Michigan Math. J. 65 (2016), 833–854

Universality in Measure in the Bulk for Varying Weights

Eli Levin & Doron S. Lubinsky

Abstract. We prove that universality holds in measure for sequences $\{g_n e^{-2nQ}\}$ of varying weights, where e^{-Q} is an exponential weight, and the functions $\{g_n\}$ admit suitable polynomial approximations.

1. Introduction

In the theory of random Hermitian matrices, one considers a probability distribution $\mathcal{P}^{(n)}$ on the eigenvalues $x_1 \le x_2 \le \cdots \le x_n$ of $n \times n$ Hermitian matrices,

$$\mathcal{P}^{(n)}(x_1, x_2, \ldots, x_n) = ce^{-\sum_{j=1}^{n} 2nQ_n(x_j)} \prod_{i<j} (x_i - x_j)^2.$$

See [5, p. 106 ff.]. Here, c is a normalizing constant, often called the partition function, and Q_n is a given function. In the Gaussian unitary ensemble, $Q_n(x) = \frac{1}{2}x^2$.

Orthogonal polynomials play a crucial role in analyzing these. For $n \ge 1$, let μ_n be a finite positive Borel measure with support supp$[\mu_n]$ containing infinitely many points. We may define orthonormal polynomials

$$p_m(\mu_n, x) = \gamma_m(\mu_n)x^m + \cdots, \quad \gamma_m(\mu_n) > 0,$$

$m = 0, 1, 2, \ldots,$ satisfying the orthonormality conditions

$$\int p_k(\mu_n, x) p_\ell(\mu_n, x)\, d\mu_n(x) = \delta_{k\ell}.$$

Throughout we use μ_n' to denote the Radon–Nikodym derivative of μ_n. The nth reproducing kernel for μ_n is [10; 23]

$$K_n(\mu_n, x, y) = \sum_{k=0}^{n-1} p_k(\mu_n, x) p_k(\mu_n, y), \tag{1.1}$$

and the normalized kernel is

$$\tilde{K}_n(\mu_n, x, y) = \mu_n'(x)^{1/2} \mu_n'(y)^{1/2} K_n(\mu_n, x, y). \tag{1.2}$$

The nth Christoffel function is

$$\lambda_n(\mu_n, x) = K_n(\mu_n, x, x)^{-1}.$$

When

$$d\mu_n(x) = e^{-2nQ_n(x)}\, dx,$$

Received July 3, 2015. Revision received October 19, 2016.
Second author's research supported by NSF grant DMS136208.

there is the basic formula for the probability distribution $\mathcal{P}^{(n)}$ [5, p. 112]:

$$\mathcal{P}^{(n)}(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \det(\tilde{K}_n(\mu_n, x_i, x_j))_{1 \le i, j \le n}.$$

Note that this is the standard definition, which includes the factor $\mu_n'(x_1)\mu_n'(x_2) \cdot \cdots \mu_n'(x_n)$ in the right-hand side. One particularly important quantity is the m-point correlation function,

$$\begin{aligned} &R_m(\mu_n, x_1, x_2, \ldots, x_m) \\ &\quad = \frac{n!}{(n-m)!} \int \cdots \int \mathcal{P}^{(n)}(x_1, x_2, \ldots, x_n)\, dx_{m+1}\, dx_{m+2} \cdots dx_n. \end{aligned}$$

The function R_m admits the remarkable identity [5, p. 112], first proved by Freeman Dyson,

$$R_m(\mu_n, x_1, x_2, \ldots, x_m) = \det(\tilde{K}_n(\mu_n, x_i, x_j))_{1 \le i, j \le m}.$$

The *universality limit in the bulk* asserts that, for fixed $m \ge 2$, ξ in a suitable subset of the (common) supports of $\{\mu_n\}$, and real $a_1, a_2, \ldots, a_m$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\tilde{K}_n(\mu_n, \xi, \xi)^m} R_m\biggl(\mu_n, \xi + \frac{a_1}{\tilde{K}_n(\mu_n, \xi, \xi)}, \xi + \frac{a_2}{\tilde{K}_n(\mu_n, \xi, \xi)}, \ldots, \\ &\qquad \xi + \frac{a_m}{\tilde{K}_n(\mu_n, \xi, \xi)}\biggr) \\ &\quad = \det(\mathbb{S}(a_i - a_j))_{1 \le i, j \le m}, \end{aligned}$$

where

$$\mathbb{S}(t) = \frac{\sin \pi t}{\pi t}$$

is the sinc kernel. Because m is fixed in this limit, this reduces to the case $m = 2$, namely

$$\lim_{n\to\infty} \frac{\tilde{K}_n(\mu_n, \xi + \frac{a}{\tilde{K}_n(\mu_n,\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\mu_n,\xi,\xi)})}{\tilde{K}_n(\mu_n, \xi, \xi)} = \mathbb{S}(a - b). \tag{1.3}$$

Typically, this is established uniformly for a, b in compact subsets of the real line. Thus, an assertion about the distribution of eigenvalues of random matrices has been reduced to a technical limit involving orthogonal polynomials. See [1; 2; 4; 5; 6; 8; 9; 13; 15; 16; 17; 18; 19; 23; 24; 27; 28] for some references to the extensive literature on this topic.

We shall need some concepts from potential theory for external fields [22]. Let Σ be a closed set on the real line, and $W(x) = \exp(-Q(x))$ be an upper semicontinuous function on Σ that is positive on a set of positive linear Lebesgue measure. If Σ is unbounded, then we assume that

$$\lim_{|x|\to\infty, x\in\Sigma} W(x)|x| = 0. \tag{1.4}$$

We say, following Saff and Totik, [22], that W is *admissible.* Associated with Σ and Q, we may consider the extremal problem

$$\inf_{\nu}\left(\int\int \log\frac{1}{|x-t|}\,d\nu(x)\,d\nu(t)+2\int Q\,d\nu\right),$$

where the inf is taken over all positive Borel measures ν with support in Σ and $\nu(\Sigma)=1$. The inf is attained by a unique equilibrium measure ω_Q, characterized by the following conditions: let

$$U^{\omega_Q}(z)=\int \log\frac{1}{|z-t|}\,d\omega_Q(t)$$

denote the logarithmic potential for ω_Q, and let

$$\mathcal{S}_Q=\operatorname{supp}[\omega_Q]$$

denote the (compact) support of the equilibrium measure. Then [22, Theorem I.1.3, p. 27]

$$U^{\omega_Q}+Q\geq F_Q \quad \text{q.e. on } \Sigma; \tag{1.5}$$

$$U^{\omega_Q}+Q=F_Q \quad \text{q.e. in } \mathcal{S}_Q. \tag{1.6}$$

Here the number F_Q is a constant, and q.e. stands for quasi-everywhere, that is, except on a set of capacity 0. Notice that we are using ω_Q for the equilibrium measure, rather than the more standard μ_W or ν_W, to avoid confusion with μ_n or ν_n.

Our first result in [15, p. 747, Theorem 1.1] was the following. Its proof depended heavily on asymptotics of Christoffel functions that has been established by Vili Totik [26]:

THEOREM A. *Let* $W=e^{-Q}$ *be a continuous nonnegative function on the set* Σ, *which is assumed to consist of at most finitely many intervals. If* Σ *is unbounded, then we also assume* (1.4). *Let* h *be a bounded positive continuous function on* Σ, *and for* $n\geq 1$, *let*

$$d\mu_n(x)=(hW^{2n})(x)\,dx.$$

Let J *be a closed interval lying in the interior of* $\mathcal{S}_Q=\operatorname{supp}[\omega_Q]$, *where* ω_Q *denotes the equilibrium measure for* W. *Assume that* ω_Q *is absolutely continuous in a neighborhood of* J *and that* ω_Q' *and* Q' *are continuous in that neighborhood with* $\omega_Q'>0$ *there. Then, uniformly for* $\xi\in J$ *and* a,b *in compact subsets of the real line, we have*

$$\lim_{n\to\infty}\frac{\tilde{K}_n\left(\mu_n,\xi+\frac{a}{\tilde{K}_n(\mu_n,\xi,\xi)},\xi+\frac{b}{\tilde{K}_n(\mu_n,\xi,\xi)}\right)}{\tilde{K}_n(\mu_n,\xi,\xi)}=\mathbb{S}(a-b). \tag{1.7}$$

In particular, if Q' satisfies a Lipschitz condition of some positive order in a neighborhood of J, then [22, p. 199] ω_Q', is continuous there, and hence we obtain universality except near zeros of ω_Q'. Note too that when Q is convex in Σ, or $xQ'(x)$ is increasing there, the support of ω_Q consists of at most finitely many intervals, with at most one interval per component of Σ [22, p. 199].

Theorem A is a particular case of a more general result in [15, p. 748, Theorem 1.2], which allowed fairly general sequences of weights, but involved hypotheses on asymptotics of ratios of their Christoffel functions. In [15], applications were also made to universality for fixed exponential weights on the real line.

