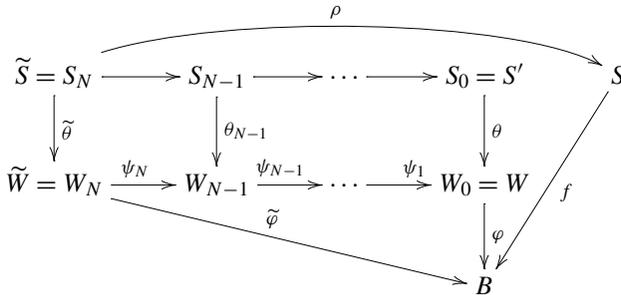
(2) The case where $h = 0$ needs a separate treatment. As we shall see later in Theorem 4.3, the inequality $K_f^2 \geq \lambda_{g,h,n} \chi_f$ also holds for $h = 0$.

We keep the notation of the previous section, and, for the time being, we do not exclude the case where $h = 0$ from the consideration. We obtain a classical n -cyclic covering $\theta_i: S_i \rightarrow W_i$ branched over R_i by setting

$$S_i = \text{Spec} \left(\bigoplus_{j=0}^{n-1} \mathcal{O}_{W_i}(-j\mathfrak{d}_i) \right).$$

Since R_i is reduced, S_i is a normal surface.

There exists a natural birational morphism $S_i \rightarrow S_{i-1}$. Set $S' = S_0$, $\theta = \theta_0$, $\mathfrak{d} = \mathfrak{d}_0$, and $f' = \varphi \circ \theta$.



From Lemma 1.5 we have

$$K_{\tilde{\varphi}} = \tilde{\psi}^* K_{\varphi} + \sum_{i=1}^N \mathbf{E}_i, \tag{2.2}$$

$$\tilde{\mathfrak{d}} = \tilde{\psi}^* \mathfrak{d} - \sum_{i=1}^N \left[\frac{m_i}{n} \right] \mathbf{E}_i, \tag{2.3}$$

where \mathbf{E}_i denotes the total transform of E_i . Since

$$K_{\tilde{S}} = \tilde{\theta}^*(K_{\tilde{W}} + (n-1)\tilde{\mathfrak{d}})$$

and

$$\chi(\mathcal{O}_{\tilde{S}}) = n\chi(\mathcal{O}_{\tilde{W}}) + \frac{1}{2} \sum_{j=1}^{n-1} j\tilde{\mathfrak{d}}(j\tilde{\mathfrak{d}} + K_{\tilde{W}}),$$

we get

$$K_{\tilde{f}}^2 = n(K_{\varphi}^2 + 2(n-1)K_{\tilde{\varphi}}\tilde{\mathfrak{d}} + (n-1)^2\tilde{\mathfrak{d}}^2), \tag{2.4}$$

$$\chi_{\tilde{f}} = n\chi_{\tilde{\varphi}} + \frac{1}{2} \sum_{j=1}^{n-1} j\tilde{\mathfrak{d}}(j\tilde{\mathfrak{d}} + K_{\tilde{\varphi}}). \tag{2.5}$$

Similarly, we have

$$\omega_{f'}^2 = n(K_{\varphi}^2 + 2(n-1)K_{\varphi}\mathfrak{d} + (n-1)^2\mathfrak{d}^2), \tag{2.6}$$

$$\chi_{f'} = n\chi_{\varphi} + \frac{1}{2} \sum_{j=1}^{n-1} j\mathfrak{d}(j\mathfrak{d} + K_{\varphi}), \tag{2.7}$$

where $\omega_{f'}$ denotes the relative dualizing sheaf of f' . Combining Eqs. (2.2)–(2.7), we obtain

$$\omega_{f'}^2 - K_{\tilde{f}}^2 = n \sum_{i=1}^N \left((n-1) \left[\frac{m_i}{n} \right] - 1 \right)^2, \quad (2.8)$$

$$\chi_{f'} - \chi_{\tilde{f}} = \frac{1}{12} n(n-1) \sum_{i=1}^N \left[\frac{m_i}{n} \right] \left((2n-1) \left[\frac{m_i}{n} \right] - 3 \right). \quad (2.9)$$

LEMMA 2.3. *Assume that*

$$g \geq \frac{2n-1}{n+1} (2hn + n - 1)$$

when $h > 0$. Then $\omega_{f'}^2 \geq \lambda_{g,h,n} \chi_{f'}$, where $\lambda_{g,h,n}$ is the rational number in (2.1).

Proof. Suppose that $h = 0$. Then R is numerically equivalent to $(-r/2)K_\varphi + M\Gamma$ for some $M \in \frac{1}{2}\mathbb{Z}$ since $\varphi: W \rightarrow B$ is a \mathbb{P}^1 -bundle and $K_W\Gamma = -2$, $R\Gamma = \tilde{R}\tilde{\Gamma} = r$. Moreover, $K_\varphi^2 = 0$ and $\chi_\varphi = 0$. Hence, we have

$$\begin{aligned} \omega_{f'}^2 &= \frac{4(g-1)(n-1)}{n} M, \\ \chi_{f'} &= \frac{2(2n-1)(g-1) + n(n+1)}{6n} M \end{aligned}$$

from (1.2), (2.6) and (2.7). Thus, we get $\omega_{f'}^2 = \lambda_{g,0,n} \chi_{f'}$.

If $h = 1$, then $K_\varphi^2 = 0$ and $\chi_\varphi = \chi(\mathcal{O}_W)$ since φ is a relatively minimal elliptic surface. Then we have

$$\begin{aligned} \omega_{f'}^2 - \lambda_{g,1,n} \chi_{f'} &= n \left(K_\varphi^2 - \frac{12(n-1)}{2n-1} \chi_\varphi \right) + \frac{n(n-1)(n+1)}{2n-1} K_\varphi \mathfrak{d} \\ &= \frac{n(n-1)}{2n-1} ((n+1)K_\varphi \mathfrak{d} - 12\chi_\varphi). \end{aligned} \quad (2.10)$$

By the canonical bundle formula, K_φ is numerically equivalent to $\chi(\mathcal{O}_W)\Gamma + \sum_{i=1}^l (1 - 1/k_i)\Gamma$, where $\{k_i \mid i = 1, \dots, l\}$ denotes the set of multiplicities of all multiple fibers of φ , $k_i \geq 2$. Hence, we have

$$K_\varphi \mathfrak{d} \geq \chi(\mathcal{O}_W)\Gamma \mathfrak{d} = \frac{2(g-1)}{n(n-1)} \chi(\mathcal{O}_W).$$

Thus,

$$\begin{aligned} &\frac{n(n-1)}{2n-1} ((n+1)K_\varphi \mathfrak{d} - 12\chi_\varphi) \\ &\geq \frac{n(n-1)}{2n-1} \left(\frac{2(n+1)(g-1)}{n(n-1)} - 12 \right) \chi(\mathcal{O}_W) \\ &= \frac{2}{2n-1} ((n+1)g - (2n-1)(3n-1)) \chi(\mathcal{O}_W). \end{aligned} \quad (2.11)$$

From (2.10), (2.11), and the hypothesis $g \geq (2n-1)(3n-1)/(n+1)$ we have $\omega_{f'}^2 \geq \lambda_{g,1,n}\chi_{f'}$.

Let $h \geq 2$. We compute $\omega_{f'}^2 - \lambda_{g,h,n}\chi_{f'}$ by (2.6) and (2.7) as follows:

$$\begin{aligned} & \omega_{f'}^2 - \lambda_{g,h,n}\chi_{f'} \\ &= n(K_\varphi^2 + 2(n-1)K_\varphi\vartheta + (n-1)^2\vartheta^2) \\ & \quad - \lambda_{g,h,n}\left(n\chi_\varphi + \frac{1}{4}n(n-1)K_\varphi\vartheta + \frac{1}{12}n(n-1)(2n-1)\vartheta^2\right) \\ &= n(K_\varphi^2 - \lambda_{g,h,n}\chi_\varphi) + \frac{n(n-1)}{4}(8 - \lambda_{g,h,n})K_\varphi\vartheta \\ & \quad + \frac{n(n-1)}{12}(12(n-1) - (2n-1)\lambda_{g,h,n})\vartheta^2. \end{aligned} \tag{2.12}$$

Since the slope inequality of φ gives us

$$K_\varphi^2 \geq \frac{4(h-1)}{h}\chi_\varphi,$$

we have

$$\begin{aligned} K_\varphi^2 - \lambda_{g,h,n}\chi_\varphi &= \frac{h\lambda_{g,h,n}}{4(h-1)}\left(K_\varphi^2 - \frac{4(h-1)}{h}\chi_\varphi\right) + \left(1 - \frac{h\lambda_{g,h,n}}{4(h-1)}\right)K_\varphi^2 \\ &\geq \left(1 - \frac{h\lambda_{g,h,n}}{4(h-1)}\right)K_\varphi^2. \end{aligned} \tag{2.13}$$

