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Intersection of Positive Closed Currents of Higher Bidegrees

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ABSTRACT. Let X be a compact Kähler manifold of dimension n. Let T and S be two positive closed currents on X of bidegrees (p, p) and (q, q), respectively, with $p + q \leq n$. Assume that T has a continuous superpotential. We prove that the wedge product $T \wedge S$, defined by Dinh and Sibony, is a positive closed current.

1. Introduction

Let X be a compact Kähler manifold of dimension n. Let T and S be two positive closed currents on X of bidegrees (p, p) and (q, q), respectively, with $p+q \leq n$. Demailly [4] asked the question to define the intersection $T \wedge S$. The theory of intersections of currents of bidegrees $(1, 1)$ is well developed (see, e.g., [1; 3; 5; 10]). So the question of Demailly concerns currents of higher *degree.*

The problem was recently solved by Dinh and Sibony [9] using their theory of superpotentials (see also [7]). Assume that T has continuous superpotentials (see [9] or Section 2 for the terminology). Then the wedge product $T \wedge S$ is well defined. It is known that this product is the difference of two positive closed currents. The operator satisfies basic properties like the commutativity and associativity when several currents intersect. The Hodge cohomology class of $T \wedge S$ is the cup product of those of T and S. Moreover, $T \wedge S$ depends continuously on S. Therefore, it is positive when S can be approximated by smooth positive closed forms. The last property of approximation is satisfied when X is a homogeneous manifold and also in the case of some dynamical Green currents. The purpose of this work is to prove the positivity of $T \wedge S$ in the general setting. We have the following theorem.

THEOREM 1.1. *Let X be a compact Kähler manifold of dimension n. Let T and S be two positive closed currents on X of bidegrees (p, p) and (q, q), respectively, with $p + q \leq n$. Assume that T has a continuous superpotential. Then the intersection current $T \wedge S$ is a positive closed current of bidegrees $(p+q, p+q)$.*

In Section 2, we recall some basic properties of positive closed currents and their superpotentials. In Section 3, we introduce an alternative definition of $T \wedge S$, which is a positive closed current. We then show that this definition is equivalent to that by Dinh and Sibony. The theorem then follows immediately. We will present now the main idea.

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Suppose first that T and S are positive closed smooth forms of X. Let π_j $(j = 1, 2)$ be the projections from $X \times X$ to the first and second components, respectively. We have $T \otimes S = \pi_1^*(T) \wedge \pi_2^*(S)$. This is a positive closed smooth form on $X \times X$. Then we can compute $T \wedge S$ via the formula

$$T \wedge S = (\pi_j)_*(T \otimes S \wedge [\Delta]) \quad \text{for } j = 1, 2, \tag{1.1}$$

where $[\Delta]$ is the current of integration on the diagonal Δ of $X \times X$.

Observe that because of $[\Delta]$, formula (1.1) cannot be extended to general singular currents T and S. We can, however, use the theory of intersection with $(1, 1)$-currents if, instead of Δ, we have a hypersurface. This is the reason why we consider the blow-up $\widehat{X \times X}$ of $X \times X$ along Δ. Let Π be the natural projection from $\widehat{X \times X}$ to $X \times X$, and $\widehat{\Delta} = \Pi^{-1}(\Delta)$ be the exceptional hypersurface. Recall from [2; 15] that the blow-up of a compact Kähler manifold along a submanifold is also Kähler. Let $\widehat{\omega}$ be a Kähler form of $\widehat{X \times X}$. Observe that $\Pi_*(\widehat{\omega}^{n-1} \wedge [\widehat{\Delta}])$ is a nonzero positive closed current of $X \times X$ supported on Δ and has the same dimension as Δ. Therefore, it equals a constant times $[\Delta]$ (see, e.g., [5]). By normalizing $\widehat{\omega}$ we can suppose that

$$\Pi_*(\widehat{\omega}^{n-1} \wedge [\widehat{\Delta}]) = [\Delta]. \tag{1.2}$$

Put $\widehat{T \otimes S} = \Pi^*(T \otimes S)$ and $\Pi_j = \pi_j \circ \Pi$ $(j = 1, 2)$. Then (1.1) can be rewritten as

$$T \wedge S = (\Pi_j)_*(\widehat{T \otimes S} \wedge \widehat{\omega}^{n-1} \wedge [\widehat{\Delta}]). \tag{1.3}$$

In general, when T and S are only positive closed currents, we still can define $\widehat{T \otimes S}$ as a positive closed current outside $\widehat{\Delta}$ and extend it by 0 through $\widehat{\Delta}$. We can show that $\widehat{T \otimes S} \wedge \widehat{\omega}^{n-1} \wedge [\widehat{\Delta}]$ is well defined, provided that T has a continuous superpotential. In this case, we can use (1.3) as an alternative definition of $T \wedge S$, which gives a positive closed current; see Corollary 3.5. Proposition 3.7 shows that this definition is equivalent to that of Dinh and Sibony.