Our goal in this paper is to prove that universality holds in measure for varying weights under more general hypotheses than those in Theorem A. Our first result is an equiconvergence result. Throughout, meas denotes the linear Lebesgue measure. We also say that a sequence of real-valued functions $\{f_n\}$ defined on a compact set Σ converges in measure to a function g if for each $\varepsilon > 0$,

$$\text{meas}\{x \in \Sigma : |f_n(x) - g(x)| > \varepsilon\} \to 0 \quad \text{as } n \to \infty.$$

THEOREM 1.1. *Let Σ be a closed set, and $\{\nu_n\}$ and $\{\mu_n\}$ be sequences of positive measures on Σ. Assume that, for $n \geq 1$,*

$$\mu_n'(x) = h(x) e^{-2nQ(x)}, \quad x \in \Sigma, \tag{1.8}$$

where h is bounded, positive, and continuous on Σ, and e^{-Q} is admissible, and also, as $n \to \infty$,

$$\frac{1}{n} K_n(\mu_n, x, x)\mu_n'(x) \to \omega_Q' \quad \textit{in measure in } \mathcal{S}_Q. \tag{1.9}$$

Assume also that, for $n \geq 1$,

$$\nu_n'(x) = g_n(x)\mu_n'(x), \quad x \in \Sigma, \tag{1.10}$$

and for $n \geq 1$, there exists a polynomial S_n of degree ℓ_n that is positive in Σ, with

$$\ell_n = o(n), \quad n \to \infty, \tag{1.11}$$

and

$$S_n^2(x) g_n(x) \leq 1, \quad \textit{a.e. } x \in \Sigma, \tag{1.12}$$

while as $n \to \infty$,

$$S_n^2 g_n \to 1 \quad \textit{in measure in } \mathcal{S}_Q. \tag{1.13}$$

Let $R, \varepsilon > 0$. Then for large enough n, there exists a set $\mathcal{E}_n$ of linear Lebesgue measure $< \varepsilon$ such that

$$\sup_{\substack{|a|,|b| \leq R, \\ \xi \in \mathcal{S}_Q \setminus \mathcal{E}_n}} \frac{1}{n} \bigg| K_n\left(\nu_n, \xi + \frac{a}{n}, \xi + \frac{b}{n}\right) - S_n\left(\xi + \frac{a}{n}\right) S_n\left(\xi + \frac{b}{n}\right) K_n\left(\mu_n, \xi + \frac{a}{n}, \xi + \frac{b}{n}\right) \bigg| \nu_n'(\xi) = o(1). \tag{1.14}$$

The set $\mathcal{E}_n$ does not depend on R.

REMARKS. (a) Note that in (1.14), a, b are complex numbers.

(b) In (1.9), we could equivalently assume that the convergence in measure takes place in Σ.

(c) Note that (1.9) holds under the conditions of Theorem A; see [26].

(d) A more explicit form of Theorem 1.1 is given in Theorem 4.3.

(e) The conclusion of Theorem 1.1 is in fact almost uniform convergence, as used in Egorov's theorem, and so is a little stronger than convergence in measure.

One case where we can find suitable approximating polynomials $\{S_n\}$ is in the following result, where Σ is assumed to be a union of finitely many compact intervals. Let $\alpha \in (0,\infty)$, and for $n \geq 1$, let $\varphi_n : \Sigma \to \mathbb{R}$. We say that $\{\varphi_n\}$ is *uniformly smooth of order* α if

(i) when $\alpha = k$, where k is a positive integer, $\varphi_n^{(k-1)}$ is absolutely continuous for each $n \geq 1$, with

$$\sup_n \|\varphi_n^{(k)}\|_{L_\infty(\Sigma)} < \infty,$$

(ii) when $\alpha = k + \Delta$, where k is a nonnegative integer, and $0 < \Delta < 1$, $\varphi_n^{(k)}$ satisfies a uniform Lipschitz condition of order Δ:

$$\sup_n \sup_{x,y\in\Sigma} \frac{|\varphi_n^{(k)}(x) - \varphi_n^{(k)}(y)|}{|x-y|^\Delta} < \infty.$$

THEOREM 1.2. *Let Σ consist of finitely many compact intervals, and let*

$$\nu_n'(x) = h(x) e^{\tau_n \varphi_n(x)} \left(\prod_{j=1}^N |x - b_{nj}|^{\beta_{nj}} \right)^n e^{-2nQ(x)}, \quad x \in \Sigma, \tag{1.15}$$

where h is a positive continuous function on Σ, and where Q is continuous, while Q' is continuous in Σ, except perhaps at finitely many points. Assume, moreover, that the equilibrium measure ω_Q is absolutely continuous and that ω_Q' is positive and continuous in $\mathcal{S}_Q^o$, except perhaps at finitely many points. Assume also that $N \geq 1$ and all $\{b_{nj}\}_{n,j}$ lie in some compact subset of the real line, while

$$\max_{1\leq j\leq N} \beta_{nj} \to 0 \quad \text{as } n \to \infty. \tag{1.16}$$

In addition, assume that $\alpha \in (0,\infty)$ and $\{\varphi_n\}$ is a sequence of real-valued functions on Σ that is uniformly smooth of order α, whereas for $n \geq 1$,

$$\|\varphi_n\|_{L_\infty(\Sigma)} \leq 1 \tag{1.17}$$

and

$$\tau_n = o(n^{\frac{\alpha}{1+\alpha}}), \quad n \to \infty. \tag{1.18}$$

Then, given $\varepsilon > 0$,

$$\operatorname{meas}\left\{ x \in \mathcal{S}_Q : \sup_{|a|,|b|\leq R} \left| \frac{\tilde{K}_n(\nu_n, \xi + \frac{a}{n\omega_Q'(\xi)}, \xi + \frac{b}{n\omega_Q'(\xi)})}{n\omega_Q'(\xi)} - \mathbb{S}(a-b) \right| > \varepsilon \right\}$$
$$\to 0 \quad \text{as } n \to \infty. \tag{1.19}$$

REMARKS. (a) Note that in (1.19), we restrict x to lie in $\mathcal{S}_Q$ since we cannot have universality (at least in the sense above) outside this set. Moreover, a, b are real since $\tilde{K}_n$ is only defined in Σ.

(b) The requirement that Σ is compact can be removed if we assume that the measures decay exponentially at ∞, allowing us to apply restricted range inequalities from, for example, [14; 22].

(c) We could allow more general rates of growth of $\{\tau_n\}$ than (1.18). Moreover, if all $\{\varphi_n\}$ are analytic and uniformly bounded in an open set containing Σ that is independent of n, we could replace (1.18) by

$$\tau_n = o\left(\frac{n}{\log n}\right), \quad n \to \infty.$$

(d) We could multiply by other factors, such as $e^{o(n)U^{\rho_n}}$, where $\{\rho_n\}$ are a sequence of measures supported on $\Sigma = [0, 1]$ that admit polynomial approximation of the type described in [25, Theorem 8.1, p. 49].

2. An Integral Estimate

We start with the following simple comparison inequality.

LEMMA 2.1. *Let $m, n \geq 1$, and μ_n and ν_n be positive measures on the real line. Assume that S_n is a polynomial of degree $\ell < n$ and*

$$n = m + \ell. \tag{2.1}$$

Assume also that on $\mathbb{R}$,

$$S_n^2 \, d\nu_n \leq d\mu_n. \tag{2.2}$$

Then

$$\begin{aligned} \Gamma_n &:= \frac{1}{n} \int\int (K_n(\nu_n, x, y) - S_n(x)S_n(y)K_n(\mu_n, x, y))^2 \, d\nu_n(x) \, d\nu_n(y) \\ &\leq 4\left(1 + \frac{\ell}{n} - \frac{1}{n}\int S_n^2(y)K_n(\mu_n, y, y) \, d\nu_n(y)\right). \end{aligned} \tag{2.3}$$

Proof. Using the reproducing kernel property and (2.1), followed by (2.2), we have

$$\begin{aligned} \Delta_n &:= \int\int (K_n(\nu_n, x, y) - S_n(x)S_n(y)K_m(\mu_n, x, y))^2 \, d\nu_n(x) \, d\nu_n(y) \\ &= \int\int K_n(\nu_n, x, y)^2 \, d\nu_n(x) \, d\nu_n(y) \\ &\quad - 2\int S_n(y)\left[\int S_n(x)K_m(\mu_n, x, y)K_n(\nu_n, x, y) \, d\nu_n(x)\right] d\nu_n(y) \\ &\quad + \int S_n(y)^2\left[\int S_n(x)^2 K_m(\mu_n, x, y)^2 \, d\nu_n(x)\right] d\nu_n(y) \\ &= \int K_n(\nu_n, y, y) \, d\nu_n(y) - 2\int S_n^2(y)K_m(\mu_n, y, y) \, d\nu_n(y) \\ &\quad + \int S_n(y)^2\left[\int S_n(x)^2 K_m(\mu_n, x, y)^2 \, d\nu_n(x)\right] d\nu_n(y) \end{aligned}$$

$$\leq n - 2\int S_n^2(y) K_m(\mu_n, y, y)\, d\nu_n(y)$$
$$+ \int\int K_m(\mu_n, x, y)^2\, d\mu_n(x)\, d\mu_n(y)$$
$$= n + m - 2\int S_n^2(y) K_m(\mu_n, y, y)\, d\nu_n(y). \tag{2.4}$$

