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Twisting of Composite Torus Knots

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Abstract. We prove that the family of connected sums of torus knots $T(2,p) \# T(2,q) \# T(2,r)$ is nontwisted for any odd positive integers $p,q,r \geq 3$, partially answering in the positive a conjecture of Teragaito [19].

1. Introduction

Let K be a knot in the 3-sphere S^3, and D^2 a disk intersecting K in its interior. Let n be an integer. A $(-\frac{1}{n})$-Dehn surgery along $C = \partial D^2$ changes K into a new knot K_n in S^3. Let $\omega = \mathrm{lk}(\partial D^2, L)$. We say that K_n is obtained from K by (n,ω)-*twisting* (or simply *twisting*). Then we write $K \overset{(n,\omega)}{\rightarrow} K_n$ or $K \overset{(n,\omega)}{\rightarrow} K(n,\omega)$. We say that K_n is an (n,ω)-twisted knot (or simply a twisted knot) if K is the unknot (see Figure 1).

An easy example is depicted in Figure 2, where we show that the right-handed trefoil $T(2,3)$ is obtained from the unknot $T(2,1)$ by a $(+1,2)$-twisting (in this case, $n = +1$ and $\omega = +2$). A less obvious example is given in Figure 3, where it is shown that the composite knot $T(2,3) \# T(2,5)$ can be obtained from the unknot by a $(+1,4)$-twisting (in this case, $n = +1$ and $\omega = +4$); see [10]. Here, $T(2,q)$ denotes the $(2,q)$-torus knot (see [11]).

Active research on twisting of knots started around 1990. One pioneer was the author's Ph.D. thesis advisor Y. Mathieu, who asked the following questions in [13].

Question 1.1. Is every knot in S^3 twisted? If not, what is the minimal number of twisting disks?

Question 1.2. Is every twisted knot in S^3 prime?

To answer Question 1.1, Miyazaki and Yasuhara [15] were the first to give an infinite family of knots that are nontwisted. In particular, they showed that the granny knot, that is, the product of two right-handed trefoil knots, is the smallest nontwisted knot. In his Ph.D. thesis [3], the author showed that $T(5,8)$ is the smallest nontwisted torus knot. This was followed by a joint work with Yasuhara [4], in which we gave an infinite family of nontwisted torus knots (i.e., $T(p, p+7)$ for any $p \geq 7$) using some techniques derived from old gauge theory. On the other hand, Ohyama [16] showed that any knot in S^3 can be untied by (at most) two disks.

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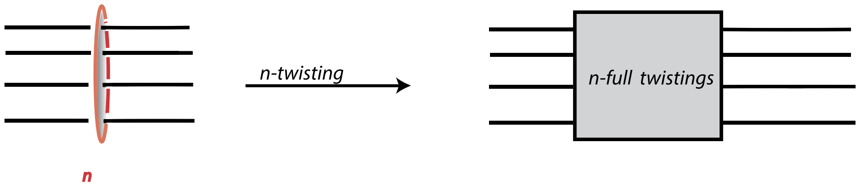