2. Superpotential of Positive Closed Currents

We will recall now some basic facts and refer to [9] for details. Let X be a compact Kähler manifold of dimension n, and ω be a Kähler form on X. It is well known that the de Rham cohomology of currents and smooth forms are canonically equal (see [12, Chap. 3]). Denote them by $H^r(X, \mathbb{C})$ with $0 \le r \le n$. For any closed current T of degree r, we denote by $\{T\}$ its cohomology class in $H^r(X, \mathbb{C})$. Let $H^{p,p}(X, \mathbb{R})$ be the vector subspace of $H^{p,p}(X, \mathbb{C})$ spanned by the classes of closed real $2p$-forms. Since a closed positive (p, p)-current is real, its class belongs to $H^{p,p}(X, \mathbb{R})$. If V is an analytic subset of X of dimension $n - p$, then it defines a positive closed current $[V]$ of bidegrees (p, p) by integration over V. We denote its class by $\{V\}$ for simplicity.

Let $\mathcal{C}_p$ be the convex cone of positive closed (p, p)-currents on X, and $\mathcal{D}_p$ be the real vector space generated by $\mathcal{C}_p$. Since the Kähler form ω is strictly positive, the set $\mathcal{D}_p$ contains all real closed smooth (p, p)-forms. Let $\mathcal{D}_p^0$ be the subspace

of $\mathcal{D}_p$ of currents belonging to the class 0 in $H^{p,p}(X,\mathbb{R})$. We recall the notion of $*$-norm on $\mathcal{D}_p$. Consider first a positive closed current S in $\mathcal{D}_p$. Define its $*$-norm by

$$\|S\|_* = |\langle S, \omega^{n-p}\rangle|,$$

which is equal to the mass of S. In general, since any $S \in \mathcal{D}_p$ can be written as the difference of two positive closed currents, define

$$\|S\|_* = \inf(\|S^+\|_* + \|S^-\|_*),$$

where the infimum is taken over all $S^+, S^- \in \mathcal{C}_p$ such that $S = S^+ - S^-$. By the compactness property of positive closed currents, this infimum is attained for some S^+ and S^-. We say that S_k converges to S in $\mathcal{D}_p$ for the $*$-topology if S_k converges to S weakly as current and $\|S_k\|_*$ is bounded independently of k. The following result is due to Dinh and Sibony; see [9, Th. 2.4.4] and also [6, Th. 1.1].

PROPOSITION 2.1. *There is a positive constant c such that for all $S \in \mathcal{D}_p$, there exist smooth forms $S_k \in \mathcal{D}_p$ with $k \in \mathbb{N}$ such that S_n converges weakly to S and $\|S_k\|_* \le c\|S\|_*$ for all k.*

Let T be in $\mathcal{D}_p$, and R be in $\mathcal{D}^0_q$. By the dd^c-lemma for currents (see [11, Th. 1.2.1]) there is a real $(q-1,q-1)$-current U_R such that $dd^c U_R = R$. We call U_R a potential of R. Consider the following important example of R. Let V be a hypersurface of X, and β_0 be a smooth form of the same cohomology class with $[V]$. Then $R = [V] - \beta_0$ is in $\mathcal{D}^0_1$. We can construct an explicit potential U_R as follows. Consider the holomorphic line bundle of X associated with V and a holomorphic section σ whose divisor is V. Take a smooth Hermitian metric on this line bundle and denote by $|\cdot|$ the norm induced by this metric. By the Poincaré–Lelong formula, there is a smooth form β_1 such that

$$dd^c \log|\sigma| = [V] - \beta_1.$$

Since $\{\beta_0\} = \{V\} = \{\beta_1\}$, there is a smooth function f on X such that $dd^c f = \beta_0 - \beta_1$. The function $U_R := \log|\sigma| - f$ is a potential of R. Note that U_R is smooth outside V and if σ' is a holomorphic function on an open neighborhood W of a point of V such that its divisor is $V \cap W$, then