Next,

$$\int S_n^2(y)(K_n - K_m)(\mu_n, y, y)\, d\nu_n(y)$$
$$\leq \int (K_n - K_m)(\mu_n, y, y)\, d\mu_n(y) = n - m.$$

Substituting into (2.4) and using $n - m = \ell$ give

$$\Delta_n \leq n + m - 2\int S_n^2(y) K_n(\mu_n, y, y)\, d\nu_n(y) + 2(n - m)$$
$$= 2n + \ell - 2\int S_n^2(y) K_n(\mu_n, y, y)\, d\nu_n(y). \tag{2.5}$$

Using the inequality $(x + y)^2 \leq 2(x^2 + y^2)$, we see that

$$n\Gamma_n \leq 2\Delta_n + 2\int\int (K_n(\mu_n, x, y) - K_m(\mu_n, x, y))^2 S_n^2(x) S_n^2(y)\, d\nu_n(x)\, d\nu_n(y)$$
$$\leq 2\Delta_n + 2\int\int (K_n(\mu_n, x, y) - K_m(\mu_n, x, y))^2\, d\mu_n(x)\, d\mu_n(y)$$
$$= 2\Delta_n + 2(n - m) = 2\Delta_n + 2\ell.$$

Now use (2.5). □

Under the hypotheses of Theorem 1.1, we can estimate Γ_n defined by (2.3).

LEMMA 2.2. *Assume the hypotheses of Theorem 1.1. Then*

$$\lim_{n\to\infty} \Gamma_n = 0. \tag{2.6}$$

Proof. By our hypotheses,

$$\frac{1}{n} K_n(\mu_n, x, x)\nu_n'(x) S_n^2(x) = \frac{1}{n} K_n(\mu_n, x, x)\mu_n'(x)[S_n^2(x) g_n(x)] \to \omega_Q'$$

in measure in $\mathcal{S}_Q$ as $n \to \infty$. Let $\varepsilon > 0$ and

$$\mathcal{E}_n = \left\{x \in \mathcal{S}_Q : \left|\frac{1}{n} K_n(\mu_n, x, x)\nu_n'(x) S_n^2(x) - \omega_Q'(x)\right| > \varepsilon\right\},$$

so that $\mathrm{meas}(\mathcal{E}_n) \to 0$ as $n \to \infty$. Then, as $n \to \infty$,

$$\frac{1}{n}\int S_n^2(x) K_n(\mu_n, x, x)\, d\nu_n(x)$$
$$\geq \int_{\mathcal{S}_Q \setminus \mathcal{E}_n} (\omega_Q'(x) - \varepsilon)\, dx$$
$$\to \int_{\mathcal{S}_Q} (\omega_Q'(x) - \varepsilon)\, dx = 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\liminf_{n\to\infty} \frac{1}{n} \int S_n^2(x) K_n(\mu_n, x, x)\, d\nu_n(x) \geq 1$$

because of integrability of ω'_Q. Together with our hypothesis (1.11) and (2.3), this gives (2.6). □

3. Potential Theoretic Estimates

Next, we turn to some potential theoretic estimates. Given a positive measure ρ with compact support in the plane, recall that [20]

$$U^\rho(z) = \int \log \frac{1}{|z-t|}\, d\rho(t)$$

denotes the associated potential. We need a general growth estimate.

LEMMA 3.1. *Let ρ be a measure with compact support in the real line and with total mass ≤ 1. Let $\varepsilon > 0$. There is a set $\mathcal{F} \subset \mathbb{R}$ with* $\operatorname{meas}(\mathcal{F}) < \varepsilon$ *such that, for $\xi \in \mathbb{R}\backslash\mathcal{F}$ and all $u \in \mathbb{C}$,*

$$U^\rho(\xi) - U^\rho(\xi + u) \leq C_0|u|/\varepsilon. \tag{3.1}$$

Here C_0 is independent of $\rho, \xi, \varepsilon, u$.

Proof. This is essentially the same as that of Lemma 4.1 and Corollary 4.2 [18, pp. 232–233]. There estimates were made for the Green's function, but replacing the equilibrium measure by the measure ρ, we obtain essentially what we need. Nevertheless, we provide the details. Recall that the maximal function of the measure ρ is

$$\mathcal{M}[\rho](x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} d\rho.$$

Moreover, $\mathcal{H}^*$ denotes the maximal Hilbert transform of ρ defined by

$$\mathcal{H}^*[\rho](x) = \sup_{\varepsilon>0} \left| \int_{|t-x|\geq\varepsilon} \frac{1}{t-x}\, d\rho(t) \right|.$$

Step 1: *A growth estimate.* We first show that

$$U^\rho(\xi) - U^\rho(\xi+u) \leq 26|u|\mathcal{M}[\rho](\xi) + |\operatorname{Re} u|\mathcal{H}^*[\rho](\xi). \tag{3.2}$$

Write $u = x + iy$. Now, for real ξ,

$$\begin{aligned} &U^\rho(\xi) - U^\rho(\xi+u) \\ &\quad = \frac{1}{2} \int \log\left[1 + \frac{2x}{\xi - t} + \frac{|u|^2}{(\xi-t)^2}\right] d\rho(t). \end{aligned}$$

Let $\mathcal{S}_1$ denote the set of t for which

$$\left|\frac{2x}{\xi-t}\right| \leq \frac{2|u|^2}{(\xi-t)^2} \Longleftrightarrow |\xi - t| \leq \frac{|u|^2}{|x|}.$$

Let $\mathcal{S}_2$ denote the complementary range. In the case where $x = 0$, of course, $\mathcal{S}_2$ is empty. Let us assume that $x \neq 0$; the case $x = 0$ is easier. We see that

$$\begin{aligned}
&\int_{\mathcal{S}_1} \log\left[1 + \frac{2x}{\xi - t} + \frac{|u|^2}{(\xi - t)^2}\right] d\rho(t) \\
&\quad \leq \int_{|\xi - t| \leq \frac{|u|^2}{|x|}} \log\left[1 + \frac{3|u|^2}{(\xi - t)^2}\right] d\rho(t) \\
&\quad \leq \sum_{k=0}^{\infty} \int_{2^{-k-1}\frac{|u|^2}{|x|} \leq |\xi - t| \leq 2^{-k}\frac{|u|^2}{|x|}} \log\left[1 + \frac{12x^2}{|u|^2} 2^{2k}\right] d\rho(t) \\
&\quad \leq \sum_{k=0}^{\infty} \log\left[1 + \frac{12x^2}{|u|^2} 2^{2k}\right] 2^{-k+1} \frac{|u|^2}{|x|} \mathcal{M}[\rho](\xi) \\
&\quad \leq \frac{|u|^2}{|x|} \mathcal{M}[\rho](\xi) 4 \int_0^{\infty} \log\left[1 + \frac{12x^2}{|u|^2 t^2}\right] dt \\
&\quad = |u| \mathcal{M}[\rho](\xi) 8\sqrt{3}\pi,
\end{aligned} \tag{3.3}$$

cf. [12, p. 525, no. 4.222.1]. Next, in $\mathcal{S}_2$, we have $|\xi - t| \geq |u|^2/|x|$, so using the inequality $\log(1 + t) \leq t$, $t \geq -1$, we obtain

$$\begin{aligned}
&\int_{\mathcal{S}_2} \log\left[1 + \frac{2x}{\xi - t} + \frac{|u|^2}{(\xi - t)^2}\right] d\rho(t) \\
&\quad \leq \int_{|\xi - t| \geq |u|^2/|x|} \left[\frac{2x}{\xi - t} + \frac{|u|^2}{(\xi - t)^2}\right] d\rho(t) \\
&\quad \leq 2|x| |\mathcal{H}^*[\rho](\xi)| + |u|^2 \int_{|\xi - t| \geq |u|^2/|x|} \frac{1}{(\xi - t)^2}\, d\rho(t).
\end{aligned} \tag{3.4}$$

Here,

$$\begin{aligned}
&\int_{|\xi - t| \geq |u|^2/|x|} \frac{1}{(\xi - t)^2}\, d\rho(t) \\
&\quad \leq \sum_{k=0}^{\infty} \int_{2^{k+1}|u|^2/|x| \geq |\xi - t| \geq 2^k |u|^2/|x|} \frac{1}{(2^k |u|^2/|x|)^2}\, d\rho(t) \\
&\quad \leq \sum_{k=0}^{\infty} \frac{x^2}{|u|^4} 2^{-2k} \cdot 2^{k+2} \frac{|u|^2}{|x|} \mathcal{M}[\rho](\xi) \\
&\quad = \frac{|x|}{|u|^2} 8\mathcal{M}[\rho](\xi).
\end{aligned}$$

Combining this with (3.3) and (3.4) gives

$$\begin{aligned}
&U^{\rho}(\xi) - U^{\rho}(\xi + u) \\
&\quad \leq 4\sqrt{3}\pi\, |u| \mathcal{M}[\rho](\xi) + |x| |\mathcal{H}^*[\rho](\xi)| + |x| 4\mathcal{M}[\rho](\xi).
\end{aligned}$$

Estimating the constants gives (3.2).

