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Potential Theory for Quaternionic Plurisubharmonic Functions

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Abstract. In this paper, we establish quaternionic versions of the potential description of various "small" sets related to quaternionic plurisubharmonic functions in the n-dimensional quaternionic space $\mathbb{H}^n$. We use the quaternionic capacity introduced in [31] to characterize the $(-\infty)$-sets of plurisubharmonic functions as the sets of vanishing capacity. The latter requirement is also equivalent to the negligibility of a set. We also prove the Josefson theorem on the equivalence of the locally and globally quaternionic polar sets in $\mathbb{H}^n$, following the Bedford–Taylor method.

1. Introduction

The pluripotential theory, which is a nonlinear complex counterpart of classical potential theory, has occupied an important place in mathematics. Although relatively young, the pluripotential theory has attracted considerable interest among analysts. The central part of the pluripotential theory is occupied by maximal plurisubharmonic functions and the generalized complex Monge–Ampère operator. Decisive progress in this field has been made by Bedford and Taylor [6; 7; 8; 9], Demailly [14; 15; 16; 17], Cegrell [10; 11; 12; 13], to mention a few. Cegrell's book [10] provides an excellent in-depth study of capacities in $\mathbb{C}^n$. See also [20] for a detailed discussion on various types of small sets in $\mathbb{C}^n$.

The potential theory for the Hessian equation has also been intensively studied in recent years. Labutin [21] studied the potential estimates for the real k-Hessian equation and used a special capacity to investigate the typical questions of potential theory: local behavior, removability of singularities, and polar, negligible, and thin sets. See [23; 24; 25; 26; 28] and references therein for other potential results for the complex Hessian and real Hessian equation.

In the n-dimensional quaternionic space $\mathbb{H}^n$, at present, little is known about the quaternionic pluripolar sets and the zero sets of the quaternionic capacities. The purpose of this paper is to give a potential-theoretic description of various "small" sets related to the quaternionic Monge–Ampère operator.

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Let Ω be an open set in $\mathbb{H}^n$. The quaternionic Monge–Ampère operator is defined as the Moore determinant of the quaternionic Hessian of u:

$$\det(u) = \det\left[\frac{\partial^2 u}{\partial q_j\, \partial \bar{q}_k}(q)\right].$$

Alesker [2] proved a quaternionic version of Chern–Levine–Nirenberg estimate and extended the definition of quaternionic Monge–Ampère operator to continuous quaternionic plurisubharmonic functions. Since it is inconvenient to use the Moore determinant, the study of the quaternionic Monge–Ampère operator is much more difficult than that of the complex Monge–Ampère operator.

To define the quaternionic Monge–Ampère operator on general quaternionic manifolds, Alesker [5] introduced an operator in terms of the Baston operator $\triangle$, which is the first operator of the quaternionic complex on quaternionic manifolds. The nth-power of this operator is exactly the quaternionic Monge–Ampère operator when the manifold is flat. On the flat space $\mathbb{H}^n$, the Baston operator $\triangle$ is the first operator of the 0-Cauchy–Fueter complex:

$$0 \to C^\infty(\Omega,\mathbb{C}) \xrightarrow{\triangle} C^\infty\Big(\Omega, \bigwedge^2 \mathbb{C}^{2n}\Big) \xrightarrow{D} C^\infty\Big(\Omega, \bigwedge^3 \mathbb{C}^{2n}\Big) \to \cdots. \tag{1.1}$$

Wang [32] wrote down explicitly each operator of the k-Cauchy–Fueter complex in terms of real variables.

Motivated by this, Wang and the first author introduced in [30] two first-order differential operators d_0 and d_1 acting on the quaternionic version of differential forms. The second operator D in (1.1) can be written as $D := \binom{d_0}{d_1}$. The behavior of d_0, d_1, and $\triangle = d_0 d_1$ is very similar to ∂, $\overline{\partial}$, and $\partial\overline{\partial}$ in several complex variables. The quaternionic Monge–Ampère operator can be defined as $(\triangle u)^n$ and has a simple explicit expression, which is much more convenient to use than the previous definition by using the Moore determinant.

By introducing the quaternionic version of differential forms Wang and the first author defined in [30] the notions of closed positive forms and closed positive currents in the quaternionic case, and our definition of closedness well matches positivity. We proved that $\triangle u$ is a closed positive 2-current for any plurisubharmonic function u and showed that when functions $u_1, \dots, u_k$ are locally bounded, $\triangle u_1 \wedge \cdots \wedge \triangle u_k$ is a well-defined closed positive current and is continuous on decreasing sequences.

Based on these observations, Zhang and the first author established in [31] several useful quaternionic versions of results in the complex pluripotential theory, which play key roles in this paper. We showed that quasi-continuity, one of the most important properties of complex plurisubharmonic functions, holds also for quaternionic plurisubharmonic functions in $\mathbb{H}^n$. We also proved an equivalent characterization of the maximal plurisubharmonic functions and comparison theorems, which are connected to the uniqueness of the Dirichlet problem of quaternionic Monge–Ampère equations [3; 33]. We also established other quaternionic versions of results in the complex pluripotential theory [27; 29].

In this paper, we are concerned with the quaternionic capacities and with the description of exceptional sets related to the quaternionic Monge–Ampère operator in $\mathbb{H}^n$.

Now we introduce two types of exceptional sets in $\mathbb{H}^n$. A set $E \subset \mathbb{H}^n$ is said to be *locally quaternionic polar* (locally Q-polar for short) if for each point $a \in E$, there are a neighborhood $B(a,r)$ and a function $u \in \mathrm{PSH}(B(a,r))$ such that $u|_{E\cap B(a,r)} = -\infty$. A set $E \subset \Omega$ in $\mathbb{H}^n$ is said to be *globally quaternionic polar* (globally Q-polar for short) in Ω if there exists a function $u \in \mathrm{PSH}(\Omega)$ such that $E \subset \{u = -\infty\}$.

We show Josefson's theorem on the equivalence of the locally and globally Q-polar sets in $\mathbb{H}^n$ following the proof in pluripotential theory on $\mathbb{C}^n$ given in [9]. The original proof was given by Josefson [19] basing on complicated estimates for polynomials.

Theorem 1.1. *If $P \subset \mathbb{H}^n$ is locally Q-polar, then there exists $v \in \mathrm{PSH}(\mathbb{H}^n)$ with $P \subset \{v = -\infty\}$, that is, P is globally Q-polar in $\mathbb{H}^n$.*

We also consider the so-called *negligible sets,* which are those of the form

$$N = \{q \in \Omega, u(q) < u^*(q)\}, \tag{1.2}$$

where $u = \sup_\alpha u_\alpha$ is the upper envelope of a family of functions $(u_\alpha) \subset \mathrm{PSH}(\Omega)$ that are locally bounded from above in Ω, and u^* is the upper semicontinuous regularization of u, that is, $u^*(q) = \limsup_{q'\to q} u(q')$ for $q \in \Omega$. We show in Theorem 1.2 that the negligible sets are precisely the Q-polar sets.

In order to prove the quasi-continuity theorem (Lemma 2.3 in Sect. 2), the first author introduced in [31] the quaternionic capacities for quaternionic plurisubharmonic functions. These capacities are defined in the same way as the capacities introduced by Bedford and Taylor [9] for plurisubharmonic functions in $\mathbb{C}^n$.

Let Ω be a bounded open set of $\mathbb{H}^n$. If K is a compact subset of Ω, we define the (*relative*) *quaternionic capacity* of K in Ω by

$$C(K,\Omega) = \sup\left\{\int_K (\triangle u)^n : u \in \mathrm{PSH}(\Omega), 0 \le u \le 1\right\}. \tag{1.3}$$

For $E \subset \Omega$, we define

$$C(E,\Omega) = \sup\{C(K,\Omega) : K \text{ is a compact subset of } E\}. \tag{1.4}$$

If $E \subset \Omega$ is a Borel set, then we have

$$C(E,\Omega) = \sup\left\{\int_E (\triangle u)^n : u \in \mathrm{PSH}(\Omega), 0 \le u \le 1\right\}. \tag{1.5}$$

For any set $E \subset \Omega$, the *outer capacity* of E is defined by

$$C^*(E,\Omega) = \inf\{C(\omega,\Omega) : \omega \text{ is open}, E \subset \omega \subset \Omega\}. \tag{1.6}$$

The main result of this paper is the following theorem giving a characterization of exceptional sets in terms of the outer capacity.