We consider the intersection matrix of $\{K_\varphi, \vartheta, \Gamma\}$,

$$\begin{pmatrix} K_\varphi^2 & K_\varphi\vartheta & K_\varphi\Gamma \\ K_\varphi\vartheta & \vartheta^2 & \vartheta\Gamma \\ K_\varphi\Gamma & \vartheta\Gamma & 0 \end{pmatrix}. \tag{2.14}$$

By Arakerov's theorem we have $K_\varphi^2 \geq 0$, and then the matrix is not negative definite. Hence, the determinant of (2.14) is nonnegative, that is, we have

$$2(K_\varphi\vartheta)(\vartheta\Gamma)(K_\varphi\Gamma) - \vartheta^2(K_\varphi\Gamma)^2 - (\vartheta\Gamma)^2K_\varphi^2 \geq 0 \tag{2.15}$$

by the Hodge index theorem. Since

$$\vartheta\Gamma = \frac{r}{n} = \frac{2(g-1-n(h-1))}{n(n-1)}$$

and

$$K_\varphi\Gamma = 2(h-1),$$

inequality (2.15) is equivalent to

$$\begin{aligned} & 2(g-1-n(h-1))K_\varphi\vartheta - n(n-1)(h-1)\vartheta^2 \\ & \geq \frac{1}{n(n-1)(h-1)}(g-1-n(h-1))^2K_\varphi^2. \end{aligned} \tag{2.16}$$

On the other hand, by the definition of $\lambda_{g,h,n}$ we have

$$\begin{aligned} & \frac{n(n-1)}{4}(8 - \lambda_{g,h,n}) \\ &= \frac{n(n-1)(n+1)}{2(2n-1)(g-1) - n(n+1)(h-1)} 2(g-1 - n(h-1)) \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} & \frac{n(n-1)}{12}(12(n-1) - (2n-1)\lambda_{g,h,n}) \\ &= \frac{n(n-1)(n+1)}{2(2n-1)(g-1) - n(n+1)(h-1)} (-n(n-1)(h-1)). \end{aligned} \quad (2.18)$$

Hence, combining (2.12), (2.13), (2.16), (2.17), and (2.18), we get

$$\begin{aligned} & \omega_{f'}^2 - \lambda_{g,h,n}\chi_{f'} \\ & \geq n \left(1 - \frac{h\lambda_{g,h,n}}{4(h-1)} \right) K_\varphi^2 \\ & \quad + \frac{(n+1)(g-1 - n(h-1))^2}{(h-1)(2(2n-1)(g-1) - n(n+1)(h-1))} K_\varphi^2. \end{aligned} \quad (2.19)$$

Since

$$1 - \frac{h\lambda_{g,h,n}}{4(h-1)} = \frac{2(g-1)(-hn + 2h - 2n + 1) - n(n+1)(h-1)^2}{(h-1)(2(2n-1)(g-1) - n(n+1)(h-1))},$$

the right-hand side of inequality (2.19) is

$$\frac{(g-1)((n+1)g - (2hn + n - 1)(2n-1))}{(h-1)(2(2n-1)(g-1) - n(n+1)(h-1))} K_\varphi^2. \quad (2.20)$$

By the assumption $g \geq (2n-1)(2hn + n - 1)/(n+1)$, (2.20) is nonnegative. \square

Proof of Theorem 2.1. Let ε be the number of blow-ups of $\rho: \tilde{S} \rightarrow S$. Then we have

$$K_{\tilde{f}}^2 = K_f^2 - \varepsilon. \quad (2.21)$$

Using (2.8), (2.9), (2.17), and (2.18), we can calculate $K_{\tilde{f}}^2 - \lambda_{g,h,n}\chi_{\tilde{f}}$ as follows:

$$\begin{aligned} & K_{\tilde{f}}^2 - \lambda_{g,h,n}\chi_{\tilde{f}} \\ &= K_{\tilde{f}}^2 - \lambda_{g,h,n}\chi_{\tilde{f}} + \varepsilon \\ &= \omega_{f'}^2 - n \sum_{i=1}^N \left((n-1) \left[\frac{m_i}{n} \right] - 1 \right)^2 - \lambda_{g,h,n}\chi_{f'} \\ & \quad + \lambda_{g,h,n} \frac{1}{12} n(n-1) \sum_{i=1}^N \left[\frac{m_i}{n} \right] \left((2n-1) \left[\frac{m_i}{n} \right] - 3 \right) + \varepsilon \\ &= \omega_{f'}^2 - \lambda_{g,h,n}\chi_{f'} + \frac{1}{12} n(n-1) ((2n-1)\lambda_{g,h,n} - 12(n-1)) \sum_{i=1}^N \left[\frac{m_i}{n} \right]^2 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{4}n(n-1)(8 - \lambda_{g,h,n}) \sum_{i=1}^N \left[\frac{m_i}{n} \right] - nN + \varepsilon \\
 = & \omega_{f'}^2 - \lambda_{g,h,n} \chi_{f'} + \frac{n^2(n-1)^2(n+1)(h-1)}{2(2n-1)(g-1) - n(n+1)(h-1)} \sum_{i=1}^N \left[\frac{m_i}{n} \right]^2 \\
 & + \frac{2n(n-1)(n+1)(g-1 - n(h-1))}{2(2n-1)(g-1) - n(n+1)(h-1)} \sum_{i=1}^N \left[\frac{m_i}{n} \right] - nN + \varepsilon. \tag{2.22}
 \end{aligned}$$

When $h \geq 1$, the right-hand side of (2.22) increases with respect to the multiplicity m_i . So we may assume that $[m_i/n] = 1$ for all $i = 1, \dots, N$. Then,

$$\begin{aligned}
 & \frac{n^2(n-1)^2(n+1)(h-1)}{2(2n-1)(g-1) - n(n+1)(h-1)} \sum_{i=1}^N \left[\frac{m_i}{n} \right]^2 \\
 & + \frac{2n(n-1)(n+1)(g-1 - n(h-1))}{2(2n-1)(g-1) - n(n+1)(h-1)} \sum_{i=1}^N \left[\frac{m_i}{n} \right] - nN \\
 = & \frac{n^2(n-2)N}{2(2n-1)(g-1) - n(n+1)(h-1)} ((n+1)(n-2)(h-1) + 2(g-1)) \\
 \geq & 0.
 \end{aligned}$$

Combining this with Lemma 2.3, we conclude the proof. □

REMARK 2.4. From the proofs of Theorem 2.1 and Lemma 2.3, $K_f^2 = \lambda_{g,h,n} \chi_f$ if and only if the following three conditions are satisfied:

- (1) Either R is nonsingular, that is, $\tilde{R} = R$, or $n = 2$ and R has only negligible singularities.
- (2) The equality sign of the slope inequality of φ holds: $K_\varphi^2 = \frac{4(h-1)}{h} \chi_\varphi$.
- (3) The intersection matrix of $\{K_\varphi, \vartheta, \Gamma\}$ is singular, that is, $K_\varphi, \vartheta, \Gamma$ are linearly dependent in $\text{NS}(W)_\mathbb{R}$.

This also holds for $h = 0$ as we shall see in Theorem 4.3. Condition (2) means that φ is locally trivial or hyperelliptic with the Horikawa index 0 (see [9; 15]). We will give an example of primitive cyclic covering fibrations of type (g, h, n) with the slope $\lambda_{g,h,n}$ and φ being locally trivial in Example 2.5.

EXAMPLE 2.5. Let B and Γ be smooth curves of genus b and h , respectively. Let ϑ_1 and ϑ_2 be divisors on B and Γ of degrees N and M , respectively. For N and M sufficiently large, the divisor $\vartheta := p_1^* \vartheta_1 + p_2^* \vartheta_2$ on $B \times \Gamma$ gives us a base point free linear system, where p_1 and p_2 are the natural projections from $B \times \Gamma$ to B and to Γ , respectively. Thus, we can take a smooth divisor $R \in |n\vartheta|$ for any $n > 0$ by Bertini’s theorem. Hence, we may construct a classical n -cyclic covering $\theta: S \rightarrow B \times \Gamma$ branched over R . Let $f: S \rightarrow B$ be the composite of p_1 and θ . We will compute K_f^2 and χ_f . Let F be a general fiber of f , and g the genus of F .

