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Topological Obstructions for Rational Cuspidal Curves in Hirzebruch Surfaces

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Abstract. We study rational cuspidal curves in Hirzebruch surfaces. We provide two obstructions for the existence of rational cuspidal curves in Hirzebruch surfaces with prescribed types of singular points. The first result comes from Heegaard Floer theory and is a generalization of a result by Livingston and the first author. The second criterion is obtained by comparing the spectrum of a suitably defined link at infinity of a curve with spectra of its singular points.

1. Introduction

Let C be a reduced and irreducible algebraic curve in a smooth complex surface X. A singular point p on C is called a *cusp* if it is locally irreducible. The curve is called *cuspidal* if all its singularities are cusps.

Cuspidal curves in the projective plane have been investigated in classical algebraic geometry and have been subject of intense study the past three decades. The renewed interest in these curves in the 1980s came after results by Lin and Zaidenberg [19] and Matsuoka and Sakai [21]. Moreover, two questions about plane cuspidal curves were asked by Sakai in 1994 (see [15]), and ever since, several attempts have been made to describe and classify rational cuspidal curves in the projective plane (see [6; 10; 11; 12; 13; 18; 20; 32; 33; 34; 36; 37]).

In [23] the second author turned the attention to cuspidal curves in Hirzebruch surfaces and found that many of the results for plane cuspidal curves could be extended to curves in Hirzebruch surfaces (see [24; 25]). Indeed, this does not come as a surprise since the Hirzebruch surfaces are linked to each other and the projective plane by birational transformations and since such transformations clearly transform rational curves to rational curves. However, the picture is somewhat more complicated; in general, a cuspidal curve might acquire some multibranched singular points under a birational transformation. Therefore, there is no direct correspondence between rational cuspidal curves in $\mathbb{C}P^2$ and rational cuspidal curves in Hirzebruch surfaces.

In the present article we continue this work and extend two results from the plane case to the case of cuspidal curves in Hirzebruch surfaces. The first result, given in Theorem 1.1, is a consequence of Heegaard Floer theory, and it is a generalization of the result by Livingston and the first author [3]. We refer to Section 2 for explaining notation used in the theorem and especially to Section 2.2 for the definition of the function R.

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THEOREM 1.1. *Let C be a rational cuspidal curve of type (a, b) in a Hirzebruch surface X_e with $e \geq 0$. Let $g = (a-1)(b-1) + \frac{1}{2}b(b-1)e$. Then for every $m \in [-g, g]$ and for any presentation $m + g = s_1 b + s_2(a + be) + 1$, where s_1 and s_2 are integers, we have*

$$R(m+g) \geq P(s_1, s_2), \tag{1.2}$$

where R is the counting function for the semigroups of the singular points of C, and

$$P(s_1, s_2) = (s_1+1)(s_2+1) + \frac{1}{2}s_2(s_2+1)e.$$

Notice that if $m + g - 1$ is not divisible by $\gcd(a, b)$, then Theorem 1.1 does not provide a direct restriction on the value of $R(m+g)$.

In Section 3.5 we show an alternative, algebraic proof of inequality (1.2), following the ideas of [11]. As a matter of fact, Theorem 3.16 gives a lower bound for a function R for any rational cuspidal curve in any algebraic surface. It is natural to conjecture that the d-invariants estimate used in the proof of Theorem 1.1 will give the same bound.

We also remark, that if $m + g \notin [-g, g]$, then the value of $R(m+g)$ is fixed: it is 0 if $m + g < 0$ and m if $m + g > 2g$.

Our second result is about the semicontinuity property of the spectrum. It puts restrictions on the spectrum of singular points of a rational cuspidal curve in X_e.

THEOREM 1.3. *Let C be a rational cuspidal curve of type (a, b) in X_e. Let $Sp_1, \dots, Sp_n$ be the spectra of its singular points. Let $Sp^{\infty}_{a,b}$ be the spectrum at infinity given in Table 1. Then for every $x \in (0, 1)$ such that $x \notin Sp^{\infty}_{a,b}$, we have*

$$\begin{aligned} \sum_{j=1}^{n} \#Sp_j \cap (x, x+1) &\leq \#Sp^{\infty}_{a,b} \cap (x, x+1), \\ \sum_{j=1}^{n} \#Sp_j \setminus (x, x+1) &\leq \#Sp^{\infty}_{a,b} \setminus (x, x+1). \end{aligned} \tag{1.4}$$

These two results, Theorem 1.1 and Theorem 1.3, give two restrictions for possible configurations of singular points on a cuspidal curve. As we show in Section 6, the two results differ in nature. Indeed, for unicuspidal curves, the semigroup distribution property obstructs cases where the multiplicities are large (close to b), whereas the spectrum semicontinuity is effective in obstructing curves with low multiplicities.

1.1. Structure

In this article we first set up the notation in Section 2. In Section 3 we use Heegaard Floer theory to establish Theorem 1.1. In Section 4 we study the link at infinity of curves in Hirzebruch surfaces. In Section 5 we show how the spectrum of the link at infinity can be computed, and the result is as shown in Table 1. Our

Table 1 Spectrum at infinity of a type (a, b) curve. Here $w = a + be$, and p, q are assumed to be integers, $1 \le p \le w - 1$, $1 \le q \le b - 1$. The number x is in the interval $[0, 2]$

	The value of x	Multiplicity of x in $Sp^{\infty}_{a,b}$
1.	$x = 1$	$a + b - 1$
2.	$x = \frac{p}{w}$ and $x = \frac{q}{b}$ for some p and q	$\lfloor \frac{pb}{w} \rfloor + \lfloor \frac{qa}{b} \rfloor - 1$
3.	$x = 1 + \frac{p}{w}$ and $x = 1 + \frac{q}{b}$ for some p and q	$a + b - 1 - \lfloor \frac{pb}{w} \rfloor - \lfloor \frac{qa}{b} \rfloor$
4.	$x = \frac{p}{w}$ for some p but $x \ne \frac{q}{b}$ for any q	$\lfloor \frac{pb}{w} \rfloor$
5.	$x = 1 + \frac{p}{w}$ for some p but $x \ne 1 + \frac{q}{b}$ for any q	$b - 1 - \lfloor \frac{pb}{w} \rfloor$
6.	$x = \frac{q}{b}$ for some q but $x \ne \frac{p}{w}$ for any p	$\lfloor \frac{qa}{b} \rfloor$
7.	$x = 1 + \frac{q}{b}$ for some q but $x \ne 1 + \frac{p}{w}$ for any p	$a - 1 - \lfloor \frac{qa}{b} \rfloor$
8.	For all other x	0

main result in this section, giving new restrictions for cuspidal curves, is Theorem 1.3 that compares the spectrum of singular points of the curve to the spectrum of the link at infinity. Finally, in Section 6 we give some examples of possible applications.

2. Generalities

2.1. Hirzebruch Surfaces

Let X_e, $e \ge 0$, be a Hirzebruch surface, regarded as a projectivization of a rank 2 bundle $\mathcal{O} \oplus \mathcal{O}(-e)$ over $\mathbb{C}P^1$. Let L be a fiber, and M_0 the special section, so that the intersections and self-intersections are as follows:

$$L^2 = 0, \qquad M_0^2 = -e, \qquad L \cdot M_0 = 1.$$

We define $M = eL + M_0$, so $M^2 = e$. Then L and M generate $H_2(X_e; \mathbb{Z})$, and the intersection matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & e \end{pmatrix}.$$

Definition 2.1. For integers $a \ge 0$ and $b > 0$ ($b \ge 0$ when $e = 0$), a curve $C \subset X_e$ is *of type* (a, b) if it is irreducible and its homology class is $aL + bM \in H_2(X_e)$.

Remark 2.2. Unless stated otherwise, we shall suppose that C is rational and cuspidal.

We denote

$$d = C^2 = (aL + bM)^2 = 2ab + b^2 e.$$

Furthermore, let

$$c = \gcd(a, b), \qquad a' = a/c, \qquad b' = b/c.$$

The arithmetic genus of C is given by the formula (see [23, Cor. 3.1.4]).

$$g = (a-1)(b-1) + \frac{1}{2}b(b-1)e. \tag{2.3}$$

2.2. *Singular Points and Semigroups*

(We refer to [38, Chap. 4] for more details about semigroups of singular points.)

Let z be a cuspidal singular point of C. We can associate with z a semigroup S_z of nonnegative integers. For a quasi-homogeneous singularity given by $x^p - y^q = 0$ with p, q coprime, the semigroup is generated by p and q. We always assume that $0 \in S_z$.

Given any semigroup in $\mathbb{Z}_{\geq 0}$, we define the function $R_S\colon \mathbb{Z} \to \mathbb{Z}$ by

$$R_S(t) = \begin{cases} \#S \cap [0, t), & t > 0, \\ 0, & t \leq 0. \end{cases} \tag{2.4}$$

We have the following fact.

LEMMA 2.5. *If S is a semigroup of a singular point z with Milnor number μ (the genus of the link of the singular point is then $\mu/2$), then for all $m \geq 0$, we have $R_S(m + \mu) = m + \mu/2$.*

Proof. The complement $\mathbb{Z}_{\geq 0} \setminus S$ has precisely $\mu/2$ elements, and the largest is $\mu - 1$; see [38, Chap. 4]. □

Given any two functions $R_1, R_2\colon \mathbb{Z} \to \mathbb{Z}$ bounded from below, we define their *infimum convolution* to be

$$R_1 \diamond R_2(t) = \min_{k \in \mathbb{Z}} R_1(k) + R_2(t-k).$$

The infimum convolution is clearly commutative and associative.

DEFINITION 2.6. Let $C \subset X_e$ be a cuspidal curve (not necessarily rational). Then the R-function of C is defined as

$$R = R_{S_1} \diamond R_{S_2} \diamond \cdots \diamond R_{S_n},$$

where $S_1, \ldots, S_n$ are semigroups corresponding to singular points of C.

We have the following corollary to Lemma 2.5.

COROLLARY 2.7. *If g is the sum of genera of the links of singular points of a cuspidal curve C, then $R(2g + m) = g + m$ for any $m \geq 0$.*

3. A Criterion from the Complement of C and Related Invariants

Suppose that C is a rational cuspidal curve in X_e. We consider N, a tubular neighborhood of C in X_e. Let Y be the boundary of N with *reversed* orientation, and let $W = X_e \setminus N$. We have

$$\partial W = Y.$$

The main goal of this section is to give a proof of Theorem 1.1. The key result in our proof is the theorem by Ozsváth and Szabó; see [30, Thm. 9.6]. It gives a lower bound on the d-invariant of a three-manifold bounding a smooth negative-definite (that is, having negative-definite intersection form on second homologies) four-manifold. The d-invariant, also known as the correction term, is a rational number associated to any closed three-dimensional manifold equipped with a spin^c structure whose first Chern class is torsion. The d-invariant is defined in [30] using Heegaard Floer theory.

In our context, that is, when Y is a rational homology sphere and W is a negative-definite smooth four-manifold, [30, Thm. 9.6] can be formulated as follows.

Theorem 3.1. *For any* spin^c *structure* $\mathfrak{s}$ *on* Y *extending to a* spin^c *structure* $\mathfrak{t}$ *on* W, *we have*

$$d(Y, \mathfrak{s}) \geq \frac{1}{4}(c_1^2(\mathfrak{t}) - 3\sigma(W) - 2\chi(W)). \tag{3.2}$$

In order to use this result, we need to decrypt the information encoded in inequality (3.2). We will do this in the following steps, following the pattern used in [1; 2; 3].

- Describe Y as a surgery on a knot in S^3 and compute its d-invariants.
- Study homological properties of W, in particular, show that the intersection form is negative definite.
- Check which spin^c structures on Y extend over W.
- Compute $c_1^2(\mathfrak{t})$ for such structures.
- Compute $d(Y, \mathfrak{s})$.

3.1. The Manifold Y and its d-Invariants

We shall need the following characterization of Y.

Proposition 3.3. *Let* $K_1, \dots, K_n$ *be the links of the singularities on the curve* C, *and* $K = K_1 \# \cdots \# K_n$. *Then* $-Y$ *is a surgery on* S^3 *along* K *with surgery coefficient* $C^2 = 2ab + b^2 e = d$.

