Michigan Math. J. 65 (2016), 715–747

Spectral Triples from Stationary Bratteli Diagrams

J. Kellendonk & J. Savinien

Abstract. We define spectral triples for stationary Bratteli diagrams and study associated zeta functions, traces of heat kernels, and their spectral states. We observe that the zeta functions are periodic with purely imaginary periods and that the Seeley coefficients are $\log(t)$ periodic. We interpret these as a sign of self-similarity. We describe several examples and emphasize the case of substitution tiling spaces. For such tilings, the spectral measure turns out to be the unique measure that is ergodic under the translation action.

1. Introduction

Even though noncommutative geometry [4] was invented to describe (virtual) noncommutative spaces, it turned out also to provide new perspectives on (classical) commutative spaces. In particular, Connes' idea of spectral triples aiming at a spectral description of geometry has generated new concepts, or shed new light on existing ones, for topological spaces: dimension spectrum, Seeley type coefficients, spectral state are notions derived from the spectral triple, and we will talk about them here. Indeed, we study in this paper certain spectral triples for commutative algebras that are associated with stationary Bratteli diagrams, that is, with the space of infinite paths on a finite oriented graph. Such Bratteli diagrams occur in systems with self-similarity such as the tiling systems defined by substitutions.

Our construction follows from earlier ones for metric spaces, which go under the name "direct sum of point pairs" [3] or "approximating graph" [15], suitably adapted to incorporate the self-similar symmetry. The construction is therefore more rigid. The so-called Dirac operator D of the spectral triple will depend on a parameter ρ, which is related to the self-similar scaling. We observe a new feature that, we believe, ought to be interpreted as a sign of self-similarity: The zeta function is periodic with purely imaginary period $\frac{2\pi i}{\log \rho}$. Correspondingly, what corresponds to the Seeley coefficients (in the case of manifolds) in the expansion of the trace of the heat-kernel e^{-tD^2} is here given by functions of $\log t$ that are $\frac{2\pi}{\log \rho}$-periodic. This has consequences for the usual formulae for tensor products of spectral triples. If we take the tensor product of two such triples and compare the spectral states $\mathcal{T}_{1,2}$ for the individual factors with the spectral state $\mathcal{T}$ of the tensor product, then a formula like $\mathcal{T}(A_1 \otimes A_2) = \mathcal{T}_1(A_1)\mathcal{T}_2(A_2)$ will not always hold due to resonance phenomena of the involved periodicities.

Received January 17, 2015. Revision received August 30, 2016.
Work supported by the ANR grant SubTile no. NT09 564112.

Our main application will be to the tiling space of a substitution tiling. In this case, the finite oriented graph defining the spectral triple is the substitution graph. Moreover, the spectral triple is essentially described by the tensor product of two spectral triples of the type mentioned, one for the transversal and one for the longitudinal direction.

Summary of Results

After a quick introduction to spectral triples, we are first concerned with the properties of their zeta functions in the case that the expansion of the trace of the heat kernel e^{-tD^2} is not simply an expansion into powers of t but of the type

$$\mathrm{Tr}(e^{-tD^2}) \overset{t\to 0}{\sim} f(-\log t)t^{\alpha} \tag{1}$$

with $\Re(\alpha) < 0$, where $f : \mathbb{R}_+ \to \mathbb{R}$ a bounded locally integrable function such that $\lim_{s\to 0^+} s\mathcal{L}[f](s)$ exists and is nonzero, where $\mathcal{L}$ is the Laplace transform. A nonconstant f in that expansion has consequences that we did not first expect. We are led in Section 2.2 to study classes of operators on $\mathcal{B}(\mathfrak{H})$ that have a compatible behavior. An operator $A \in \mathcal{B}(\mathfrak{H})$ is *weakly regular* if there exists a bounded locally integrable function $f_A : \mathbb{R}_+ \to \mathbb{R}$ for which $\lim_{s\to 0^+} s\mathcal{L}[f_A](s)$ exists and is nonzero, such that

$$\mathrm{Tr}(e^{-tD^2}A) \overset{t\to 0}{\sim} f_A(-\log t)t^{\alpha}, \tag{2}$$

where α is the same as in equation (1). For such operators, the spectral state does not depend on a choice of a Dixmier trace and is given by

$$\mathcal{T}(A) = \lim_{s\to 0^+} \frac{\mathcal{L}[f_A](s)}{\mathcal{L}[f](s)},$$

where f is the same as in equation (1); see Lemma 2.3. We also define *strongly regular* operators, for which we have in particular $f_A = \mathcal{T}(A)f$ in equation (2) (see Lemma 2.4). Regular operators have an interesting behavior under tensor product, which we will use in the applications to tilings. If the spectral triple is a tensor product, that is, $(\mathcal{A}, \mathfrak{H}, D) = (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathfrak{H}_1 \otimes \mathfrak{H}_2, D_1 \otimes \mathbf{1} + \chi \otimes D_2)$, where χ is a grading on $(\mathcal{A}_1, \mathfrak{H}_1, D_1)$, then we have:

$$\mathcal{T}(A_1 \otimes A_2) = \lim_{s\to 0^+} \frac{\mathcal{L}[f_{A_1} f_{A_2}](s)}{\mathcal{L}[f_1 f_2](s)},$$

where f_i is as in (1) for D_i, and f_{A_i} as in (2) for A_i, for each factor $i = 1, 2$ of the tensor product; see Lemma 2.7. In general, only if both A_1 and A_2 are strongly regular for the individual spectral triples, the state will factorize as $\mathcal{T}(A_1 \otimes A_2) = \mathcal{T}_1(A_1)\mathcal{T}_2(A_2)$. Here $\mathcal{T}_i$ denotes the spectral state of $(\mathcal{A}_i, \mathfrak{H}_i, D_i)$, $i = 1, 2$ (see Corollary 2.8). It is easy to build examples for which this equality fails for more general operators: for instance, we could have $\mathcal{T}_1(A_1) = 0$ and $\mathcal{T}(A_1 \otimes A_2) \neq 0$; see the end of Section 3.3 for such a counterexample.