THEOREM 1.2. *Let Ω be a bounded open set of $\mathbb{H}^n$, and $E \subset \Omega$. The following three statements are equivalent*:

(1) *E is Q-polar*;
(2) *E is negligible*;
(3) $C^*(E, \Omega) = 0$.

In particular, if Ω is a strongly pseudo-convex smooth open set in $\mathbb{H}^n$ and $E \Subset \Omega$, then each of (1)–(3) *is equivalent to*

(4) $u_E^* = 0$.

Here u_E^* is the upper semicontinuous regularization of the *relative extremal function u_E* defined as ($E \Subset \Omega$)

$$u_E(q) = u_{E,\Omega}(q) = \sup\{u(q) : u \in \mathrm{PSH}(\Omega), u \leq 0, u|_E \leq -1\}, \quad q \in \Omega. \quad (1.7)$$

The function u_E^* is a powerful tool to give the connection between the outer capacities and the Q-polar sets.

Finally we prove that the outer capacity $C^*(\cdot, \Omega)$ is a generalized capacity in the sense of Choquet (Theorem 3.1 in Section 3).

Although we use ideas of Bedford and Taylor [9] from the pluripotential theory in $\mathbb{C}^n$, our potential results for the quaternionic Monge–Ampère operator are completely new. The theory of quaternionic closed positive currents established recently in [30] allows us to treat the quaternionic Monge–Ampère operator as an operator of divergence form, and so we can integrate by parts. Since this can avoid the inconvenience in using Moore determinant, we established several useful quaternionic versions of results in the complex pluripotential theory in [31]. All these preparations play key roles in this paper.

2. Preliminaries on Quaternionic Monge–Ampère Measure

Recall that an upper semicontinuous function u on $\mathbb{H}^n$ is said to be *quaternionic plurisubharmonic* if u is subharmonic on each right quaternionic line. Denote by *PSH* the class of all quaternionic plurisubharmonic functions (see [2; 3; 4; 5] for more information about quaternionic plurisubharmonic functions).

As in [30], we use the conjugate embedding

$$\tau : \mathbb{H}^n \cong \mathbb{R}^{4n} \hookrightarrow \mathbb{C}^{2n\times 2},$$
$$(q_0, \ldots, q_{n-1}) \mapsto \mathbf{z} = (z^{j\alpha}) \in \mathbb{C}^{2n\times 2},$$

$q_j = x_{4j} + \mathbf{i}x_{4j+1} + \mathbf{j}x_{4j+2} + \mathbf{k}x_{4j+3}$, $j = 0, 1, \ldots, 2n-1$, $\alpha = 0, 1$, with

$$\begin{pmatrix} z^{00} & z^{01} \\ z^{10} & z^{11} \\ \vdots & \vdots \\ z^{(2l)0} & z^{(2l)1} \\ z^{(2l+1)0} & z^{(2l+1)1} \\ \vdots & \vdots \\ z^{(2n-2)0} & z^{(2n-2)1} \\ z^{(2n-1)0} & z^{(2n-1)1} \end{pmatrix} := \begin{pmatrix} x_0 - \mathbf{i}x_1 & -x_2 + \mathbf{i}x_3 \\ x_2 + \mathbf{i}x_3 & x_0 + \mathbf{i}x_1 \\ \vdots & \vdots \\ x_{4l} - \mathbf{i}x_{4l+1} & -x_{4l+2} + \mathbf{i}x_{4l+3} \\ x_{4l+2} + \mathbf{i}x_{4l+3} & x_{4l} + \mathbf{i}x_{4l+1} \\ \vdots & \vdots \\ x_{4n-4} - \mathbf{i}x_{4n-3} & -x_{4n-2} + \mathbf{i}x_{4n-1} \\ x_{4n-2} + \mathbf{i}x_{4n-1} & x_{4n-4} + \mathbf{i}x_{4n-3} \end{pmatrix}. \tag{2.1}$$

Pulling back to the quaternionic space $\mathbb{H}^n \cong \mathbb{R}^{4n}$ by the embedding (2.1), we define on $\mathbb{R}^{4n}$ the first-order differential operators $\nabla_{j\alpha}$ as follows:

$$\begin{pmatrix} \nabla_{00} & \nabla_{01} \\ \nabla_{10} & \nabla_{11} \\ \vdots & \vdots \\ \nabla_{(2l)0} & \nabla_{(2l)1} \\ \nabla_{(2l+1)0} & \nabla_{(2l+1)1} \\ \vdots & \vdots \\ \nabla_{(2n-2)0} & \nabla_{(2n-2)1} \\ \nabla_{(2n-1)0} & \nabla_{(2n-1)1} \end{pmatrix} := \begin{pmatrix} \partial_{x_0} + \mathbf{i}\partial_{x_1} & -\partial_{x_2} - \mathbf{i}\partial_{x_3} \\ \partial_{x_2} - \mathbf{i}\partial_{x_3} & \partial_{x_0} - \mathbf{i}\partial_{x_1} \\ \vdots & \vdots \\ \partial_{x_{4l}} + \mathbf{i}\partial_{x_{4l+1}} & -\partial_{x_{4l+2}} - \mathbf{i}\partial_{x_{4l+3}} \\ \partial_{x_{4l+2}} - \mathbf{i}\partial_{x_{4l+3}} & \partial_{x_{4l}} - \mathbf{i}\partial_{x_{4l+1}} \\ \vdots & \vdots \\ \partial_{x_{4n-4}} + \mathbf{i}\partial_{x_{4n-3}} & -\partial_{x_{4n-2}} - \mathbf{i}\partial_{x_{4n-1}} \\ \partial_{x_{4n-2}} - \mathbf{i}\partial_{x_{4n-1}} & \partial_{x_{4n-4}} - \mathbf{i}\partial_{x_{4n-3}} \end{pmatrix}. \tag{2.2}$$

Let $\bigwedge^{2k} \mathbb{C}^{2n}$ be the complex exterior algebra generated by $\mathbb{C}^{2n}$, $0 \le k \le n$. Fix a basis $\{\omega^0, \omega^1, \ldots, \omega^{2n-1}\}$ of $\mathbb{C}^{2n}$. Let Ω be a domain in $\mathbb{R}^{4n}$. We define $d_0, d_1 : C_0^\infty(\Omega, \bigwedge^p \mathbb{C}^{2n}) \to C_0^\infty(\Omega, \bigwedge^{p+1} \mathbb{C}^{2n})$ by

$$\begin{aligned} d_0 F &= \sum_{k,I} \nabla_{k0} f_I \omega^k \wedge \omega^I, \\ d_1 F &= \sum_{k,I} \nabla_{k1} f_I \omega^k \wedge \omega^I, \\ \triangle F &= d_0 d_1 F, \end{aligned}$$

for $F = \sum_I f_I \omega^I \in C_0^\infty(\Omega, \bigwedge^p \mathbb{C}^{2n})$, where the multiindex $I = (i_1, \ldots, i_p)$, and $\omega^I := \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$. Although d_0, d_1 are not exterior differentials, their behavior is similar to the exterior differential: $d_0 d_1 = -d_1 d_0$; $d_0^2 = d_1^2 = 0$; for $F \in C_0^\infty(\Omega, \bigwedge^p \mathbb{C}^{2n})$ and $G \in C_0^\infty(\Omega, \bigwedge^q \mathbb{C}^{2n})$, we have

$$\begin{aligned} d_\alpha (F \wedge G) &= d_\alpha F \wedge G + (-1)^p F \wedge d_\alpha G, \quad \alpha = 0, 1, \\ d_0 \triangle &= d_1 \triangle = 0, \end{aligned} \tag{2.3}$$

and (1.1) is a complex since $D\triangle = 0$.