Proof. The proof is the same as in the case where X_e is the projective plane; see [3]. □

As a corollary, we can write down homologies of Y.

Corollary 3.4. *We have* $H_1(Y) = \mathbb{Z}/d\mathbb{Z}$ *and* $H_2(Y) = 0$.

LEMMA 3.5. *The genus of the knot K is equal to g (defined in (2.3)).*

Proof. This follows immediately from the genus formula (2.3) and the fact that C is rational. □

REMARK 3.6. We notice that $d-2g=2a+2b+be-2>0$, unless $a=1$, $b=0$ or $b=1$, $a=e=0$. The last two cases imply that C is a line ($a=1$, $b=0$ imply that $C=L$ is the fiber, and $b=1$, $a=e=0$ imply that $C=M$ is the section in $\mathbb{C}P^1\times\mathbb{C}P^1$). In the future we will ignore these trivial cases.

Since Y is presented as an integer surgery on a knot with slope d, it has an enumeration of spin^c structures $\mathfrak{s}_m$, where $m\in[-d/2,d/2)$. Details are presented in Section 3.3. Given that result, we have the following:

PROPOSITION 3.7 (see [3, Thm. 5.1]). *The d-invariant $d(Y,\mathfrak{s}_m)$ is equal to*

$$-d(Y,\mathfrak{s}_m)=\frac{(d-2m)^2-d}{4d}-2(R(m+g)-m),$$

where R is the R-function from Definition 2.6.

Proof. The proof of Proposition 3.7 consists of two steps. First, the knot K (see Proposition 3.3) is a connected sum of algebraic knots (or, more generally, L-space knots). Therefore, we can compute the Heegaard Floer chain complex $\mathrm{CFK}^\infty(K)$ using the Alexander polynomials of $K_1,\dots,K_n$. The Alexander polynomial of an algebraic knot is tightly related to the semigroup of the singular point (see [38]), so the R-function enters the formula.

The second part is expressing Heegaard Floer homology of $+d$ surgery on K (that is, of $-Y$) in terms of $\mathrm{CFK}^\infty(K)$. This part uses the fact that $d>2g$; see Remark 3.6. Lastly, we note that by [30] reversing the orientation amounts to reversing the sign of the d-invariant. □

3.2. Homological Properties of W

We begin with the following result.

LEMMA 3.8. *We have $H_2(W)=\mathbb{Z}$ and $H_1(W)=\mathbb{Z}/c\mathbb{Z}$, where we recall that $c=\gcd(a,b)$.*

Proof. Set $X=X_e$. Consider the long exact sequence of the pair (X,W). By excision, $H_*(X,W)\cong H_*(N,Y)$. The latter group is $\mathbb{Z}$ in degrees 2, 4 and 0 otherwise by Thom isomorphism. Hence, $H_3(W)=\mathbb{Z}$, and the long exact sequence of the pair gives the two following exact sequences:

$$0\longrightarrow H_4(X)\cong\mathbb{Z}\longrightarrow H_4(X,W)\cong\mathbb{Z}\longrightarrow H_3(W)\longrightarrow 0,$$
$$0\longrightarrow H_2(W)\longrightarrow H_2(X)\longrightarrow H_2(X,W)\cong\mathbb{Z}\longrightarrow H_1(W)\longrightarrow 0.$$

It follows that $b_3(W)=0$. To study $H_2(W)$, we observe that the map from $H_2(X)\to H_2(X,W)\cong\mathbb{Z}$ can be described explicitly. Namely, for $z\in H_2(X)$ represented by a (real) surface Z intersecting C transversally, z is mapped to $Z\cdot C$

times the generator. In particular, L is mapped to $L \cdot C = b$, and M is mapped to $M \cdot C = a + be$. The image of $H_2(X)$ in $H_2(X, W)$ is therefore generated by a, b. Hence,

$$H_2(W) = \mathbb{Z} \quad \text{and} \quad H_1(W) = \mathbb{Z}/c\mathbb{Z}. \qquad \square$$

Our aim is to compute the intersection form on W. Notice that the class $H = (a' + b'e)L - b'M$ intersects trivially with C, so it belongs to the kernel of $H_2(X) \to H_2(X, W)$. In particular, it descends to a class in $H_2(W)$ (by a slight abuse of notation we will still denote it by H) with self-intersection equal to

$$(-2a'b' - b'b'e) = -\frac{d}{c^2}.$$

LEMMA 3.9. *The class H generates $H_2(W)$.*

Proof. The intersection form on W, that is, the map $H_2(W) \to \mathrm{Hom}(H_2(W), \mathbb{Z})$, is given by the sequence of maps

$$H_2(W) \to H_2(W, Y) \xrightarrow{\cong} H^2(W) \xleftarrow{\cong} \mathrm{Hom}(H_2(W), \mathbb{Z}).$$

Here the first map is the exact sequence of the pair (W, Y), the second map is the Poincaré duality isomorphism, and the third follows from the universal coefficient theorem. In particular, $\mathrm{coker}(H_2(W) \to \mathrm{Hom}(H_2(W), \mathbb{Z}))$ is isomorphic to $\mathrm{coker}(H_2(W) \to H_2(W, Y))$. With $Y = \partial W$, we have

$$0 \to \mathrm{coker}(H_2(W) \to H_2(W, \partial W)) \to H_1(Y) \to H_1(W) \to 0.$$

All the groups in the short exact sequence are finite, so taking the cardinalities, we obtain

$$|H_1(Y)| = |H_1(W)| \cdot |\mathrm{coker}(H_2(W) \to H_2(W, \partial W))|.$$

It follows that

$$|\mathrm{coker}(H_2(W) \to \mathrm{Hom}(H_2(W), \mathbb{Z}))| = d/c^2.$$

Since $H_2(W)$ has rank one, the cardinality of the cokernel is precisely the absolute value of the self-intersection of a generator. If H were a nontrivial multiple of a generator, the cokernel of the intersection form would be smaller than d/c^2. $\square$

We notice that by the universal coefficient theorem, $H^2(W) \cong \mathbb{Z} \oplus \mathbb{Z}/c\mathbb{Z}$; in particular, $H^2(W; \mathbb{Q}) \cong \mathbb{Q}$. The classes L and M can be regarded as classes in $H^2(W)$ under the composition $H_2(X) \to H^2(X) \to H^2(W)$, where the first map is the Poincaré duality, and the second is the restriction homomorphism. We have $L \cdot H = -b'$ and $M \cdot H = a'$, so in $H^2(W; \mathbb{Q})$, L is $-b'$ times the generator, and M is a' times the generator. Since the intersection form on $H^2(W)$ is the inverse of the intersection form on the nontorsion part of $H_2(W)$, the classes in $H^2(W; \mathbb{Z})$ represented by L and M have the following intersections:

$$L^2 = -\frac{b^2}{d}, \qquad M^2 = -\frac{a^2}{d}, \qquad L \cdot M = -\frac{ab}{d}. \tag{3.10}$$

3.3. Spin^c *Structures on Y and W*

A spin^c structure on a manifold U is a choice of a complex line bundle L over U and of a spin structure on the bundle $TN \otimes L^{-1}$. When speaking of restricting or prolonging spin^c structures from one submanifold to another, the intuition that the spin^c structures are "like line bundles" is very convenient.

If U is a three-dimensional rational homology sphere, then spin^c structures are in a bijective correspondence with elements in $H_1(U,\mathbb{Z}) \cong H^2(U,\mathbb{Z})$. If U is represented as a integral surgery along a knot in S^3, then we have a simple description of the spin^c structures on U; see [31, Sect. 4].

PROPOSITION 3.11. *Let K be a knot in S^1, and $d > 0$ be an integer. Let Z be a 4-manifold obtained by attaching a two-handle to a ball B^4 along K with framing d (in this way, $\partial Z = S^3_d(K) =: U$). Then*

- *Any* spin^c *structure on U extends to Z.*
- *For any $m \in [-d/2, d/2)$, there is a* unique spin^c *structure on Z, which extends to a* spin^c *structure $\mathfrak{t}_m$ over Z such that*

$$\langle c_1(\mathfrak{t}_m), \Sigma\rangle + 2m = d,$$

where Σ is a generator of $H_2(Z)$ consisting of the core of the two handle capped with a Seifert surface for K.

This proposition allows us to characterize the spin^c structures for $-Y$ that extend over W because of the description $-Y = S^3_d(K)$ in Proposition 3.3.

DEFINITION 3.12. For any $m \in [-d/2, d/2)$, the spin^c structure $\mathfrak{s}_m$ on $-Y$ is the spin^c structure that extends to $\mathfrak{t}_m$ over N.

Notice that a spin^c structure on $-Y$ induces a spin^c structure on Y. We ask, which spin^c structures on Y extend over W and what is the first Chern class of such an extended spin^c structure? To answer this question, we note that if a spin^c structure $\mathfrak{s}_m$ on Y extends over W, then it can be glued with $\mathfrak{t}_m$ on N to form a spin^c structure on the whole Hirzebruch surface X_e. Conversely, a spin^c structure on X_e can be restricted to W. To study which spin^c structures on Y extend over W, it is enough to study restrictions of spin^c structures on X_e to W.

By [14, Sect. 1.4.2], the first Chern class induces an isomorphism between the set of spin^c structures $\mathrm{Spin}^c(X_e)$ and the set of characteristic elements in $H^2(X_e;\mathbb{Z})$. We recall that $x \in H^2(X_e;\mathbb{Z})$ is *characteristic* if for any $w \in H_2(X_e;\mathbb{Z})$, we have $\langle x, w\rangle = w \cdot w \bmod 2$. It is clear that characteristic elements on X_e are classes $r_1L + r_2M$ such that r_2 is even and r_1 is congruent to e modulo 2.

So let us consider a class $r_1L + r_2M$ with r_1, r_2 as before. The corresponding spin^c structure on X_e restricts to the spin^c structure $\mathfrak{t}$ on N with

$$\langle c_1(\mathfrak{t}), C\rangle = (r_1L + r_2M) \cdot C = r_1b + r_2a + r_2be.$$

Let us define

$$k = r_1 b + r_2 a + r_2 be \quad \text{and} \quad m = (d-k)/2.$$

If $m \in [-d/2, d/2)$, then it follows that $\mathfrak{t}$ restricts to the class $\mathfrak{s}_m$ on W. Using (3.10), we can compute the square of the first Chern class of $\mathfrak{t}_m$ on W. Let us summarize this discussion in the following result.

LEMMA 3.13. *Let $m \in [-d/2, d/2)$. If $m = (d-k)/2$ is such that k can be presented as $r_1 b + r_2 a + r_2 be$ for r_2 even and $r_1 \equiv e$* mod 2, *then the* spin^c *structure $\mathfrak{s}_m$ on Y extends to a* spin^c *structure $\mathfrak{t}_k$ on W with*

$$c_1^2(\mathfrak{t}_k) = -\frac{(r_1 b - r_2 a)^2}{d}.$$

Often, k can be presented as $r_1 b + r_2 a + r_2 be$ with r_2 even and $r_1 \equiv e$ mod 2 in more than one way. This means that the extension of $\mathfrak{s}_m$ is not unique, and we eventually get more than one restriction coming from Theorem 3.1.

3.4. *Proof of Theorem 1.1*

Let us now choose two integers s_1 and s_2. Set

$$r_1 = 2a + e - 2 - 2s_1 \quad \text{and} \quad r_2 = 2b - 2 - 2s_2.$$

As before, we write $k = r_1 b + r_2(a + be)$ and $m = (d-k)/2$. We have

$$m + g = s_2 b + s_1(a + be) + 1.$$

Therefore, Lemma 3.13 and Theorem 3.1, together with computations of d-invariants in Proposition 3.7, give us

$$\frac{k^2 - (r_1 b - r_2 a)^2}{8d} \leq R(m+g) - m. \tag{3.14}$$

After straightforward but tedious computations, we obtain

$$R(s_1 b + s_2(a+be)+1) \geq (s_1+1)(s_2+1) + \frac{1}{2} s_2(s_2+1)e =: P(s_1, s_2). \tag{3.15}$$

To conclude the proof, we need to find the range for which (3.15) holds. We have $m \in [-d/2, d/2)$; hence, $m + g \in [g - d/2, g + d/2)$. By Remark 3.6 we have $d > 2g$, so $g - d/2 < 0$ and $g + d/2 > 2g$. Thus, $[0, 2g] \subset [g - d/2, g + d/2]$. The values of $R(k)$ for k outside of $[0, 2g]$ are well understood; see Corollary 2.7.