Applying the Hurwitz formula to $\theta|_F: F \rightarrow \Gamma = \{t\} \times \Gamma$, we have

$$\begin{aligned} 2g - 2 &= n(2h - 2) + (n - 1)n(p_1^* \mathfrak{d}_1 + p_2^* \mathfrak{d}_2) p_1^* t \\ &= n(2h - 2) + (n - 1)nM. \end{aligned}$$

Hence,

$$M = \frac{2(g - 1 - n(h - 1))}{n(n - 1)}.$$

Since $K_{p_1} = p_2^* K_\Gamma$ and $\chi_{p_1} = 0$, we obtain

$$\begin{aligned} K_f^2 &= (\theta^*(K_{p_1} + (n - 1)\mathfrak{d}))^2 \\ &= n(p_1^*(n - 1)\mathfrak{d}_1 + p_2^*((n - 1)\mathfrak{d}_2 + K_\Gamma))^2 \\ &= 2n(n - 1)N((n - 1)M + 2(h - 1)) \end{aligned}$$

and

$$\begin{aligned} \chi_f &= n\chi_{p_1} + \frac{1}{2} \sum_{j=1}^{n-1} j\mathfrak{d}(j\mathfrak{d} + K_{p_1}) \\ &= \frac{1}{4}n(n - 1)\mathfrak{d}K_{p_1} + \frac{1}{12}n(n - 1)(2n - 1)\mathfrak{d}^2 \\ &= \frac{1}{4}n(n - 1)N(2h - 2) + \frac{1}{12}n(n - 1)(2n - 1)2NM \\ &= \frac{1}{6}n(n - 1)N(3(h - 1) + (2n - 1)M). \end{aligned}$$

Thus, we get

$$\begin{aligned} \frac{K_f^2}{\chi_f} &= \frac{12(2(h - 1) + (n - 1)M)}{3(h - 1) + (2n - 1)M} \\ &= \frac{24(n - 1)(g - 1)}{2(2n - 1)(g - 1) - n(n + 1)(h - 1)} \\ &= \lambda_{g,h,n}. \end{aligned}$$

This example implies that the bound of Theorem 2.1 on the slope is sharp when h and n are fixed arbitrarily.

3. Primitive Cyclic Covering Fibrations of a Ruled Surface

From now on, we concentrate on the case $h = 0$ and consider primitive cyclic covering fibrations of type $(g, 0, n)$.

In this section, we study singular points on the branch locus R to know how \tilde{f} -vertical (-1) -curves in the ramification divisor appear. As we have already seen, this amounts to observing how $\tilde{\varphi}$ -vertical $(-n)$ -curves in \tilde{R} appear.

LEMMA 3.1. *There exists a relatively minimal model $\varphi: W \rightarrow B$ of $\tilde{\varphi}$ such that if $n = 2$ and g is even, then*

$$\text{mult}_x(R) \leq \frac{r}{2} = g + 1$$

for all $x \in R$, and otherwise,

$$\text{mult}_x(R_h) \leq \frac{r}{2} = \frac{g}{n-1} + 1$$

for all $x \in R_h$, where R_h denotes the horizontal part of R , that is, the sum of all φ -horizontal components of R .

Proof. We take a relatively minimal model $\varphi: W \rightarrow B$ of $\tilde{\varphi}$ arbitrarily. Suppose that there exists a point x of R at which R has the multiplicity greater than $r/2$. Let Γ be the fiber of φ through x . We will perform the elementary transformation at x as follows: Let $\psi_1: W_1 \rightarrow W$ be the blow-up at x , $E_1 = \psi_1^{-1}(x)$ the exceptional (-1) -curve, and E'_1 the proper transform of Γ . Then, E'_1 is also a (-1) -curve. Let $\psi'_1: W_1 \rightarrow W'$ be the contraction of E'_1 , and $\varphi': W' \rightarrow B$ the induced fibration. Set $x' = \psi'_1(E'_1)$ and $\Gamma' = \psi'_1(\Gamma)$. Then, Γ' is the fiber of φ' over $t = \varphi(\Gamma)$. Let m denotes the multiplicity of R at x . Put $R_1 = \psi_1^*R - n[m/n]E_1$ and $R' = (\psi'_1)_*R_1$. Let m' denote the multiplicity of R' at x' . Then it follows from Lemma 1.5 that $R_1 = (\psi'_1)^*R' - n[m'/n]E'_1$. Since $R\Gamma = R_1\psi_1^*\Gamma = R_1(E_1 + E'_1)$, we have

$$\left[\frac{m}{n} \right] + \left[\frac{m'}{n} \right] = \frac{r}{n}.$$

Moreover, since $m > r/2$, we get

$$\left[\frac{m'}{n} \right] \leq \frac{r}{2n} \leq \left[\frac{m}{n} \right].$$

Hence, we have $m' \leq r/2 + 1$ from Lemma 1.5. Moreover, $m' \leq r/2$ when $n = 2$ and g is even. If $m' = r/2 + 1$, then we have $m' \in n\mathbb{Z} + 1$, and hence E'_1 is contained in R_1 . In particular, Γ is contained in R . Since $r/2 \in n\mathbb{Z}$, we have $m = r/2 + 1$. Hence, the multiplicity of R_h at x is $r/2$. If $m' \leq r/2$, then we replace the relatively minimal model W with W' . Since the number of singularities of R are finite, we obtain a relatively minimal model satisfying the condition inductively. \square

In the sequel, we tacitly assume that our relatively minimal model $\varphi: W \rightarrow B$ of $\tilde{\varphi}$ enjoys the property of the lemma.

REMARK 3.2. Recall that r is a multiple of n . If $n \geq 3$ and R has a singular point of multiplicity $m \geq 2$, then $m \leq r/2 + 1$ from Lemma 3.1, whereas $m \in n\mathbb{Z}$ or $m \in n\mathbb{Z} + 1$ by Lemma 1.5. It follows that $r \geq 2n$ when $n \geq 3$ and R has a singular point. If $r = n \geq 3$, then R is nonsingular, and we have $K_f^2 = \lambda_{g,0,n}\chi_f$ by Lemma 2.3. If $n = 2$, then $r = 2g + 2 \geq 6$ since $g \geq 2$.

Recall that the gonality of a nonsingular projective curve is the minimum of the degree of surjective morphisms to \mathbb{P}^1 . The gonality of a fibered surface is defined to be that of a general fiber (see [11]).

LEMMA 3.3. *The gonality of a primitive cyclic covering fibration of type $(g, 0, n)$ is n when $r \geq 2n$.*

Proof. We use the so-called Castelnuovo–Severi inequality. Assume contrary that the general fiber F has a morphism onto \mathbb{P}^1 of degree $k < n$. This, together with the natural map $F \rightarrow F/\langle \sigma|_F \rangle \simeq \mathbb{P}^1$, defines a morphism $\Phi : F \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. If Φ is of degree m onto the image $\Phi(F)$, then m is a common divisor of n, k , and $\Phi(F)$ is a divisor of bidegree $(n/m, k/m)$. In particular, the arithmetic genus of $\Phi(F)$ is given by $(n/m - 1)(k/m - 1)$. Now, let F' be the normalization of $\Phi(F)$. Since the cyclic n -covering $F \rightarrow \mathbb{P}^1$ factors through F' , we see that the induced covering $F' \rightarrow \mathbb{P}^1$ of degree n/m is a totally ramified covering branched over r points. Then, by the Hurwitz formula we have $2g(F') = (n/m - 1)(r - 2)$. Since the genus $g(F')$ of F' is not greater than the arithmetic genus of $\Phi(F)$, we get $r \leq 2(k/m)$, which is impossible since $r \geq 2n$ and $k < n$. A more careful study shows that the gonality pencil is unique when $r \geq 3n$ and that the gonality of F is not less than $n/2$ when $r = n$. \square

As we saw in Section 1, \tilde{f} -vertical (-1) -curves in $\text{Fix}(\tilde{\sigma})$ are in one-to-one correspondence with $\tilde{\varphi}$ -vertical $(-n)$ -curves in \tilde{R} via $\tilde{\theta}$. So we need to know how $\tilde{\varphi}$ -vertical $(-n)$ -curves in \tilde{R} appear during the process of “modulo n resolution” $\tilde{\psi} : \tilde{W} \rightarrow W$ of R .

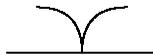
Let $L \simeq \mathbb{P}^1$ be a $\tilde{\varphi}$ -vertical curve contained in \tilde{R} . Then we have $\tilde{\theta}^*L = nD$ for some \tilde{f} -vertical $D \simeq \mathbb{P}^1$ contained in $\text{Fix}(\tilde{\sigma})$. If $D^2 = -a$, then $L^2 = -an$. The image of L by the natural morphism $\tilde{\psi} : \tilde{W} \rightarrow W$ is either a point or a fiber of φ . In either case, we define a curve C and a number c as follows.

- If $\tilde{\psi}(L)$ is a point, then L is the proper transform of an exceptional (-1) -curve, say, E_j . Since $E_j^2 = -1$ and $L^2 = -an$, E_j is blown up $an - 1$ times in total to get L . We put $C = E_j$ and $c = an - 1$.
- If $\tilde{\psi}(L)$ is a fiber Γ of φ , then L is the proper transform of Γ . Since $\Gamma^2 = 0$ and $L^2 = -an$, Γ is blown up an times in total to get L . We put $C = \Gamma$ and $c = an$.