Step 2: *Estimate the maximal function and Hilbert transform.* Next, we use the fact that both the maximal function and the maximal Hilbert transform are weak type (1, 1). That is, [21, p. 137, Theorem 7.4],

$$\operatorname{meas}\{\xi : \mathcal{M}[\rho](\xi) > \lambda\} \leq \frac{3}{\lambda}\int d\rho \leq \frac{3}{\lambda},$$

and [3, p. 139], [11, p. 128 ff.]

$$\operatorname{meas}\{\xi : \mathcal{H}^*[\rho](\xi) > \lambda\} \leq \frac{C_1}{\lambda}\int d\rho \leq \frac{C_1}{\lambda}.$$

Here C_1 is independent of ρ and λ. Note that in these references the bound for $\mathcal{H}^*[\rho]$ is proved assuming that ρ is absolutely continuous. However, it is true without this restriction. Indeed, John Garnett emailed to the authors about [11]: "The proof of Theorem 2.1 of Chapter III about the maximal conjugate function goes through with the L_1 function replaced by a finite measure. Also, on the line it is easy to estimate the difference between the truncated Hilbert transform of $|x| > \varepsilon$ and the conjugate function at i/ε by the ordinary maximal function". Choosing $\lambda = \frac{2}{\varepsilon}\max\{3, C_1\}$, we obtain a set $\mathcal{F}$ of measure $\leq \varepsilon$ such that, for $\xi \in \mathbb{R}\backslash\mathcal{F}$ and all complex u, (3.1) holds with a constant C_0 independent of $\rho, \xi, u, \varepsilon$. □

LEMMA 3.2. *Let Σ be a closed set, let $Q : \Sigma \to [0, \infty)$ be such that e^{-Q} is admissible, and let ω_Q denote the equilibrium measure for e^{-Q}, with support $\mathcal{S}_Q$. Let L be a compact subset of Σ, and assume that L is the closure of its (nonempty) interior. Let $n > \ell \geq 1$. Let S_n be a polynomial of degree ℓ that has no zeros on Σ. Let $\varepsilon > 0$, and $R : \mathbb{C} \to \mathbb{C}$ be a function such that* $\log|R|$ *is subharmonic in $\mathbb{C}$. Assume also that*

$$\lim_{|z|\to\infty} (\log|R(z)| - (n+\ell)\log|z|) \tag{3.5}$$

exists and is finite and that

$$|R(x)|e^{-nQ(x)}/|S_n(x)| \leq 1 \quad \text{for } x \in L. \tag{3.6}$$

Then there exists a set $\mathcal{F}_n \subset \mathbb{R}$ with $\operatorname{meas}(\mathcal{F}_n) < \varepsilon$ *such that, for $\xi \in (L \cap \mathcal{S}_Q)\backslash\mathcal{F}_n$ and all $u \in \mathbb{C}$,*

$$|R(\xi+u)|e^{-nQ(\xi)}/|S_n(\xi)| \leq e^{C_0 n|u|/\varepsilon}, \quad u \in \mathbb{C}. \tag{3.7}$$

The set $\mathcal{F}_n$ depends on $Q, \varepsilon, n, \ell, S_n$ but not on the particular R. Also, C_0 is independent of $Q, R, n, \xi, \varepsilon, u$.

Proof. Recall that

$$U^{\omega_Q} + Q = F_Q \quad \text{q.e. in } \mathcal{S}_Q.$$

Now balayage that part of ω_Q with support in $\mathcal{S}_Q\backslash L^o$ onto L. More precisely, we apply Theorem II.4.7 in [22, p. 116] with the domain $G = \mathbb{C}\backslash L$ giving the balayage measure

$$\nu = \widehat{\omega_{Q|G}}.$$

It has the property that, for some constant C_0,

$$U^{\nu}(x) = U^{\omega_{Q|G}}(x) + C_0, \quad \text{q.e. } x \in L.$$

If we let

$$\tilde{\omega}_Q = \omega_{Q|L} + \nu$$

(the sum of ν and the restriction of ω_Q to L), then for q.e. $x \in L \cap \mathcal{S}_Q$,

$$U^{\tilde{\omega}_Q}(x) = U^{\omega_Q}(x) + C_0 = -Q(x) + F_Q + C_0 \tag{3.8}$$

(recall (1.6)). Note that $\tilde{\omega}_Q$ has support in L. Next, we may assume that S_n is monic (a constant can be absorbed into R). Then we can write

$$S_n(x) = \prod_{j=1}^{\ell}(x - b_j),$$

and if

$$\rho = \sum_{j=1}^{\ell} \delta_{b_j}$$

is a measure with compact support in $\mathbb{C}\backslash\Sigma$, then we have

$$-\log|S_n(z)| = U^{\rho}(z).$$

Let $\tilde{\rho}$ denote the balayage measure of ρ onto $L \cap \mathcal{S}_Q$, so that by [22, Theorem II.4.7, p. 116] it has total mass ℓ, support in $L \cap \mathcal{S}_Q$, and for some constant C_1,

$$U^{\tilde{\rho}}(x) = U^{\rho}(x) + C_1 \quad \text{q.e. in } L \cap \mathcal{S}_Q. \tag{3.9}$$

Let

$$C_2 = n(F_Q + C_0) + C_1.$$

Then for q.e. $x \in L \cap \mathcal{S}_Q$, we have from (3.8) and (3.9)

$$e^{nU^{\tilde{\omega}_Q}(x) + U^{\tilde{\rho}}(x) - C_2} = e^{-nQ(x)}/|S_n(x)|, \tag{3.10}$$

so we can recast (3.6) as

$$|R(x)|e^{nU^{\tilde{\omega}_Q}(x) + U^{\tilde{\rho}}(x) - C_2} \le 1 \quad \text{for q.e. } x \in L \cap \mathcal{S}_Q.$$

Moreover, since $\tilde{\omega}_Q$ and $\tilde{\rho}$ have supports in $L \cap \mathcal{S}_Q$, their potentials are harmonic outside this set, so $\log|R(z)| + nU^{\tilde{\omega}_Q}(z) + U^{\tilde{\rho}}(z) - C_2$ is subharmonic in $\mathbb{C}\backslash(L \cap \mathcal{S}_Q)$, with a finite limit at ∞ in view of (3.5). By the maximum principle for subharmonic functions, we have

$$|R(z)|e^{nU^{\tilde{\omega}_Q}(z) + U^{\tilde{\rho}}(z) - C_2} \le 1, \quad z \in \mathbb{C}. \tag{3.11}$$

Now let

$$\omega := \frac{1}{n+\ell}[n\tilde{\omega}_Q + \tilde{\rho}],$$

which is a measure of total mass 1 with support in $L \cap \mathcal{S}_Q$. From (3.11) we have

$$|R(z)|e^{(n+\ell)U^{\omega}(z) - C_2} \le 1, \quad z \in \mathbb{C}.$$

Then for $\xi \in \mathbb{R}$ and complex u, we have

$$|R(\xi + u)|e^{(n+\ell)U^{\omega}(\xi) - C_2} \le e^{(n+\ell)[U^{\omega}(\xi) - U^{\omega}(\xi+u)]},$$

and now we can apply Lemma 3.1. Note that the set $\mathcal{F}_n$ in Lemma 3.1 does not depend on the particular R since ω does not depend on R. Finally, by (3.10), in $L \cap \mathcal{S}_Q$, we have

$$e^{(n+\ell)U^{\omega}(\xi)-C_2} = e^{-nQ(\xi)}/|S_n(\xi)|. \qquad \square$$

4. Pointwise Estimates and Proof of Theorem 1.1

Throughout, we assume the hypotheses of Theorem 1.1 and the notation of the previous sections. In particular, Γ_n is defined by (2.3). We begin with the following:

LEMMA 4.1. *Let $\varepsilon > 0$. For $n \geq n_0(\varepsilon)$, there exists a set $\mathcal{G}_n$ of measure $\leq \Gamma_n^{1/2} + \varepsilon$ such that, for $\xi \in \mathcal{S}_Q \backslash \mathcal{G}_n$ and for all $u, z \in \mathbb{C}$, we have*

$$\begin{aligned} &\frac{1}{n}|K_n(\nu_n, \xi+u, z) - S_n(\xi+u)S_n(z)K_n(\mu_n, \xi+u, z)|\nu_n'(\xi) \\ &\qquad \leq 4\left[\frac{1}{n}K_{n+\ell}(\nu_n, z, \bar{z})\nu_n'(\xi)\right]^{1/2}\Gamma_n^{1/4}e^{C_0 n|u|/\varepsilon}. \end{aligned} \tag{4.1}$$

Proof. Step 1: *An integral estimate and exceptional set.* Let $\mathcal{E}_n$ denote the set of $x \in \Sigma$ for which

$$\frac{1}{n}\int (K_n(\nu_n, x, y) - S_n(x)S_n(y)K_n(\mu_n, x, y))^2\, d\nu_n(y)\nu_n'(x) > \sqrt{\Gamma_n}.$$