Figure 1

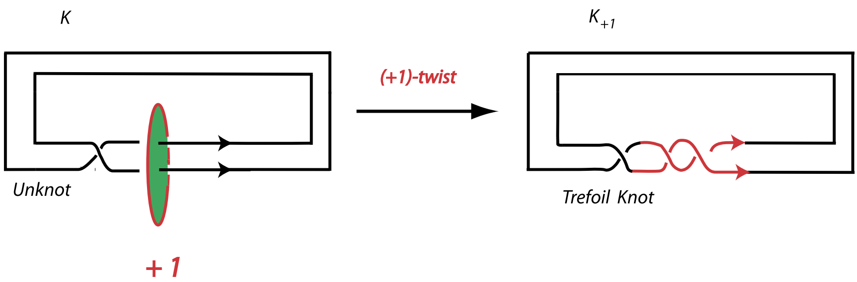

Figure 2

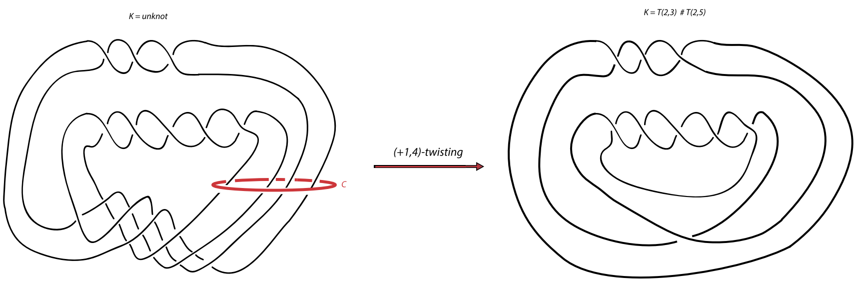

Figure 3

To answer Question 1.2, Hayashi and Motegi [10] and M. Teragaito [20] independently found examples of composite twisted knots (see Figure 3). In particular, Goodman-Strauss [8] showed that any composite knot of the form $T(p,q) \# T(-q, p+q)$ is a twisted knot for any coprime positive integers $1 < p < q$. More generally, Hayashi and Motegi [10] and Goodman-Strauss [8] proved independently that only single twisting (i.e., $|n| = 1$) can yield a composite knot. The tools used were combinatorial methods as in CGLS [5]. Moreover, Goodman-Strauss [8] proved that K_1 and K_{-1} cannot both be composite and classified all composite knots of the form $K_1 \# K_2$, where K_1 and K_2 are both prime knots (for an extensive list of twisted composite knots, we refer the reader to the appendix of Goodman-Strauss's paper [8]). However, there is no known twisted knot with three or more factors, that is, $k = k_1 \# k_2 \# \cdots \# k_m$, where k_i is a prime knot for $i = 1, 2, \ldots, m$, and $m \geq 3$, which motivates the still open Teragaito's conjecture.

CONJECTURE 1.1 (Teragaito [19]). *Any composite knot with three or more factors is nontwisted.*

In this paper, we prove the following theorem.

THEOREM 1.1. *$T(2, p) \# T(2, q) \# T(2, r)$ is not twisted for any odd positive integers $p, q, r \geq 3$.*

2. Preliminaries

In what follows, let X be a smooth, closed, oriented, simply connected 4-manifold. Then the second homology group $H_2(X; \mathbb{Z})$ is finitely generated (for details, we refer to the book by Milnor and Stasheff [14]). The ordinary form $q_X : H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) \longrightarrow \mathbb{Z}$ given by the intersection pairing for 2-cycles such that $q_X(\alpha, \beta) = \alpha \cdot \beta$ is a symmetric unimodular bilinear form. The signature of this form, denoted $\sigma(X)$, is the difference of the numbers of positive and negative eigenvalues of a matrix representing q_X. Let $b_2^+(X)$ (resp. $b_2^-(X)$) be the rank of the positive (resp. negative) part of the intersection form of X. The second Betti number is $b_2(X) = b_2^+(X) + b_2^-(X)$, and the signature is $\sigma(X) = b_2^+(X) - b_2^-(X)$. From now on, a homology class in $H_2(X - B^4, \partial; \mathbb{Z})$ is identified with its image by the homomorphism

$$H_2(X - B^4, \partial(X - B^4); \mathbb{Z}) \cong H_2(X - B^4; \mathbb{Z}) \longrightarrow H_2(X; \mathbb{Z}).$$

Recall that $\mathbb{CP}^2$ is the closed 4-manifold obtained by the free action of $\mathbb{C}^* = \mathbb{C} - \{0\}$ on $\mathbb{C}^3 - \{(0, 0, 0)\}$ defined by $\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$, where $\lambda \in \mathbb{C}^*$, that is, $\mathbb{CP}^2 = (\mathbb{C}^3 - \{(0, 0, 0)\})/\mathbb{C}^*$. An element of $\mathbb{CP}^2$ is denoted by its homogeneous coordinates $[x : y : z]$, which are defined up to the multiplication by $\lambda \in \mathbb{C}^*$. The fundamental class of the submanifold $H = \{[x : y : z] \in \mathbb{CP}^2 | x = 0\}$ ($H \cong \mathbb{CP}^1$) generates the second homology group $H_2(\mathbb{CP}^2; \mathbb{Z})$ (see Gompf and Stipsicz [8]). Since $H \cong \mathbb{CP}^1$, the standard generator of $H_2(\mathbb{CP}^2; \mathbb{Z})$ is denoted, from now on, by $\gamma = [\mathbb{CP}^1]$. Therefore, the standard generator of $H_2(\mathbb{CP}^2 - B^4; \mathbb{Z})$ is $\mathbb{CP}^1 - B^2 \subset \mathbb{CP}^2 - B^4$ with complex orientations.