In the former case, since E_j is contained in R_j , the multiplicity of R_{j-1} at the point obtained by contracting E_j is in $n\mathbb{Z} + 1$. We will drop j and simply write R instead of R_j for the time being.

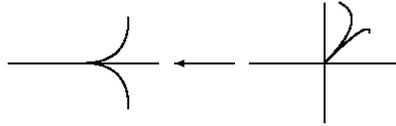
Let x_1, \dots, x_l be all the singular points of R on C , and m_i the multiplicity of R at x_i . Clearly, we have $1 \leq l \leq c$. We consider a local analytic branch D of $R - C$ around x_i whose multiplicity at x_i is $m \geq 2$ (i.e., D has a cusp at x_i). Then we have one of the following:

- (i) D is not tangent to C at x_i . If we blow x_i up, then the proper transform of D does not meet that of C . Hence, we have $(DC)_{x_i} = m$, where $(DC)_{x_i}$ denotes the local intersection number of D and C at x_i .

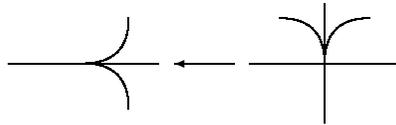


- (ii) D is tangent to C at x_i . If we blow x_i up, then one of the following three cases occurs.

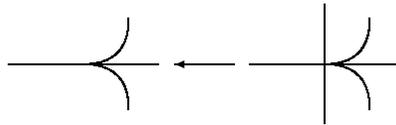
- (ii.1) The proper transform of D is tangent to neither that of C nor the exceptional (-1) -curve.



(ii.2) The proper transform of D is tangent to the exceptional (-1) -curve. Then the multiplicity m' of the proper transform of D at the singular point is less than m , and we have $(DC)_{x_i} = m + m'$.



(ii.3) The proper transform of D is still tangent to that of C .



We perform blowing-ups at x_i and points infinitely near to it. Then case (ii.3) may occur repeatedly, but at most a finite number of times. Suppose that the proper transform of D becomes not tangent to that of C just after k th blow-up. If the proper transform of D is as in (ii.1) after the k th blow-up (or D is as in (i) when $k = 0$), then we have $(DC)_{x_i} = (k + 1)m$. If the proper transform of D is as in (ii.2) after the k th blow-up, then we have $(DC)_{x_i} = km + m'$. In either case, it is convenient to consider as if D consists of m local branches D_1, \dots, D_m smooth at x_i and such that $(D_j C)_{x_i} = k + 1$ for $j = 1, \dots, m$ in the former case and

$$(D_j C)_{x_i} = \begin{cases} k & \text{for } j = 1, \dots, m - m', \\ k + 1 & \text{for } j = m - m' + 1, \dots, m \end{cases}$$

in the latter case. We call D_j a virtual local branch of D .

NOTATION 3.4. For a positive integer k , we let $s_{i,k}$ be the number of such virtual local branches D_\bullet satisfying $(D_\bullet C)_{x_i} = k$, among those of all local analytic branches of $R - C$ around x_i . Here, when $\text{mult}_{x_i}(D) = 1$, we regard D itself as a virtual local branch. We let i_{\max} be the greatest integer k satisfying $s_{i,k} \neq 0$.

We put $x_{i,1} = x_i$ and $m_{i,1} = m_i$. Let $\psi_{i,1}: W_{i,1} \rightarrow W$ be the blow-up at $x_{i,1}$ and put $E_{i,1} = \psi_{i,1}^{-1}(x_{i,1})$ and $R_{i,1} = \psi_{i,1}^* R - n[m_{i,1}/n]E_{i,1}$. Inductively, we define $x_{i,j}, m_{i,j}, \psi_{i,j}: W_{i,j} \rightarrow W_{i,j-1}, E_{i,j}$, and $R_{i,j}$ to be the intersection point of the proper transform of C and $E_{i,j-1}$, the multiplicity of $R_{i,j-1}$ at $x_{i,j}$, the blow-up of $W_{i,j-1}$ at $x_{i,j}$, the exceptional curve for $\psi_{i,j}$, and $R_{i,j} = \psi_{i,j}^* R_{i,j-1} - n[m_{i,j}/n]E_{i,j}$, respectively. Put $i_{\text{bm}} = \max\{j \mid m_{i,j} > 1\}$, that is, the number of blowing-ups occurring over x_i . We may assume that $i_{\text{bm}} \geq (i + 1)_{\text{bm}}$ for $i = 1, \dots, l - 1$ after rearranging the index if necessary.

LEMMA 3.5. *With this notation and assumption, the following hold.*

- (1) *If $n \geq 3$, then $i_{\text{bm}} = i_{\text{max}}$ for all i . If $n = 2$, then $i_{\text{bm}} = i_{\text{max}}$ (resp. $i_{\text{bm}} = i_{\text{max}} + 1$) if and only if $m_{i, i_{\text{max}}} \in 2\mathbb{Z}$ (resp. $m_{i, i_{\text{max}}} \in 2\mathbb{Z} + 1$).*
- (2) *$m_{i, 1} = \sum_{k=1}^{i_{\text{max}}} s_{i, k} + 1$ and $m_{i, i_{\text{bm}}} \in n\mathbb{Z}$.*
- (3) *$m_{i, j} \in n\mathbb{Z}$ (resp. $n\mathbb{Z} + 1$) if and only if $m_{i, j+1} = \sum_{k=j+1}^{i_{\text{max}}} s_{i, k} + 1$ (resp. $\sum_{k=j+1}^{i_{\text{max}}} s_{i, k} + 2$).*
- (4) *$((R - C)C)_{x_i} = \sum_{k=1}^{i_{\text{max}}} k s_{i, k}$.*
- (5) *$c = \sum_{i=1}^l i_{\text{bm}}$.*

Proof. Item (1) is clear from the definitions of i_{max} and i_{bm} . Since $m_{i, j}$ is the number of virtual local branches of $R_{i, j-1}$ through $x_{i, j}$, we get the first equality of (2) and (3) by Lemma 1.5. In (2), “+1” is the contribution of C . If $m_{i, i_{\text{max}}} \in n\mathbb{Z} + 1$, then $x_{i, i_{\text{max}}+1}$ is a double point, which is impossible when $n \geq 3$ by Lemma 1.5. Thus, we have shown (2). Item (4) is clear from the definition of $s_{i, k}$. For each i , we blow up $x_i \in C$ and its infinitely near points on the proper transforms of C exactly i_{max} times. Hence, we have (5). \square

Put $t = (R - C)C$. It is the number of branch points r if $\tilde{\varphi}(L)$ is a fiber of φ , whereas it is the multiplicity at the point x to which C is contracted if $\tilde{\varphi}(L)$ is a point. By Lemma 3.5(4) we get

$$t = \sum_{i=1}^l \sum_{k=1}^{i_{\text{max}}} k s_{i, k}.$$

Let c_i be the cardinality of $\{j \mid m_{i, j} \in n\mathbb{Z} + 1\}$. Clearly, we have $0 \leq c_i \leq i_{\text{bm}} - 1$. Set $d_{i, j} = [m_{i, j}/n]$.

PROPOSITION 3.6. *We have the following equalities:*

$$t + c + \sum_{i=1}^l c_i = \sum_{i=1}^l \sum_{j=1}^{i_{\text{bm}}} m_{i, j}, \quad \frac{t + c}{n} = \sum_{i=1}^l \sum_{j=1}^{i_{\text{bm}}} d_{i, j}.$$

Proof. It follows from Lemma 3.5 that

$$t = \sum_{i=1}^l \sum_{k=1}^{i_{\text{max}}} k s_{i, k} = \sum_{i=1}^l \left(\sum_{j=1}^{i_{\text{bm}}} m_{i, j} - i_{\text{bm}} - c_i \right) = \sum_{i=1}^l \left(\sum_{j=1}^{i_{\text{bm}}} m_{i, j} - c_i \right) - c.$$

Hence, we have the first equality. The second is clear from the first. \square

LEMMA 3.7. *The following hold:*

- (1) *When $n \geq 3$, $m_{i, j} \geq m_{i, j+1}$ with equality if and only if $s_{i, j} = 0$ when $m_{i, j} \in n\mathbb{Z}$ or $s_{i, j} = 1$ when $m_{i, j} \in n\mathbb{Z} + 1$. When $n = 2$, $m_{i, j} + 1 \geq m_{i, j+1}$ with equality only if $m_{i, j} \in 2\mathbb{Z} + 1$ and $m_{i, j+1} \in 2\mathbb{Z}$.*
- (2) *If $m_{i, j-1} \in n\mathbb{Z} + 1$ and $m_{i, j} \in n\mathbb{Z}$, then $m_{i, j} > m_{i, j+1}$.*
- (3) *If $m_{i, j} = nd_{i, j} + 1 \in n\mathbb{Z} + 1$, then $d_{i, j} - d_{i, j+1} \geq n - 3$.*