$$U_R(x) - \log|\sigma'| \quad \text{is smooth on } W. \tag{2.1}$$

Consider now a current $R \in \mathcal{D}^0_{n-p+1}$ and an $(n-p,n-p)$-current U_R that is a potential of R. Let $\alpha = (\alpha_1,\dots,\alpha_h)$ with $h = \dim H^{p,p}(X,\mathbb{R})$ be a fixed family of real smooth closed (p,p)-forms such that the family of classes $\{\alpha\} = (\{\alpha_1\},\dots,\{\alpha_h\})$ is a basis of $H^{p,p}(X,\mathbb{R})$. By adding to U_R a suitable closed smooth form we can assume that $\langle U_R, \alpha_i\rangle = 0$ for $i = 1,\dots,h$. We say that U_R is α-normalized.

DEFINITION 2.2 ([9, Def. 3.2.2]). Let T be a current in $\mathcal{D}_p$ as before. The α-normalized superpotential $\mathcal{U}_T$ of T is the function defined on smooth forms $R \in \mathcal{D}^0_{n-p+1}$ and given by

$$\mathcal{U}_T(R) = \langle T, U_R\rangle,$$

where U_R is an α-normalized smooth potential of R. We say that T has a continuous superpotential if $\mathcal{U}_T$ can be extended to a function on $\mathcal{D}^0_{n-p+1}$ that is continuous with respect to the $*$-topology. In this case, the extension is also denoted by $\mathcal{U}_T$.

By [9, Lemma 3.2.1], $\mathcal{U}_T(R)$ does not depend on the choice of an α-normalized U_R. The continuity of $\mathcal{U}_T$ also does not depend on α. Observe that when $\{T\} = 0$, the α-normalized superpotential of T does not depend on α. Indeed, in this case, it is the restriction of any potential U_T of T to the set of smooth forms in $\mathcal{D}^0_{n-p+1}$. Assume that T has a continuous superpotential. Take any current $S \in \mathcal{D}_q$. Let $(a_1, \dots, a_h)$ be the coefficients of $\{T\}$ in the basis $\{\alpha\}$. Define $T \wedge S$ to be the real $(p+q, p+q)$-current satisfying

$$\langle T \wedge S, \Phi\rangle := \mathcal{U}_T(dd^c\Phi \wedge S) + \sum_{1\le j\le h} a_j \langle \alpha_j, \Phi \wedge S\rangle \tag{2.2}$$

for any real smooth $(n-p-q, n-p-q)$-form Φ.

3. Alternative Definition for the Intersection of Currents

Let X, $\widehat{X \times X}$, ω, $\widehat{\omega}$, Π, Π_j, π_j, Δ, $\widehat{\Delta}$ be as in the previous sections. Consider two currents $T \in \mathcal{D}_p$ and $S \in \mathcal{D}_q$ as before with $p+q \le n$. Let h, a_j, and α_j with $1 \le j \le h$ be as in the last section. From now on, assume that T is positive and has a continuous superpotential. Note that $\Pi_j = \pi_j \circ \Pi$ are submersions; for a proof, see [9] or the proof of our Lemma 3.2. Define $\widehat{T} = \Pi_1^*(T)$ and $\widehat{S} = \Pi_2^*(S)$. They are positive closed currents on $\widehat{X \times X}$. Put $\widehat{\alpha}_j = \Pi_1^*(\alpha_j)$ for $1 \le j \le h$.

LEMMA 3.1. *The current $\widehat{T}$ has a continuous superpotential.*

Proof. Suppose that the classes $\{\widehat{\alpha}_j\}$ are linearly dependent. Then there exist real numbers b_j with $1 \le j \le h$ that are not simultaneously equal to zero and a smooth form $\widehat{\gamma}$ such that $\sum_{j=1}^h b_j\widehat{\alpha}_j = d(\widehat{\gamma})$. Taking the wedge product with $\widehat{\omega}^n$ in the last equality and then using the push-forward by $(\Pi_1)_*$ give

$$\sum_{j=1}^h b_j\alpha_j \wedge (\Pi_1)_*(\widehat{\omega}^n) = d((\Pi_1)_*(\widehat{\gamma} \wedge \widehat{\omega}^n)). \tag{3.1}$$

Note that $(\Pi_1)_*\widehat{\omega}^n$ is actually a nonzero constant since $\widehat{\omega}^n$ is closed and positive. We deduce that the left-hand side of (3.1) is a nontrivial linear combination of α_j, $1 \le j \le h$. However, this contradicts the fact that $\{\alpha_j\}$ are linearly independent. Hence, the classes $\{\widehat{\alpha}_j\}$ are linearly independent. Complete them to be basis $\widehat{\alpha}'$ of $H^{p,p}(\widehat{X \times X}, \mathbb{R})$. Let $\mathcal{U}_{\widehat{T}}$ be the $\widehat{\alpha}'$-normalized superpotential of $\widehat{T}$.