Then

$$\begin{aligned} &\operatorname{meas}(\mathcal{E}_n)\sqrt{\Gamma_n} \\ &\quad \leq \frac{1}{n}\int\int (K_n(\nu_n, x, y) - S_n(x)S_n(y)K_n(\mu_n, x, y))^2\, d\nu_n(y)\, d\nu_n(x) = \Gamma_n \\ &\quad \Rightarrow \operatorname{meas}(\mathcal{E}_n) \leq \sqrt{\Gamma_n}. \end{aligned} \tag{4.2}$$

Step 2: *A Christoffel function estimate.*

We use the inequality

$$|P(z)|^2 \leq K_{n+\ell}(\nu_n, z, \bar{z})\int |P|^2\, d\nu_n,$$

valid for all polynomials P of degree $< n+\ell$ and all $z \in \mathbb{C}$, applied to

$$P(z) = \frac{1}{n}(K_n(\nu_n, x, z) - S_n(x)S_n(z)K_n(\mu_n, x, z))$$

with $x \in \Sigma \backslash \mathcal{E}_n$ fixed. We then obtain from Step 1, for $x \in \Sigma \backslash \mathcal{E}_n$ and all $z \in \mathbb{C}$,

$$\begin{aligned} &\frac{1}{n^2}|K_n(\nu_n, x, z) - S_n(x)S_n(z)K_n(\mu_n, x, z)|^2\nu_n'(x) \\ &\quad = |P(z)|^2\nu_n'(x) \\ &\quad \leq K_{n+\ell}(\nu_n, z, \bar{z})\left(\int |P|^2\, d\nu_n\right)\nu_n'(x) \leq \frac{1}{n}K_{n+\ell}(\nu_n, z, \bar{z})\Gamma_n^{1/2}. \end{aligned}$$

Now take square roots: for $x \in \Sigma \backslash \mathcal{E}_n$ and all $z \in \mathbb{C}$,

$$\frac{1}{n}|K_n(\nu_n, x, z) - S_n(x) S_n(z) K_n(\mu_n, x, z)| \nu_n'(x)^{1/2} \leq \left(\frac{1}{n} K_{n+\ell}(\nu_n, z, \bar{z})\right)^{1/2} \Gamma_n^{1/4}. \tag{4.3}$$

Step 3: *Fix z and apply Lemma* 3.2.

Fix $n \geq 1$, fix $z \in \mathbb{C}$ with $S_n(z) p_{n-1}(z) \neq 0$, let $L_n = \mathcal{S}_Q \backslash \mathcal{E}_n$ (which is independent of z), and denote

$$A_n = \left[\frac{1}{n} K_{n+\ell}(\nu_n, z, \bar{z})\right]^{1/2} \Gamma_n^{1/4};$$
$$R(t) = \frac{1}{n A_n}(K_n(\nu_n, t, z) - S_n(t) S_n(z) K_n(\mu_n, t, z)).$$

Then $\log|R(t)|$ is subharmonic in the plane, and (assuming that S_n has full degree $\ell \geq 1$, as we may) $\log|R(t)| - (n+\ell-1)\log|t|$ has a finite limit as $|t| \to \infty$ since we assumed that $(S_n p_{n-1})(z) \neq 0$. Moreover, for $x \in L_n$, we can recast (4.3) as

$$|R(x)| h(x)^{1/2} e^{-nQ(x)} g_n(x)^{1/2} \leq 1.$$

(Recall (1.8) and (1.10).) Since h is bounded above and below by positive constants in Σ and does not depend on n, we can simply assume that $h \equiv 1$. Next, by our hypothesis that $g_n S_n^2 \to 1$ in measure in $\mathcal{S}_Q$, for large enough n, we may replace $g_n^{1/2}$ in this last inequality by $\frac{1}{2}|S_n|^{-1}$ except on a set of small measure. Absorbing that set of small measure into $\mathcal{E}_n$ (perhaps by replacing ε by $\varepsilon/2$), we obtain, for large enough n,

$$\frac{1}{2}|R(x)| e^{-nQ(x)} |S_n(x)|^{-1} \leq 1, \quad x \in L_n.$$

Since the left-hand side is continuous, we may assume that L_n is closed and (by removing isolated points) is even the closure of its interior. Moreover, the new L_n is still independent of z above. Then by Lemma 3.2 there exists a set $\mathcal{F}_n \subset \mathbb{R}$ with $\text{meas}(\mathcal{F}_n) < \varepsilon$ such that, for $\xi \in L_n \backslash \mathcal{F}_n = \mathcal{S}_Q \backslash (\mathcal{E}_n \cup \mathcal{F}_n)$ and all $u \in \mathbb{C}$,

$$\frac{1}{2n A_n}|K_n(\nu_n, \xi + u, z) - S_n(\xi + u) S_n(z) K_n(\mu_n, \xi + u, z)| e^{-nQ(\xi)} / |S_n(\xi)| \leq e^{C_0 n|u|/\varepsilon}.$$

Note that the set $\mathcal{F}_n$ did not depend on the particular R, so it works for all complex z. We may replace $e^{-nQ(\xi)}/|S_n(\xi)|$ by $\nu_n'(\xi)^{1/2}$ in this last inequality, at the cost of adding a set of small measure to $\mathcal{E}_n \cup \mathcal{F}_n$. Finally, we set $\mathcal{G}_n = \mathcal{E}_n \cup \mathcal{F}_n$, multiply by $\nu_n'(\xi)^{1/2}$, and take into account the definition of A_n to obtain the result, at least when $(S_n p_{n-1})(z) \neq 0$. Since this last condition holds except at finitely many z, we can use continuity in z to deduce (4.1) for such z also. □

Next, we estimate $K_{n+\ell}$. We shall assume that in μ_n', $h \equiv 1$. The general case involves trivial modifications.

LEMMA 4.2. *Let $\varepsilon > 0$. There exists a set $\mathcal{H}_n$ of measure at most 2ε such that, for $\xi \in \mathcal{S}_Q \backslash \mathcal{H}_n$ and all complex v,*

$$\frac{1}{n}|K_{n+\ell}(\nu_n, \xi + v, \xi + \bar{v})|\nu_n'(\xi) \le \frac{4}{\varepsilon} e^{nC_1|v|/\varepsilon}, \tag{4.4}$$

where C_1 is independent of n, ξ, v, ε.

Proof. We shall abbreviate $p_k(\nu_n, z)$ as $p_k(z)$. First, we note that, for $n \ge 1$,

$$\log K_n(\nu_n, z, \bar{z}) = \log\left(\sum_{k=0}^{n-1} |p_k(z)|^2\right)$$

is subharmonic in the plane. Indeed, if we fix z and $r > 0$, we can choose unimodular constants $\{\alpha_k\}$ such that

$$\begin{aligned}\log K_n(\nu_n, z, \bar{z}) &= \log\left|\sum_{k=0}^{n-1} \alpha_k p_k(z)^2\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\sum_{k=0}^{n-1} \alpha_k p_k(z + re^{i\theta})^2\right| d\theta \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sum_{k=0}^{n-1} |p_k(z + re^{i\theta})|^2\right) d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log K_n(\nu_n, z + re^{i\theta}, \overline{z + re^{i\theta}})\, d\theta.\end{aligned}$$

In the second last line, we used the subharmonicity of absolute values of analytic functions. Moreover, we see that, as $|z| \to \infty$,

$$\log K_n(\nu_n, z, \bar{z}) = (2n-2)\log|z| + 2\log\gamma_{n-1} + o(1).$$

Now, for small $\varepsilon > 0$, let

$$\mathcal{I}_n = \left\{t \in \mathcal{S}_Q : \frac{1}{n}K_{n+\ell}(\nu_n, t, t)\nu_n'(t) \ge \frac{2}{\varepsilon}\right\}.$$

Observe that

$$\operatorname{meas}(\mathcal{I}_n)\frac{2}{\varepsilon} \le \int \frac{1}{n}K_{n+\ell}(\nu_n, t, t)\, d\nu_n(t) < 2$$

since $n > \ell$, so

$$\operatorname{meas}(\mathcal{I}_n) \le \varepsilon. \tag{4.5}$$

Since $S_n^2 g_n \to 1$ in measure, we may absorb a set of small measure into $\mathcal{I}_n$ and obtain

$$\frac{1}{2n}K_{n+\ell}(\nu_n, t, t)e^{-2nQ(t)}/S_n^2(t) \le \frac{2}{\varepsilon}, \quad t \in \mathcal{S}_Q \backslash \mathcal{I}_n.$$

We now let $L_n = \mathcal{S}_Q \backslash \mathcal{I}_n$. Since the left-hand side in this last inequality is continuous, we may assume that L_n is closed and (by removing isolated points) is also the closure of its interior. Next, by Lemma 3.2 with $R(t) = K_{n+\ell}(\nu_n, t, \bar{t})$, with