For $u_1, \dots, u_n \in C^2$, from (2.3) it easily follows that $\triangle u_1 \wedge \cdots \wedge \triangle u_n$ satisfies the following remarkable identities:

$$\begin{aligned}
&\triangle u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n \\
&\quad = d_0(d_1 u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n) \\
&\quad = -d_1(d_0 u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n) = d_0 d_1(u_1 \triangle u_2 \wedge \cdots \wedge \triangle u_n) \\
&\quad = \triangle(u_1 \triangle u_2 \wedge \cdots \wedge \triangle u_n).
\end{aligned}$$

For $u \in C^2$, we define

$$\triangle_{ij} u := \frac{1}{2}(\nabla_{i0}\nabla_{j1} u - \nabla_{i1}\nabla_{j0} u).$$

Then, for $u_1, \dots, u_n \in C^2$,

$$\begin{aligned}
\triangle u_1 \wedge \cdots \wedge \triangle u_n &= \sum_{i_1, j_1, \dots} \triangle_{i_1 j_1} u_1 \dots \triangle_{i_n j_n} u_n \omega^{i_1} \wedge \omega^{j_1} \wedge \cdots \wedge \omega^{i_n} \wedge \omega^{j_n} \\
&= \sum_{i_1, j_1, \dots} \delta^{i_1 j_1 \dots i_n j_n}_{01 \dots (2n-1)} \triangle_{i_1 j_1} u_1 \dots \triangle_{i_n j_n} u_n \Omega_{2n},
\end{aligned}$$

where Ω_{2n} is defined as

$$\Omega_{2n} := \omega^0 \wedge \omega^1 \wedge \cdots \wedge \omega^{2n-2} \wedge \omega^{2n-1}, \tag{2.4}$$

and $\delta^{i_1 j_1 \dots i_n j_n}_{01 \dots (2n-1)}$:= the sign of the permutation from $(i_1, j_1, \dots, i_n, j_n)$ to $(0, 1, \dots, 2n-1)$ if $\{i_1, j_1, \dots, i_n, j_n\} = \{0, 1, \dots, 2n-1\}$; otherwise, $\delta^{i_1 j_1 \dots i_n j_n}_{01 \dots (2n-1)} = 0$. In particular, when $u_1 = \cdots = u_n = u$, $\triangle u_1 \wedge \cdots \wedge \triangle u_n$ coincides with $(\triangle u)^n := \bigwedge^n \triangle u$.

Although a $2n$-form is not an authentic differential form and we cannot integrate it, we can define $\int_\Omega F := \int_\Omega f \, dV$ if we write $F = f \Omega_{2n} \in L^1(\Omega, \bigwedge^{2n} \mathbb{C}^{2n})$, where dV is the Lebesgue measure, and Ω_{2n} is given by (2.4). In particular, if F is a positive $2n$-form, then $\int_\Omega F \geq 0$. For a $2n$-current $F = \mu \Omega_{2n}$ where the coefficient is a measure μ, define

$$\int_\Omega F := \int_\Omega \mu.$$

We proved that $\triangle u$ is a closed positive 2-current for any $u \in \mathrm{PSH}(\Omega)$. Inductively, for $u_1, \dots, u_p \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$, we showed that

$$\triangle u_1 \wedge \cdots \wedge \triangle u_p := \triangle(u_1 \triangle u_2 \dots \wedge \triangle u_p) \tag{2.5}$$

is a closed positive $2p$-current. In particular, for $u_1, \dots, u_n \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$, $\triangle u_1 \wedge \cdots \wedge \triangle u_n = \mu \Omega_{2n}$ for a well-defined positive Radon measure μ. See [30] for the detailed information about the closed positive currents in $\mathbb{H}^n$.

LEMMA 2.1 (Thm. 3.1 in [30]). *Let* $v^1, \dots, v^k \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$, *and let* $\{v^1_j\}_{j \in \mathbb{N}}, \dots, \{v^k_j\}_{j \in \mathbb{N}}$ *be decreasing sequences of PSH functions in* Ω *such that* $\lim_{j \to \infty} v^t_j = v^t$ *pointwise in* Ω *for each* t. *Then the currents* $\triangle v^1_j \wedge \cdots \wedge \triangle v^k_j$ *converge weakly to* $\triangle v^1 \wedge \cdots \wedge \triangle v^k$ *as* $j \to \infty$.

Alesker [5, Prop. 6.3] gave a quaternionic version of Chern–Levine–Nirenberg estimate, and we gave an elementary and simpler proof in [30].

LEMMA 2.2 (Chern–Levine–Nirenberg-type estimate; see Prop. 3.10 in [30]). *Let Ω be a domain in $\mathbb{H}^n$. Let K, L be compact subsets of Ω such that L is contained in the interior of K. Then there exists a constant C depending only on K, L, Ω such that, for any $v \in \mathrm{PSH}(\Omega)$ and $u_1, \dots, u_n \in \mathrm{PSH} \cap C^2(\Omega)$, we have*

$$\|\triangle u_1 \wedge \cdots \wedge \triangle u_k\|_L \le C\|u_1\|_{L^\infty(K)} \cdots \|u_k\|_{L^\infty(K)}, \tag{a}$$

$$\|\triangle u_1 \wedge \cdots \wedge \triangle u_k\|_L \le C\|u_1\|_{L^1(K)}\|u_2\|_{L^\infty(K)} \cdots \|u_k\|_{L^\infty(K)}, \tag{b}$$

$$\|v\triangle u_1 \wedge \cdots \wedge \triangle u_k\|_L \le C\|v\|_{L^1(K)}\|u_1\|_{L^\infty(K)} \cdots \|u_k\|_{L^\infty(K)}. \tag{c}$$

We can get (b) by following the proof of (a) in Proposition 3.10 in [30] and using the fact that

$$\int_{L\cap\overline{B}'_j} \triangle w \wedge \beta^{n-1} \le \int_{B_j} \chi\triangle w \wedge \beta^{n-1} = \int_{B_j} w\triangle\chi \wedge \beta^{n-1} \le C\|w\|_{L^1(K)} \tag{2.6}$$

for any $w \in \mathrm{PSH}(\Omega)$. Estimate (c) can be proved by following the proof of the complex Monge–Ampère case in Demailly's book ([16], p. 149, Prop. 3.11). This estimate also holds for any $u_1, \dots, u_k \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$.

Analogous classical results for subharmonic functions also hold for the quaternionic plurisubharmonic functions. We list these properties here without proofs; all of them can be derived from the subharmonic case (see Chap. 2 in [20]).

PROPOSITION 2.1. *Let Ω be an open subset of $\mathbb{H}^n$.*

(1) *The family* $\mathrm{PSH}(\Omega)$ *is a convex cone, that is, if α, β are nonnegative numbers and $u, v \in \mathrm{PSH}(\Omega)$, then $\alpha u + \beta v \in \mathrm{PSH}(\Omega)$ and* $\max\{u, v\} \in \mathrm{PSH}(\Omega)$.
(2) *If Ω is connected and $\{u_j\} \subset \mathrm{PSH}(\Omega)$ is a decreasing sequence, then $u =$* $\lim_{j\to\infty} u_j \in \mathrm{PSH}(\Omega)$ *or $u \equiv -\infty$.*
(3) *Let $\{u_\alpha\}_{\alpha\in A} \subset \mathrm{PSH}(\Omega)$ be such that its upper envelope $u = \sup_{\alpha\in A} u_\alpha$ is locally bounded above. Then the upper semicontinuous regularization $u^* \in$* $\mathrm{PSH}(\Omega)$.
(4) *Let ω be a nonempty proper open subset of Ω, $u \in \mathrm{PSH}(\Omega)$, $v \in \mathrm{PSH}(\omega)$, and* $\limsup_{q\to\zeta} v(q) \le u(\zeta)$ *for each $\zeta \in \partial\omega \cap \Omega$. Then*

$$w := \begin{cases} \max\{u, v\} & \text{in } \omega \\ u & \text{in } \Omega\backslash\omega \end{cases} \in \mathrm{PSH}(\Omega).$$

(5) *Let F be a closed subset of Ω of the form $F = \{q \in \Omega, v(q) = -\infty\}$, where* $v \in \mathrm{PSH}(\Omega)$. *If $u \in \mathrm{PSH}(\Omega\backslash F)$ is bounded above, then*

$$\widetilde{u}(q) := \begin{cases} u(q), & q \in \Omega\backslash F \\ \limsup_{q'\notin F, q'\to q} u(q'), & q \in F \end{cases} \in \mathrm{PSH}(\Omega).$$

LEMMA 2.3 (Quasicontinuity theorem; see Thm. 1.1 in [31]). *Let Ω be an open subset of $\mathbb{H}^n$, and let u be a locally bounded PSH function. Then, for each $\varepsilon > 0$,* *there exists an open subset ω of Ω such that $C(\omega) < \varepsilon$ and u is continuous on $\Omega\backslash\omega$.*

After appropriate results are later proved in Section 3, it will be clear in Remark 3.1 that the assumption of local boundedness of the function u is superfluous.