3.5. *Alternative Proof of Theorem 1.1 via Bézout-like Argument*

The following result is a direct generalization of [11, Prop. 2], and it provides an algebraic proof of Theorem 1.1.

THEOREM 3.16. *Let X be a projective algebraic surface, and C a rational cuspidal curve on it with singular points $z_1, \dots, z_n$. Let $\mathcal{L}$ be a line bundle on X such* *that* $t := \langle c_1(\mathcal{L}), C\rangle > 0$, *the space of global sections* $\Gamma(X, \mathcal{L})$ *has positive dimension, and* $\mathcal{L}$ *have no section vanishing entirely on* C. *Then for the* R*-function as in Definition 2.6, we have*

$$R(t+1) \geq \dim \Gamma(X, \mathcal{L}).$$

Proof. Denote $U = \Gamma(X, \mathcal{L})$. For a section $u \in U$, we denote by $m_j(u)$ the intersection multiplicity of $u^{-1}(0)$ and C at z_j. If u does not vanish at z_j, then we set $m_j(u) = 0$.

Given nonnegative integers $p_1, \dots, p_n$, set

$$V = V(p_1, \dots, p_n) := \{u \in U : \forall_j m_j(u) \geq p_j\}.$$

Then V is a linear subspace of U.

LEMMA 3.17. *We have*

$$\operatorname{codim} V \leq \sum \#S_i \cap [0, p_j),$$

where S_j *is the semigroup associated with the singular point* z_j.

Given this result, we finish the proof of Theorem 3.16 by contradiction. Namely, fix $p > t$ and suppose that $p_1, \dots, p_n$ satisfy $p_1 + \dots + p_n = p$, but $\sum \#S_j \cap [0, p_j) < \dim U$. By Lemma 3.17 the space $V(p_1, \dots, p_n)$ has positive dimension; hence, there exists a section u of $\mathcal{L}$ such that $D := u^{-1}(0)$ intersects C at points z_j with multiplicity at least p_j. Since $D \cdot C = \langle c_1(\mathcal{L}), C\rangle = t$ by assumption, we must have $C \subset D$. But this contradicts the assumption that no section of $\mathcal{L}$ vanishes entirely on C. This implies that $\sum \#S_j \cap [0, p_j) \geq \dim U$. With notation as in Section 2.2, we write this as

$$\sum R_j(p_j) \geq \dim U.$$

Taking the infimum over all possible numbers p_j satisfying $\sum p_j = p$, we infer that $R(p) \geq \dim U$. This is exactly the statement of Theorem 3.16. □

Proof of Lemma 3.17. Notice that by definition of the semigroup we have $m_j(u) \in S_j$ for any j. It follows that it is enough to show the lemma only for $p_1 \in S_1, \dots, p_n \in S_n$. We proceed by induction on $\nu := \sum \#S_j \cap [0, p_j)$. If $\nu = 0$, then it follows that $p_1 = \dots = p_n = 0$, so there is nothing to prove. Suppose that the lemma holds for $p_1, \dots, p_n$ and let p_1' be the next element in S_1 after p_1 (the argument works for arbitrary z_j; we fix $j = 1$ for simplicity). If $\nu \geq \dim U - 1$, then the statement of Lemma 3.17 is that

$$\operatorname{codim} V(p_1', p_2, \dots, p_n) \leq \nu + 1 = \dim U,$$

which holds always if V is not empty. So the only relevant case is $\nu < \dim U - 1$. The condition $m_1(u) > p_1$ defines a subspace of $V(p_1, \dots, p_n)$ of codimension at most 1. But $m_1(u) > p_1$ implies that $m_1(u) \geq p_1'$. This means that

$$\operatorname{codim}(V(p_1', p_2, \dots, p_n) \subset V(p_1, \dots, p_n)) \leq 1$$

and the codimension of $V(p_1', p_2, \dots, p_n)$ in U is at most one greater than the codimension of $V(p_1, \dots, p_n)$. Since $\#S_1 \cap [0, p_1') = 1 + \#S_1 \cap [0, p_1)$, the induction step is accomplished. $\square$

EXAMPLE 3.18. As an application of Theorem 3.16, let us discuss the case where $X = X_e$ is a Hirzebruch surface and C is of type (a, b). Let us choose $s_1 > 0$ and $s_2 \geq 0$ such that $s_1 < a$, $s_2 < b$. Let $\mathcal{L} = \mathcal{O}(s_1 L + s_2 M)$. It is clear that no section of $\mathcal{L}$ vanishes on C. Otherwise, the sheaf $\mathcal{O}((s_1 - a)L + (s_2 - b)M)$ would have a section, but this is impossible since the assumptions $s_1 < a$, $s_2 < b$ guarantee that $\mathcal{O}((a - s_1)L + (b - s_2)M)$ is very ample; see [16, Cor. V.2.18]. We have

$$t = \langle c_1(\mathcal{L}), C\rangle = (s_1 L + s_2 M) \cdot (aL + bM) = s_1 b + s_2(a + be).$$

Since $s_1 > 0$ and $s_2 \geq 0$, we also have

$$\dim \Gamma(X_e, \mathcal{L}) = \chi(\mathcal{L}).$$

The last quantity can be computed using the Riemann–Roch theorem. We obtain

$$\chi(\mathcal{L}) = P(s_1, s_2).$$

Theorem 3.16 now gives the same inequality as Theorem 1.1.

REMARK 3.19. In Example 3.18 the range of s_1, s_2 is slightly different than the range in Theorem 1.1. It is not hard to extend the range of s_1, s_2 using, for instance, [8, Prop. 4.3.3] or formula in [23, p. 18] to compute $\dim \Gamma(X, \mathcal{L})$ and to check that $\mathcal{L}$ does not admit a section vanishing entirely on C. We did not extend it in Example 3.18 because it is only an alternative proof of Theorem 1.1.

4. The Link at Infinity for Hirzebruch Surfaces

The main goal of Sections 4 and 5 is to establish Theorem 1.3. We first pass to describing the link of a curve at infinity.

Let C be a curve of type (a, b) in a Hirzebruch surface X_e. Let L be a vertical line, and M_0 the special section. Then $X_e \setminus (L \cup M_0) = \mathbb{C}^2$. Let N be a tubular neighborhood of $L \cup M_0$. Then $\partial N \cong S^3$, and the intersection of C with ∂N is a candidate for a link of C at infinity. In our computations we assume for simplicity that C intersects both L and M_0 transversally. Making the intersection of C transverse to L is simple: we can choose L to be a transverse fiber, and the set of transverse fibers is open-dense. Yet, in order to make C transverse to M_0, we might need to perturb M_0 in the *smooth category*. This will not affect our reasoning because the link at infinity will be shown to be a topological invariant and the proof of the semicontinuity of the spectrum works even in the topological category.

In this approach we will follow the ideas of [28].

4.1. The Link at Infinity: A First Glance in the Plane

To illustrate our computations, we begin with a rather standard example of computing the link at infinity of a quasi-homogeneous curve in $\mathbb{C}^2$.

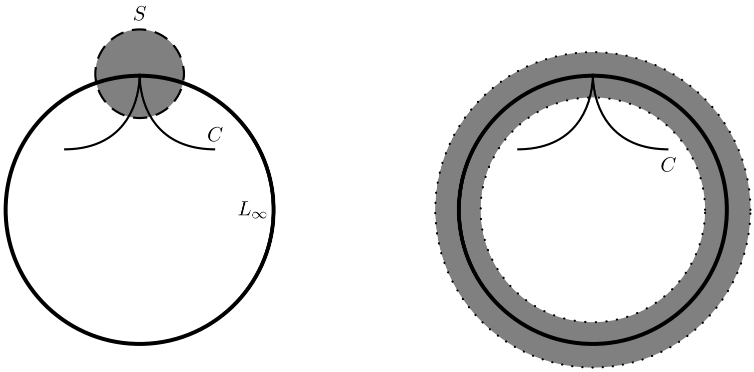

Figure 1 Left: The line at infinity L_∞ and the link of the singularity of C at infinity. A similar picture appears in [28, p. 462]. Right: Adding a two-handle to the neighborhood of the point at infinity yields a tubular neighborhood of the line at infinity. Its boundary is a large sphere in $\mathbb{C}^2$

So let C be given by $x^p - y^q = 0$ in $\mathbb{C}^2$ with $q > p$ and $\gcd(p, q) = 1$. Its link at infinity is clearly the torus knot $T(p, q)$. We will show an alternative approach for computing this link, following essentially [28].

First, let us study the intersection of C with the line at infinity. We choose the coordinates $z = \frac{1}{x}$ and $u = \frac{y}{x}$, and the line at infinity in these coordinates is given by $z = 0$. The equation $x^p - y^q = 0$ transforms into

$$\frac{1}{z^p} - \frac{u^q}{z^q} = 0,$$

that is,

$$z^{q-p} - u^q = 0.$$

Take a small ball B with a center at $(z, u) = (0, 0)$ and let $S = \partial B$. The line at infinity intersects S along an unknot, whereas C intersects S along the torus knot $T(q, q-p)$, that is, the link of the singularity on the curve $z^{q-p} - u^q = 0$; see Figure 1 (left). The unknot and the torus knot are presented schematically in Figure 2. This torus knot has the linking number q with the unknot, corresponding to the fact that the intersection number of C with L is equal to q.

Adding a two-handle to B along the unknot with framing 1 yields the tubular neighborhood of the line at infinity; see Figure 1 (right), the framing is exactly the self-intersection of the line at infinity. The boundary of this tubular neighborhood is a large sphere in $\mathbb{C}^2$, that is, the complement of the line of infinity seen from outside. On the other hand, this sphere is also the result of a $+1$ surgery on the unknot.

The $+1$ surgery on the unknot, that is, blowing down the unknot with framing $+1$ means that the torus knot $T(q-p, q)$ acquires a negative Dehn twist along the longitude, so it becomes the torus knot $T(-p, q)$ in the large sphere. This

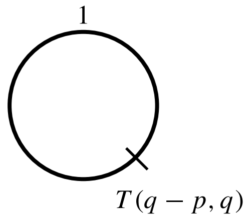

Figure 2 A schematic presentation of the link from Figure 1 (left part). The unknot is the intersection of the line at infinity with S, whereas the torus knot, represented here by a segment, is the intersection of C with S

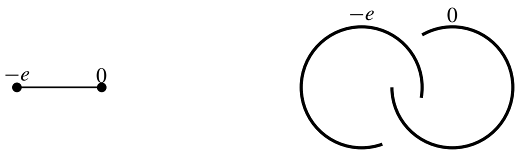

Figure 3 The plumbing diagram of the regular neighborhood of a "line at infinity" of a Hirzebruch surface and the corresponding surgery presentation of the three sphere

sphere is seen from outside, so we need to reverse the orientation obtaining the link $T(p,q)$. This is the link of C at infinity.

4.2. Plumbing at Infinity and the Nagata Transform

On a Hirzebruch surface we need to find an analogue to the notion of a line at infinity. We choose one particular fiber L and a horizontal section M', which is a smooth rational curve isotopic to the special section; for example, we can take M' to be a smooth perturbation of the special section M_0. We have $L^2 = 0$, $L \cdot M' = 1$, and $M'^2 = -e$. Consider N, the tubular neighborhood of $L \cup M'$ and $Y = \partial N$. The complement $X_e \setminus N$ is a four-ball, and ∂N is a three-sphere. As a boundary of a neighborhood of $L \cup M'$, Y is a plumbed 3-manifold with a plumbing graph as depicted in Figure 3 (left). It will be more convenient to use a surgery description of a plumbed manifold as on the right-hand side of Figure 3.

There exists a birational map between X_e and X_{e-1}, which is a sequence of one (-1) blow-up and (-1) blow-down. In algebraic geometry, this birational map is called the *Nagata transform* (sometimes referred to as an *elementary transformation*); see Figure 4 for an illustration. Successive Nagata transforms of the Hirzebruch surface X_e yield, after a finite number of steps, the situation in Figure 5. Contracting the -1 curve yields an unknot with framing 1 (this corresponds to the well-known fact that X_1 is $\mathbb{C}P^2$ blown up in one point). Contracting this unknot, we obtain S^3.