Put $\alpha_T = \sum_{j=1}^h a_j\alpha_j$ and $\widehat{\alpha}_T = \Pi_1^*\alpha_T$. Remark that α_T and $\widehat{\alpha}_T$ are in the same cohomology classes with T and $\widehat{T}$, respectively. Let $U_{T-\alpha_T}$ be a potential of $T - \alpha_T$. Then $U_{\widehat{T}-\widehat{\alpha}_T} := \Pi_1^*U_{T-\alpha_T}$ is a potential of $\widehat{T} - \widehat{\alpha}_T$. By definition, for

any smooth form $\tilde{R} \in \mathcal{D}^0_{2n-p+1}(\widehat{X \times X})$, we have

$$\mathcal{U}_{\widehat{T}}(\tilde{R}) = \langle \widehat{T}, U_{\tilde{R}} \rangle = \langle \widehat{T} - \widehat{\alpha}_T, U_{\tilde{R}} \rangle = \langle U_{\widehat{T}-\widehat{\alpha}_T}, \tilde{R} \rangle.$$

By our choice of potentials, the last quantity equals

$$\langle U_{T-\alpha_T}, (\Pi_1)_* \tilde{R} \rangle = \mathcal{U}_T((\Pi_1)_* \tilde{R}).$$

The continuity of $\mathcal{U}_T$ now implies immediately the same property for $\mathcal{U}_{\widehat{T}}$. The proof is finished. □

Thanks to Lemma 3.1, we can define $\widehat{T} \wedge \widehat{S}$ as in (2.2). Recall that $T \otimes S$ is a positive closed $(p+q, p+q)$-current on $X \times X$ depending continuously on T and S. Its action on smooth forms can be described as follows. Let x be local coordinates of X. They induce naturally local coordinates (x, y) on $X \times X$. For a smooth form $\Phi(x, y)$ of $X \times X$, we have

$$\langle T \otimes S, \Phi \rangle = \langle T, S(\Phi(x, \cdot)) \rangle = \langle S, T(\Phi(\cdot, y)) \rangle. \tag{3.2}$$

Let Π' be the restriction of Π to $\widehat{X \times X} \backslash \widehat{\Delta}$. The current

$$\widehat{T \otimes S} = \Pi'^*(T \otimes S)$$

is well defined and positive closed on $\widehat{X \times X} \backslash \widehat{\Delta}$ because Π' is biholomorphic. By Proposition 5.1 of [8] the mass of $\widehat{T \otimes S}$ is bounded. Hence, it can be extended by zero to be a positive closed current of $\widehat{X \otimes X}$ through $\widehat{\Delta}$; see [5; 13; 14]. We still denote by $\widehat{T \otimes S}$ the extended current. Take a smooth closed $(1,1)$-form $\widehat{\beta}$ with $\{\widehat{\beta}\} = \{\widehat{\Delta}\}$. Since $\widehat{\Delta}$ is a hypersurface, choose a potential

$$\hat{u} = U_{[\widehat{\Delta}]-\widehat{\beta}} \tag{3.3}$$

of $[\widehat{\Delta}] - \widehat{\beta}$ as in Section 2. It is smooth outside $\widehat{\Delta}$, and its behavior near $\widehat{\Delta}$ is described by (2.1). By adding a constant to $\hat{u}$ if necessary we can assume that $\hat{u} \leq -1$.

LEMMA 3.2. *The current $\hat{u}\widehat{S}$ is well defined. Moreover, if smooth forms $S_k \in \mathcal{D}_q$ converge to S in the $*$-topology, then $\hat{u}\widehat{S}_k$ converge weakly to $\hat{u}\widehat{S}$.*

Proof. We prove the first assertion. For any smooth $(2n-q, 2n-q)$-form $\hat{\eta}$ on $\widehat{X \times X}$, we will show that $(\Pi_2)_*(\hat{u}\hat{\eta})$ is a smooth form on X. This allows us to define

$$\langle \hat{u}\widehat{S}, \hat{\eta} \rangle = \langle S, (\Pi_2)_*(\hat{u}\hat{\eta}) \rangle. \tag{3.4}$$