We have already seen in Lemma 2.1 that the quaternionic Monge–Ampère operator is continuous on decreasing sequences of locally bounded PSH functions. It turns out that this operator also behaves well on increasing sequences just as the complex Monge–Ampère operator. See [9] for an analogous result in the complex case.

LEMMA 2.4 (Prop. 4.1 in [31]). *Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathrm{PSH}\cap L^{\infty}_{\mathrm{loc}}(\Omega)$ that increases to $u \in \mathrm{PSH}\cap L^{\infty}_{\mathrm{loc}}(\Omega)$ almost everywhere in Ω (with respect to the Lebesgue measure). Then the currents $(\triangle u_j)^n$ converge weakly to $(\triangle u)^n$ as $j \to \infty$.*

By using the quasi-continuity theorem and the convergence result we proved in [31] the following comparison theorem, which will be a useful tool in this paper. This comparison result implies the minimum principle results in [2], which are essential to the uniqueness of the Dirichlet problem of quaternionic Monge–Ampère equations (see [3; 33]).

LEMMA 2.5 (See Thm. 1.2 in [31]). *Let Ω be a bounded open set of $\mathbb{H}^n$. Let $u, v \in \mathrm{PSH}\cap L^{\infty}(\Omega)$. If for any $\zeta \in \partial\Omega$,*

$$\liminf_{\zeta\leftarrow q\in\Omega}(u(q) - v(q)) \geq 0,$$

then

$$\int_{\{u<v\}}(\triangle v)^n \leq \int_{\{u<v\}}(\triangle u)^n. \tag{2.7}$$

3. Capacity and Description of Exceptional Sets

Let Ω be a bounded open set of $\mathbb{H}^n$. The Chern–Levine–Nirenberg-type estimate (Lemma 2.2) shows that the capacity defined by (1.3) and (1.4) satisfies $C(E,\Omega) < +\infty$ for any $E \subset \Omega$. We give some elementary properties of capacity.

PROPOSITION 3.1. (1) *If $E_1 \subset E_2 \subset \Omega$, then $C(E_1,\Omega) \leq C(E_2,\Omega)$.*
(2) *If $E \subset \Omega_1 \subset \Omega_2$, then $C(E,\Omega_1) \geq C(E,\Omega_2)$.*
(3) *If $E_1, E_2, \ldots$ are subsets of Ω, then*

$$C\Big(\bigcup_{j=1}^{\infty} E_j, \Omega\Big) \leq \sum_{j=1}^{\infty} C(E_j,\Omega).$$

(4) *If* $E_1 \subset E_2 \subset \cdots$ *are Borel subsets of* Ω, *then*

$$C\left(\bigcup_{j=1}^{\infty} E_j, \Omega\right) = \lim_{j\to\infty} C(E_j, \Omega).$$

(5) *If* $\Omega_1 \subset \Omega_2 \Subset H^n$ *and* $\omega \Subset \Omega_1$, *then there exists a constant* $A > 0$ *such that, for all Borel subsets* $E \subset \omega$, *we have* $C(E, \Omega_1) \leq AC(E, \Omega_2)$.

Proof. Note that (1)–(4) are direct consequences of definitions (1.3), (1.4), and (1.5). We give a proof of property (5) here. Without loss of generality, we may suppose that $\omega \Subset \Omega_1$ are concentric balls, say, $\Omega_1 = B(0,r)$ and $\omega = B(0, r-\varepsilon)$, $\varepsilon > 0$. For each $u \in \mathrm{PSH}(\Omega_1)$ with $0 \leq u \leq 1$, define

$$\widetilde{u}(q) = \begin{cases} \max\{u(q), \lambda(\|q\|^2 - r^2) + 2\} & \text{on } \Omega_1, \\ \lambda(\|q\|^2 - r^2) + 2 & \text{on } \Omega_2 \backslash \Omega_1. \end{cases}$$

Take a constant λ sufficiently large such that $\lambda((r-\varepsilon)^2 - r^2) + 2 \leq 0$. Then $\widetilde{u} = u$ on ω, and $u(q) < \lambda(\|q\|^2 - r^2) + 2$ on $\partial\Omega_1$. By Proposition 2.1 we have $\widetilde{u} \in \mathrm{PSH}(\Omega_2)$. Since $0 \leq \widetilde{u} \leq M$ for some constant $M > 0$, $0 \leq \frac{\widetilde{u}}{M} \leq 1$. For any Borel subset $E \subset \omega$, we have

$$\int_E (\triangle u)^n = \int_E (\triangle \widetilde{u})^n \leq M^n C(E, \Omega_2).$$

Hence, $C(E, \Omega_1) \leq M^n C(E, \Omega_2)$. □

Lemma 3.1. *Let* Ω *be a bounded open set of* $\mathbb{H}^n$. *Let* K *be a compact subset of* Ω, *and* $\omega \Subset \Omega$ *a neighborhood of* K. *There is a constant* $A > 0$ *such that, for each* $v \in \mathrm{PSH}(\Omega)$,

$$C(K \cap \{v < -m\}, \Omega) \leq \frac{1}{m} A \|v\|_{L^1(\overline{\omega})}.$$

Proof. For each $u \in \mathrm{PSH}(\Omega)$ such that $0 \leq u \leq 1$, by Lemma 2.2 we have

$$\int_{K\cap\{v<-m\}} (\triangle u)^n \leq \frac{1}{m} \int_K |v|(\triangle u)^n \leq \frac{1}{m} C_{K,\overline{\omega}} \|v\|_{L^1(\overline{\omega})}.$$ □

Note that the outer capacity $C^*(\cdot, \Omega)$ given by (1.6) also satisfies properties (1)–(3) in Proposition 3.1. For any $E \subset \Omega$, the definitions imply $C(E, \Omega) \leq C^*(E, \Omega)$, and from the definitions and Proposition 3.1 it follows that $C(E, \Omega) = C^*(E, \Omega)$ for all open sets $E \subset \Omega$.

Corollary 3.1. *Let* Ω *be a bounded open set of* $\mathbb{H}^n$. *If* P *is globally* Q*-polar in* Ω, *then* $C^*(P, \Omega) = 0$.

Proof. By the definition of globally Q-polar we assume that $P \subset \{v = -\infty\}$ with $v \in \mathrm{PSH}(\Omega)$. Let $\Omega = \bigcup_{j\geq 1} \Omega_j$ with $\Omega_j \Subset \Omega$. By Lemma 3.1 there is an open set $G_j = \Omega_j \cap \{v < -m_j\}$ with $C(G_j, \Omega) < \varepsilon 2^{-j}$. So we have $\{v = -\infty\} \subset G := \bigcup G_j$ and $C(G, \Omega) < \varepsilon$. □

The main tool in the proof of our main result (Theorem 1.2) is the relative extremal function u_E defined by (1.7). Its upper semicontinuous regularization u_E^* is PSH in Ω (by Prop. 2.1), and $-1 \le u_E^* \le 0$ in Ω. If Ω is strongly pseudo-convex, then $u_E^*(q) \to 0$ as $q \to \partial\Omega$. This function is defined in the same way as the relative extremal function given by Demailly [17] for the complex case (see also [20] for a detailed discussion). In the complex case, u_E^* is sometimes called the PSH measure of E relative to Ω [22] or the regularized relative extremal function.

LEMMA 3.2. *Fix a ball $\overline{B} \subset \Omega$. For any $g \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$, there exists a PSH function $\widetilde{g}$ such that $\widetilde{g} \ge g$ on Ω, $\widetilde{g} = g$ on $\Omega \backslash B$, and $(\triangle \widetilde{g})^n = 0$ on B.*

Proof. First consider the case where g is continuous in Ω. We shall use the Perron–Bremermann function defined by Alesker (Sect. 6 in [3]):

$$u = \sup\left\{ v : v \text{ is finite PSH on } B, \limsup_{q \to \zeta} v(q) \le g(\zeta), \forall \zeta \in \partial B \right\}.$$

Then by Theorem 6.1 in [3], $u \in \mathrm{PSH}(B)$ is continuous on $\overline{B}$ and $u = g$ on ∂B. Note that u is maximal in B. By Theorem 1.3 in [31] we have $(\triangle u)^n = 0$ on B. Let

$$\widetilde{g} = \begin{cases} u & \text{on } B, \\ g & \text{on } \Omega \backslash B. \end{cases}$$

Then $\widetilde{g} \ge g$ on B. Since $\widetilde{g}$ is the decreasing limit of PSH functions

$$g_k = \begin{cases} \max\{u, g + \frac{1}{k}\} & \text{on } B, \\ g + \frac{1}{k} & \text{on } \Omega \backslash B, \end{cases}$$

we get $\widetilde{g} \in \mathrm{PSH}(\Omega)$ by Proposition 2.1.