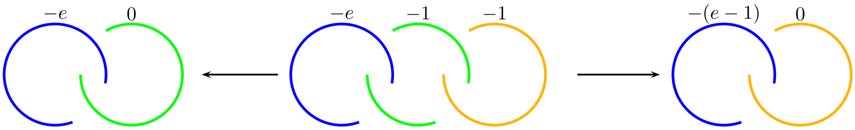

Figure 4 Nagata transform at the level of links at infinity. The surgery presentation on the left corresponds to X_e, it is blown up to the diagram at the center, and then the middle component is contracted, yielding a presentation corresponding to X_{e-1}

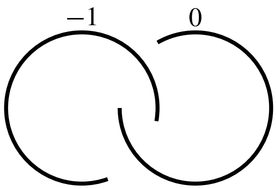

Figure 5 Last stage of the Nagata transform

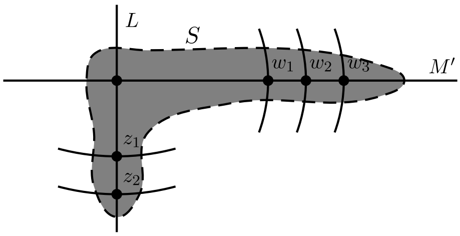

Figure 6 Figure 1 revisited. This time the ball B (the shaded region) captures all the intersection points of C with $L \cup M'$, as well as $L \cap M'$

4.3. The Link at Infinity of Curves in Hirzebruch Surfaces

Now we want to use the Nagata transform to describe the link at infinity of a given curve in a Hirzebruch surface. For simplicity, we restrict to the case where C is a type (a, b) curve in X_e intersecting L and M' transversally. By definition in $H_2(X_e; \mathbb{Z})$ we have $C = aL + bM = aL + b(M_0 + eL)$; hence, $C \cdot L = b$ and $C \cdot M' = C \cdot M_0 = a + be - be = a$. Let $z_1, \ldots, z_b = C \cap L$ and $w_1, \ldots, w_a = C \cap M'$. Choose a ball B lying in a small tubular neighborhood of $L \cap M'$ and containing all the points $z_1, \ldots, z_b$, $w_1, \ldots, w_a$ and $L \cap M'$ as in Figure 6. Let $S = \partial B$. The intersection of $L \cup M'$ with S forms a Hopf link. On gluing two-handles to B along this link, with framings respectively L^2 and M'^2 we obtain a tubular neighborhood of $L \cup M'$. The boundary of this tubular neighborhood is a large sphere in $\mathbb{C}^2 = X_e \setminus (L \cup M')$ seen from outside. This large sphere is thus the effect of a $(0, -e)$ surgery on the Hopf link; compare with Figure 3.

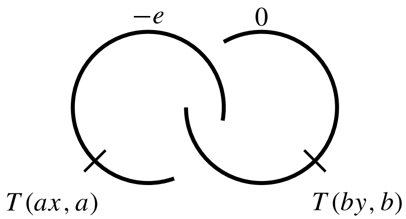

Figure 7 The link consisting of a meridians going around $-e$ framed circle with x twists and b meridians of the 0 framed circle with y full twists

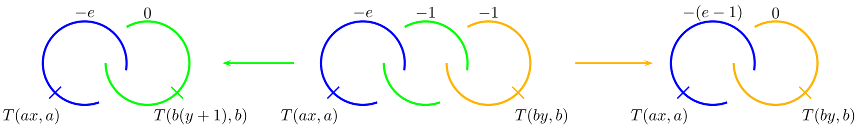

Figure 8 The Nagata transform and its effect on the link consisting of a meridians of the $-e$ framed circle and b meridians of the 0 framed circle

The intersection $C \cap S$ consists of $a + b$ components, corresponding to points $z_1, \dots, z_b$ and $w_1, \dots, w_a$. Since C is smooth at all these points and it is transverse to $L \cup M'$, each component of the intersection is an unknot. The unknots corresponding to $z_1, \dots, z_b$ have linking number 1 with the link $L \cap S$, whereas the unknots corresponding to $w_1, \dots, w_a$ have linking number 1 with $M' \cap S$. To be consistent with Section 4.1, we remark that the first set of meridians forms the torus link $T(0, b)$, whereas the second set forms the torus link $T(0, a)$. We present these torus links schematically in Figure 7, where we introduced also integer parameters x and y. At the beginning we have $x = y = 0$, but later we will have to allow that $x, y \neq 0$.

Our aim is to see these two torus links in a standard sphere, that is, to contract the framed Hopf link in Figure 3 (right). This is performed inductively, and the induction step is the Nagata transform. We will explain now how the two torus links in Figure 7 change under the Nagata transform.

Start with the intermediate stage of the Nagata transform; in Figure 8 it is the middle link. Blowing down either the middle circle or the right circle in this link (this corresponds to going to the right link and to the left link, respectively) does not affect $T(ax, a)$. If the middle circle is blown down, then the link $T(by, b)$ also remains unchanged. However, if we blow down the right circle, then the meridians of the right circle get an additional twist, as well as a clasp with the middle circle, so that $T(by, b)$ becomes $T(b(y + 1), b)$. The Nagata transform corresponds to going from the left link to the right link in Figure 8, so that $T(b(y+1), y)$ becomes $T(by, b)$, $T(ax, a)$ is unchanged, and the framing of the left circle is changed from $-e$ to $-(e - 1)$.

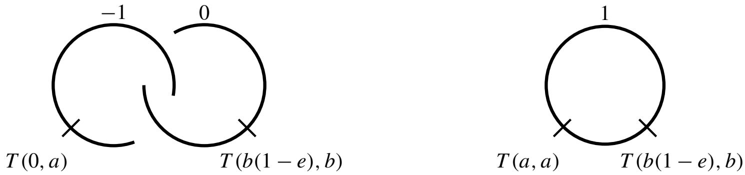

Figure 9 On the left: the link of C at infinity after $e-1$ Nagata transforms. On the right, this link after blowing down the -1 curve

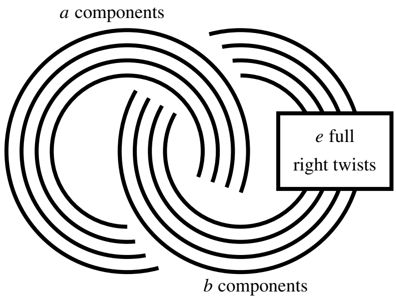

Figure 10 The link at infinity of a curve of degree (a,b)

After $e-1$ subsequent Nagata transforms, the framing of the left circle becomes -1. We blow down the circle obtaining a link, with only one circle (with framing $+1$) and torus links $T(a,a)$ and $T((1-e)b,b)$ linked to it; see Figure 9.

Blowing down the circle with framing $+1$ on the right of Figure 9 means that the two torus links will acquire a negative twist. That is, the $T(a,a)$ torus link will become the $T(0,a)$ torus link again, whereas $T(b(1-e),b)$ will become the $T(-be,b)$ torus link. The two torus links are now clasped: each of the components of $T(0,a)$ with each of the components of $T(-be,b)$ will form a negative Hopf link. As in the case of the link at infinity of a curve in $\mathbb{C}P^2$, we have to reverse the orientation. In this way the $T(-be,b)$ will become $T(be,e)$, whereas $T(0,a)$ will remain $T(0,a)$. The negative Hopf link will become the positive Hopf link. The final result is shown in Figure 10.

Definition 4.1. The link in Figure 10 is called the *link at infinity* of a curve of degree (a,b). We denote this link by $L_{a,b}$.

5. Hermitian Variation Structure of the Link at Infinity

In this section we determine the spectrum of the curve C at infinity using the language of Hermitian Variation Structures for links as introduced in [5]. Hermitian Variation Structure is encoded in so-called Hodge numbers $p^k_\lambda(u)$ and q^k_λ, $k=1,2,\dots$, where $\lambda\neq 0$ is a complex number in the unit disk (and $p^k_\lambda=0$ if $|\lambda|<1$, whereas $q^k_\lambda=0$ if $|\lambda|=1$), and $u\in\{\pm1\}$. Even a sketchy definition of

p_λ^k and q_λ^k is beyond the scope of the paper. To give a flavor, suppose that L is a fibered link and let h be the monodromy operator. Then $p_\lambda^k(1) + p_\lambda^k(-1)$ is precisely the number of Jordan blocks of h with eigenvalue λ and size k, and q_λ^k is the number of Jordan blocks of size k and eigenvalue λ. The sign $u = \pm 1$ in $p_\lambda^k(u)$ roughly corresponds to the value of the jump of the Tristram–Levine signature at λ. We now state the following rigorous results proved in [5]; for simplicity, we assume that $q_\lambda^k = 0$. This is so in case of all algebraic links because the Alexander polynomial has no roots outside the unit circle.

PROPOSITION 5.1. *Let L be a link whose Alexander polynomial has no roots outside the unit circle. Then*

$$\Delta_L = \prod_{\lambda \in S^1} (t-\lambda)^{\sum_{k,u} k p_\lambda^k(u)}.$$

Moreover, let $\sigma_L(\xi)$ be the Tristram–Levine signature function of L with $\xi \in S^1$. If $\xi \neq 1$, then

$$\sigma_L(\xi) = -\sum_{\substack{\lambda<\xi \\ k \text{ odd} \\ u=\pm 1}} u p_\lambda^k(u) + \sum_{\substack{\lambda>\xi \\ k \text{ odd} \\ u=\pm 1}} u p_\lambda^k(u) + \sum_{\substack{k \text{ even} \\ u=\pm 1}} u p_\xi^k(u),$$

where for $\xi = e^{2\pi i x}$ and $\lambda = e^{2\pi i y}$ with $x, y \in (0,1)$, we say that $\xi > \lambda$ if $x > y$.

In the following we will gather enough data about $p_\lambda^k(u)$ to be able to compute the spectrum of the link at infinity. We will do this in four steps. First, we will compute the Alexander polynomial of $L_{a,b}$. Then we use the fact that $L_{a,b}$ is a splice link to show via the monodromy theorem that the $p_\lambda^k(u)$ vanish for $k > 2$ and for $k = 2$ if $\lambda = 1$. Next, we compute the equivariant signatures of the link at infinity. Then we use the equivariant signatures and the Alexander polynomial to gather enough conditions on the Hodge numbers to recover the noninteger part of the spectrum. In the latter step the Hodge numbers for $\lambda = 1$ will be determined from the linking matrix of the components of $L_{a,b}$ as in [4, Sect. 3].

The proof of Theorem 1.3 is presented at the end of this section.

5.1. Splice Presentation of $L_{a,b}$ and its First Consequences

After a series of observations about the link $L_{a,b}$, our first result on the Hodge numbers is given in Lemma 5.6.

We begin with the following observation.

LEMMA 5.2. *The link at infinity $L_{a,b}$ is a graph link. Its Eisenbud–Neumann diagram is as on Figure 11.*

We denote by L^1 and L^2 the splice components of $L_{a,b}$. On Figure 11 the component L^1 is presented on the left, whereas the component L^2 is on the right. For these components, we have the following estimates.

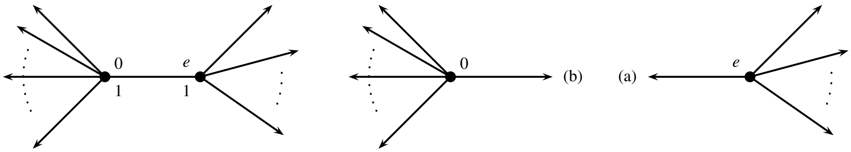

Figure 11 The link $L_{a,b}$ and its splice components. There are a arrowheads on the left and b arrowheads on the right

LEMMA 5.3. *The multiplicity of the multilink L^1 is equal to b, whereas the multiplicity of L^2 is equal to $w = a + be$.*

Proof. The results follow from straightforward computations; see, for example, [26]. □

Then we make the following observation about $L_{a,b}$.

COROLLARY 5.4. *The link $L_{a,b}$ is a fibered link.*

Proof. Because the multiplicity of every node is nonnegative, this follows from [9, Thm. 11.2]. □

Moreover, [9, Thm. 11.3] allows us to explicitly compute the Alexander polynomial of $L_{a,b}$.