To see that $(\Pi_2)_*(\hat{u}\hat{\eta})$ is smooth, we just need to work locally. Consider local coordinates $(W, x = (x_1, \ldots, x_n))$ on a chart W of X. Without loss of generality, we can suppose that W is diffeomorphic to the unit ball $\mathbb{B}_1$ in $\mathbb{C}^n$. Consider induced local coordinates (x, y) on $W \times W$. We have $\Delta \cap (W \times W) = \{x = y\}$. Define new local coordinates (x', y) on $W \times W$ by putting $x' := x - y$. Hence, Δ is given by the equation $x' = 0$. The set $\Pi^{-1}(W \times W)$ is biholomorphic to the manifold M in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{P}^{n-1}$ defined by

$$M = \{(x', y, [v]) : y \in \mathbb{B}_1, x' + y \in \mathbb{B}_1, [v] \in \mathbb{P}^{n-1} \text{ and } x' \in [v]\},$$

where $[v] = [v_1 : v_2 : \cdots : v_n]$ denotes the homogeneous coordinates of $\mathbb{P}^{n-1}$. Let M_j $(1 \le j \le n)$ be the open subset of M containing all points $(x', y, [v]) \in M$ with $v_j \neq 0$. They form an open covering of M. For $(x', y, [v]) \in M_1$, we have $x_1' v_j = x_j' v_1$. Choose $v_1 = 1$. Then $x_j' = x_1' v_j$. We deduce that $(x_1', v_2, \ldots, v_n, y)$ are coordinates on M_1 and $\widehat{\Delta} \cap M_1 = \{x_1' = 0\}$. Since $\Pi_2(x_1', v_2, \ldots, v_n, y) = y$, we see that

$$\begin{aligned}(\Pi_2)_*(\hat{u}\hat{\eta}) &= \int_{x_1', v_2, \ldots, v_n} \hat{u}(x_1', v_2, \ldots, v_n, y)\hat{\eta}(x_1', v_2, \ldots, v_n, y) \\ &= \int_{x_1', v_2, \ldots, v_n} \log|x_1'|\hat{\eta}(x_1', v_2, \ldots, v_n, y) \\ &\quad + \int_{x_1', v_2, \ldots, v_n} \hat{u}'(x_1', v_2, \ldots, v_n, y)\hat{\eta}(x_1', v_2, \ldots, v_n, y),\end{aligned}$$

where $\hat{u}'(x_1', v_2, \ldots, v_n, y)$ is a smooth function; see (2.1). This implies that the last integral defines a smooth form in y. It is also clear that the integral involving $\log|x_1'|$ depends smoothly in y. The proof of the first assertion is finished. The second assertion is a direct consequence of identity (3.4). The proof is finished. □

PROPOSITION 3.3. *We have* $\widehat{T} \wedge \widehat{S} = \widehat{T \otimes S}$.

Proof. Consider first the case where S is smooth. So $\widehat{T} \wedge \widehat{S}$ is the usual wedge product of a current with a smooth form. We then see that $\widehat{T} \wedge \widehat{S} = \Pi^*(T \otimes S) = \widehat{T \otimes S}$ outside $\widehat{\Delta}$. Observe that the fibers of the submersion Π_1 are transverse to $\widehat{\Delta}$. Therefore, $\widehat{T}$ has no mass on $\widehat{\Delta}$. Hence, $\widehat{T} \wedge \widehat{S}$ has no mass on $\widehat{\Delta}$. We deduce that $\widehat{T} \wedge \widehat{S} = \widehat{T \otimes S}$ in this case because $\widehat{T \otimes S}$ has no mass on $\widehat{\Delta}$ by definition.

In general, by Proposition 2.1 there is a sequence of smooth forms $S_k \in \mathcal{D}_q$ converging to S in the $*$-topology. The first case and the continuity on S imply that $\widehat{T} \wedge \widehat{S} = \widehat{T \otimes S}$ outside $\widehat{\Delta}$. It remains to show that the restriction $\mathbf{1}_{\widehat{\Delta}}(\widehat{T} \wedge \widehat{S})$ of $\widehat{T} \wedge \widehat{S}$ vanishes. This is equivalent to say that

$$\int_{\widehat{\Delta}} \widehat{T} \wedge \widehat{S} \wedge \widehat{\Phi} = 0 \tag{3.5}$$

for any smooth form $\widehat{\Phi}$ of bidegrees $2n - p - q$. By Proposition 2.1 we can write $S = S^+ - S^-$ where S^+ and S^- are approximable by smooth positive closed forms. Since $\widehat{T} \wedge \widehat{S} = \widehat{T} \wedge \widehat{S}^+ - \widehat{T} \wedge \widehat{S}^-$, we only need to verify that $\mathbf{1}_{\widehat{\Delta}}(\widehat{T} \wedge \widehat{S}^{\pm}) = 0$. Therefore, without loss of generality, assume that $\widehat{T} \wedge \widehat{S}$ is positive. Consequently, it suffices to prove (3.5) for $\widehat{\Phi} = \widehat{\omega}^{2n-p-q}$.