Proof. If $m_{i,j} < m_{i,j+1}$, then we have $s_{i,j} = 0$ and $m_{i,j} + 1 = m_{i,j+1}$ since $m_{i,j} - m_{i,j+1} = s_{i,j} - 1, s_{i,j}$, or $s_{i,j} + 1$. Then $m_{i,j} \in n\mathbb{Z} + 1$, and we get $m_{i,j+1} \in n\mathbb{Z} + 2$ by Lemma 3.5(2). This contradicts Lemma 1.5 when $n \geq 3$. Hence, $m_{i,j} \geq m_{i,j+1}$. The rest of (1) follows from Lemma 3.5(2). If $m_{i,j-1} \in n\mathbb{Z} + 1$ and $m_{i,j} \in n\mathbb{Z}$, then $m_{i,j} = \sum_{k=j}^{i_{\max}} s_{i,k} + 2$ and $m_{i,j+1} = \sum_{k=j+1}^{i_{\max}} s_{i,k} + 1$ by Lemma 3.5(2). Then, $m_{i,j} - m_{i,j+1} = s_{i,j} + 1 > 0$, and hence (2) follows. Suppose that $m_{i,j} = nd_{i,j} + 1 \in n\mathbb{Z} + 1$. Let C' be the exceptional curve $E_{i,j}$ and define $x'_{i,j}, m'_{i,j}, d'_{i,j}, c'$, et cetera on C' similarly to C . Since C' becomes a $(-a'n)$ -curve by blowing up for some $a' \geq 1$, we have $c' = a'n - 1$. We may assume that $m_{i,j+1} = m'_{i,j+1}$. Then we have

$$\frac{m_{i,j} + c'}{n} = \sum_{p=1}^{l'} \sum_{q=1}^{p'_{\max}} d'_{p,q} \geq d_{i,j+1} + c' - 1.$$

Hence, we get

$$\begin{aligned} d_{i,j} - d_{i,j+1} &\geq a'(n - 1) - 2 \\ &\geq n - 3, \end{aligned}$$

and thus (3) follows. □

PROPOSITION 3.8. (1) *If $g < (n - 1)(an(n - 1)/2 - 1)$, then there are no $\tilde{\varphi}$ -vertical $(-an)$ -curves in \tilde{R} .*

(2) *If $(n - 1)(an(n - 1)/2 - 1) \leq g < (n - 1)(an^2 - (a + 1)n - 1)$, then any $\tilde{\varphi}$ -vertical $(-an)$ -curve in \tilde{R} is the proper transform of a fiber of φ .*

(3) *Let x be a singular point of multiplicity $m \in n\mathbb{Z} + 1$. If the proper transform L of the exceptional curve obtained by blowing up at x is a $(-an)$ -curve, then $[m/n] \geq a(n - 1) - 1$.*

Proof. Let L be a $\tilde{\varphi}$ -vertical $(-an)$ -curve in \tilde{R} . Since

$$\frac{t + c}{n} = \sum_{i=1}^l \sum_{j=1}^{i_{\max}} d_{i,j} \geq c$$

by Proposition 3.6, we get $t \geq (n - 1)c$.

If L is the proper transform of a fiber of φ , then $t = r$ and $c = an$. Hence, $r \geq an(n - 1)$. Then

$$g \geq (n - 1) \left(\frac{an(n - 1)}{2} - 1 \right)$$

by (1.2).

If L is the proper transform of an exceptional curve E appeared in $\tilde{\psi}$, then $m \geq (an - 1)(n - 1)$, where m denotes the multiplicity of the branch locus at the point obtained by contracting E . Hence, (3) follows. Moreover, we have $m \leq r/2 + 1$ by Lemmas 3.1 and 3.5(1). Then we get

$$g \geq (n - 1)(an^2 - (a + 1)n - 1)$$

from the previous two inequalities and (1.2).

By an easy computation we can show that

$$(n - 1)(an^2 - (a + 1)n - 1) \geq (n - 1) \left(\frac{an(n - 1)}{2} - 1 \right)$$

with equality if and only if $(a, n) = (1, 3)$. In sum, we get (1) and (2). □

When $a = 1$, we get the following:

- COROLLARY 3.9. (1) *If $g < (n - 2)(n - 1)(n + 1)/2$, then any irreducible component of \tilde{R} is $\tilde{\varphi}$ -horizontal.*
 (2) *If $(n - 2)(n - 1)(n + 1)/2 \leq g < (n - 1)(n^2 - 2n - 1)$, then any $\tilde{\varphi}$ -vertical $(-n)$ -curve in \tilde{R} is the proper transform of a fiber of φ .*
 (3) *The multiplicity of any singular point of the branch locus of type $n\mathbb{Z} + 1$ is greater than or equal to $(n - 1)^2$.*

4. Slope Equality, Horikawa Index and the Local Signature

Let $f: S \rightarrow B$ be a primitive cyclic covering fibration of type $(g, 0, n)$. Firstly, we introduce singularity indices.

DEFINITION 4.1 (Singularity index α). Let k be a positive integer. For $p \in B$, we consider all the singular points (including infinitely near ones) of R on the fiber Γ_p of $\varphi: W \rightarrow B$ over p . We let $\alpha_k(F_p)$ be the number of singular points of multiplicity either kn or $kn + 1$ among them and call it the k th singularity index of F_p , the fiber of $f: S \rightarrow B$ over p . Clearly, we have $\alpha_k(F_p) = 0$ except for a finite number of $p \in B$. We put $\alpha_k = \sum_{p \in B} \alpha_k(F_p)$ and call it the k th singularity index of f .

We also define 0th singularity index $\alpha_0(F_p)$ as follows. Let D_1 be the sum of all $\tilde{\varphi}$ -vertical $(-n)$ -curves contained in \tilde{R} and put $\tilde{R}_0 = \tilde{R} - D_1$. Then, $\alpha_0(F_p)$ is the ramification index of $\tilde{\varphi}|_{\tilde{R}_0}: \tilde{R}_0 \rightarrow B$ over p , that is, the ramification index of $\tilde{\varphi}|_{(\tilde{R}_0)_h}: (\tilde{R}_0)_h \rightarrow B$ over p minus the sum of the topological Euler number of irreducible components of $(\tilde{R}_0)_v$ over p . Then $\alpha_0(F_p) = 0$ except for a finite number of $p \in B$, and we have

$$\sum_{p \in B} \alpha_0(F_p) = (K_{\tilde{\varphi}} + \tilde{R}_0)\tilde{R}_0$$

by definition. We put $\alpha_0 = \sum_{p \in B} \alpha_0(F_p)$ and call it the 0th singularity index of f .

REMARK 4.2. The singularity indices just defined are somewhat different from those defined in [16] for $n = 2$ because all singular points are “essential” if $n \geq 3$. We can check that the value of $\alpha_k(F_p)$ does not depend on the choice of the relatively minimal model of $\tilde{W} \rightarrow B$ satisfying Lemma 3.1 by the same argument for $n = 2$ in [16].

Let $\varepsilon(F_p)$ be the number of (-1) -curves contained in F_p and put $\varepsilon = \sum_{p \in B} \varepsilon(F_p)$. This is no more than the number of blowing-ups appearing in $\rho: \tilde{S} \rightarrow S$.

Now, we compute the numerical invariants of f using singularity indices. Recall that R is numerically equivalent to $-rK_\varphi/2 + M\Gamma$ for some half-integer M . We express M in terms of singularity indices by calculating $(K_{\tilde{\varphi}} + \tilde{R})\tilde{R}$ in two ways. From (2.2) and (2.3) we have

$$\begin{aligned} (K_{\tilde{\varphi}} + \tilde{R})\tilde{R} &= \left(\tilde{\psi}^*(K_\varphi + R) + \sum_{i=1}^N \left(1 - n \left[\frac{m_i}{n} \right] \right) \mathbf{E}_i \right) \left(\tilde{\psi}^*R - n \left[\frac{m_i}{n} \right] \mathbf{E}_i \right) \\ &= (K_\varphi + R)R - \sum_{i=1}^N n \left[\frac{m_i}{n} \right] \left(n \left[\frac{m_i}{n} \right] - 1 \right) \\ &= \left(\left(1 - \frac{r}{2} \right) K_\varphi + M\Gamma \right) \left(-\frac{r}{2} K_\varphi + M\Gamma \right) - \sum_{k \geq 1} nk(nk - 1)\alpha_k, \end{aligned}$$

and, thus,

$$(K_{\tilde{\varphi}} + \tilde{R})\tilde{R} = 2(r-1)M - n \sum_{k \geq 1} k(nk - 1)\alpha_k. \quad (4.1)$$

On the other hand, we have

$$(K_{\tilde{\varphi}} + \tilde{R})\tilde{R} = (K_{\tilde{\varphi}} + \tilde{R}_0)\tilde{R}_0 + D_1(K_{\tilde{\varphi}} + D_1) = \alpha_0 - 2\varepsilon. \quad (4.2)$$

Hence,

$$M = \frac{1}{2(r-1)} \left(\alpha_0 + n \sum_{k \geq 1} k(nk - 1)\alpha_k - 2\varepsilon \right) \quad (4.3)$$

by (4.1) and (4.2).