For an arbitrary function $g \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$, its regularization $g_l := g * \rho_{\frac{1}{l}} \searrow g$. The function $\widetilde{g} := \lim_{l \to +\infty} \widetilde{g_l}$ has all required properties. □

LEMMA 3.3 (Choquet's lemma). *Every family (u_α) has a countable subfamily $(u_{\alpha(j)})$ whose upper envelope v satisfies $v \le u \le u^* = v^*$, where u is the upper envelope of (u_α).*

PROPOSITION 3.2. *Let Ω be an open set in $\mathbb{H}^n$, and let $K \subset \Omega$ be compact. Then $(\triangle u_K^*)^n = 0$ on $\Omega \backslash K$.*

Proof. By Lemma 3.3 there exists a sequence $\{v_j\} \subset \mathrm{PSH}(\Omega)$ such that $v_j \le 0$ on Ω, $v_j \le -1$ on K and $v^* = u_K^*$. Replacing v_j by $\max\{-1, v_1, \dots, v_j\}$, we can assume that $\{v_j\}$ is increasing and $v_j \ge -1$ for all j.

Fix a ball $B \subset \Omega \backslash K$. Let $\widetilde{v_j}$ be as in Lemma 3.2. We have $\widetilde{v_j} \le 0$ on Ω and $\widetilde{v_j} \le -1$ on K. Then $v_j \le \widetilde{v_j} \le u_K$ and $\widetilde{v} = \lim_j \widetilde{v_j}$ such that $v^* = \widetilde{v}^* = u_K^*$ and $\lim \widetilde{v_j} = \lim v_j = u_K^*$ a.e. in Ω. Since $(\triangle \widetilde{v_j})^n = 0$ on B, by Lemma 2.4 we have $(\triangle u_K^*)^n = 0$ on B. Since B is taken arbitrary, it follows that $(\triangle u_K^*)^n = 0$ on $\Omega \backslash K$. □

COROLLARY 3.2. *For arbitrary sets $E \Subset \Omega$, we have the following properties of the regularized relative extremal functions u_E^*:*

(1) *If $E_1 \subset E_2 \subset \Omega_1 \subset \Omega_2$, then $u^*_{E_1,\Omega_1} \geq u^*_{E_2,\Omega_1} \geq u^*_{E_2,\Omega_2}$.*

(2) *$u_E^* = u_E = -1$ on E^0 and $(\triangle u_E^*)^n = 0$ on $\Omega \backslash \overline{E}$; so, $(\triangle u_E^*)^n$ is supported by ∂E.*

Proof. (1) is obvious. From the definition it follows directly that $u_E^* = u_E = -1$ on E^0, and hence $(\triangle u_E^*)^n = 0$ on E^0. By Proposition 3.2, $(\triangle u_E^*)^n = 0$ on $\Omega\backslash\overline{E}$. So, $(\triangle u_E^*)^n$ is supported by ∂E. □

PROPOSITION 3.3. *Let $\Omega \subset \mathbb{H}^n$ be a strongly pseudo-convex smooth open set. For arbitrary sets $E \Subset \Omega$, we have $C^*(E,\Omega) = \int_\Omega (\triangle u_E^*)^n$.*

Proof. First, we show that, for a compact set $K \subset \Omega$,

$$C(K,\Omega) = \int_\Omega (\triangle u_K^*)^n = \int_K (\triangle u_K^*)^n. \tag{3.1}$$

The second equality in (3.1) directly follows from Proposition 3.2. Since $-1 \leq u_K^* \leq 0$ on Ω, $C(K,\Omega) \geq \int_K (\triangle u_K^*)^n$ by definition. Let $\psi < 0$ be a smooth strictly PSH exhaustion function of Ω. Then we have $A\psi \leq -1$ on K for A large enough.

As in the proof of Proposition 3.2, there exists an increasing sequence $\{v_j\} \subset$ PSH(Ω) such that $-1 \leq v_j \leq 0$ on Ω, $v_j \leq -1$ on K, and $v^* = u_K^*$. We can assume that $v_j \geq A\psi$ on Ω (otherwise, we can replace v_j by $\max\{v_j, A\psi\}$). Take $\varepsilon \in (0,1)$ and $\omega \in$ PSH(Ω) such that $0 \leq \omega \leq 1-\varepsilon$. Now we have

$$K \subset \{v_j \leq \omega - 1\} \subset \{A\psi \leq \omega - 1\} \subset \{A\psi \leq -\varepsilon\}.$$

Note that $v_j \geq A\psi > -\varepsilon > v - 1$ near $\partial\Omega$ sufficiently. By the comparison theorem (Lemma 2.5) we have

$$\int_K (\triangle\omega)^n \leq \int_{\{v_j \leq \omega-1\}} (\triangle\omega)^n \leq \int_{\{v_j \leq \omega-1\}} (\triangle v_j)^n \leq \int_{\{A\psi \leq -\varepsilon\}} (\triangle v_j)^n.$$

By Lemma 2.4, $(\triangle v_j)^n$ converges weakly to $(\triangle u_K^*)^n$ as $j \to \infty$. Thus,

$$\int_K (\triangle\omega)^n \leq \int_{\{A\psi \leq -\varepsilon\}} (\triangle u_K^*)^n = \int_K (\triangle u_K^*)^n,$$

where the last identity follows from Proposition 3.2. Note that

$$C(K,\Omega) = (1-\varepsilon)^{-n} \sup\left\{\int_K (\triangle\omega)^n : \omega \in \mathrm{PSH}(\Omega), 0 \leq \omega \leq 1-\varepsilon\right\}.$$

Then (3.1) follows.

Now, for every open set $G \Subset \Omega$, we are going to show that

$$C^*(G,\Omega)(= C(G,\Omega)) = \int_{\overline{G}} (\triangle u_G^*)^n = \int_\Omega (\triangle u_G^*)^n = \int_\Omega (\triangle u_G)^n. \tag{3.2}$$

Let $K_1 \subset K_2 \subset \cdots$ be compact subsets of G with $K_j \subset K_{j+1}^0$ and $\bigcup_j K_j = G$. Then $u_{K_j}^* = -1$ on $K_j^0 \supset K_{j-1}$, and $\lim_j u_{K_j}^* = -1$ on G. Since $K_j \subset G$, from Corollary 3.2(1) it directly follows that $u_G^* \leq u_{K_j}^*$. Then $u_G^* \leq \lim_j u_{K_j}^*$. On the other hand, $\lim_j u^*_{K_j} \le u_G$ by definition (1.7). Therefore, $u^*_G \le \lim_j u^*_{K_j} \le u_G \le u^*_G$. Then (3.2) follows from (3.1) and Proposition 3.1(4).

Now let $E \Subset \Omega$ be arbitrary, and let $\psi < 0$ be a strictly PSH exhaustion function of Ω. For every open set $E \subset G \Subset \Omega$, we have $u^*_G \ge A\psi$ and $u^*_E \ge u^*_G$ by Corollary 3.2(1). Note that $0 \ge u^*_G \ge A\psi$ and $u^*_G(q) \to 0$ as $q \to \partial\Omega$. By the comparison theorem (Lemma 2.5) we have

$$\int_\Omega (\triangle u^*_E)^n \le \int_\Omega (\triangle u^*_G)^n = C(G, \Omega).$$

Thus, $\int_\Omega (\triangle u^*_E)^n \le C^*(E, \Omega)$ by definition (1.6).