Let $\alpha = S^2 \times \{\star\}$ and $\beta = \{\star\} \times S^2$ denote the standard generators of $H_2(S^2 \times S^2; \mathbb{Z})$ such that $\alpha^2 = \beta^2 = 0$, $\alpha \cdot \beta = 1$, and let γ (resp. $\bar{\gamma}$) be the standard generators of $H_2(\mathbb{CP}^2; \mathbb{Z})$ (resp. $H_2(\overline{\mathbb{CP}^2}; \mathbb{Z})$) with $\gamma^2 = +1$ (resp. $\bar{\gamma}^2 = -1$).

A second homology class $\xi \in H_2(X; \mathbb{Z})$ is said to be characteristic if ξ is dual to the second Stiefel–Whitney class $w_2(X)$ or, equivalently,

$$\xi \cdot x \equiv x \cdot x \pmod 2$$

for any $x \in H_2(X; \mathbb{Z})$ (we leave the details to Milnor and Stasheff [14]).

EXAMPLE 2.1. $(a, b) \in H_2(S^2 \times S^2; \mathbb{Z})$ is characteristic if and only if a and b are both even.

EXAMPLE 2.2. $d\gamma \in H_2(\mathbb{CP}^2; \mathbb{Z})$ is characteristic if and only if d is odd.

The following theorems give obstructions on the genus of an embedded surface representing either a characteristic class or bounding a knot in a punctured 4-manifold. Recall that the Arf invariant of a knot K is denoted by $\mathrm{Arf}(K)$, $\sigma_p(K)$ denotes the Tristram p-signature [21], and $e_2(K)$ denotes the minimum number of generators of $H_2(X_K; \mathbb{Z})$, where X_K is the 2-fold branched covering of S^3 along K.

THEOREM 2.1 (Acosta [1]). *Suppose that ξ is a characteristic homology class in an indefinite smooth oriented 4-manifold of genus g. Let $m = \min(b_2^+(X), b_2^-(X))$.*

(1) *If $\xi^2 \equiv \sigma(X) \bmod 16$, then either $\xi^2 = \sigma(X)$ or, if not,*
   (a) *If $\xi^2 = 0$ or ξ^2 and $\sigma(X)$ have the same sign, then $|\xi^2 - \sigma(X)|/8 \leq m + g - 1$.*
   (b) *If $\sigma(X) = 0$ or ξ^2 and $\sigma(X)$ have opposite signs, then $|\xi^2 - \sigma(X)|/8 \leq m + g - 2$.*
(2) *If $\xi^2 \equiv \sigma(X) + 8 \bmod 16$, then*
   (a) *If $\xi^2 = -8$ or $\xi^2 + 8$ and $\sigma(X)$ have the same sign, then $|\xi^2 + 8 - \sigma(X)|/8 \leq m + g + 1$.*
   (b) *If $\sigma(X) = 0$ or $\xi^2 + 8$ and $\sigma(X)$ have opposite signs, then $|\xi^2 + 8 - \sigma(X)|/8 \leq m + g$.*

THEOREM 2.2 (Gilmer [7] and Viro [22]). *Let X be an oriented compact 4-manifold with $\partial X = S^3$, and K a knot in ∂X. Suppose that K bounds a surface of genus g in X representing an element ξ in $H_2(X; \partial X)$.*

(1) *If ξ is divisible by an odd prime d, then $|(d^2 - 1)/(2d^2)\xi^2 - \sigma(X) - \sigma_d(K)| \leq \dim H_2(X; \mathbb{Z}_d) + 2g$.*
(2) *If ξ is divisible by 2, then $|\xi^2/2 - \sigma(X) - \sigma(K)| \leq \dim H_2(X; \mathbb{Z}_2) + 2g$.*

THEOREM 2.3 (Robertello [17]). *Let X be an oriented compact 4-manifold with $\partial X = S^3$, and K a knot in ∂X. Suppose that K bounds a disk in X representing a characteristic element ξ in $H_2(X; \partial X)$. Then $(\xi^2 - \sigma(X))/8 \equiv \mathrm{Arf}(K) \pmod 2$.*