Next, we compute K_f^2 and χ_f . We have

$$\begin{aligned} \tilde{\mathfrak{d}}^2 &= \mathfrak{d}^2 - \sum_{i=1}^N \left[\frac{m_i}{n} \right]^2 = \frac{2r}{n^2}M - \sum_{k \geq 1} k^2\alpha_k, \\ \tilde{\mathfrak{d}}K_{\tilde{\varphi}} &= \mathfrak{d}K_\varphi + \sum_{i=1}^N \left[\frac{m_i}{n} \right] = -\frac{2M}{n} + \sum_{k \geq 1} k\alpha_k, \\ K_{\tilde{\varphi}}^2 &= K_\varphi^2 - N = -\sum_{k \geq 1} \alpha_k. \end{aligned}$$

Thus, we get

$$\begin{aligned} K_f^2 &= -n \sum_{k \geq 1} \alpha_k + 2n(n-1) \left(-\frac{2M}{n} + \sum_{k \geq 1} k\alpha_k \right) \\ &\quad + n(n-1)^2 \left(\frac{2rM}{n^2} - \sum_{k \geq 1} k^2\alpha_k \right) \\ &= \frac{2(n-1)((n-1)r - 2n)}{n} M - n \sum_{k \geq 1} ((n-1)k - 1)^2 \alpha_k \end{aligned}$$

and

$$\begin{aligned}\chi_{\tilde{f}} &= \frac{1}{12}n(n-1)(2n-1)\left(\frac{2rM}{n^2} - \sum_{k \geq 1} k^2 \alpha_k\right) + \frac{1}{4}n(n-1)\left(-\frac{2M}{n} + \sum_{k \geq 1} k \alpha_k\right) \\ &= \frac{n-1}{6n}(r(2n-1) - 3n)M - \frac{n(n-1)}{12} \sum_{k \geq 1} ((2n-1)k^2 - 3k) \alpha_k\end{aligned}$$

by (2.4) and (2.5). Hence, substituting (4.3), we obtain

$$\begin{aligned}K_f^2 &= \frac{n-1}{r-1} \left(\frac{(n-1)r - 2n}{n} (\alpha_0 - 2\varepsilon) + (n+1) \sum_{k \geq 1} k(-nk + r) \alpha_k \right) \\ &\quad - n \sum_{k \geq 1} \alpha_k + \varepsilon, \\ \chi_f &= \frac{n-1}{12(r-1)} \left(\frac{(2n-1)r - 3n}{n} (\alpha_0 - 2\varepsilon) + (n+1) \sum_{k \geq 1} k(-nk + r) \alpha_k \right).\end{aligned}$$

These give us

$$e_f = 12\chi_f - K_f^2 = (n-1)\alpha_0 + n \sum_{k \geq 1} \alpha_k - (2n-1)\varepsilon \quad (4.4)$$

by Noether's formula. Furthermore, we have

$$\begin{aligned}K_f^2 - \lambda_{g,0,n} \chi_f &= \frac{n}{(2n-1)r - 3n} \sum_{k \geq 1} ((n+1)(n-1)(-nk^2 + rk) - (2n-1)r + 3n) \alpha_k \\ &\quad + \varepsilon,\end{aligned}$$

where $\lambda_{g,0,n}$ is the rational number defined in (2.1). In this expression, the coefficient of α_k is nonnegative since $r \geq 2n$. In fact, the quadratic function $(n+1)(n-1)(-nk^2 + rk) - (2n-1)r + 3n$ in k is monotonically increasing in the interval $[1, r/2n]$, and its value at $k = 1$ is $n(n-2)(r-n-2) \geq 0$.

We put

$$\text{Ind}(F_p) = n \sum_{k \geq 1} \left(\frac{(n+1)(n-1)(r-nk)k}{(2n-1)r - 3n} - 1 \right) \alpha_k(F_p) + \varepsilon(F_p). \quad (4.5)$$

Let $\mathcal{A}_{g,0,n}$ be the set of all fiber germs of primitive cyclic covering fibrations of type $(g, 0, n)$. Then (4.5) defines a well-defined function $\text{Ind} : \mathcal{A}_{g,0,n} \rightarrow \mathbb{Q}_{\geq 0}$, called the Horikawa index (see [1]). Note that we have $\text{Ind}(F_p) = 0$ when either $r = n \geq 3$ (in this case, R is smooth) or $p \in B$ is general. We have shown the following:

THEOREM 4.3. *Let $f : S \rightarrow B$ be a primitive cyclic covering fibration of type $(g, 0, n)$. Then*

$$K_f^2 = \lambda_{g,0,n} \chi_f + \sum_{p \in B} \text{Ind}(F_p),$$

where $\lambda_{g,0,n}$ is the rational number in (2.1), and $\text{Ind}: \mathcal{A}_{g,0,n} \rightarrow \mathbb{Q}_{\geq 0}$ denotes the Horikawa index defined by (4.5).

REMARK 4.4. As we saw in Lemma 3.3, the gonality of f is n when $r \geq 2n$. Therefore, the lower bound of the slope of n -gonal fibrations cannot exceed

$$\lambda_{g,0,n} = \frac{12(n-1)}{2n-1} \left(1 - \frac{n(n+1)}{2(2n-1)(g-1) + n(n+1)} \right).$$

When $n = 2$, we have $\lambda_{g,0,2} = 4 - 4/g$, and, therefore, the theorem recovers the slope equality for hyperelliptic fibrations. When $n = 3$, we have $\lambda_{g,0,3} = 24(g-1)/(5g+1)$, which coincides with the lower bound of the slope of semistable trigonal fibrations in [14]. This is expected to hold also for unstable ones (see [10, p. 20]), and ours serves a new evidence for that. When $n = 4$, we have $\lambda_{g,0,4} = 36(g-1)/(7g+3)$, which is strictly greater than the bound $24(g-1)/(5g+3)$ given in [5] for semistable (nonfactorized) tetragonal fibrations.

Now, we state a topological application of the slope equality. For an oriented compact real four-dimensional manifold X , the signature $\text{Sign}(X)$ is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the intersection form on $H^2(X)$. Using the singularity indices, we observe the local concentration of $\text{Sign}(S)$ on a finite number of fiber germs.

COROLLARY 4.5. *Let $f: S \rightarrow B$ be a primitive cyclic covering fibration of type $(g, 0, n)$. Then,*

$$\text{Sign}(S) = \sum_{p \in B} \sigma(F_p),$$

where $\sigma: \mathcal{A}_{g,0,n} \rightarrow \mathbb{Q}$ is defined by

$$\begin{aligned} \sigma(F_p) &= \frac{-(n-1)(n+1)r}{3n(r-1)} \alpha_0(F_p) \\ &+ \sum_{k \geq 1} \left(\frac{(n-1)(n+1)(-nk^2 + rk)}{3(r-1)} - n \right) \alpha_k(F_p) \\ &+ \frac{1}{3n(r-1)} ((n+2)(2n-1)r - 3n) \varepsilon(F_p), \end{aligned}$$

which is called the local signature of F_p .

Proof. Once we have the Horikawa index, we can define the local signature according to [1] as follows. By the index theorem (see [7, p. 126]) we have

$$\text{Sign}(S) = \sum_{p+q \equiv 0 \pmod{2}} h^{p,q}(S) = K_f^2 - 8\chi_f.$$

Put $\lambda = \lambda_{g,0,n}$. Then $\lambda < 12$. From the slope equality $K_f^2 = \lambda\chi_f + \text{Ind}$, where $\text{Ind} = \sum_{p \in B} \text{Ind}(F_p)$, and Noether's formula $12\chi_f = K_f^2 + e_f$ we get

$$K_f^2 = \frac{12}{12 - \lambda} \text{Ind} + \frac{\lambda}{12 - \lambda} e_f$$

$$\chi_f = \frac{1}{12 - \lambda} \text{Ind} + \frac{1}{12 - \lambda} e_f.$$

Then

$$\text{Sign}(S) = \frac{4}{12 - \lambda} \text{Ind} - \frac{8 - \lambda}{12 - \lambda} e_f.$$

Substituting (4.4) and (4.5), the desired equality follows by the definition of $\sigma(F_p)$. □

5. Upper Bound of the Slope

Let $f: S \rightarrow B$ be a primitive cyclic covering fibration of type $(g, 0, n)$. In this section, we give an upper bound of the slope of f . Let the situation be as in the previous section.