On the other hand, Lemma 3.3 shows that there exists a sequence $\{v_j\} \subset$ PSH with $-1 \le v_j \le 0$, $v_j \ge A\psi$ and $\lim v_j = u_E$ a.e. in Ω. Consider the open sets $G_j = \{q \in \Omega, (1 + \frac{1}{j})v_j(q) < -1\}$. Then $G_j \supset E$, G_j is decreasing, and $(1 + \frac{1}{j})v_j \le u_{G_j} \le u^*_{G_j}$. Noting that $u^*_{G_j} \nearrow u^*_E$, we have $C^*(E, \Omega) \le \lim_j C(G_j, \Omega) = \lim_j \int_\Omega (\triangle u^*_{G_j})^n = \int_\Omega (\triangle u^*_E)^n$. The proposition is finally proved. □

COROLLARY 3.3. *If $\Omega \subset \mathbb{H}^n$ is open, then, for all $\omega \Subset \Omega$ and $u \in \mathrm{PSH}(\Omega)$,*

$$\lim_{j\to\infty} C(\{u < -j\} \cap \omega, \Omega) = 0.$$

Proof. Assume that $u < 0$ on ω. Cover $\overline{\omega}$ by a finite union of balls in Ω. By Proposition 3.1 we can assume that Ω is strongly pseudo-convex. Set $P_j = \{u < -j\} \cap \omega$. Then we have $\max\{\frac{u}{j}, -1\} \le u_{P_j} \le 0$. Therefore, $\lim_{j\to\infty} u_{P_j} = 0$ almost everywhere in Ω. By (3.2) and Lemma 2.4 we have $\lim_{j\to\infty} C(P_j, \Omega) = 0$. □

REMARK 3.1. By Corollary 3.3 we can generalize the quasi-continuity theorem (Lemma 2.3) to the unbounded case, that is, for arbitrary $u \in \mathrm{PSH}(\Omega)$ and any $\varepsilon > 0$, there exists an open subset $\omega \subset \Omega$ with $C(\omega) < \varepsilon$ such that u is continuous on $\Omega \backslash \omega$.

LEMMA 3.4. *Let Ω be a bounded open subset of $\mathbb{H}^n$. Let $u, v \in \mathrm{PSH} \cap L^\infty(\Omega)$. If $\limsup_{\zeta \leftarrow \partial\Omega} |u(\zeta) - v(\zeta)| = 0$ and $(\triangle u)^n = (\triangle v)^n$ in Ω, then $u \equiv v$ in Ω.*

Proof. It suffices to prove that $u \ge v$. Let $\varphi < 0$ be a smooth strictly PSH function in Ω. Suppose that $\{u < v\}$ is not empty. Then the set $S = \{u < v + \varepsilon\varphi\}$ is also nonempty for some proper $\varepsilon > 0$. Since u and $v + \varepsilon\varphi$ are both subharmonic, by the classical result for the subharmonic functions we know that the set S must have positive Lebesgue measure. By Lemma 2.5,

$$\int_S (\triangle u)^n \ge \int_S (\triangle(v + \varepsilon\varphi))^n \ge \int_S (\triangle v)^n + \varepsilon^n \int_S (\triangle\varphi)^n.$$

The last integral over S is strictly positive, so we get a contradiction. □

By Proposition 3.3 and Lemma 3.4 we have the following conclusion.

COROLLARY 3.4. *Let Ω be a strongly pseudo-convex smooth open set in $\mathbb{H}^n$, and let $E \Subset \Omega$. Then $C^*(E,\Omega)=0$ if and only if $u_E^*=0$.*

PROPOSITION 3.4. *Let Ω be a connected bounded open set in $\mathbb{H}^n$, and let $E \subset \Omega$. The following statements are equivalent*:

(1) $u_E^* \equiv 0$;
(2) *there exists* $v \in \mathrm{PSH}(\Omega)$, $v \le 0$, *such that* $E \subset \{v=-\infty\}$.

Proof. The implication (2) $\Rightarrow$ (1) is obvious. If v is as in (2), then for each $\varepsilon > 0$, $\varepsilon v \le u_E$. So, $u_E = 0$ on $\Omega \backslash \{v=-\infty\}$. It follows that $u_E^* \equiv 0$.

Now assume that $u_E^* \equiv 0$. By Lemma 3.3 there exists a sequence $\{v_j\} \subset \mathrm{PSH}(\Omega)$, $-1 \le v_j \le u_E$, converging increasingly a.e. in Ω to u_E^*. We can extract a subsequence such that $\int_\Omega |v_j|\,d\lambda < 2^{-j}$. Since $v_j \le 0$ and $v_j \le -1$ on E, the function $v := \sum v_j \le 0$, and $v=-\infty$ on E. Since v is the limit of the decreasing sequence of its partial sums and $v \not\equiv -\infty$ in Ω, we have $v \in \mathrm{PSH}(\Omega)$ by Proposition 2.1. □

Now we prove the Josefson theorem on $\mathbb{H}^n$ following the proof given in [9] in pluripotential theory on $\mathbb{C}^n$.

Proof of Theorem 1.1. By the definition of locally Q-polar we can find sets P_j and Ω_j with Ω_j strongly pseudo-convex smooth open such that $P_j \Subset \Omega_j \Subset \mathbb{H}^n$, $\bigcup_{j\ge1} P_j = P$, and P_j is contained in the $-\infty$ poles of a single plurisubharmonic function in Ω_j. By Propositions 3.3 and 3.4 we have $C^*(P_j,\Omega_j)=0$.

Let $i_1, i_2, \dots$ be a listing of the positive integers such that each one appears infinitely many times. For a sequence $c_1 < c_2 < \cdots$, $c_j \to +\infty$, set $B_j = \{q \in \mathbb{H}^n, \|q\| < c_j\}$. We can choose c_j large enough such that $\Omega_{i_j} \Subset B_j$ and $|q|-c_j < -1$ on P_{ij}. It follows from $C^*(P_{i_j},\Omega_{i_j})=0$ and $\Omega_{i_j} \Subset B_j$ that $C^*(P_{i_j},B_j)=0$. Hence, by Corollary 3.4 the extremal function $u_{P_{ij}}^*$ in B_j is zero, and we can find $v_j \in \mathrm{PSH}(B_j)$ with $v_j \le 0$ on B_j, $v_j \le -1$ on P_{i_j}, and $\int_{B_j} |v_j|\,dV < 2^{-j}$. Define

$$\widetilde{v_j}(q) = \begin{cases} \|q\| - c_j, & q \in \mathbb{H}^n \backslash B_j, \\ \max\{v_j(q), \|q\|-c_j\}, & q \in B_j. \end{cases}$$

Then $\widetilde{v_j} \le -1$ on P_{i_j} and $\widetilde{v_j} \in \mathrm{PSH}(\mathbb{H}^n)$ by Proposition 2.1. Since $\widetilde{v_j} < 0$ on B_j and $\int_{B_j} |v_j|\,dV < 2^{-j}$, $v = \sum_{j=1}^\infty \widetilde{v_j}$ is a PSH function on $\mathbb{H}^n$. Since $\widetilde{v_j} = -1$ on P_{i_j} and each P_i repeated infinitely many times, it follows that $v=-\infty$ on $P = \bigcup_{j\ge1} P_j$. This completes the proof. □

LEMMA 3.5. *Let $\Omega \Subset \mathbb{H}^n$, and let $K_1 \supset K_2 \supset \cdots$, $K = \bigcap_j K_j$ be compact subsets of Ω. Then*

(a) $\lim C(K_j,\Omega) = C(K,\Omega)$;
(b) $C^*(K,\Omega) = C(K,\Omega)$.

In particular, $C^*(K,\Omega)=C(K,\Omega)$ for any compact set $K \subset \Omega$.

Proof. (a) follows from Lemma 2.4 and Proposition 3.3. Note that K_j are neighborhoods of K, and (b) follows directly from (a). □

As an application of the quasi-continuity theorem, we can prove an interesting inequality for the quaternionic Monge–Ampère operator. Here we follow the proof of the complex case in Demailly [17].