L taken as L_n, Q replaced by $2Q$, and S_n by S_n^2, there is a set $\mathcal{J}_n$ of measure at most ε such that, for $\xi \in L_n \backslash \mathcal{J}_n = \mathcal{S}_Q \backslash (\mathcal{I}_n \cup \mathcal{J}_n)$ and all complex v,

$$\frac{\varepsilon}{4n} K_{n+\ell}(\nu_n, \xi + v, \overline{\xi + v}) e^{-2nQ(\xi)} / S_n^2(\xi) \leq e^{nC_1|v|/\varepsilon}. \tag{4.6}$$

Here C_1 is independent of n, v, ε, ξ. Setting $\mathcal{H}_n = \mathcal{I}_n \cup \mathcal{J}_n$, we see that $\text{meas}(\mathcal{H}_n) < 2\varepsilon$, and for $\xi \in \mathcal{S}_Q \backslash \mathcal{H}_n$ and all complex v, we have (4.4), except that ν_n' is replaced by $e^{-2nQ(\xi)}/S_n^2(\xi)$. Again, we can add a small set to $\mathcal{H}_n$ and replace $e^{-2nQ(\xi)}/S_n^2(\xi)$ by $\nu_n'(\xi)$. □

Now we put it all together:

Proof of Theorem 1.1. We apply Lemmas 4.1 and 4.2: let $\mathcal{E}_n = \mathcal{G}_n \cup \mathcal{H}_n$, so that $\text{meas}(\mathcal{E}_n) < 3\varepsilon + 2\Gamma_n^{1/2}$ and, for $\xi \in \mathcal{S}_Q \backslash \mathcal{E}_n$ and all $u, v \in \mathbb{C}$,

$$\begin{aligned} &\frac{1}{n} |K_n(\nu_n, \xi + u, \xi + v) - S_n(\xi + u) S_n(\xi + v) K_n(\mu_n, \xi + u, \xi + v)| \nu_n'(\xi) \\ &\quad \leq \frac{8}{\sqrt{\varepsilon}} \Gamma_n^{1/4} e^{C_0 n(|u|+|v|)/\varepsilon}. \end{aligned}$$

Finally, we know that (2.6) holds, so we can just choose $u = a/n$, $v = b/n$, and replace ε by $\varepsilon/4$. □

For future use, we record this last inequality as the following:

THEOREM 4.3. *Assume the hypotheses of Theorem* 1.1. *Let* $\varepsilon > 0$. *Then, for* $n \geq 1$, *there exists a set* $\mathcal{E}_n$ *of measure* $\leq \varepsilon$ *such that, for* $\xi \in S_Q \backslash \mathcal{E}_n$ *and all complex* a, b,

$$\begin{aligned} &\frac{1}{n} \left| K_n\left(\nu_n, \xi + \frac{a}{n}, \xi + \frac{b}{n}\right) \right. \\ &\qquad \left. - S_n\left(\xi + \frac{a}{n}\right) S_n\left(\xi + \frac{b}{n}\right) K_n\left(\mu_n, \xi + \frac{a}{n}, \xi + \frac{b}{n}\right) \right| \nu_n'(\xi) \\ &\quad \leq \frac{8}{\sqrt{\varepsilon}} \Gamma_n^{1/4} e^{C_0(|a|+|b|)/\varepsilon}. \end{aligned} \tag{4.7}$$

Here C_0 *is independent of* $n, \xi, a, b, \varepsilon$. *We may replace* $\nu_n'(\xi)$ *by* $\mu_n'(\xi)/S_n^2(\xi)$ *in this last left-hand side.*

5. Proof of Theorem 1.2

Our main task is to find approximating polynomials $\{S_n\}$ satisfying hypotheses (1.11)–(1.13) of Theorem 1.1. We begin with the following:

LEMMA 5.1. *Let* $\{\delta_n\} \subset (0, \infty)$ *with*

$$\lim_{n \to \infty} \delta_n = 0. \tag{5.1}$$

Let $r > 0$. *There exist nonnegative polynomials* $\{P_n\}$ *with*

$$\deg(P_n) = o(n), \quad n \to \infty, \tag{5.2}$$

and, *as* $n \to \infty$, *uniformly in compact subsets of* $(-r, r)\backslash\{0\}$,

$$|x|^{n\delta_n} P_n(x) \to 1, \tag{5.3}$$

while for $n \geq 1$,

$$\sup_{x\in[-r,r]} |x|^{n\delta_n} P_n(x) \leq 1. \tag{5.4}$$

Proof. First, note that by dilating the variable we may assume that $r = 1$. Next, it suffices to find nonnegative polynomials $\{P_n\}$ of degree $o(n)$ satisfying (5.2) and also, as $n \to \infty$,

$$x^{n\delta_n} P_n(x) \to 1 \quad \text{uniformly in compact subsets of } (0, 1), \tag{5.5}$$

whereas, for $n \geq 1$,

$$0 \leq x^{n\delta_n} P_n(x) \leq 1 \quad \text{in } [0, 1]. \tag{5.6}$$

Indeed, we can always replace $P_n(x)$ satisfying these last two conditions by $P_n(x^2)$, after also replacing δ_n by $\delta_n/2$.

Next, we recall classical results from the theory of incomplete polynomials. Let $\theta \in (0, \frac{1}{4})$, and $f_\theta : [0, 1] \to [0, 1]$ be continuous with $f_\theta = 0$ in $[0, \theta^2] \cup \{1\}$. Then by [22, p. 283] there exist polynomials $R_{m,\theta}$ of degree $\leq m$ such that

$$\lim_{m\to\infty} x^{m\theta/(1-\theta)} R_{m,\theta}(x) = f_\theta(x) \tag{5.7}$$

uniformly in [0, 1]. We may also assume that $R_{m,\theta} \geq 0$ in [0, 1]. (If not consider $R^2_{m,\theta}$ after modifying the value of θ). By dividing by $R_{m,\theta}$ by $1 + \eta_m$, where η_m decreases to 0 sufficiently slowly, we can also assume that

$$0 \leq x^{m\theta/(1-\theta)} R_{m,\theta}(x) < 1 \quad \text{in } [0, 1]. \tag{5.8}$$

We shall choose our f_θ so that $0 \leq f_\theta \leq 1$ in [0, 1] and $f_\theta = 1$ in $[\theta, 1 - \theta]$.

Next, fix $\varepsilon \in (0, \frac{1}{2})$ and set $\theta = \frac{\varepsilon}{2}$. For large enough n, write

$$\frac{1-\theta}{\theta}\delta_n n = m_n + \Delta_n,$$

where m_n is a positive integer, and $\Delta_n \in [0, 1)$. We shall set, for some suitable polynomial V_n,

$$P_n(x) = R_{m_n,\theta}(x)V_n(x),$$

so that

$$x^{\delta_n n} P_n(x) = (x^{m_n\theta/(1-\theta)} R_{m_n,\theta}(x))(x^{\frac{\theta}{1-\theta}\Delta_n} V_n(x)).$$

Assume now that $\deg(V_n) \leq \log n$ and

$$0 \leq x^{\frac{\theta}{1-\theta}\Delta_n} V_n(x) \leq 1 \quad \text{in } [0, 1], \tag{5.9}$$

while uniformly in compact subsets of (0, 1],

$$\lim_{n\to\infty} x^{\frac{\theta}{1-\theta}\Delta_n} V_n(x) = 1. \tag{5.10}$$

Then P_n has degree $\leq m_n + \log n \leq \frac{1-\theta}{\theta}\delta_n n + \log n$, and by (5.8) and (5.9)

$$0 \leq x^{\delta_n n} P_n(x) \leq 1,$$

while uniformly in $[\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}]$, by (5.7), (5.10), and our choice of f_θ,

$$\lim_{n\to\infty} x^{\delta_n n} P_n(x) = 1.$$

(Recall that $f_\theta = 1$ in $[\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}] = [\theta, 1-\theta]$.) The conclusion of the lemma then follows by choosing a sufficiently slowly decreasing sequence of values for ε and corresponding $\{P_n\}$ (with "long" subsequences of $\{P_n\}$ arising from each of the values of ε).