LEMMA 2.1. *If K is a knot obtained by a $(-1, \omega)$-twisting from the unknot K_0, then K bounds a properly embedded smooth disk $(D, \partial D) \subset (\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4))$ such that $[D] = \omega\gamma \in H_2(\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4); \mathbb{Z})$.*

Recall, for convenience of the reader, a proof of Lemma 2.1. As shown in Figure 4, let D be a disk on which the $(-1, \omega)$-twisting is performed. Note that the $(+1)$-Dehn surgery on $\partial D = C$ changes K_0 to K. Regard K_0 and D as contained in the boundary of a four-dimensional 0-handle h^0. Then attach a 2-handle h^2 to h^0 along ∂D with framing $+1$. Since $\mathbb{CP}^2 = h^0 \cup h^2 \cup h^3$ with $h^0 \cong B^4$ and $h^3 \cong$

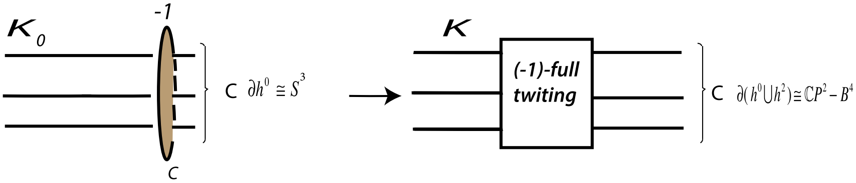

Figure 4

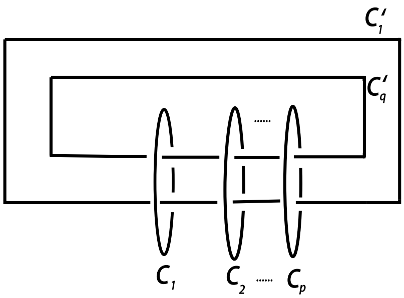

Figure 5 The link $L(p, q)$

B^4, the resulting 4-manifold $h^0 \cup h^2$ is diffeomorphic to $\mathbb{CP}^2 - B^4$ (see [12]). Let $(\Delta, \partial\Delta) \subset (B^4, \partial B^4 \cong S^3)$ be a compact and orientable disk with $\partial\Delta = K_0$. Since $\mathrm{lk}(K_0, \partial D) = \omega$, we can check that $[\Delta] = \omega\gamma \in H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$, where γ is the standard generator of $H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$.

Lemma 2.2 (Nakanishi [15]). *Suppose that K is obtained from a trivial knot K_0 by (n, ω)-twisting. If ω is even, then $e_2(K) \leq 2$.*

Lemma 2.3 (Ait Nouh [2]). *The d-signature of a $(2, q)$-torus knot $T(2, q)$ is given by the formula*

$$\sigma_d(T(2, q)) = -(q - 1) - \left[\frac{q}{2d}\right].$$

To prove Theorem 1.1, we recall the definition of band surgery.

Let L be a c-component oriented link. Let $B_1, \ldots, B_b$ be mutually disjoint oriented bands in S^3 such that $B_i \cap L = \partial B_i \cap L = \alpha_i \cup \alpha_i'$, where $\alpha_1, \alpha_1', \ldots, \alpha_b, \alpha_b'$ are disjoint connected arcs. The closure of $L \cup \partial B_1 \cup \cdots \cup \partial B_b$ is also a link L'.

Definition 2.1. If L' has the orientation compatible with the orientation of $L - \bigcup_{i=1,\ldots,b} \alpha_i \cup \alpha_i'$ and $\bigcup_{i=1,\ldots,b} (\partial B_i - \alpha_i \cup \alpha_i')$, then L' is called the link obtained by the *band surgery* along the bands $B_1, \ldots, B_b$. If $c = b + 1$, then this operation is called a *fusion.*

Example 2.3. Let $L(p, q) = C_1 \cup \cdots \cup C_p \cup C_1' \cup \cdots \cup C_q'$ denote the $((p, 0), (q, 0))$-cable on the Hopf link with linking number 1 (see Figure 5). Then