LEMMA 5.5. *The Alexander polynomial of $L_{a,b}$ is equal to*

$$\Delta(t) = (t-1)(t^w - 1)^{b-1}(t^b - 1)^{a-1}.$$

The fact that $L_{a,b}$ is a graph link affects the Hodge numbers in the following way.

LEMMA 5.6 (The monodromy theorem; see [9, Sect. 13] or [26]). *We have $p_\lambda^k(u) = 0$ for $k > 2$ and for $k = 2$ and $\lambda = 1$. Moreover, the only positive values of $p_\lambda^k(u)$ can appear if λ is a root of unity of order w or b.*

The Alexander polynomial is the characteristic polynomial of the monodromy h acting on $H_1(F)$, where F is the fiber of the fibration of the complement to $L_{a,b}$. By Lemma 5.6 the monodromy has Jordan block of size at most 2. We can compute the Jordan blocks of size 2 from the characteristic polynomial of the monodromy operator restricted to the image $h^c - 1$, where $c = \gcd(w, b)$. The algorithm in [9, Sect. 14] or [26] allows us to compute this as well. We obtain

$$\Delta_2(t) = \frac{t^c - 1}{t - 1}.$$

This polynomial affects the Hermitian Variation Structure related to the link $L_{a,b}$, but it does not affect its spectrum.

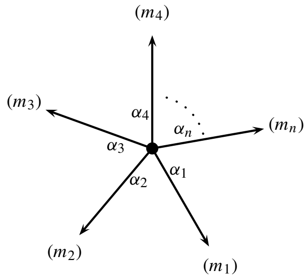

Figure 12 Notation used in proof of Lemma 5.7

5.2. The Equivariant Signatures of the Link at Infinity

The equivariant signatures of a link are integer numbers associated to any $\lambda \in S^1$. For $\lambda \neq 1$, the equivariant signature is half the jump of the Tristram–Levine signatures, that is, for $\lambda = e^{2\pi i x}$ and $x \in (0, 1)$, we have

$$2\sigma_\lambda = \lim_{y \to x^+} \sigma(e^{2\pi i y}) - \lim_{y \to x^-} \sigma(e^{2\pi i y}).$$

We refer to [22] for discussion of various definitions of signatures and relations between them.

In this part we use Neumann's algorithm (see [27]) to compute the equivariant signatures of L^1 and L^2 for $\lambda \in S^1 \setminus \{1\}$. The equivariant signatures are splice additive, and hence the signature of $L_{a,b}$ is the sum of the signatures of L^1 and L^2. For $x \in (0, 1)$, we denote by σ_x^1, respectively σ_x^2, the equivariant signature of L^1, respectively L^2, corresponding to the value $\lambda = e^{2\pi i x}$.

The following lemma is the core result.

Lemma 5.7. *For $p = 1, \ldots, w - 1$, the signature is equal to*

$$\sigma_{p/w}^1 = 2\left\lfloor \frac{pb}{w} \right\rfloor - (b - 1) - \delta,$$

where $\delta = 1$ if w divides pb, otherwise it is 0. Similarly, for $q = 1, \ldots, b - 1$, the signature

$$\sigma_{q/b}^2 = 2\left\lfloor \frac{qa}{b} \right\rfloor - (a - 1) - \delta',$$

where $\delta' = 1$ if b divides qa, otherwise it is 0.

Proof. The two parts (for L^1 and L^2) are analogous; we will prove only the first one. The second part can be deduced from the first by swapping the roles of a and b and setting $e = 0$.

The algorithm in [27] is the following. Consider a splice component as in Figure 12. A general splice component can have leaf type vertices, but the output is the same if the leaf vertices are replaced by arrowheads with multiplicity $m_i = 0$.

One chooses $\beta_1, \dots, \beta_n$ to be integers satisfying

$$\beta_j \alpha_1 \cdots \widehat{\alpha_j} \cdots \alpha_n \equiv 1 \bmod \alpha_j.$$

The elements $\beta_1, \dots, \beta_n$ are not uniquely defined, but the ambiguity does not affect the result. The *multiplicity* of the splice component is

$$m = \sum m_j \alpha_1 \cdots \widehat{\alpha_j} \cdots \alpha_n.$$

The numbers $s_j = (m_j - \beta_j m)/\alpha_j$ are integers. For any $p = 1, \dots, m-1$, the equivariant signature $\sigma_{p/m}$ is equal to

$$2 \sum_{j=1}^{n} \left\langle \frac{s_j p}{w} \right\rangle,$$

where $x \mapsto \langle x \rangle$ is the sawtooth function, that is,

$$\langle x \rangle = \begin{cases} \{x\} - \frac{1}{2}, & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases} \tag{5.8}$$

Application of this algorithm to L^1 is straightforward. As depicted in Figure 11, the (multi)link L^1 has b arrowheads with multiplicity 1 and one arrowhead with multiplicity a. That gives $m_1 = \cdots = m_b = 1$, $m_{b+1} = a$. Furthermore, $\alpha_1 = \cdots = \alpha_b = 1$ and $\alpha_{b+1} = e$, We have $\beta_1 = \cdots = \beta_b = 0$ and $\beta_{b+1} = 1$. The multiplicity is $w = a + be$. We have $s_1 = \cdots = s_b = 1$ and $s_{b+1} = -b$.

According to [27], the equivariant signature at $\lambda \in S^1$ is zero unless $\lambda = e^{2\pi i p/w}$ for some $p = 1, \dots, w-1$. In the latter case the equivariant signature is equal to

$$\sigma^1_{p/w} = 2 \sum_{j=1}^{b+1} \left\langle \frac{s_j p}{w} \right\rangle = 2b \left\langle \frac{p}{w} \right\rangle - 2 \left\langle \frac{pb}{w} \right\rangle.$$

The expression for $\sigma^1_{p/w}$ can be rewritten as

$$\sigma^1_{p/w} = 2 \left\lfloor \frac{pb}{w} \right\rfloor - (b-1) - \delta, \tag{5.9}$$

where $\delta = 1$ if $w | pb$ (that is, if $b\frac{p}{w}$ is an integer), otherwise $\delta = 0$. This proves the first part. $\square$

We then observe that since b divides qa if and only if b divides qw, the case $\delta = 1$ is equivalent to $\delta' = 1$. Hence, this happens only if x can be written as p/w and x can be written as q/b for some integers p and q. Note that this observation is reflected in the structure of Table 1: the contribution of an element x that can be written both as $\frac{p}{w}$ and as $\frac{q}{b}$ (the second row in Table 1) is not merely a sum of the contribution from the fourth and sixth rows.

5.3. *The Part* $\lambda \neq 1$ *of the Spectrum*

Now we pass to the third step in our process, where we relate the Hodge numbers to the Alexander polynomial and to the equivariant signatures for $\lambda \neq 1$.

REMARK 5.10. For the reader's convenience, in Section 5.5 we discuss the sign conventions used throughout this section. We also present a sample computation of the spectrum.

We begin with the following observation (see [5, Sect. 4.1]):

$$\sum_{u=\pm 1} (p_\lambda^1(u) + 2p_\lambda^2(u)) = \operatorname{ord}_{t=\lambda} \Delta(t)$$
$$= \begin{cases} b-1 & \text{if } \lambda = e^{2\pi i p/w}, \\ a-1 & \text{if } \lambda = e^{2\pi i q/b}, \\ a+b-2 & \text{if } \lambda = e^{2\pi i q/b} = e^{2\pi i p/w}, \end{cases} \tag{5.11}$$

where we should interpret the expressions like $\lambda = e^{2\pi i q/b}$ as "there exists $q \in \mathbb{Z}$ such that $\lambda = e^{2\pi i q/b}$".

On the other hand, the equivariant signature can be computed from Hodge numbers as

$$p_\lambda^1(-1) - p_\lambda^1(+1) = \sigma_x^1 + \sigma_x^2, \tag{5.12}$$

where x is such that $e^{2\pi i x} = \lambda$. To complete the picture, we note that

$$p_\lambda^2(+1) + p_\lambda^2(-1) = \operatorname{ord}_{t=\lambda} \Delta_2(t),$$

but we will not need this formula.

Even though the last three formulae are insufficient to determine the full Hermitian Variation Structure of $L_{a,b}$ corresponding to the eigenvalue λ, the terms $p_\lambda^2(+1)$ and $p_\lambda^2(-1)$ give the same contribution to all the three formulae. Luckily, their contribution to the spectrum is also the same. The first two formulae are enough to recover the spectrum. Let us cite a result from [5, Sect. 2.3].

PROPOSITION 5.13. *Let* $x \in (0, 1)$ *and* $\lambda = e^{2\pi i x}$. *Then the multiplicity of* x *in the spectrum is equal to*

$$A_x := p_\lambda^1(-1) + p_\lambda^2(+1) + p_\lambda^2(-1),$$

and the multiplicity of $1 + x$ *in the spectrum is equal to*

$$B_x := p_\lambda^1(+1) + p_\lambda^2(+1) + p_\lambda^2(-1).$$

We now observe that $A_x + B_x$ is the order at λ of the Alexander polynomial, whereas $A_x - B_x$ is equal to the equivariant signature. Hence, knowing the Alexander polynomial from Lemma 5.5 and the equivariant signature from Lemma 5.7, we can compute explicitly A_x and B_x, that is, find the spectrum. The results of the computations is presented in Table 1. We omit the details here.

5.4. *The Part* $\lambda = 1$ *of the Spectrum*

It remains to discuss the case of $\lambda = 1$. By Lemma 5.6 we have that $p_1^2(\pm 1) = 0$. We also have that

$$p_1^1(-1) + p_1^1(+1) = \operatorname{ord}_{t=1} \Delta(t) = a + b - 1.$$

LEMMA 5.14. *We have* $p_1^1(-1) = 0$.

Proof. The argument is the same as in [4, Prop. 3.4.2]; we will show that the linking form on the subspace of $H_1(F)$ (recall that F is the fiber of the fibration of the complement to $L_{a,b}$; in particular, it is a Seifert surface for L) spanned by the components of $L_{a,b}$ is negative definite. Let us denote by $L_1^1, \dots, L_a^1$, respectively $L_1^2, \dots, L_b^2$, the components of $L_{a,b}$ lying on a splice component L^1, respectively L^2. We regard them as elements in $H_1(F)$. In $H_1(F)$ these cycles are subject to one relation, namely $L_1^1 + \dots + L_b^2 = 0$. Let us consider the element L in the space spanned by $L_1^1, \dots, L_b^2$, that is,

$$L = \sum \gamma_i L_i^1 + \sum \delta_j L_j^2.$$

By the arguments of [26, Sect. 3] or [4, Sect. 3.7] the self-linking of L is equal to

$$\begin{aligned} -\sum_{i<i'} (\gamma_i - \gamma_{i'})^2 \operatorname{lk}(L_i^1, L_{i'}^1) - \sum_{j<j'} (\delta_j - \delta_{j'})^2 \operatorname{lk}(L_j^2, L_{j'}^2) \\ - \sum_{i,j} (\gamma_i - \delta_j)^2 \operatorname{lk}(L_i^1, L_j^2). \end{aligned}$$

This expression is nonpositive because the linking numbers are nonnegative. If it is zero, then for any i, j, we have $\gamma_i = \delta_j$ because $\operatorname{lk}(L_i^1, L_j^2) = 1$ for any i, j. This implies that $\gamma_1 = \dots = \gamma_a = \delta_1 = \dots = \delta_b$, that is, the link represents $0 \in H_1(F)$. It follows that the linking form is negative definite, so the argument in [26, proof of Thm. 3.4] implies that $p_1^1(-1) = 0$. $\square$

This shows the following result.

PROPOSITION 5.15. *The number* 1 *enters the spectrum with multiplicity* $a + b - 1$, *whereas the number* 2 *does not belong to the spectrum.*

In this way the computation of the spectrum of the link at infinity is completed.