Let χ be a convex increasing smooth function on $\mathbb{R}$ such that $\chi(t) = 0$ if $t \le -1/4$, $\chi(t) = t$ for $t \ge 1/4$ and $0 \le \chi' \le 1$. For each positive integer k, put

$$\hat{u}_k = \chi(\hat{u} + k) - k.$$

This is a smooth negative quasi-p.s.h. function since $\widehat{u} \le -1$. The functions $\hat{u}_k$ decrease to $\hat{u}$, and $-\hat{u}_k/k$ decrease to the characteristic function $\mathbf{1}_{\widehat{\Delta}}$ of $\widehat{\Delta}$ as

$k \to \infty$. The first property implies that $\widehat{S} \wedge dd^c \hat{u}_k$ converges weakly to $\widehat{S} \wedge dd^c \hat{u}$; see Lemma 3.2. We also have

$$dd^c \hat{u}_k = [\chi''(\hat{u}+k)]d\hat{u} \wedge d^c \hat{u} + \chi'(\hat{u}+k)dd^c \hat{u} \geq \chi'(\hat{u}+k)dd^c \hat{u} \geq -c\widehat{\omega}$$

for some positive constant c. This yields that $dd^c \hat{u}_k = (dd^c \hat{u}_k + c\widehat{\omega}) - c\widehat{\omega}$, which is the difference of two positive closed currents in the same cohomology class $c\{\widehat{\omega}\}$. We deduce that $dd^c \hat{u}_k$ is $*$-bounded uniformly in k, and then so is $\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}$ because we have

$$\|\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}\|_* \leq c\|S\|_* \|dd^c \hat{u}_k\|_* \tag{3.6}$$

for a positive constant c depending only on (X, ω). It follows that

$$\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q} \to \widehat{S} \wedge dd^c \hat{u} \wedge \widehat{\omega}^{2n-p-q}$$

in the $*$-topology. Equality (3.5) with $\widehat{\Phi} = \widehat{\omega}^{2n-p-q}$ is equivalent to

$$\left\langle \widehat{T} \wedge \widehat{S}, -\frac{\hat{u}_k}{k} \cdot \widehat{\omega}^{2n-p-q} \right\rangle \to 0 \quad \text{as } k \to \infty. \tag{3.7}$$

Applying the formula (2.2) to $\widehat{T} \wedge \widehat{S}$ gives

$$\begin{aligned} &\left\langle \widehat{T} \wedge \widehat{S}, -\frac{\hat{u}_k}{k} \cdot \widehat{\omega}^{2n-p-q} \right\rangle \\ &\quad = -\frac{1}{k}\mathcal{U}_{\widehat{T}}(\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}) - \frac{1}{k}\langle \widehat{\alpha}_T, \hat{u}_k \widehat{S} \wedge \widehat{\omega}^{2n-p-q} \rangle, \end{aligned}$$

where $\widehat{\alpha}_T = \sum_{j=1}^h a_j \widehat{\alpha}_j$. The last quantity converges to 0 as $k \to \infty$ since the mass norm of $\hat{u}_k \widehat{S}$ is bounded independently of k by Lemma 3.2. On the other hand, the continuity of $\mathcal{U}_{\widehat{T}}$ gives

$$\mathcal{U}_{\widehat{T}}(\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}) \to \mathcal{U}_{\widehat{T}}(\widehat{S} \wedge dd^c \hat{u} \wedge \widehat{\omega}^{2n-p-q}),$$

which is finite, as $k \to \infty$. Hence, we get (3.7). The proof is finished. □

LEMMA 3.4. *The current $\hat{u}(\widehat{T} \wedge \widehat{S})$ is well defined. Denote it by $\hat{u}\widehat{T} \wedge \widehat{S}$ for simplicity. For any closed real smooth form $\widehat{\Phi}$ of $\widehat{X \times X}$ of the right bidegrees, we have*

$$\langle \hat{u}\widehat{T} \wedge \widehat{S}, \widehat{\Phi} \rangle = \mathcal{U}_{\widehat{T}}(dd^c(\hat{u}\widehat{S} \wedge \widehat{\Phi})) + \sum_{j=1}^h a_j \langle \widehat{S}, \hat{u}\widehat{\alpha}_j \wedge \widehat{\Phi} \rangle. \tag{3.8}$$

In particular, $\langle \hat{u}\widehat{T} \wedge \widehat{S}, \widehat{\Phi} \rangle$ depends continuously on S.