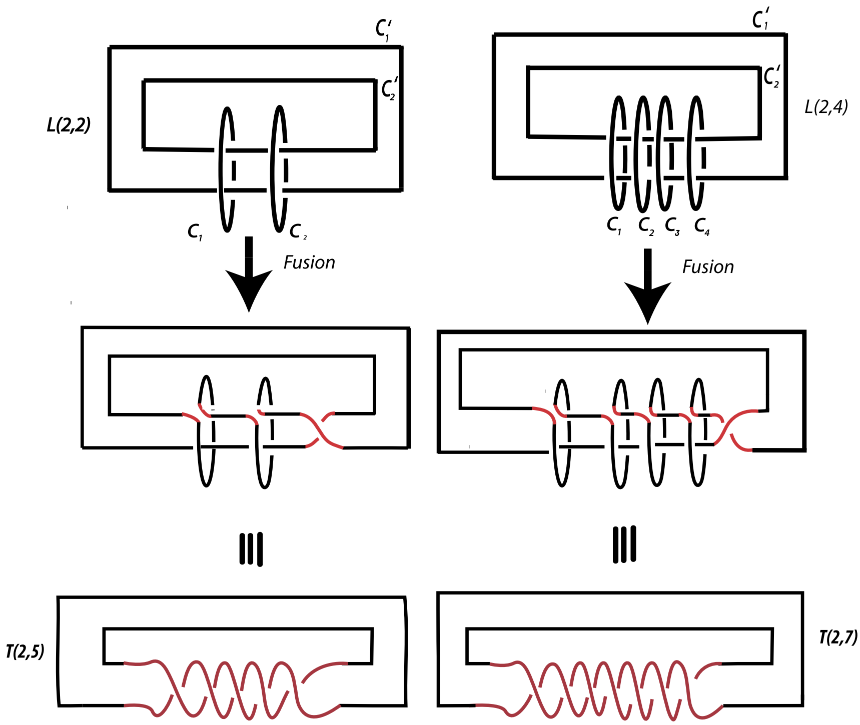

Figure 6

$T(2,5)$ (resp. $T(2,7)$) can be obtained from $L(2,2)$ (resp. $L(2,4)$) by fusion (see Figure 6).

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following proposition.

PROPOSITION 3.1. *$T(2,p)\,\#\,T(2,q)\,\#\,T(2,r)$ is obtained from $L(2,g^*+\ell)$ by adding $b=g^*+\ell+5$ bands, where g^* denotes the 4-ball genus of $T(2,p)\,\#\,T(2,q)\,\#\,T(2,r)$, and ℓ is the number of integers in the set $\{p,q,r\}$ that are congruent to 3 modulo 4. In particular, there is a cobordism of genus two between $L(2,g^*+\ell)$ and $T(2,p)\,\#\,T(2,q)\,\#\,T(2,r)$, where $g^*+\ell$ is always even.*

Proof. Figure 7 shows that if $p\equiv 1 \pmod 4$ (resp. $p\equiv 3 \pmod 4$), then $T(2,p)$ is obtained from $L(2,\frac{p-1}{2})$ (resp. $L(2,\frac{p+1}{2})$) by fusion, that is, by adding $\frac{p-1}{2}+1$ (resp. $\frac{p+1}{2}+1$) bands. Therefore, to prove the proposition, there are four cases to distinguish:

Case I. $p\equiv q\equiv r\equiv 1 \pmod 4$.
Case II. $p\equiv 3$ and $q\equiv r\equiv 1 \pmod 4$.

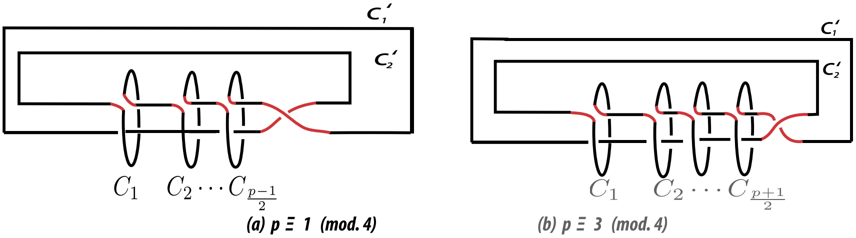

Figure 7

Case III. $p \equiv q \equiv 3 \pmod 4$ and $r \equiv 1 \pmod 4$.
Case IV. $p \equiv q \equiv r \equiv 3 \pmod 4$.