For a vertical divisor T and $p \in B$, we denote by $T(p)$ the greatest subdivisor of T consisting of components of the fiber over p . Then $T = \sum_{p \in B} T(p)$. We consider a family $\{L^i\}_i$ of vertical irreducible curves in \tilde{R} over p satisfying:

- (i) L^1 is the proper transform of the fiber Γ_p or a (-1) -curve E^1 .
- (ii) For $i \geq 2$, L^i is the proper transform of an exceptional (-1) -curve E^i that contracts to a point x^i on C^k or its proper transform for some $k < i$, where we let C^1 to be E^1 or Γ_p according to whether L^1 is the proper transform of which curve, and $C^j = E^j$ for $j < i$.
- (iii) $\{L^i\}_i$ is the largest among those satisfying (i) and (ii).

The set of all vertical irreducible curves in \tilde{R} over p is decomposed into the disjoint union of such families uniquely. We denote it as

$$\tilde{R}_v(p) = D^1(p) + \cdots + D^{\eta_p}(p), \quad D^t(p) = \sum_{k \geq 1} L^{t,k},$$

where η_p denotes the number of the decomposition, and $\{L^{t,i}\}_i$ satisfies (i), (ii), (iii). Let $C^{t,k}$ be the exceptional (-1) -curve or the fiber Γ_p the proper transform of which is $L^{t,k}$.

DEFINITION 5.1. Let $j_a^i(F_p)$ be the number of irreducible curves with self-intersection number $-an$ contained in $D^i(p)$. Put $j_a(F_p) := \sum_{t=1}^{\eta_p} j_a^t(F_p)$, $j^t(F_p) := \sum_{a \geq 1} j_a^t(F_p)$, and $j(F_p) := \sum_{t=1}^{\eta_p} j^t(F_p)$.

Let $\alpha_0^+(F_p)$ be the ramification index of $\tilde{\varphi}: \tilde{R}_h \rightarrow B$ over p . It is clear that $\varepsilon(F_p) = j_1(F_p)$ and $\alpha(F_p) = \alpha_0^+(F_p) - 2 \sum_{a \geq 2} j_a(F_p)$.

Let $\iota^t(F_p)$ and $\kappa^t(F_p)$ be the number of singular points over p of types $n\mathbb{Z}$ and $n\mathbb{Z} + 1$, respectively, at which two proper transforms of $C^{t,k}$ meet. We put $\iota(F_p) = \sum_{t=1}^{\eta_p} \iota^t(F_p)$ and $\kappa(F_p) = \sum_{t=1}^{\eta_p} \kappa^t(F_p)$.

LEMMA 5.2. *The following hold:*

- (1) $\iota(F_p) = j(F_p) - \eta_p$.
- (2) $\alpha_0^+(F_p) \geq (n - 2)(j(F_p) - \eta_p + 2\kappa(F_p))$.
- (3) $\sum_{k \geq 1} \alpha_k(F_p) \geq \sum_{a \geq 1} (an - 2)j_a(F_p) + 2\eta_p - \kappa(F_p)$.

Proof. We consider the following graph \mathbf{G}^t : The vertex set $V(\mathbf{G}^t)$ is defined by the symbol set $\{v^{t,k}\}_{k=1}^{j^t(F_p)}$. The edge set $E(\mathbf{G}^t)$ is defined by the symbol set $\{e_x\}_x$, where x moves among all singularities contributing to $\iota^t(F_p)$. If $C^{t,k}$ or a proper transform of it meets that of $C^{t,k'}$ at a singularity x of type $n\mathbb{Z}$, the edge e_x connects $v^{t,k}$ and $v^{t,k'}$. By the definition of the decomposition $\tilde{R}_v(p) = D^1(p) + \dots + D^{\eta_p}(p)$, the graph \mathbf{G}^t is connected for any $t = 1, \dots, \eta_p$. Since this graph \mathbf{G}^t has no loops, we have $\iota^t(F_p) = j^t(F_p) - 1$. Thus, we get (1) by summing it up for t .

When $L^{t,k}$ is a $(-an)$ -curve, $L^{t,k}$ is obtained by blowing $C^{t,k}$ up $an - 1$ (or an when $k = 1$ and $C^{t,1} = \Gamma_p$) times. Thus, disregarding overlaps, $D^t(p)$ is obtained by blowing up

$$\sum_{a \geq 1} (an - 1)j_a^t(F_p) \quad \left(\text{or } \sum_{a \geq 1} (an - 1)j_a^t(F_p) + 1 \right)$$

times. Thus, the number of singular points needed to obtain $D^t(p)$ can be expressed as

$$\sum_{a \geq 1} (an - 1)j_a^t(F_p) + 1 - \iota^t(F_p) - \kappa^t(F_p) = \sum_{a \geq 1} (an - 2)j_a^t(F_p) + 2 - \kappa^t(F_p).$$

Then, the number of singular points needed to obtain $\tilde{R}_v(p)$ is

$$\sum_{t=1}^{\eta_p} \left(\sum_{a \geq 1} (an - 2)j_a^t(F_p) + 2 - \kappa^t(F_p) \right) = \sum_{a \geq 1} (an - 2)j_a(F_p) + 2\eta_p - \kappa(F_p).$$

This gives us (3).

It remains to show (2). Let $\tilde{\Gamma}_p = \sum_i m_i G_i$ be the irreducible decomposition. Then we have

$$\begin{aligned} \alpha_0^+(F_p) &= r - \#(\text{Supp}(\tilde{R}_h) \cap \text{Supp}(\tilde{\Gamma}_p)) \\ &= \sum_i m_i \tilde{R}_h G_i - \# \left(\text{Supp}(\tilde{R}_h) \cap \text{Supp} \left(\bigcup_i G_i \right) \right) \\ &\geq \sum_i (m_i - 1) \tilde{R}_h G_i. \end{aligned}$$

Let $x^{t,1}, \dots, x^{t,\iota^t(F_p)}$ be all singular points over p of type $n\mathbb{Z}$ at which two proper transforms of $C^{t,k}$ meet. Let $E^{t,k}$ be the exceptional curve obtained by blowing up at $x^{t,k}$ and $m^{t,k}$ the multiplicity of the fiber over p along $E^{t,k}$. Let us estimate $\sum_{t=1}^{\eta_p} \sum_{k=1}^{\iota^t(F_p)} (m^{t,k} - 1) \tilde{R}_h \hat{E}^{t,k}$. If there exists a singular point of type $n\mathbb{Z}$ on $E^{t,k}$, then we replace $E^{t,k}$ to the exceptional curve E obtained by blowing up at this point. Repeating this procedure, we may assume that there exist no singular points

of type $n\mathbb{Z}$ on $E^{t,k}$. If there exists a singular point of type $n\mathbb{Z} + 1$ on $E^{t,k}$, then the proper transform of the exceptional (-1) -curve obtained by blowing up at this point belongs to other $D^u(p)$ and becomes $L^{u,1}$ in $D^u(p)$. Since the multiplicity of $\tilde{\Gamma}_p$ along it is not less than $m^{t,k} > 1$, we do not have to consider this situation. Thus, we may assume that there exist no singularities on $E^{t,k}$, and we have $\tilde{R}_h \widehat{E}^{t,k} \geq n - 2$. On the other hand, we see that $\sum_{k=1}^{\iota^t(F_p)} m^{t,k} \geq 2\iota^t(F_p) + 2\kappa^t(F_p)$. Hence, we have

$$\begin{aligned} \sum_i (m_i - 1) \tilde{R}_h G_i &\geq \sum_{t=1}^{\eta_p} \sum_{k=1}^{\iota^t(F_p)} (m^{t,k} - 1) \tilde{R}_h \widehat{E}^{t,k} \\ &\geq (n - 2) \sum_{t=1}^{\eta_p} \sum_{k=1}^{\iota^t(F_p)} (m^{t,k} - 1) \\ &\geq (n - 2) \sum_{t=1}^{\eta_p} (\iota^t(F_p) + 2\kappa^t(F_p)) \\ &= (n - 2)(\iota(F_p) + 2\kappa(F_p)). \end{aligned}$$

Since $\iota(F_p) = j(F_p) - \eta_p$, we get (2). \square

Using Lemma 5.2, we give an upper bound of the slope.