5.5. *Some Examples*

It might be quite difficult not to get lost in various conventions. To show that the sign conventions are consistent, we look at the positive trefoil, whose signature function is -2 for $z \in (e^{2\pi i/6}, e^{2\pi 5i/6})$ and zero outside of the closure of this interval. The input for Neumann's algorithm for the equivariant signature is $m_1 = 0$, $m_2 = 0$, $m_3 = 1$, $\alpha_1 = 2$, $\alpha_2 = 3$, and $\alpha_3 = 1$, so that $w = 6$ and $s_1 = -3$, $s_2 = -4$, $s_3 = 1$. The equivariant signature $2 \sum \langle s_j p/q \rangle$ is equal to -1 for $p/q = 1/6$ and $+1$ for $p/q = 5/6$. The spectrum should be $\{\frac{5}{6}, \frac{7}{6}\}$ (this is the spectrum of the singularity $x^2 + y^3 = 0$, and we can compute it from the Thom–Sebastiani formula), and the Hodge numbers are $p^1_{5/6}(-1) = 1$ and $p^1_{1/6}(+1) = 1$; see [5, Sect. 5.1]. In particular, the conventions we have been using are the following:

- The equivariant signatures are taken as $\sum \langle s_j p/q \rangle$, where $\langle x \rangle$ is the sawtooth function; see (5.8).
- The equivariant signature is half the jump of the Tristram–Levine signature, that is, half the difference between the right and left limits of the function $x \mapsto \sigma(e^{2\pi i x})$.
- The Hodge numbers $p^1_\lambda(+1)$ correspond to negative equivariant signature, and the Hodge numbers $p^1_\lambda(-1)$ correspond to positive values of the equivariant signature. Therefore, the equivariant signature is $p^1_\lambda(-1) - p^1_\lambda(+1)$.
- The Hodge numbers $p^1_\lambda(-1)$ correspond to values in the spectrum in the interval $(0, 1)$, whereas the Hodge numbers $p^1_\lambda(+1)$ correspond to values in $(1, 2)$.

Now let us present a computation of the spectrum.

Example 5.16. Suppose that $a = 6$, $b = 4$, and $e = 0$, so that $w = 6$. The equivariant signatures for $\sigma^1_{i/6}$ are equal to

$$-3, -1, 0, 1, 3.$$

The equivariant signatures $\sigma^2_{j/4}$ are equal to

$$-3, 0, 3.$$

By additivity, the equivariant signatures of $L_{4,6}$ at $e^{2\pi i x}$ for

$$x \in \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \right\}$$

are respectively

$$\{-3, -3, -1, 0, 1, 3, 3\};$$

the order of the root of the Alexander polynomials at each of these points is respectively

$$\{3, 5, 3, 8, 3, 5, 3\}.$$

Hence, the part of the spectrum in $(0, 1)$ is

$$\left\{ \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6} \right\},$$

and the part in $(1, 2)$ is symmetric. The value 1 appears in the spectrum with multiplicity

$$a + b - 1 = 9.$$

The total number of elements in the spectrum is

$$w(b - 1) + b(a - 1) + 1 = 18 + 20 + 1 = 39.$$

Notice that twice the genus of a curve of type $(4, 6)$ is equal to $(4-1)(2 \cdot 6 - 2) = 30$, and the difference $39 - 30 = 9 = a + b - 1$.

5.6. *The Semicontinuity of the Spectrum*

We are now ready to prove Theorem 1.3, comparing the spectrum of the link at infinity to the spectrum of the singular points of C.

Proof of Theorem 1.3. We follow the argument of [4]. So suppose that C has singular points $z_1, \dots, z_k$. The links of singularities of $z_1, \dots, z_k$ are denoted by $K_1, \dots, K_k$. The spectrum of z_j will be Sp_j, and μ_j denotes the Milnor number of z_j. Notice that μ_j is the degree of the (univariate) Alexander polynomial of K_j.

Pick L and M' as in Section 4.3. They intersect C transversally, and let N be a tubular neighborhood of $L \cup M'$. The complement $B = X_e \setminus N$ is a standard 4-ball. We may and will assume that $z_1, \dots, z_k$ are all in B. By definition $L_{a,b} = C \cap \partial B$ is the link of C at infinity. Set $C' = C \cap B$ and let C'' be a smoothing of C', that is, a smooth curve obtained by replacing a neighborhood of each of the singular points $z_1, \dots, z_k$ of C' by a corresponding Milnor fiber. For example, if C' is given by $F^{-1}(0)$ for some complex analytic function $F\colon B \to \mathbb{C}$ (such that ∇F does not vanish identically on C'), then C'' can be taken to be $F^{-1}(t)$ where t is a noncritical value of F sufficiently close to 0. The intersection of C'' with the boundary of B is isotopic to $L_{a,b}$. With this definition, C'' is isotopic to the fiber of $L_{a,b}$; see [28; 29].

Suppose $x \in [0, 1]$ is such that $\xi := e^{2\pi i x}$ is not a root of the Alexander polynomials of $L_{a,b}$. Then, by [4, Prop. 2.5.5] we have

$$\begin{aligned} -\sigma_{L_{a,b}}(\xi) + (1 - \chi(C'')) &\geq \sum_{j=1}^{k} (-\sigma_{K_j}(\xi) + \mu_j), \\ \sigma_{L_{a,b}}(\xi) + (1 - \chi(C'')) &\geq \sum_{j=1}^{k} (\sigma_{K_j}(\xi) + \mu_j), \end{aligned} \tag{5.17}$$

where $\sigma_L(\xi)$ denotes the Tristram–Levine signature of L evaluated at ξ.

Now we have that $1 - \chi(C'') = b_1(C'')$, but C'' is isotopic to the fiber of the link $L_{a,b}$. So $b_1(C'')$ is just the degree of the Alexander polynomial of $L_{a,b}$. Rewrite (5.17) as

$$\mp\sigma_{L_{a,b}} + \deg \Delta_{L_{a,b}} \geq \sum_{j=1}^{k} (\mp\sigma_{K_j}(\xi) + \deg \Delta_{K_j}). \tag{5.18}$$

By [4, Cor. 2.4.6], (5.18) is precisely the statement of Theorem 1.3. □

6. Examples and Applications

In this section we will show applications of Theorems 1.1 and 1.3 and get results about rational cuspidal curves in Hirzebruch surfaces. In particular, we show that the theorems imply that not all possible (after fundamental results, that is, the genus formula etc.) cusps can exist on such curves. Moreover, we give explicit constructions of some of the possible curves that pass the obstructions in the theorems.

6.1. A Simple Multiplicity Estimate

The following result bounding the multiplicity of a singularity on a curve in a Hirzebruch surface is a trivial consequence of Bézout's theorem. The standard proof is given, for instance, in [23, Thm. 3.1.5]. We show that for rational cuspidal curves, it can also be proven using topological methods.

PROPOSITION 6.1. *Let r be the multiplicity of a singular point on a curve (in general, not necessarily rational or cuspidal) C of type (a, b) in the Hirzebruch surface X_e. Then $r \leq b$.*

Proof. Suppose now that the curve C is rational and cuspidal. We set $s_1 = 1$ and $s_2 = 0$ in Theorem 1.1 and obtain $R(b+1) \geq 2$ for the R function associated with C (see Definition 2.6). Suppose that $z_1, \dots, z_n$ are singular points of C and that $R_1, \dots, R_n$ are the corresponding semigroup densities as in (2.4). Let us consider the point z_k for $k = 1, \dots, n$. Set $m_1 = \cdots = m_{k-1} = 0$, $m_{k+1} = \cdots = m_n = 0$, and $m_k = b + 1$, so that $\sum m_j = b + 1$. By the definition of the infimum convolution we get

$$R_1(m_1) + \cdots + R_n(m_n) \geq R(m_1 + \cdots + m_n).$$

But $R_1(m_1) = \cdots = R_{k-1}(m_{k-1}) = R_{k+1}(m_{k+1}) = \cdots = R_n(m_n) = 0$. Eventually, we obtain

$$R_k(b + 1) \geq 2.$$

But if z_k has multiplicity greater than b, then zero is the only element in the semigroup of z_k that is smaller than $b+1$, which leads to the desired contradiction. □

6.2. Singular Points with Multiplicity 3 *on* $(4, 4)$ *Curves*

Consider curves of type $(4, 4)$ in the Hirzebruch surface X_e. Such a curve has genus $6e + 9$. We ask whether such a curve can have a singularity of type $(3, 6e + 10)$? Note that the genus formula implies that if a $(4, 4)$ curve has such a singularity, then it is rational and cuspidal.

PROPOSITION 6.2. *If e is even, then a $(4, 4)$ curve in X_e cannot have a $(3, 6e+10)$ singularity.*

Proof. Set $s_1 = 1 + e/2$ and $s_2 = 1$. We obtain $s_1 b + s_2(a + be) + 1 = 6e + 9$. Theorem 1.1 implies that $R(6e+9) \geq 2e+4$. But $R(6e+9)$ is the number of the elements in the interval $[0, 6e + 8]$ that belong to the semigroup generated by 3 and $6e+10$. This is exactly the set of all integers in $[0, 6e+8]$ divisible by 3. Its cardinality is $2e+3$, so we get a contradiction. □

The result does not say anything about the case where e is odd. In fact, Theorem 1.1 will not obstruct the existence of a $(3, 6e+10)$ singularity on a $(4,4)$ curve in X_e for e odd.

6.3. *Curves of Type* $(6,6)$ *with One Cusp with One Single Puiseux Pair in* X_0

We now compute the spectra for the link at infinity and the link at the cusp for rational unicuspidal curves of type $(6,6)$ with a single Puiseux pair in X_0. We do this for all curves with Puiseux pair that fits the genus formula, and there are three such curves. The computations show that Theorem 1.3 obstructs the existence of one of these curves. Moreover, we provide a sketch of the construction of one of the other curves. Note that we cannot say anything about the existence of the remaining curve.

Let C be a unicuspidal curve of type $(6,6)$ in X_0, and let the cusp have a single Puiseux pair. We have $a=b=6$, $g=(6-1)(6-1)=25$, and $2g=50$. The list of theoretically possible Puiseux pairs is $(2,51)$, $(3,26)$, and $(6,11)$. Using Table 1, we find:

$$Sp^{\infty}_{6,6}=\left\{\frac{1}{6}^{1},\frac{1}{3}^{3},\frac{1}{2}^{5},\frac{2}{3}^{7},\frac{5}{6}^{9},1^{11},\frac{7}{6}^{9},\frac{4}{3}^{7},\frac{3}{2}^{5},\frac{5}{3}^{3},\frac{11}{6}^{1}\right\},$$

where the exponent denotes the multiplicity of the element.

$(2,51)$ For this Puiseux pair, we choose $x=\frac{1}{2}+\epsilon$ for some ϵ such that $\frac{1}{51}>\epsilon>0$. Then

$$\#Sp^{\infty}_{6,6}\cap(x,x+1)=48.$$

On the other hand,

$$Sp_{2,51}\cap(x,x+1)=\left\{\frac{1}{2}+\frac{1}{51},\ldots,\frac{1}{2}+\frac{50}{51}\right\};$$

hence,

$$\#Sp_{2,51}\cap(x,x+1)=50.$$

By Theorem 1.3 this is not possible, so such a cusp cannot exist on a curve of type $(6,6)$.

In the next two cases the spectrum does not obstruct the existence. To show this, we need to verify finitely many values of x, a task which can be done for instance using a computer. We present calculation for one specific value of x, the one close to the inverse of the multiplicity of the singular point. This is motivated by an observation that in most cases if Theorem 1.3 gives an obstruction, then this obstruction can be seen for that value of x (this is not a rigorous statement, it is rather a hint for applying Theorem 1.3).

$(3,26)$ For this Puiseux pair, we choose $x=\frac{1}{3}+\epsilon$ for some ϵ such that $\frac{1}{26}>\epsilon>0$. Then

$$\#Sp^{\infty}_{6,6}\cap(x,x+1)=48.$$

On the other hand,

$$Sp_{3,26} \cap (x, x+1) = \left\{ \frac{1}{3} + \frac{1}{26}, \dots, \frac{1}{3} + \frac{25}{26}, \frac{2}{3} + \frac{1}{26}, \dots, \frac{2}{3} + \frac{17}{26} \right\};$$

hence,

$$\#Sp_{3,26} \cap (x, x+1) = 42.$$

This does not violate Theorem 1.3.