By a band surgery with $b = 2$, $L(2, g^* + \ell)$ can be turned into a connected sum of $L(2, \frac{p\pm1}{2})$, $L(2, \frac{q\pm1}{2})$, $L(2, \frac{r\pm1}{2})$, which has $g^* + \ell + 4$ components. Since each of the summands can be turned into $T(2, p)$, $T(2, q)$, $T(2, r)$, respectively, by a fusion, we have that $T(2, p) \# T(2, q) \# T(2, r)$ can be obtained from $L(2, g^* + \ell)$ by a band surgery with $b = g^* + \ell + 5$. Since the proofs of these cases are similar, we provide more details for the case $\ell = 0$.

Case I. $p \equiv q \equiv r \equiv 1 \pmod 4$.

This is equivalent to $\ell = 0$. As shown in Figures 7 and 8, $k = T(2, p) \# T(2, q) \# T(2, r)$ can be obtained from the link $L(2, \frac{p-1}{2} + \frac{q-1}{2} + \frac{r-1}{2}) = L(2, g^*)$ by adding the number of bands equal to

$$\begin{aligned} b &= \frac{p-1}{2} + \frac{q-1}{2} + \frac{r-1}{2} + 5 \\ &= g^* + 5. \end{aligned}$$

Note that $c = \frac{p-1}{2} + \frac{q-1}{2} + \frac{r-1}{2} + 2$ or, equivalently, $c = g^* + 2$. Since $g_c = \frac{1-c+b}{2}$, we have that $g_c = 2$ and $g^* + \ell = g^*$ is even.

Note that in all four cases, $b = g^* + \ell + 5$ and $c = g^* + \ell + 2$, and, therefore, there is a cobordism of genus $g_c = \frac{1-c+b}{2}$ $(= 2)$ (see [6]) between $L(2, g^* + 3)$ and k. □

Proof of Theorem 1.1. Assume for a contradiction that $K \cong T(2, p) \# T(2, q) \# T(2, r)$ can be obtained by (n, ω)-twisting from an unknot K_0. Since $e_2(T(2, p) \# T(2, q) \# T(2, r)) > 2$, by Lemma 2.2, ω is odd. Since K is a composite knot, $n = \pm1$ (see [10; 9]). The following proofs are based on the gluing of two punctured standard 4-manifolds, as depicted in Figure 9.

Case I. Assume that $n = +1$. Then $\bar{K} = T(-2, p) \# T(-2, q) \# T(-2, r)$ can be obtained by $(-1, \omega)$-twisting along an unknot $\bar{K}_0$, the inverse of the mirror-image of K_0 (see [3]). By Lemma 2.1 this yields that $\bar{K}$ bounds a disk $(D, \partial D) \subset (\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4) \cong S^3)$ such that $[D] = \omega\gamma \in H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$, where γ denotes the standard generator of $H_2(\mathbb{CP}^2; \mathbb{Z})$ with $\gamma^2 = +1$.

On the other hand, there exist a 4-ball J and a mutually disjoint union of $g^* + \ell + 2$ properly embedded 2-disks $\Delta_1, \Delta_2, \ldots, \Delta_{g^*+\ell+2}$ such that $\Delta =$