THEOREM 5.3. *Let $f: S \rightarrow B$ be a primitive cyclic covering fibration of type $(g, 0, n)$ and assume that $n \geq 4$. Put*

$$r := \frac{2g}{n-1} + 2, \quad \delta := \begin{cases} 0 & \text{if } r \in 2n\mathbb{Z}, \\ 1 & \text{if } r \notin 2n\mathbb{Z}. \end{cases}$$

Then, the following hold:

(1) *If $n \leq r < n(n-1)$, then*

$$K_f^2 \leq \left(12 - \frac{48n^2(r-1)}{(n-1)(n+1)(r^2 - \delta n^2)} \right) \chi_f.$$

(2) *If $r \geq n(n-1)$, then*

$$K_f^2 \leq \left(12 - \frac{48n(n-1)(r-1)}{n(n+1)r^2 - 8(2n-1)r + 24n - \delta n^3(n+1)} \right) \chi_f.$$

Proof. First, assume that $r \geq n(n-1)$. We put

$$\mu = \frac{48n(n-1)(r-1)}{n(n+1)r^2 - 8(2n-1)r + 24n - \delta n^3(n+1)}, \quad \mu' = \frac{n-1}{12(r-1)}\mu.$$

By the formulae for χ_f and e_f obtained in the previous section, we get

$$\begin{aligned} (12 - \mu)\chi_f - K_f^2 \\ &= e_f - \mu\chi_f \\ &= (n-1)\alpha_0 + n \sum_{k \geq 1} \alpha_k - (2n-1)\varepsilon \end{aligned}$$

$$\begin{aligned}
& -\mu' \left(\frac{r(2n-1)-3n}{n} \alpha_0 + (n+1) \sum_{k \geq 1} (-nk^2 + rk) \alpha_k \right. \\
& \quad \left. - \frac{2(r(2n-1)-3n)}{n} \varepsilon \right) \\
& = \left(n-1 - \frac{r(2n-1)-3n}{n} \mu' \right) \alpha_0 \\
& \quad + \sum_{k \geq 1} \left(Q(k) + n - \frac{(n+1)(r^2 - \delta n^2)}{4n} \mu' \right) \alpha_k \\
& \quad - \left(2n-1 - \frac{2(r(2n-1)-3n)}{n} \mu' \right) \varepsilon,
\end{aligned}$$

where

$$Q(k) = \mu' \left(n(n+1) \left(k - \frac{r}{2n} \right)^2 - \frac{n(n+1)\delta}{4} \right) \geq 0.$$

Therefore,

$$(12 - \mu) \chi_f - K_f^2 \geq A_n \alpha_0 + B_n \sum_{k \geq 1} \alpha_k - (2A_n + 1) \varepsilon, \quad (5.1)$$

where

$$A_n = n-1 - \frac{r(2n-1)-3n}{n} \mu', \quad B_n = n - \frac{(n+1)(r^2 - \delta n^2)}{4n} \mu'.$$

By the definitions of μ and μ' we see that A_n and B_n are positive rational numbers satisfying

$$-2A_n + nB_n - 1 = 0. \quad (5.2)$$

We will show that the right-hand side of (5.1) is nonnegative by estimating it on each fiber F_p . By Lemma 5.2 we have

$$\begin{aligned}
& A_n \alpha_0(F_p) + B_n \sum_{k \geq 1} \alpha_k(F_p) - (2A_n + 1) \varepsilon(F_p) \\
& \geq \sum_{a \geq 2} ((n-4)A_n + (an-2)B_n) j_a(F_p) \\
& \quad + ((n-4)A_n + (n-2)B_n - 1) j_1(F_p) \\
& \quad - ((n-2)A_n - 2B_n) \eta_p + (2(n-2)A_n - B_n) \kappa(F_p) \\
& = \sum_{a \geq 2} (-2A_n + anB_n) j_a(F_p) + (-2A_n + nB_n - 1) j_1(F_p) \\
& \quad + ((n-2)A_n - 2B_n) (j(F_p) - \eta_p) + (2(n-2)A_n - B_n) \kappa(F_p).
\end{aligned}$$

When $n \geq 4$, we can show that $(n-2)A_n - 2B_n > 0$, and thus all coefficients of $j_a(F_p)$, $j(F_p) - \eta_p$, $\kappa(F_p)$ are nonnegative by (5.2). Hence, we get (1).

Assume that $n \leq r < n(n-1)$. We put

$$\mu = \frac{48n^2(r-1)}{(n-1)(n+1)(r^2 - \delta n^2)}, \quad \mu' = \frac{n-1}{12(r-1)} \mu$$

and

$$A_n = n - 1 - \frac{r(2n-1) - 3n}{n} \mu', \quad B_n = n - \frac{(n+1)(r^2 - \delta n^2)}{4n} \mu'.$$

Clearly, $A_n > 0$ and $B_n = 0$. By Corollary 3.9 we get $j(F_p) = 0$ for any $p \in B$. Thus, we get

$$(12 - \mu)\chi_f - K_f^2 \geq A_n \alpha_0^+ + B_n \sum_{k \geq 1} \alpha_k = A_n \alpha_0^+ \geq 0,$$

which is the desired inequality. \square

Appendix

Let $f : S \rightarrow B$ be a fibration of genus $g \geq 2$. Put $e_f(F_p) = e(F_p) - e(F)$ for any fiber F_p , where F is a general fiber. It is well known that $e_f(F_p) \geq 0$ with equality if and only if F_p is a smooth curve of genus g and that $e_f = \sum_{p \in B} e_f(F_p)$. In Section 4, we have defined the Horikawa index and the local signature for primitive cyclic covering fibrations of type $(g, 0, n)$ using singularity indices. However, in general, it is not known whether Horikawa indices or local signatures, if they exist, are unique (see [1]). As to primitive cyclic covering fibrations of type $(g, 0, n)$, we have obtained another local concentration of e_f in (4.4). Therefore, we may have two apparently distinct expressions of $\text{Sign}(S)$ in this case from the proof of Corollary 4.5. In this appendix, we show that two expressions coincide. Namely:

PROPOSITION A.1. *Let $f : S \rightarrow B$ be a primitive cyclic covering fibration of type $(g, 0, n)$. Then we have*

$$e_f(F_p) = (n-1)\alpha_0(F_p) + n \sum_{k \geq 1} \alpha_k(F_p) - (2n-1)\varepsilon(F_p)$$

for any $p \in B$.

Proof. It is sufficient to show that

$$e_{\tilde{f}}(\tilde{F}_p) = (n-1)\alpha_0^+(F_p) + n \sum_{k \geq 1} \alpha_k(F_p) - 2(n-1)j(F_p).$$

Let $N = \sum_{k \geq 1} \alpha_k(F_p)$ be the number of blow-ups on Γ_p , and $\tilde{\Gamma}_p = \sum_{i=0}^N m_i \Gamma_i$ be the irreducible decomposition. We may assume that Γ_i and \tilde{R}_h are transverse for simplicity. Put $r_i = \Gamma_i \tilde{R}$ and $F_i = \tilde{\theta}^* \Gamma_i$. Then,

$$F_p = \sum_{i=0}^N m_i F_i = \sum_{r_i > 0} m_i F_i + \sum_{r_i=0} m_i (F_{i,1} + \cdots + F_{i,n}) + \sum_{\Gamma_i \subset \tilde{R}} m_i n F'_i,$$

where $F_{i,j}$ and F'_i are smooth rational curves. For $r_i > 0$, the restriction map $F_i \rightarrow \Gamma_i$ is an n -cyclic covering. From the Hurwitz formula, we have $2g(F_i) - 2 = -2n + (n-1)r_i$. Let N_1 , N_2 , and N_3 be the numbers of intersection points of two Γ_i and Γ_j that is contained in \tilde{R}_h , not contained in \tilde{R}_h , and that one Γ_i

is contained in \tilde{R} , respectively. Clearly, it follows that $N = N_1 + N_2 + N_3$. Let $J = j(F_p)$, and K be the number of Γ_i such that $r_i = 0$. Then, we have

$$\begin{aligned} e(\tilde{F}_p) &= \sum_{r_i > 0} e(F_i) + 2nK + 2J - N_1 - nN_2 - N_3 \\ &= \sum_{r_i > 0} (2n - (n-1)r_i) + 2nK + 2J - N_1 - nN_2 - N_3 \\ &= 2n(N+1) - 2(n-1)J - (n-1) \sum_{r_i > 0} r_i - N - (n-1)N_2. \end{aligned}$$

Since $e(\tilde{F}) = 2n - (n-1)r$, we have

$$e_{\tilde{F}}(\tilde{F}_p) = (2n-1)N - 2(n-1)J + (n-1) \left(r - \sum_{r_i > 0} r_i \right) - (n-1)N_2.$$

On the other hand, we have

$$\begin{aligned} \alpha_0^+(F_p) &= r - \#(\text{Supp}(\tilde{\Gamma}_p) \cap \text{Supp}(\tilde{R})) \\ &= r - \sum_{r_i > 0} r_i + N_1 + N_3. \end{aligned}$$

Combining these equalities, the assertion follows. \square

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