(6, 11) For this Puiseux pair, we choose $x = \frac{1}{6} + \epsilon$ for some ϵ such that $\frac{1}{11} > \epsilon > 0$. Then

$$\#Sp^{\infty}_{6,6} \cap (x, x+1) = 44.$$

On the other hand,

$$\begin{aligned} Sp_{6,11} \cap (x, x+1) = \Bigg\{ & \frac{1}{6} + \frac{1}{11}, \dots, \frac{1}{6} + \frac{10}{11}, \\ & \frac{1}{3} + \frac{1}{11}, \dots, \frac{1}{3} + \frac{9}{11}, \\ & \frac{1}{2} + \frac{1}{11}, \dots, \frac{1}{2} + \frac{7}{11}, \\ & \frac{2}{3} + \frac{1}{11}, \dots, \frac{2}{3} + \frac{5}{11}, \\ & \frac{5}{6} + \frac{1}{11}, \dots, \frac{5}{6} + \frac{3}{11} \Bigg\}; \end{aligned}$$

hence,

$$\#Sp_{6,11} \cap (x, x+1) = 34.$$

This curve passes the criterion from Theorem 1.3, and it can in fact be constructed by a simple transformation of the plane unicuspidal curve $y^5 z - x^6 = 0$. Indeed, blow up two points on the line $y = 0$, which is tangent to the curve at the cusp with coordinates $(0 : 0 : 1)$, and contract its strict transform; see Example 6.4 and [23].

6.4. *A Finiteness Theorem*

As an application of Theorem 1.3, we shall prove the following result.

THEOREM 6.3. *Suppose $a, b > 0$ are fixed. Then there is only a finite number of triples (r, s, e) such that*

- *$r, s \geq 2$ are coprime, $e \geq 0$;*
- *A singularity of type (r, s) occurs on a rational cuspidal curve of type (a, b) in X_e;*
- *$r < b$ and $r \neq b - 1$ if b is even.*

Put differently, for given (a, b) and sufficiently large e, the only possible rational cuspidal curves of type (a, b) in X_e with one singular point having one Puiseux pair (r, s) are those with $b = r$ or $b = r + 1$ and b even.

Before we give the proof of Theorem 6.3, we construct two families of curves in Hirzebruch surfaces with $b = r$. This shows that Theorem 6.3 is close to optimal.

EXAMPLE 6.4. It is possible to construct two series of curves from Theorem 6.3 using the Cremona transformations of the plane curve C_d given by the defining polynomial $x^{d-1}z - y^d$. Note that the curves in the below series exist for all $d \geq 3$ and $e, k \geq 0$, except $(e, k) = (0, 0)$.

(1) Curves of type (kd, d) in X_e with one cusp and Puiseux pair $(d, (e + 2k)d - 1)$.
(2) Curves of type $(k(d - 1) + 1, d - 1)$ in X_e with one cusp and Puiseux pair $(d - 1, (e + 2k)(d - 1) + 1)$.

Sketch of construction. The constructions of the two series of curves in Example 6.4 are very similar; hence, we sketch only the construction of the first curves and indicate the initial blow up for the latter curves. For similar constructions with details, see [24].

Given the plane unicuspidal curve C_d as before, observe that C_d is unicuspidal and that the cusp has multiplicity sequence $[d - 1]$; see [24] for notation. We blow up a point on the tangent line, say T, to the curve at the cusp. This gives in X_1 a rational unicuspidal curve of type $(0, d)$ with a cusp with multiplicity sequence $[d - 1]$. The fiber through the cusp, that is, the strict transform of T, intersects the curve only at the cusp, with intersection multiplicity d. Now, performing subsequent elementary transformations with center on the special section and the fiber through the cusp gives the series in X_e for $e \geq 1$ and $k = 0$.

Performing a similar elementary transformation of the curve in X_1, this time with center outside the special section, gives in X_0 a curve of type (d, d) with one cusp with multiplicity sequence $[d, d - 1]$. Since the fiber through the cusp does not intersect the curve outside the cusp, it is again possible to construct the series for $e \geq 0$ and $k = 1$.

It can be shown by induction that a similar construction works for all $e, k \geq 0$, except $(e, k) = (0, 0)$; see [24]. Ultimately, we end up with unicuspidal curves of type (kd, d), where the cusp has multiplicity sequence $[d_{e+2k-1}, d - 1]$. By [7, Thm. 12, p. 516], this multiplicity sequence corresponds to the Puiseux pair $(d, (e + 2k)d - 1)$.

Note that the second series of curves is constructed from the same plane curves, in this case by blowing up the smooth intersection point of the curve and a generic line through the cusp. □

Proof of Theorem 6.3. We shall use Theorem 1.3 for $x = \frac{1}{2}$ and study the asymptotic of both sides of (1.4) as e goes to infinity. Unfortunately, it turns out that the number of elements of the spectrum of the singular point of type (r, s) that are contained in $(\frac{1}{2}, \frac{3}{2})$ and the number of elements of the spectrum at infinity that are contained in the same interval both grow like $\frac{3}{4}g$, where $g = \frac{1}{2}(r - 1)(s - 1) = (a - 1)(b - 1) + \frac{1}{2}b(b - 1)e$ is the arithmetic genus of the curve. Therefore, a more careful analysis of both terms of inequality (1.4) has to be conducted.

In what follows we will assume that e is large compared to a and b. Assume that C is a rational unicuspidal curve in X_e of type (a, b), where the cusp has one Puiseux pair (r, s) with $r < s$. We use the notation

$$S_{r,s} = \#Sp_{r,s} \cap \left(\frac{1}{2}, \frac{3}{2}\right),$$
$$S_\infty = \#Sp^\infty_{a,b} \cap \left(\frac{1}{2}, \frac{3}{2}\right),$$

where $Sp_{r,s}$ is the spectrum of the singular point of type (r, s). In our computation we shall focus ourselves on terms linear in e, neglecting elements that are of lower order with respect to e. We shall write $x \simeq y$ (where x and y are some expressions depending on e) if $\frac{x-y}{e}$ tends to zero as e goes to infinity.

For example, we have

$$2g = 2(a-1)(b-1) + b(b-1)e \simeq b(b-1)e \simeq (a+be)(b-1) = w(b-1).$$

Furthermore, by Proposition 6.1 we have that $r \leq b$; hence, $(s-1)(r-1) = (r-1)s - (r-1) \simeq (r-1)s$.

REMARK 6.5. It might happen that $\frac{1}{2}$ belongs to $Sp^\infty_{a,b}$, so we cannot use Theorem 1.3 directly. In that case, we use Theorem 1.3 for a value of x sufficiently close to $\frac{1}{2}$ (and sufficiently close is a notion depending on e: if e is large, then $|x - \frac{1}{2}|$ must be very small). The difference between the number of those elements in the spectrum (we consider $Sp^\infty_{a,b}$ and $Sp_{r,s}$) that are in $(x, 1+x)$ and those that are in $(\frac{1}{2}, \frac{3}{2})$ is equal to the multiplicity of $\frac{1}{2}$ in the spectrum, so it is $\simeq 0$ as e goes to infinity. Since we are interested in the asymptotics only, we will work with the interval $(\frac{1}{2}, \frac{3}{2})$.

Let us first deal with $S_{r,s}$. Notice that $S_{r,s}$ is twice the number of elements in $Sp_{r,s}$ in the interval $(\frac{1}{2}, 1)$, that is,

$$\begin{aligned}
\frac{1}{2} S_{r,s} &= \sum_{i=1}^{r-1} \sum_{j=1}^{s-1} \begin{cases} 1 & \text{if } \frac{i}{r} + \frac{j}{s} \in (\frac{1}{2}, 1), \\ 0 & \text{otherwise} \end{cases} \\
&= \sum_{i=1}^{r-1} \#\left\{ j \colon j \in \left(\frac{s}{2} - \frac{is}{r}, s - \frac{is}{r}\right) \cap \{1, 2, \dots, s-1\} \right\} \\
&= \sum_{i=1}^{\lfloor r/2 \rfloor} \#\left\{ j \in \left(\frac{s}{2} - \frac{is}{r}, s - \frac{is}{r}\right) \cap \{1, 2, \dots, s-1\} \right\} \\
&\quad + \sum_{i=\lfloor r/2 \rfloor + 1}^{r-1} \#\left\{ j \in \left(0, s - \frac{is}{r}\right) \cap \{1, 2, \dots, s-1\} \right\}
\end{aligned}$$

$$\overset{(1)}{\simeq} \sum_{i=1}^{\lfloor r/2 \rfloor} \frac{s}{2} + \sum_{i=\lfloor r/2 \rfloor+1}^{r-1} \left(s - \frac{is}{r}\right)$$
$$= \frac{s}{2}\left\lfloor \frac{r}{2} \right\rfloor + \left(r - 1 - \left\lfloor \frac{r}{2} \right\rfloor\right)s - \frac{s}{2r}\left(r(r-1) - \left\lfloor \frac{r}{2} \right\rfloor\left(\left\lfloor \frac{r}{2} \right\rfloor + 1\right)\right)$$
$$= s\left(\frac{r}{2} - \frac{1}{2}\left\lfloor \frac{r}{2} \right\rfloor - \frac{1}{2} + \frac{1}{2r}\left\lfloor \frac{r}{2} \right\rfloor^2 + \frac{1}{2r}\left\lfloor \frac{r}{2} \right\rfloor\right).$$

The asymptotic equality (1) holds because the difference between the number of integer elements in an interval is equal to its length up to adding or subtracting 1. On replacing the number of integers by the length of the interval, the error we make is at most ± 1 at most $r-1$ times, but $r \leq b$ is small when compared to w.

If r is odd, then we get $S_{r,s} \simeq 2s(\frac{3}{8}r - \frac{1}{4} - \frac{1}{8r})$. If r is even, then we get $S_{r,s} = 2s(\frac{3}{8}r - \frac{1}{4})$.

We now deal with the spectrum at infinity. Our first observation is that

$$S_\infty \simeq 2 \sum_{p=w/2}^{w-1} \left\lfloor \frac{pb}{w} \right\rfloor.$$

Indeed, in Table 1 we have listed the elements in the spectrum. But the contribution from all the items but the fourth and fifth in the table is bounded by a function depending only on a, b, and not on w. We write

$$\sum_{p=w/2}^{w-1} \left\lfloor \frac{pb}{w} \right\rfloor \cong \sum_{p=w/2}^{w} \left(\frac{pb}{w} - \frac{1}{2}\right) - \sum_{p=w/2}^{w} \left\langle \frac{pb}{w} \right\rangle, \tag{6.6}$$

where $\langle \cdot \rangle$ is the sawtooth function (see (5.8)), and "$\cong$" means that we neglect the contribution of at most b instances of p, where $\frac{pb}{w}$ is an integer.

We have $\sum_{p=w/2}^{w-1} p = \frac{3}{8}w^2 +$ lower order terms in w. Therefore, the first sum in (6.6) gives a value asymptotically equal to $w(\frac{3}{8}b - \frac{1}{4})$. As for the second sum, in (6.6) we shall use the following lemma.

LEMMA 6.7. *For fixed $b > 1$, we have*

$$\lim_{w\to\infty} \frac{1}{w} \sum_{p=w/2}^{w-1} \left\langle \frac{pb}{w} \right\rangle = \begin{cases} 0, & b \text{ is even,} \\ \frac{1}{8b}, & b \text{ is odd.} \end{cases}$$

The proof of Lemma 6.7 is postponed until Section 6.6. Now we resume the proof of Theorem 6.3. We observe that

$$s(r-1) \simeq 2g \simeq w(b-1). \tag{6.8}$$

Our computations insofar show that

$$S_{r,s} \simeq \begin{cases} \frac{3}{4}g + \frac{1}{4}s, & r \text{ is even,} \\ \frac{3}{4}g + (\frac{1}{4} - \frac{1}{4r})s, & r \text{ is odd.} \end{cases}$$

As for S_∞, we get

$$S_\infty \simeq \begin{cases} \frac{3}{4}g + \frac{1}{4}w, & b \text{ is even,} \\ \frac{3}{4}g + (\frac{1}{4} - \frac{1}{4b})w, & b \text{ is odd.} \end{cases}$$

Depending on the parity of b and r, we have four cases.