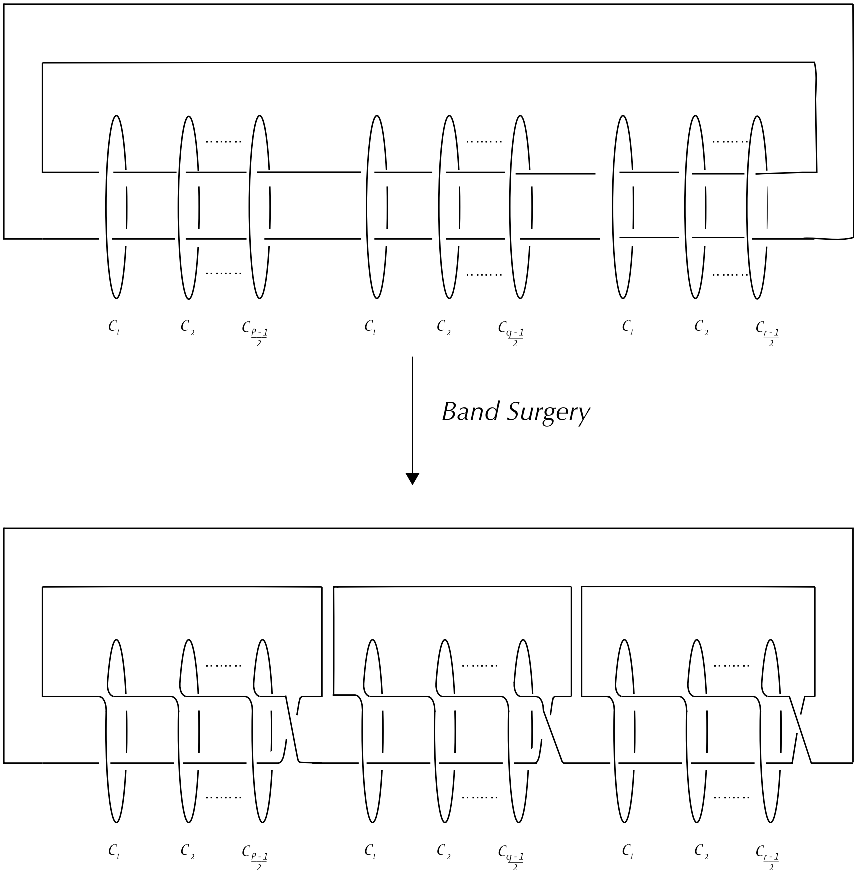

Figure 8 *Case* I: $p \equiv q \equiv r \equiv 1 \pmod 4$

$\bigcup_{i=1}^{i=g^*+\ell+2} \Delta_i$ bounds $L(2, g^* + \ell)$ with $0 \le \ell \le 3$ in $S^2 \times S^2 - J$ and $[\Delta] = 2\alpha + (g^* + \ell)\beta \in H_2(S^2 \times S^2 - J, \partial(S^2 \times S^2 - J) \cong S^3; \mathbb{Z})$, where α, β denote the standard generators of $H_2(S^2 \times S^2; \mathbb{Z})$ with $\alpha^2 = \beta^2 = 0$, $\alpha \cdot \beta = 1$, and g^* denotes the 4-ball genus of K.

Since K is obtained from $L(2, g^* + \ell)$ by the band surgery described in Proposition 3.1, there exists a $(g^* + \ell + 3)$-punctured genus-two surface $\hat{F}$ in $S^3 \times [0, 1] \subset J$ such that we can identify this band surgery with $\hat{F} \cap (S^3 \times \{1/2\})$, $\partial \hat{F} = L(2, g^* + \ell) \cup k$ with $L(2, g^* + \ell)$ lies in $S^3 \times \{0\} \cong \partial J \times \{0\}$, and K lies in $S^3 \times \{1\} \cong \partial J \times \{1\}$. The 3-sphere $S^3 \times \{1\}$ ($\cong \partial J \times \{1\}$) bounds a 4-ball $B^4 \subset J$. The surface $F = \Delta \cup \hat{F}$ is a smooth genus-two surface properly embedded in $S^2 \times S^2 - B^4$ and with boundary K such that

$$[F] = 2\alpha + (g^* + \ell)\beta \in H_2(S^2 \times S^2 - B^4, \partial(S^2 \times S^2 - B^4) \cong S^3; \mathbb{Z}).$$

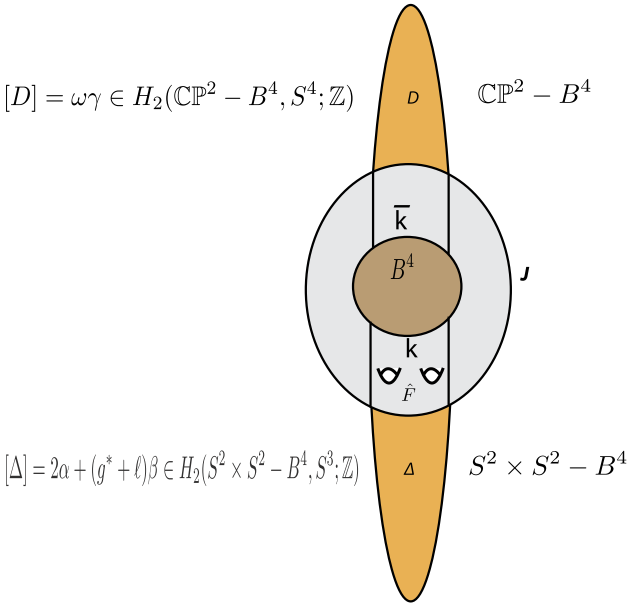

Figure 9

The genus-two smooth and closed surface $\Sigma = F \cup D$ satisfies

$$[\Sigma] = 2\alpha + (g^* + \ell)\beta + \omega\gamma \in H_2(S^2 \times S^2 \# \mathbb{CP}^2; \mathbb{Z}).$$