It remains to prove the existence of $\{V_n\}$ satisfying (5.9) and (5.10). Let

$$A_\ell(t) = -\sum_{j=1}^{\ell} \frac{t^j}{j}$$

denote the ℓth partial sum of the Maclaurin series of $\log(1-t)$, so that

$$A_\ell(t) \geq \log(1-t), \quad t \in [0, 1), \tag{5.11}$$

and

$$\sup_{t\in[0,1]} |A_\ell(t)| \leq 1 + \log \ell. \tag{5.12}$$

Also, let B_ℓ denote the ℓth partial sum of the Maclaurin series of $\exp(-t)$, so that, by straightforward calculations,

$$\sup_{|t|\leq \ell/10} |B_\ell(t)\exp(t) - 1| = o(1) \quad \text{as } \ell \to \infty. \tag{5.13}$$

We let $\ell = [\sqrt{\log n}]$ (where $[x]$ denotes the greatest integer $\leq x$) and

$$V_n(x) = B_{[\sqrt{\log n}]}\left(-\frac{\theta}{1-\theta}\Delta_n A_{[\sqrt{\log n}]}(1-x)\right),$$

so that, in view of (5.11)–(5.13), uniformly for $x \in [0, 1]$,

$$\begin{aligned} 0 &\leq x^{\frac{\theta}{1-\theta}\Delta_n} V_n(x) \\ &= (1+o(1))\exp\left(\frac{\theta}{1-\theta}\Delta_n\{\log(1-(1-x)) - A_{[\sqrt{\log n}]}(1-x)\}\right) \\ &\leq 1 + o(1). \end{aligned}$$

Moreover, the desired uniform convergence in compact subsets of $(0, 1]$ in (5.10) follows from the convergence of the partial sums $\{A_\ell\}$ and $\{B_\ell\}$. The factor $1 + o(1)$ may be replaced by 1 by dividing by $1 + \eta_n$ for some slowly decreasing sequence $\{\eta_n\}$ with limit 0. □

Next, we handle the factors $e^{\tau_n \varphi_n}$.

LEMMA 5.2. *Let* α, $\{\tau_n\}$, *and* $\{\varphi_n\}$ *be as in Theorem* 1.2. *Then there exist polynomials* $\{R_n\}$ *with*

$$\deg R_n = o(n), \quad n \to \infty; \tag{5.14}$$

$$0 \leq R_n(x)e^{\tau_n\varphi_n(x)} \leq 1, \quad x \in \Sigma; \tag{5.15}$$

$$\lim_{n\to\infty} R_n(x)e^{\tau_n\varphi_n(x)} = 1 \quad \textit{uniformly in } \Sigma. \tag{5.16}$$

Proof. For the purposes of this lemma, we can assume that Σ is a single closed interval since we can extend the domain of definition of the functions $\{\varphi_n\}$ without increasing the sup norms of φ_n and their derivatives or Lipschitz norms. Let

$$\ell_n = [n^{\frac{1}{1+\alpha}}], \quad n \geq 1, \tag{5.17}$$

so that by (1.18)

$$\lim_{n\to\infty} \tau_n/\ell_n^\alpha = 0. \tag{5.18}$$

By Jackson's theorem [7, Theorem 6.2, p. 219] and our hypothesis of uniform smoothness of order α we can find a polynomial A_n of degree $\leq \ell_n$ such that

$$\|\varphi_n - A_n\|_{L_\infty(\Sigma)} \leq C/\ell_n^\alpha.$$

Here C is independent of n. Then

$$\|\tau_n\varphi_n - \tau_n A_n\|_{L_\infty(\Sigma)} \leq C\tau_n/\ell_n^\alpha \to 0, \quad n \to \infty. \tag{5.19}$$

By adding a sequence of positive numbers with limit 0 we may assume that

$$\tau_n\varphi_n \leq \tau_n A_n \quad \text{in } \Sigma. \tag{5.20}$$

Next, let $B_n(t)$ denote the nth partial sum of the Maclaurin series of e^t. We use (5.13) and choose

$$R_n(x) = B_{10[\tau_n]}(-\tau_n A_n(x)).$$

Note that this has degree at most $10[\tau_n]\ell_n = o(\ell_n^{1+\alpha}) = o(n)$ by (5.17) and (5.18). Also,

$$\|\tau_n A_n\|_{L_\infty(\Sigma)} \leq \tau_n(1 + O(1/\ell_n^\alpha)) = \tau_n + o(1),$$

so (5.13), followed by (5.20), shows that, uniformly for $x \in \Sigma$,

$$R_n(x)e^{\tau_n\varphi_n(x)} = \exp(\tau_n(\varphi_n(x) - A_n(x)))(1 + o(1)) \leq 1 + o(1),$$

and by (5.19) we also have, uniformly for $x \in \Sigma$,

$$R_n(x)e^{\tau_n\varphi_n(x)} = \exp(\tau_n O(1/\ell_n^\alpha)) = 1 + o(1).$$

Now multiply R_n by a sequence that is $1 + o(1)$ to ensure inequality (5.15). □

We need one more lemma on local growth of ν_n'.

LEMMA 5.3. *Let $R, \varepsilon > 0$. There exist $C > 1$ and, for $n \geq 1$, a set $\mathcal{E}_n$ of measure $< \varepsilon$ such that*

$$C^{-1} \leq \sup_{\xi\in\mathcal{S}_Q\setminus\mathcal{E}_n, |a|\leq R} \frac{\nu_n'(\xi)}{\nu_n'(\xi + \frac{a}{n\omega_Q'(\xi)})} \leq C. \tag{5.21}$$

Proof. In view of the form (1.15) of ν_n', it suffices to show the following three estimates:

$$C^{-1} \leq \sup_{\xi\in\mathcal{S}_Q\setminus\mathcal{E}_{n,1}, |a|\leq R} e^{\tau_n[\varphi_n(\xi) - \varphi_n(\xi + \frac{a}{n\omega_Q'(\xi)})]} \leq C, \tag{5.22}$$

$$C^{-1} \leq \sup_{\xi\in\mathcal{S}_Q\setminus\mathcal{E}_{n,2}, |a|\leq R} \left|\frac{\xi - b_n}{\xi - b_n + \frac{a}{n\omega_Q'(\xi)}}\right|^{n\beta_n} \leq C, \tag{5.23}$$

$$C^{-1} \leq \sup_{\xi \in \mathcal{S}_Q \backslash \mathcal{E}_{n,3}, |a| \leq R} e^{2n[Q(\xi) - Q(\xi + \frac{a}{n\omega_Q'(\xi)})]} \leq C, \tag{5.24}$$

where $\mathcal{E}_{n,j}$, $j = 1, 2, 3$, are sets of small measure, and where $\lim_{n\to\infty} \beta_n = 0$, while $\{b_n\}$ is a sequence of real numbers. The proof of (5.22) is easy and follows from the mean value theorem or from our assumed Lipschitz condition (if $\alpha \leq 1$). Note that since Σ consists of finitely many intervals, ξ and $\xi + \frac{a}{n\omega_Q'(\xi)}$ belong to the same interval for large enough n and for ξ outside a set $\mathcal{E}_{n,1}$ of small measure (outside which ω_Q' is bounded above and below by specified constants). Then, recalling (1.18), we have

$$\begin{aligned}
&\sup_{\xi \in \mathcal{S}_Q \backslash \mathcal{E}_{n,1}, |a| \leq R} \tau_n \left| \varphi_n(\xi) - \varphi_n\left(\xi + \frac{a}{n\omega_Q'(\xi)}\right)\right| \\
&\quad \leq \tau_n C \left(\frac{R}{n \inf_{\xi \in \mathcal{S}_Q \backslash \mathcal{E}_{n,1}} \omega_Q'(\xi)} \right)^{\min\{1,\alpha\}} \\
&\quad = o(n^{\frac{\alpha}{1+\alpha} - \min\{1,\alpha\}}) \to 0, \quad n \to \infty.
\end{aligned}$$

So we have (5.22).

Next, for $|a| \leq R$,

$$\begin{aligned}
&\left| \frac{\xi - b_n}{\xi - b + \frac{a}{n\omega_Q'(\xi)}} \right|^{n\beta_n} \\
&\quad = \exp\left(-n\beta_n \log\left| 1 + \frac{a}{n\omega_Q'(\xi)(\xi - b_n)} \right| \right) \\
&\quad = \exp\left(n\beta_n O\left(\frac{R}{n\omega_Q'(\xi)(\xi - b_n)} \right) \right) = \exp(o(1)),
\end{aligned}$$

uniformly for $|a| \leq R$ and $\xi \in \mathcal{S}_Q \backslash \mathcal{E}_{n,2}$, where again $\mathcal{E}_{n,2}$ is a set of small measure, chosen so that $\omega_Q'(\xi)$ is bounded above and below outside $\mathcal{E}_{n,2}$, whereas a small interval around b_n is excluded. Then (5.23) follows.