- If b and r are even, then $S_{r,s}$ is asymptotically greater than S_∞ unless $s \simeq w$. But $s(r-1) \simeq w(b-1)$, so this exceptional case can occur only if $b = r$. Theorem 1.3 obstructs asymptotically all cases with $r < b$.
- If b and r are odd, then we compare $w\frac{b-1}{b}$ with $s\frac{r-1}{r}$. Since $s(r-1) \simeq w(b-1)$, we see that for large e, the quantity S_∞ is smaller than $S_{r,s}$ unless $b = r$.
- The case b odd and r even is completely obstructed (for e large) by Theorem 1.3.
- Suppose b is even and r is odd. We compare $s\frac{r-1}{r}$ with $w\frac{b-1}{b-1}$. If $r < b-1$, then we have asymptotically $S_{r,s} > S_\infty$, so Theorem 1.3 applies. We cannot obstruct the case $r = b-1$. □

6.5. *Dedekind Sums and Reciprocity Laws*

For the reader's convenience, we include some elementary facts about Dedekind sums. We refer to [35] or [17] for more detail.

Let p, q be positive integers. The *Dedekind sum* is the following expression:

$$s(p,q) = \sum_{i=0}^{q-1} \left\langle \frac{i}{q} \right\rangle \left\langle \frac{pi}{q} \right\rangle.$$

Dedekind sums appear in number theory in various places; see, for instance, [35]; in combinatorics, when they are used to compute the number of lattice points in polytopes; and in low-dimensional topology, where they appear in connection with lens spaces, Seifert fibered manifolds and signatures of torus knots: see [17] for more detail.

The most important result we use is the Dedekind reciprocity law.

THEOREM 6.9. *If p and q are coprime, then*

$$s(p,q) + s(q,p) = \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq} - 3 \right).$$

There are many ways to generalize the Dedekind sum, of which we use only one, the so-called Dedekind–Rademacher sum $D(p,q,r)$ defined as

$$D(p,q,r) = \sum_{i=0}^{r-1} \left\langle \frac{pi}{r} \right\rangle \left\langle \frac{qi}{r} \right\rangle.$$

The Dedekind reciprocity law generalizes to the Rademacher reciprocity law (or the "three-term law"), which is stated as follows.

THEOREM 6.10. *If p, q, r are pairwise coprime, then*

$$D(p,q,r) + D(r,p,q) + D(q,r,p) = \frac{p^2+q^2+r^2-3pqr}{12pqr}.$$

6.6. *Proof of Lemma 6.7*

Our goal is to compute the limit of the sequence $\frac{1}{w}a_w$, where

$$a_w := \sum_{p=w/2}^{w-1} \left\langle \frac{pb}{w} \right\rangle.$$

We write

$$\begin{aligned} a_w &= \sum_{p=w/2}^{w-1} \left\langle \frac{pb}{w} \right\rangle = \sum_{p=w/2}^{w-1} \left\langle \frac{pb}{w} \right\rangle \left\lfloor \frac{2p}{w} \right\rfloor \\ &= \sum_{p=0}^{w-1} \left\langle \frac{pb}{w} \right\rangle \left\lfloor \frac{2p}{w} \right\rfloor = \sum_{p=0}^{w-1} \left\langle \frac{pb}{w} \right\rangle \left(\frac{2p}{w} - \left\langle \frac{2p}{w} \right\rangle + \frac{1}{2} \right) \\ &= \sum_{p=0}^{w-1} \left\langle \frac{pb}{w} \right\rangle \frac{2p}{w} - \sum_{p=0}^{w-1} \left\langle \frac{pb}{w} \right\rangle \left\langle \frac{2p}{w} \right\rangle + \frac{1}{2} \sum_{p=0}^{w-1} \left\langle \frac{pb}{w} \right\rangle. \end{aligned} \tag{6.11}$$

We write

$$\begin{aligned} b_w &= \sum_{p=0}^{w-1} \left\langle \frac{pb}{w} \right\rangle \frac{2p}{w}, \\ c_w &= \sum_{p=0}^{w-1} \left\langle \frac{pb}{w} \right\rangle \left\langle \frac{2p}{w} \right\rangle, \\ d_w &= \frac{1}{2} \sum_{p=0}^{w-1} \left\langle \frac{pb}{w} \right\rangle, \end{aligned}$$

so that $a_w = b_w - c_w + d_w$. The computation of the limit $\frac{1}{w}a_w$ splits into three lemmas of increasing difficulty.

LEMMA 6.12. *We have $d_w = 0$.*

Proof. Suppose b and w are coprime. Then b is invertible modulo w, so after changing variables, the sum becomes $\sum_{p=0}^{w-1} \langle \frac{p}{w} \rangle$, which is zero by elementary calculations. If b and w are not coprime, then we write $c = \gcd(b, w)$, $w' = w/c$, $b' = b/c$, and the sum is equal to $c \sum_{p=0}^{w'-1} \langle pb'/w' \rangle$, so we reduce to the previous case. □

LEMMA 6.13. *We have* $\lim \frac{1}{w} b_w = \frac{1}{6b}$.

Proof. Let again $c = \gcd(b, w)$ and $w' = w/c$, $b' = b/c$. We have

$$b_w = \frac{2c}{b} \sum_{p=0}^{w'-1} \left\langle \frac{pb'}{w'} \right\rangle \frac{pb'}{w'}.$$

Like in Lemma 6.12, we write

$$\sum_{p=0}^{w'-1} \left\langle \frac{pb'}{w'} \right\rangle \frac{pb'}{w'} = \sum_{p=0}^{w'-1} \left\langle \frac{pb'}{w'} \right\rangle \left(\frac{pb'}{w'} - \frac{1}{2} \right) + \frac{1}{2} \sum_{p=0}^{w'-1} \left\langle \frac{pb'}{w'} \right\rangle = \sum_{p=0}^{w'-1} \left\langle \frac{pb'}{w'} \right\rangle^2,$$

where we have used the fact that $\sum_{p=0}^{w'-1} \langle pb'/w' \rangle = 0$. By assumption, b' and w' are coprime. Substituting for p the multiple pb'', where b'' is the inverse of b' modulo w', we obtain

$$\sum_{p=0}^{w'-1} \left\langle \frac{pb'}{w'} \right\rangle^2 = \sum_{p=0}^{w'-1} \left\langle \frac{p}{w'} \right\rangle^2 = s(1, w').$$

By Theorem 6.9 (and the elementary observation that $s(w', 1) = 0$) the expression evaluates to $\frac{1}{12}(w' + 2/w' - 3)$. The lemma follows immediately. □

LEMMA 6.14. *If b is odd, then we have* $\lim \frac{1}{w} c_w = \frac{1}{24b}$, *whereas if b is even, then* $\lim \frac{1}{w} c_w = \frac{1}{6b}$.

Proof. The term c_w is the Rademacher–Dedekind symbol $D(2, b, w)$. We would like to use the Rademacher reciprocity law, but in order to do this, we need to pass to a subsequence of w because we need that 2, b, w are pairwise coprime. The following claim allows us to compute $\lim \frac{1}{w} c_w$ by passing to a subsequence given by some arithmetic progression.

CLAIM. $|c_w - c_{w+1}| < \frac{3}{2} b + \frac{33}{4}$.

To prove the claim, notice that $\delta_{p,b,w} := \langle \frac{pb}{w} \rangle - \langle \frac{pb}{w+1} \rangle$ is equal to $\frac{pb}{w(w+1)} < \frac{b}{w}$, unless any one of the three conditions holds:

- $\lfloor \frac{pb}{w} \rfloor > \lfloor \frac{pb}{w+1} \rfloor$;
- $\lfloor \frac{pb}{w} \rfloor$ is an integer;
- $\lfloor \frac{pb}{w+1} \rfloor$ is an integer.

In each of these cases we have $\delta_{p,b,w} < 1$. The last two cases occur at most b times each. If the first case occurs, then there exists $k \in \mathbb{Z}$ such that $\frac{pb}{w} \geq k > \frac{pb}{w+1}$. This implies that $p \in [\frac{wk}{b}, \frac{(w+1)k}{b})$. Notice that $p < w$; hence, $k < b$, and the length of the interval is at most 1. Therefore, it cannot contain more than one integer. Hence, the first case occurs at most b times. Combining these cases (they are not mutually exclusive, but we can be slightly wasteful), we obtain

$$|\delta_{p,b,w}| \leq \begin{cases} 1 & \text{for at most } 3b \text{ values of } p \in \{0, 1, \dots, w-1\}, \\ \frac{1}{w} & \text{for all other values of } p. \end{cases} \tag{6.15}$$

We now write

$$\begin{aligned}
&\left|\left\langle\frac{pb}{w}\right\rangle\left\langle\frac{2p}{w}\right\rangle-\left\langle\frac{pb}{w+1}\right\rangle\left\langle\frac{2p}{w+1}\right\rangle\right| \\
&\quad\leq\left|\left\langle\frac{pb}{w}\right\rangle-\left\langle\frac{pb}{w+1}\right\rangle\right|\left|\left\langle\frac{2b}{w}\right\rangle\right|+\left|\left\langle\frac{2b}{w+1}\right\rangle-\left\langle\frac{2b}{w}\right\rangle\right|\left|\left\langle\frac{pb}{w+1}\right\rangle\right| \\
&\quad\leq\frac{1}{2}|\delta_{p,b,w}|+\frac{1}{2}|\delta_{p,2,w}|,
\end{aligned}$$

where we used the fact that $|\langle x\rangle|\leq\frac{1}{2}$ for all x. We sum up this inequality over $p\in\{0,\dots,w-1\}$. Combining this with (6.15) and adding a $\frac{1}{4}$ for the term $p=w$, which appears in the formula for c_{w+1} (and does not appear in the formula for c_w), we obtain

$$\begin{aligned}
|c_w-c_{w+1}|&\leq\frac{1}{4}+\frac{1}{2}\sum_{p=0}^{w-1}|\delta_{p,b,w}|+|\delta_{p,2,w}|\leq\frac{1}{4}+\frac{1}{2}(3b+1)+\frac{1}{2}(3\cdot 2+1) \\
&=\frac{3}{2}b+\frac{33}{4}.
\end{aligned}$$

This proves the claim. We now resume the proof of Theorem 6.3. We split it into two cases.

Case 1. b is odd. Suppose w is coprime to b and 2. By the Rademacher reciprocity law (Theorem 6.10),

$$c_w=\frac{b^2+w^2+4-6bw}{24bw}-\sum_{p=0}^{2}\left\langle\frac{pw}{2}\right\rangle\left\langle\frac{bw}{2}\right\rangle-\sum_{p=0}^{b-1}\left\langle\frac{2p}{b}\right\rangle\left\langle\frac{pw}{b}\right\rangle.$$

The two sums on the left are sums of bounded (as w goes to infinity) number of summands, each of which bounded by $\frac{1}{4}$. It follows that

$$\lim_{\substack{w\to\infty\\ w\text{ coprime with 2 and }b}}\frac{1}{w}c_w=\lim_{w\to\infty}\frac{1}{w}\frac{b^2+w^2+4-8bw}{24bw}=\frac{1}{24b}.$$

By the claim the limit of the subsequence of $\frac{1}{w}c_w$ over w coprime with 2 and b is the same as the limit of the sequence $\frac{1}{w}c_w$.

Case 2. b is even. Suppose w is even and $\gcd(b,w)=2$. Write $w'=w/2$, $b'=b/2$. Then we have

$$c_w=2\sum_{p=0}^{w'-1}\left\langle\frac{pb'}{w'}\right\rangle\left\langle\frac{p}{w'}\right\rangle=2s(b',w').$$

By the Dedekind reciprocity law (Theorem 6.9),

$$s(b',w')=\frac{1}{12}\left(\frac{w'}{b'}+\frac{w'}{b'}+\frac{1}{b'w'}-3\right)-s(w',b').$$

The expression $s(w',b')$ is a sum of b' terms, each bounded by $\frac{1}{4}$; hence, $\lim_{w'\to\infty}\frac{1}{w'}s(w',b')=0$. This means that

$$\lim_{\substack{w\to\infty\\ \gcd(b,w)=2}}\frac{1}{w}c_w=\frac{1}{6b}.$$

By the claim the limit of a subsequence of $\frac{1}{w}c_w$ is the same as the limit of the full sequence. □

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