PROPOSITION 3.5 (Demailly's inequality). *Let u, v be locally bounded PSH functions on Ω. Then we have the inequality of quaternionic Monge–Ampère measures*

$$(\triangle \max\{u, v\})^n \geq \chi_{\{u \geq v\}}(\triangle u)^n + \chi_{\{u<v\}}(\triangle v)^n. \tag{3.3}$$

Proof. By changing the roles of u and v it suffices to prove that

$$\int_K (\triangle \max\{u, v\})^n \geq \int_K (\triangle u)^n \tag{3.4}$$

for every compact set $K \subset \{u \geq v\}$. Since u, v are bounded, we may assume that $0 \leq u, v \leq 1$ and $0 \leq u_\varepsilon, v_\varepsilon \leq 1$, where $u_\varepsilon := u * \rho_\varepsilon$ is the standard regularization of u. By Lemma 2.3 we can assume that $G \subset \Omega$ is an open set of small capacity such that u, v are continuous on $\Omega \backslash G$. Then u_ε, v_ε converge uniformly to u, v on $\Omega \backslash G$, respectively, as ε tends to 0. For any $\delta > 0$, we can find an arbitrarily small neighborhood L of K such that $u_\varepsilon > v_\varepsilon - \delta$ on $L \backslash G$ for ε sufficiently small. By Lemma 2.1, $(\triangle u_\varepsilon)^n$ converges weakly to $(\triangle u)^n$. So we have

$$\begin{aligned}\int_K (\triangle u)^n &\leq \liminf_{\varepsilon\to 0} \int_L (\triangle u_\varepsilon)^n \leq \liminf_{\varepsilon\to 0}\left(\int_G (\triangle u_\varepsilon)^n + \int_{L\backslash G} (\triangle u_\varepsilon)^n\right)\\ &\leq C(G, \Omega) + \liminf_{\varepsilon\to 0} \int_{L\backslash G} (\triangle \max\{u_\varepsilon + \delta, v_\varepsilon\})^n\\ &= C(G, \Omega) + \int_{L\backslash G} (\triangle \max\{u + \delta, v\})^n.\end{aligned}$$

The third inequality follows from the definition of capacity and the fact that $\max\{u_\varepsilon + \delta, v_\varepsilon\} = u_\varepsilon + \delta$ on a neighborhood of $L \backslash G$.

By taking L very close to K and $C(G, \Omega)$ arbitrarily small we have $\int_K (\triangle u)^n \leq \int_K (\triangle \max\{u + \delta, v\})^n$. Let $\delta \to 0$ to get (3.4). □

Let $\{u_\alpha\}$ be a family of PSH functions in Ω that is locally bounded from above. Then the function $u = \sup_\alpha u_\alpha$ is not in general PSH or even upper semicontinuous. But its upper semicontinuous regularization

$$u^*(q) = \limsup_{q'\to q} u(q') \geq u(q), \quad q \in \Omega,$$

is PSH by Proposition 2.1(3). A set of the form

$$N = \{q \in \Omega : u(q) < u^*(q)\}$$

is called *negligible.* By the well-known result for the subharmonic functions, $u^* = u$ almost everywhere in Ω. So the Lebesgue measure of any negligible set N is zero.

PROPOSITION 3.6. *Let Ω be a bounded open set of $\mathbb{H}^n$. Every negligible set $N \subset \Omega$ satisfies $C^*(N, \Omega) = 0$.*

Proof. By Lemma 3.3 every negligible set is contained in a Borel negligible set $N = \{v < v^*\}$ with $v = \sup v_\alpha$, where $\{v_\alpha\}$ is an increasing sequence of PSH functions with $v_\alpha \geq -1$ for all α. By Lemma 2.3 there exists an open set $G \subset \Omega$ such that all v_α and v^* are continuous on $\Omega \backslash G$ and $C(G, \Omega) < \varepsilon$. Since G is open, $C^*(G, \Omega) = C(G, \Omega) < \varepsilon$.

Write

$$N \subset G \cup (N \cap (\Omega \backslash G)) = G \cup \left(\bigcup_{\delta, \lambda, \mu} K_{\delta\lambda\mu} \right)$$

with

$$K_{\delta\lambda\mu} = \{q \in \overline{\Omega_\delta} \backslash G, v(q) \leq \lambda < \mu \leq v^*(q)\}, \quad \lambda < \mu, \lambda, \mu \in \mathbb{Q}, \delta > 0.$$

Set $K = K_{\delta\lambda\mu}$ for short. Since v^* is continuous and v lower semicontinuous on $\Omega \backslash G$, we see that K is either compact or empty. It suffices to show that $C(K, \Omega) = 0$. Taking an open set $\omega \Subset \Omega$, we may assume that $v^* \leq 0$ on $\overline{\omega}$. Set $\lambda = -1$. Then $v_\alpha \leq 0$ on ω and $v_\alpha \leq v \leq -1$ on K. So, $v \leq u_K$, $v^* \leq u_K^*$, and $u_K^* \geq \mu > -1$ on K. By Proposition 3.5 we have

$$C(K, \omega) = \int_K (\triangle u_K^*)^n \leq \int_K (\triangle \max\{u_K^*, \mu\})^n \leq |\mu|^n C(K, \omega)$$

since $-1 \leq |\mu|^{-1} \max\{u_K^*, \mu\} \leq 0$. Since $|\mu| < 1$, we have $C(K, \omega) = 0$. By Lemma 3.5 we have $C^*(K, \Omega) = C(K, \Omega) = 0$. So, $C^*(N, \Omega) < \varepsilon$ for every $\varepsilon > 0$. □

COROLLARY 3.5. *Let Ω be a bounded open set of $\mathbb{H}^n$, and $P \subset \Omega$. Then P is Q-polar in Ω if and only if $C^*(P, \Omega) = 0$.*

Proof. If $C^*(P, \Omega) = 0$, then by Proposition 3.1 there exists $A > 0$ such that $C^*(P \cap \omega', \omega) \leq AC^*(P \cap \omega', \Omega) \leq AC^*(P, \Omega) = 0$ for all concentric balls $\omega' \Subset \omega \Subset \Omega$. It follows that $C^*(P \cap \omega', \omega) = 0$. By Corollary 3.4 we get $u^*_{P \cap \omega'} = 0$, and by Proposition 3.4 there exists $0 \geq v \in \mathrm{PSH}(\omega)$ such that $P \cap \omega' \subset \{v = -\infty\}$, that is, P is locally Q-polar. Then the conclusion follows from Theorem 1.1 and Corollary 3.1. □

PROPOSITION 3.7. *If $\Omega \subset \mathbb{H}^n$ is strongly pseudo-convex smooth open, then each Q-polar set $P \subset \Omega$ is negligible.*

Proof. Since P is Q-polar in Ω, by Corollary 3.5 we have $C^*(P, \Omega) = 0$. Since Ω is strongly pseudo-convex smooth open, it follows from Corollary 3.4 that $u_P^* = 0$ in Ω. Then by Proposition 3.4 there exists $v \in \mathrm{PSH}(\Omega)$, $v \leq 0$ such that $v|_P = -\infty$. Consider the PSH family $v_\varepsilon = \varepsilon v$, $\varepsilon > 0$, and $U = \sup_{\varepsilon > 0} v_\varepsilon$. Then $U(q) = 0$ if $v(q) \neq -\infty$, and $U|_P = -\infty$. So, $U^* = 0$ on Ω, and the set P is of the form $\{U < U^*\}$ and thus is negligible on Ω. □

Finally, we are ready to combine all results obtained to prove Theorem 1.2.

Proof of Theorem 1.2. By Corollary 3.5 it suffices to show that negligible sets are the same as Q-polar sets. Proposition 3.6 and Corollary 3.5 imply that every negligible set is locally Q-polar. Now we are in the position to show that each locally Q-polar set is negligible by using the method in the proof of Proposition 3.7. If P is locally Q-polar, then P is globally Q-polar in $\mathbb{H}^n$ by Theorem 1.1. So we can find $v \in \mathrm{PSH}(\mathbb{H}^n)$ such that $v|_P = -\infty$. Note that v is also a function in $\mathrm{PSH}(\Omega)$ and $\mathrm{PSH}(B)$ with $\Omega \subset B \subset \mathbb{H}^n$. Then, as in the proof of Proposition 3.7, we consider the PSH family on Ω: $v_\varepsilon = \varepsilon v$, $\varepsilon > 0$, and $U = \sup_{\varepsilon>0} v_\varepsilon$. Then $U(q) = 0$ if $v(q) \neq -\infty$, and $U|_P = -\infty$. So $U^* = 0$ on Ω, and the set P is of the form $\{U < U^*\}$ and thus is negligible on Ω. Finally, Theorem 1.2 follows from Corollary 3.4. □

THEOREM 3.1. *If $\Omega \subset \mathbb{H}^n$ is strongly pseudo-convex smooth open, then the function $E \to C^*(E, \Omega)$ is a generalized capacity. This means:*

(1) $C^*(\varnothing, \Omega) = 0$.