Finally, recall that Q' is assumed to be continuous except at finitely many points in Σ, whereas ω_Q' is positive and continuous except at finitely many points. Taking $\mathcal{E}_{n,3}$ consisting of small intervals centered on these points, we obtain a small set $\mathcal{E}_{n,3}$ such that

$$\begin{aligned}
&\sup_{\xi \in \mathcal{S}_Q \backslash \mathcal{E}_{n,3}, |a| \leq R} 2n \left| Q(\xi) - Q\left(\xi + \frac{a}{n\omega_Q'(\xi)}\right) \right| \\
&\quad \leq 2n \sup_{t \in \mathcal{S}_Q \backslash \mathcal{E}_{n,3}} |Q'(t)| \frac{R}{n\omega_Q'(\xi)} \leq C < \infty.
\end{aligned}$$

So we have (5.24). □

Proof of Theorem 1.2. We apply Theorem 1.1 with

$$\mu_n'(x) = h(x) e^{-2nQ(x)}$$

and

$$\nu_n'(x) = g_n(x)\mu_n'(x),$$

where

$$g_n(x) = e^{\tau_n h_n(x)} \left(\prod_{j=1}^{N} |x - \alpha_{nj}|^{\beta_{nj}} \right)^n.$$

By a result of Totik [26, Theorem 1.2, p. 326], uniformly for x in compact subsets of $\mathcal{S}_Q$ omitting discontinuities of ω_Q', we have

$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu_n, x, x)\mu_n'(x) = \omega_Q'(x), \tag{5.25}$$

so (1.9) is certainly true, and we can apply Lemma 2.2. Note also that by the aforementioned Theorem A of the authors [15, Theorem 1.1, p. 747],

$$\lim_{n\to\infty} \frac{\tilde{K}_n(\mu_n, \xi + \frac{a}{\tilde{K}_n(\mu_n,\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\mu_n,\xi,\xi)})}{\tilde{K}_n(\mu_n, \xi, \xi)} = \mathbb{S}(a-b),$$

uniformly for ξ in compact subsets of $\mathcal{S}_Q$ omitting zeros of Q' and zeros or discontinuities of ω_Q'. We can replace $\tilde{K}_n(\nu_n, \xi, \xi)$ in this limit by $n\omega_Q'(\xi)$ in view of Totik's asymptotics for Christoffel functions and the uniform convergence in a, b, giving, for the same range of ξ, a, b,

$$\lim_{n\to\infty} \frac{\tilde{K}_n(\mu_n, \xi + \frac{a}{n\omega_Q'(\xi)}, \xi + \frac{b}{n\omega_Q'(\xi)})}{n\omega_Q'(\xi)} = \mathbb{S}(a-b). \tag{5.26}$$

Now let us turn to the construction of the polynomials $\{S_n\}$ in Theorem 1.1. We apply Lemmas 5.1 and 5.2. In applying Lemma 5.1, we choose r sufficiently large and form a product of N terms of the form $P_n(x - b_{n,j})$, each with appropriately chosen δ_j. We multiply the polynomials from Lemmas 5.1 and 5.2 to obtain $\{S_n\}$ satisfying (1.11)–(1.13). In fact, Lemma 5.1 and 5.2 give much more than (1.11)–(1.13). Indeed, let $0 < \eta < \frac{1}{2N}$ and

$$\mathcal{U}_n(\eta) = \bigcup_{j=1}^{N} [b_{n,j} - \eta, b_{n,j} + \eta],$$

a set of measure $\leq 2N\eta$. Lemmas 5.1 and 5.2 give polynomials S_n of degree $o(n)$ such that, for each small enough η,

$$\sup_{\xi\in\Sigma\setminus\mathcal{U}_n(\eta)} |(g_n S_n)(\xi) - 1| \to 0, \quad n \to \infty.$$

Note that also, given $R > 0$ and $\eta > 0$,

$$\sup_{\xi\in\Sigma\setminus\mathcal{U}_n(\eta), |a|\leq R} \left| (g_n S_n)\left(\xi + \frac{a}{n}\right) - 1 \right| \to 0, \quad n \to \infty. \tag{5.27}$$

Next, let us set

$$\xi_{n,a} = \xi + \frac{a}{n\omega_Q'(\xi)}$$

with a similar notation for $\xi_{n,b}$. Also, let us replace a, b in (4.7) by $a/\omega'_Q(\xi)$ and $b/\omega'_Q(\xi)$. We add to the set $\mathcal{E}_n$ in Theorem 4.3 small intervals centered on the zeros and discontinuities of ω'_Q. We obtain for $\xi \in \mathcal{S}_Q \backslash \mathcal{E}_n$ and all complex a, b,

$$\begin{aligned}\frac{1}{n\omega'_Q(\xi)}&\left|\tilde{K}_n(\nu_n, \xi_{n,a}, \xi_{n,b}) - [(g_n S_n^2)(\xi_{n,a})(g_n S_n^2)(\xi_{n,b})]^{1/2} \tilde{K}_n(\mu_n, \xi_{n,a}, \xi_{n,b})\right| \\ &\times \frac{\nu'_n(\xi)}{[\nu'_n(\xi_{n,a})\nu'_n(\xi_{n,b})]^{1/2}} \\ &\leq \frac{8}{\sqrt{\varepsilon}\omega'_Q(\xi)} \Gamma_n^{1/4} e^{C_0(|a|+|b|)/\varepsilon}.\end{aligned}$$

Applying (5.26) and (5.27) and Lemma 5.3, we obtain for some suitable set $\mathcal{H}_n$ of small measure and for each $R > 0$,

$$\begin{aligned}&\sup_{\xi \in \mathcal{S}_Q \backslash \mathcal{H}_n, |a|,|b| \leq R} \left| \frac{1}{n\omega'_Q(\xi)} \tilde{K}_n(\nu_n, \xi_{n,a}, \xi_{n,b}) - \mathbb{S}(a-b) \right| \\ &\quad \leq \frac{C}{\sqrt{\varepsilon}} \sup_{\xi \in \Sigma \backslash \mathcal{H}_n, |a|,|b| \leq R} \Gamma_n^{1/4} \frac{e^{C_0 R/\varepsilon}}{\omega'_Q(\xi)} \to 0, \quad n \to \infty.\end{aligned}$$

□

References

[1] G. Akemann, J. Baik, and P. Di Francesco, eds., *The Oxford handbook of random matrix theory,* Oxford University Press, Oxford, 2011.

[2] J. Baik, T. Kriecherbauer, K. T.-R. McLaughlin, and P. D. Miller, *Uniform asymptotics for polynomials orthogonal with respect to a general class of discrete weights and universality results for associated ensembles,* Princeton Annals of Mathematics Studies, 2006.

[3] C. Bennett and R. Sharpley, *Interpolation of functions,* Academic Press, Orlando, 1988.

[4] P. Bleher and A. Its, *Random matrix models and their applications,* Cambridge University Press, Cambridge, 2001.

[5] P. Deift, *Orthogonal polynomials and random matrices: a Riemann–Hilbert approach,* Courant Institute Lecture Notes, 3, New York University Pres, New York, 1999.

[6] P. Deift and D. Gioev, *Random matrix theory: invariant ensembles and universality,* Courant Institute Lecture Notes, 18, New York University Pres, New York, 2009.

[7] R. DeVore and G. G. Lorentz, *Constructive approximation,* Springer, Berlin, 1993.

[8] P. J. Forrester, *Log-gases and random matrices,* Princeton University Press, Princeton, 2010.

[9] A. Foulquie Moreno, A. Martinez-Finkelshtein, and V. Sousa, *Asymptotics of orthogonal polynomials for a weight with a jump on* $[-1, 1]$, Constr. Approx. 33 (2011), 219–263.

[10] G. Freud, *Orthogonal polynomials,* Pergamon Press/Akademiai Kiado, Budapest, 1971.

[11] J. B. Garnett, *Bounded analytic functions,* Academic Press, Orlando, 1981.

[12] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series and products,* Academic Press, San Diego, 1979.

[13] A. B. Kuijlaars and M. Vanlessen, *Universality for eigenvalue correlations from the modified Jacobi unitary ensemble,* Int. Math. Res. Not. 30 (2002), 1575–1600.

[14] E. Levin and D. S. Lubinsky, *Orthogonal polynomials for exponential weights,* Springer, New York, 2001.

[15] ———, *Universality limits in the bulk for varying measures,* Adv. Math. 219 (2008), 743–779.

[16] ———, *Universality limits for exponential weights,* Constr. Approx. 29 (2009), 247–275.

[17] D. S. Lubinsky, *A new approach to universality limits involving orthogonal polynomials,* Ann. Math. 170 (2009), 915–939.

[18] ———, *Bulk universality holds in measure for compactly supported measures,* J. Anal. Math. 116 (2012), 219–253.

[19] L. Pastur and M. Shcherbina, *Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles,* J. Stat. Phys. 86 (1997), 109–147.

[20] T. Ransford, *Potential theory in the complex plane,* Cambridge University Press, Cambridge, 1995.

[21] W. Rudin, *Real and complex analysis,* Wiley, McGraw Hill, Singapore, 1987.

[22] E. B. Saff and V. Totik, *Logarithmic potentials with external fields,* Springer, New York, 1997.

[23] B. Simon, *Orthogonal polynomials on the unit circle, parts 1 and 2,* American Mathematical Society, Providence, 2005.

[24] ———, *Two extensions of Lubinsky's universality theorem,* J. Anal. Math. 105 (2008), 345–362.

[25] V. Totik, *Weighted approximation with varying weight,* Lecture Notes in Math., 1569, Springer, Berlin, 1994.

[26] ———, *Asymptotics for Christoffel functions with varying weights,* Adv. in Appl. Math. 25 (2000), 322–351.

[27] ———, *Universality and fine zero spacing on general sets,* Ark. Mat. 47 (2009), 361–391.

[28] ———, *Universality under Szegő's condition,* Canad. Math. Bull. 59 (2016), 211–224.

E. Levin
Mathematics Department
The Open University of Israel
P.O. Box 808
Raanana 43107
Israel

elile@openu.ac.il

D. S. Lubinsky
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160
USA

lubinsky@math.gatech.edu