Proof. Using the computation in the proof of Proposition 3.3, we have

$$\begin{aligned} \langle \widehat{T} \wedge \widehat{S}, \hat{u} \cdot \widehat{\omega}^{2n-p-q} \rangle = &\lim_{k \to \infty} \mathcal{U}_{\widehat{T}}(\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}) \\ &+ \langle \widehat{\alpha}_T, \hat{u}_k \widehat{S} \wedge \widehat{\omega}^{2n-p-q} \rangle, \end{aligned}$$

where $\hat{u}_k$ is defined as in Proposition 3.3. The same arguments as at the end of the proposition show that the last limit is finite. The first assertion follows. Note that each smooth closed form Φ can be written as the difference of two positive

closed forms. Hence, it suffices to prove (3.8) for positive closed forms Φ. The computations in Proposition 3.3 still hold for Φ in place of $\widehat{\omega}^{2n-p-q}$. Hence, (3.8) follows.

In order to prove the last assertion, it suffices to prove it for positive closed forms Φ by the same reason as before. Let $\{S_l\}_{l\in\mathbb{N}}$ be a sequence of currents in $\mathcal{D}_q$ that converges to S in the $*$-topology. Put $\widehat{S}_l = \Pi_2^*(S_l)$. It is clear that $\widehat{S}_l$ converges to $\widehat{S}$ in the $*$-topology. Lemma 3.2 implies that $dd^c(\hat{u}\widehat{S}_l \wedge \widehat{\Phi})$ converges weakly to $dd^c(\hat{u}\widehat{S} \wedge \widehat{\Phi})$ and

$$\lim_{k\to\infty} dd^c(\hat{u}_k\widehat{S}_l \wedge \widehat{\Phi}) = dd^c(\hat{u}\widehat{S}_l \wedge \widehat{\Phi}) \tag{3.9}$$

for any $l \in \mathbb{N}$. Applying (3.6) to S_k in place of S, we see that the mass of $dd^c(\hat{u}_k\widehat{S}_l \wedge \widehat{\Phi})$ is bounded independently of k and l. This, combined with (3.9), yields that the $*$-norm of $dd^c(\hat{u}\widehat{S}_l \wedge \widehat{\Phi})$ is bounded independently of l. We deduce that $dd^c(\hat{u}\widehat{S}_l \wedge \widehat{\Phi})$ converges to $dd^c(\hat{u}\widehat{S} \wedge \widehat{\Phi})$ in the $*$-topology. The continuity of $\mathcal{U}_{\widehat{T}}$ now implies that the right-hand side of (3.8) depends continuously on S. The proof is finished. □

COROLLARY 3.5. *Define the intersection* $\widehat{T \otimes S} \wedge [\widehat{\Delta}]$ *by putting*

$$\widehat{T \otimes S} \wedge [\widehat{\Delta}] = dd^c(\hat{u}\widehat{T \otimes S}) + \widehat{T \otimes S} \wedge \widehat{\beta}; \tag{3.10}$$

see (3.3) *for the definition of* $\widehat{\beta}$. *Then* $\widehat{T \otimes S} \wedge [\widehat{\Delta}]$ *is positive when* S *is positive.*

Proof. We only need to prove the positivity. This property is classic since the current $[\widehat{\Delta}]$ is of bidegrees $(1, 1)$. We give here a proof for the sake of the reader. Fix a small open subset $\widehat{W}$ of $\widehat{X \times X}$ biholomorphic to a ball. We can find a smooth function $\hat{v}$ on $\widehat{W}$ such that $dd^c\hat{v} = \widehat{\beta}$. Hence, the function $\hat{u}' = \hat{u} + \hat{v}$ satisfies $dd^c\hat{u}' = [\widehat{\Delta}] \geq 0$. So $\hat{u}'$ is p.s.h. on $\widehat{W}$. We then have $\widehat{T \otimes S} \wedge [\widehat{\Delta}] = dd^c(\hat{u}'\widehat{T \otimes S})$ on $\widehat{W}$. If $\hat{u}'_k$ is a sequence of smooth p.s.h. functions on $\widehat{W}$ decreasing to $\hat{u}'$, then the last current is the limit of $dd^c(\hat{u}'_k\widehat{T \otimes S})$, which is clearly positive since it equals $dd^c\hat{u}'_k \wedge \widehat{T \otimes S}$. The proof is finished. □