(2) *If $K_1 \supset K_2 \supset \cdots$ is a sequence of compact subsets of Ω, then*

$$\lim_{j\to\infty} C^*(K_j, \Omega) = C^*\left(\bigcap_{j=1}^{\infty} K_j, \Omega\right).$$

(3) *If $E_1 \subset E_2 \subset \cdots$ is a sequence of arbitrary subsets of Ω, then*

$$\lim_{j\to\infty} C^*(E_j, \Omega) = C^*\left(\bigcup_{j=1}^{\infty} E_j, \Omega\right).$$

All Suslin (in particular, all Borel) subsets E of Ω are capacitable, that is, $C^*(E, \Omega) = C(E, \Omega)$.

Proof. Property (1) is obvious, and (2) was shown in Lemma 3.5. By Proposition 3.1, for each E_j, $C^*(E_j, \Omega) \le C^*(\bigcup_{j=1}^{\infty} E_j, \Omega)$. It follows that $\lim_{j\to\infty} C^*(E_j, \Omega) \le C^*(\bigcup_{j=1}^{\infty} E_j, \Omega)$.

To prove the opposite inequality, it suffices to show this under the hypothesis that the sets $E_j \Subset \Omega$. Take $\varepsilon, \delta \in (0, 1)$. By Theorem 1.2 the sets $\widetilde{E}_j := \{q \in E_j, u^*_{E_j} > -1\}$ are Q-polar sets and satisfy $C^*(\widetilde{E}_j, \Omega) = 0$. By Proposition 3.1 the union $F := \bigcup_{j=1}^{\infty} \widetilde{E}_j$ satisfies $C^*(F, \Omega) = 0$. It follows from definition (1.6) that there exists an open set G such that $F \subset G \subset \Omega$ and $C^*(G, \Omega) < \varepsilon$. Define

$$U_j = \{q \in \Omega : u^*_{E_j} < -1 + \delta\} \quad \text{and} \quad V_j = U_j \cup G.$$

Since $u^*_{E_j}$ is upper semicontinuous, U_j and V_j are open. Note that $\frac{1}{1-\delta} u^*_{E_j} \le u^*_{U_j}$ in Ω. Then by the subadditivity of capacity $C(\cdot, \Omega)$ and Proposition 3.3 we have

$$\begin{aligned} C^*(V_j, \Omega) &\le C^*(G, \Omega) + C^*(U_j, \Omega) \le \varepsilon + \int_\Omega (\triangle u^*_{U_j})^n \\ &\le \varepsilon + (1-\delta)^{-n} \int_\Omega (\triangle u^*_{E_j})^n = \varepsilon + (1-\delta)^{-n} C^*(E_j, \Omega). \end{aligned}$$

Therefore, we have

$$C^*\Big(\bigcup_{j=1}^{\infty} E_j, \Omega\Big) \le C^*\Big(\bigcup_{j=1}^{\infty} V_j, \Omega\Big) = C\Big(\bigcup_{j=1}^{\infty} V_j, \Omega\Big)$$
$$= \lim_{j\to\infty} C(V_j, \Omega) \le \varepsilon + (1-\delta)^{-n} \lim_{j\to\infty} C^*(E_j, \Omega).$$

Finally, let $\varepsilon, \delta \to 0$ to obtain the required estimate. It is the classical Choquet result that the last conclusion of the theorem follows from (1)–(3); see, for example, Chapter 2 in [1] and Chapter 3 in [18]. □

References

[1] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory,* Grundlehren Math. Wiss., 314, Springer-Verlag, Berlin, 1996.

[2] S. Alesker, *Non-commutative linear algebra and plurisubharmonic functions of quaternionic variables,* Bull. Sci. Math. 127 (2003), no. 1, 1–35.

[3] ———, *Quaternionic Monge–Ampère equations,* J. Geom. Anal. 13 (2003), no. 2, 205–238.

[4] ———, *Valuations on convex sets, non-commutative determinants, and pluripotential theory,* Adv. Math. 195 (2005), no. 2, 561–595.

[5] ———, *Pluripotential theory on quaternionic manifolds,* J. Geom. Phys. 62 (2012), no. 5, 1189–1206.

[6] E. Bedford, *Envelopes of continuous, plurisubharmonic functions,* Math. Ann. 251 (1980), no. 2, 175–183.

[7] ———, *Extremal plurisubharmonic functions and pluripolar sets in* $\mathbf{C}^2$, Math. Ann. 249 (1980), no. 3, 205–223.

[8] ———, *Survey of pluri-potential theory,* Several complex variables (Stockholm, 1987/1988), Math. Notes, 38, pp. 48–97, Princeton Univ. Press, Princeton, NJ, 1993.

[9] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions,* Acta Math. 149 (1982), no. 1–2, 1–40.

[10] U. Cegrell, *Capacities in complex analysis,* Aspects Math., E14, Friedr. Vieweg & Sohn, Braunschweig, 1988.

[11] ———, *Pluricomplex energy,* Acta Math. 180 (1998), no. 2, 187–217.

[12] ———, *The general definition of the complex Monge–Ampère operator,* Ann. Inst. Fourier (Grenoble) 54 (2004), no. 1, 159–179.

[13] ———, *Maximal plurisubharmonic functions,* Uzbek. Mat. Zh. 1 (2009), 10–16.

[14] J.-P. Demailly, *Mesures de Monge–Ampère et mesures pluriharmoniques,* Math. Z. 194 (1987), no. 4, 519–564.

[15] ———, *Monge–Ampère operators, Lelong numbers and intersection theory,* Complex analysis and geometry, Univ. Ser. Math., pp. 115–193, Plenum, New York, 1993.

[16] ———, *Complex analytic and differential geometry,* book available online at ⟨http://www-fourier.ujf-grenoble.fr/~demailly/documents.html⟩.

[17] ———, *Potential theory in several complex variables,* 1991, preprint.

[18] L. Hörmander, *Notions of convexity,* Progr. Math., 127, Birkhäuser Boston, Inc., Boston, MA, 1994.

[19] B. Josefson, *On the equivalence between locally polar and globally polar sets for plurisubharmonic functions on* $\mathbf{C}^n$, Ark. Mat. 16 (1978), no. 1, 109–115.

[20] M. Klimek, *Pluripotential theory,* London Math. Soc. Monogr. Ser., 6, The Clarendon Press, Oxford University Press, New York, 1991, Oxford Science Publications.

[21] D. A. Labutin, *Potential estimates for a class of fully nonlinear elliptic equations,* Duke Math. J. 111 (2002), no. 1, 1–49.

[22] A. Sadullaev, *Plurisubharmonic measures and capacities on complex manifolds,* Uspekhi Mat. Nauk 36 (1981), no. 4(220), 53–105, 247.

[23] A. Sadullaev and B. Abdullaev, *Potential theory in the class of m-subharmonic functions,* Tr. Mat. Inst. Steklova 279 (2012), no. Analiticheskie i Geometricheskie Voprosy Kompleksnogo Analiza, 166–192.

[24] ______, *Capacities and Hessians in the class of m-subharmonic functions,* Dokl. Akad. Nauk 448 (2013), 515–517.

[25] N. S. Trudinger and X.-J. Wang, *Hessian measures. III,* J. Funct. Anal. 193 (2002), no. 1, 1–23.

[26] D. Wan, *Estimates for k-Hessian operator and some applications,* Czechoslovak Math. J. 63(138) (2013), no. 2, 547–564.

[27] ______, *The continuity and range of the quaternionic Monge–Ampère operator on quaternionic space,* Math. Z. 285 (2017), 461–478.

[28] D. Wan and W. Wang, *Complex Hessian operator and Lelong number for unbounded m-subharmonic functions,* Potential Anal. 44 (2016), no. 1, 53–69.

[29] ______, *Viscosity solutions to quaternionic Monge–Ampère equations,* Nonlinear Anal. 140 (2016), 69–81.

[30] ______, *On quaternionic Monge–Ampère operator, closed positive currents and Lelong–Jensen type formula on quaternionic space,* ⟨http://arxiv.org/abs/1401.5291v1⟩.

[31] D. Wan and W. Zhang, *Quasicontinuity and maximality of quaternionic plurisubharmonic functions,* J. Math. Anal. Appl. 424 (2015), 86–103.

[32] W. Wang, *The k-Cauchy–Fueter complex, Penrose transformation and Hartogs phenomenon for quaternionic k-regular functions,* J. Geom. Phys. 60 (2010), no. 3, 513–530.

[33] J. Zhu, *Dirichlet problem of Quaternionic Monge–Ampère equations,* ⟨http://arxiv.org/abs/1403.3197⟩.

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