By Lemma 2.2, ω is odd, and by Proposition 3.1, $g^* + \ell$ is even. Then, $\xi = [\Sigma]$ is a characteristic class in $H_2(S^2 \times S^2 \# \mathbb{CP}^2; \mathbb{Z})$. Furthermore, $X = S^2 \times S^2 \# \mathbb{CP}^2$ is homeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \# \mathbb{CP}^2$ (e.g., see Scorpan's book [18], p. 239, or Corollary 4.3 in Kirby's book [12], p. 11). Note that ξ^2 and $\sigma(X)$ have the same signs, $m = 1$, and $g = 2$. Therefore, by Theorem 2.1(1)(a) and Theorem 2.1(2)(a),

$$\frac{|\xi^2 - \sigma(X)|}{8} \leq 3$$

or, equivalently,

$$\frac{4(g^* + \ell) + \omega^2 - 1}{8} \leq 3.$$

This yields that the only possibilities are $g^* = 3$ or 4 and $\omega = \pm 1$; equivalently, $K = T(2, 3) \# T(2, 3) \# T(2, 3)$, then $\ell = 3$ or $K = T(2, 3) \# T(2, 3) \# T(2, 5)$, and then $\ell = 2$ with $\omega = \pm 1$. Then K would bound a disk $(D, \partial D) \subset (\overline{\mathbb{CP}^2} - B^4, \partial(\overline{\mathbb{CP}^2} - B^4))$ such that

$$\xi = [D] = \pm\bar{\gamma} \in H_2(\overline{\mathbb{CP}^2} - B^4, \partial(\overline{\mathbb{CP}^2} - B^4); \mathbb{Z}),$$

where $\bar{\gamma}$ is the standard generator of $H_2(\overline{\mathbb{CP}^2} - B^4, \partial(\overline{\mathbb{CP}^2} - B^4); \mathbb{Z})$ with $\bar{\gamma}^2 = -1$, and therefore $|\xi^2 - \sigma(X)|/8 = 0$. This contradicts Theorem 2.3 since $\mathrm{Arf}(K) = 1$.

Case II. Assume that $n = -1$. Then there are two cases to exclude.

Case II(a). If ω is divisible by a prime $d \geq 3$, then by Lemma 2.1, k bounds a smooth disk $(D, \partial D) \subset (\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4) \cong S^3)$ such that $\xi = [D] = \omega\gamma \in H_2(\mathbb{CP}^2 - B^4; S^3; \mathbb{Z})$. By Lemma 2.3 the signatures are

$$\sigma(K) = -(p+q+r-3) \quad \text{and}$$
$$\sigma_d(K) = -(p-1) - \left[\frac{p}{2d}\right] - (q-1) - \left[\frac{q}{2d}\right] - (r-1) - \left[\frac{r}{2d}\right] \quad \text{(see [2])}.$$

This contradicts Theorem 2.2.

Case II(b). If $\omega = \pm 1$, then by the same argument as in Case I, this would yield the existence of a genus-two surface that satisfies

$$\xi = [\Sigma] = 2\alpha + (g^* + \ell)\beta + \bar{\gamma} \in H_2(S^2 \times S^2 \# \overline{\mathbb{CP}^2}; \mathbb{Z}).$$

If we let $X = S^2 \times S^2 \# \overline{\mathbb{CP}^2}$, then ξ^2 and $\sigma(X)$ have opposite signs with $m = 1$ and $g = 2$. Therefore, by Theorem 2.1(1)(b) and Theorem 2.1(2)(b),

$$\frac{|\xi^2 - \sigma(X)|}{8} \leq 2$$

or, equivalently, $g^* + \ell \leq 4$. This yields that the only possibilities are $g^* = 3$ or 4; equivalently, $K = T(2,3) \# T(2,3) \# T(2,3)$, then $\ell = 3$ or $K = T(2,3) \# T(2,3) \# T(2,5)$, and then $\ell = 2$. Therefore, $g^* + \ell = 6$, a contradiction. □

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