LEMMA 3.6. *Let* Y *be a closed subset of* X. *Let* R *be a positive* (p, p)*-current of* X, *and let* R_k *be a sequence of positive* (p, p)*-currents of* X *converging weakly to* R *as currents in* $X\backslash Y$. *Assume that* R *has no mass on* Y *and the masses of* R_k *converge to that of* R. *Then* R_k *converges weakly to* R *in* X.

Proof. For each $\varepsilon > 0$, let Y_ε be the set of points in X of distance less than ε to Y. Let χ_ε be a continuous function on X such that $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon = 1$ on $X\backslash Y_{2\varepsilon}$, and $\chi_\varepsilon = 0$ on $\overline{Y}_\varepsilon$. Take any continuous real form Φ on X of bidegrees $n - p$. We need to prove that

$$R_k(\Phi) \to R(\Phi) \quad \text{as } k \to \infty. \tag{3.11}$$

Since a continuous form can be written as the difference of two continuous positive forms, we can assume that Φ is positive. The hypothesis on R_k implies that

$R_k(\chi_\varepsilon\Phi)$ converges to $R(\chi_\varepsilon\Phi)$. Hence, in order to prove (3.11), it is sufficient to show that

$$\lim_{\varepsilon\to 0}\delta_\varepsilon = 0, \tag{3.12}$$

where

$$\delta_\varepsilon = \limsup_{k\to\infty}\int_{\overline{Y}_{2\varepsilon}} R_k(\Phi).$$

Let $\mu_k = R_k\wedge\omega^{n-p}$ and $\mu = R\wedge\omega^{n-p}$ be the trace measures of R_k and R, respectively. Observe that δ_ε is less than a constant times

$$\limsup_{k\to\infty}\mu_k(\overline{Y}_{2\varepsilon}) = \|R\| - \liminf_{k\to\infty}\mu_k(X\backslash\overline{Y}_{2\varepsilon}).$$

Since the set $X\backslash\overline{Y}_{2\varepsilon}$ is an open subset of $X\backslash Y$, the last limit is greater than $\mu(X\backslash\overline{Y}_{2\varepsilon})$. Hence, we get

$$\limsup_{k\to\infty}\int_{\overline{Y}_{2\varepsilon}} R_k(\Phi) \lesssim \|R\| - \mu(X\backslash\overline{Y}_{2\varepsilon}) = \mu(\overline{Y}_{2\varepsilon}).$$

The last quantity converges to zero as $\varepsilon\to 0$ because μ has no mass on Y. The proof is finished. □

PROPOSITION 3.7. *For $j = 1$ or 2, we have*

$$T\wedge S = (\Pi_j)_*(\widehat{T\otimes S}\wedge[\widehat{\Delta}]\wedge\widehat{\omega}^{n-1}), \tag{3.13}$$

where $T\wedge S$ is defined as in (2.2).

Proof. As explained in Introduction, formula (3.13) holds for smooth forms T and S. We consider now the general case. We already know that $T\wedge S$ depends continuously on S for the $*$-topology. Let $\{S_k\}_{k\in\mathbb{N}}$ be a sequence of smooth forms in $\mathcal{D}_q$ that converges to S in the $*$-topology. Put $\widehat{S}_k = \Pi_2^*(S_k)$ and $R_k = \hat{u}\widehat{T}\wedge\widehat{S}_k$. It follows from Lemma 3.4 that the masses of R_k converge to the mass of $R = \hat{u}\widehat{T}\wedge\widehat{S}$. Moreover, R_k converges to R in $\widehat{X\times X}\backslash\widehat{\Delta}$. Applying Lemma 3.6 to $\widehat{X\times X}$ in the place of X, R_k, and R, we see that the right-hand sides of (3.13), which is defined in Corollary 3.5, also depend continuously on S for the $*$-topology. Hence, approximating S by smooth forms allows us to assume that S is smooth. Now Lemma 3.2 applied to $\widehat{T}$ in place of $\widehat{S}$ implies that the right-hand side of (3.13) is continuous in T. When S is smooth, it is clear that $T\wedge S$ depends continuously on T. Therefore, (3.13) holds since we can approximate T by closed smooth forms. The proof is finished. □

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