In Section 3, we study spectral triples associated with a stationary Bratteli diagram, that is, for the C*-algebra of continuous functions on the Cantor set of (half-)infinite paths on a finite oriented graph. These depend on the matrix

A encoding the edges between two levels in the diagram (called here a *graph matrix* and assumed to be primitive), a parameter $\rho \in (0, 1)$ to account for self-similar scaling and a horizontal structure $\hat{\mathcal{H}}$ (a set of edges linking the edges of the Bratteli diagram). We determine the spectral information of such spectral triples. In Theorem 3.3, we derive the Connes distance and show under which conditions it yields the Cantor topology on the path space. We compute the zeta function $\zeta(z) = \mathrm{Tr}(|D|^{-z})$ and the expansion of the heat kernel.

THEOREM (Theorems 3.4 and 3.6 and Remark 3.7 in the main text). *Consider a spectral triple associated with a stationary Bratteli diagram with graph matrix A and parameter $0 < \rho < 1$. Assume that A is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_p$.*

- *The zeta function ζ extends to a meromorphic function on $\mathbb{C}$ that is invariant under translation $z \mapsto z + \frac{2\pi\imath}{\log\rho}$. It has only simple poles, and these are at $(\log\lambda_j + 2\pi\imath k)/(-\log\rho)$, $k \in \mathbb{Z}$, $j = 1, \dots, p$. In particular, the spectral dimension (abscissa of convergence of ζ) is equal to $s_0 = \log\lambda_{\mathrm{PF}}/\log\rho$, where λ_{PF} is the Perron–Frobenius eigenvalue of A. The residue at the pole $(\log\lambda_j + 2\pi\imath k)/(-\log\rho)$ is given by $C^j_{\hat{\mathcal{H}}}\lambda_j/(-\log\rho)$.*
- *The Seeley expansion of the heat-kernel is given by*
$$\mathrm{Tr}(e^{-tD^2}) = \sum_{j:|\lambda_j|>1} C^j_{\hat{\mathcal{H}}}\mathfrak{p}_{-2\log\rho,\log\lambda_j}(-\log t)t^{\log\lambda_j/(2\log\rho)} + C^{j_0}_{\hat{\mathcal{H}}}\frac{-\log t}{-2\log\rho} + h(t),$$
where h is entire, $\mathfrak{p}_{r,a}$ is an r-periodic smooth function, and j_0 is such that $\lambda_{j_0} = 1$.

The constants $C^j_{\hat{\mathcal{H}}}$ are given in (9); they depend on the choice of horizontal edges $\hat{\mathcal{H}}$. The function $\mathfrak{p}_{r,a}$ is explicitly given in equations (15) and (16), and its average over a period is $\bar{\mathfrak{p}}_{r,a} = \frac{1}{r}\Gamma(\frac{a}{r})$. If A is not diagonalizable, then ζ has poles of higher order and the heat-kernel expansion is more involved (with powers of $\log(t)$ depending on the order of the poles); see Remark 3.5 and Theorem 3.6.

In Section 4, we apply our findings to substitution tiling spaces Ω_Φ. We consider geometric substitutions of the simplest form, as in [9], which are defined by a decomposition rule followed by a rescaling, that is, each prototile is decomposed into smaller tiles, which, when stretched by a common factor $\theta > 1$ (the dilation factor), are congruent to some original tile. The result of the substitution on a tile is called a supertile (and by iteration then an nth-order supertile). If we apply only the decomposition rule, then we obtain smaller tiles, which we call microtiles.

The approximating graph for Ω_Φ is constructed with the help of *doubly* infinite paths over the substitution graph. *Half*-infinite paths describe its canonical transversal. We use this structure to construct a spectral triple for $C(\Omega_\Phi)$ essentially as a tensor product of two spectral triples, one obtained from the substitution graph and the other from the reversed substitution graph. Indeed, the first of the two spectral triples describes the transversal, and the second the longitudinal part of Ω_Φ. Since the graph matrix of the reversed graph is the transpose of the original graph matrix, we will have to deal with only one set of eigenvalues $\lambda_1, \dots, \lambda_p$. It turns out wise, however, to keep two dilation parameters ρ_{tr} and ρ_{lg} as independent parameters, although they will later be related to the dilation factor θ of the substitution. We obtain the following:

THEOREM (Theorem 4.7 in the main text). *The spectral triple for $C(\Omega_\Phi)$ is finitely summable with spectral dimension*

$$s_0 = \frac{d \log \theta}{-\log \rho_{\mathrm{tr}}} + \frac{d \log \theta}{-\log \rho_{\mathrm{lg}}},$$

which is the sum of the spectral dimensions of the triples associated with the transversal and with the longitudinal part.

The zeta function $\zeta(z)$ has a simple pole at s_0 with positive residue.

The spectral measure is equal to the unique translation-invariant ergodic probability measure on Ω_Φ.

In an older version of this work [14], we investigated the problem of extension of the quadratic form $Q(f, g) = \mathcal{T}([D, f]^*[D, g])$ (defined on a properly chosen domain) to a Dirichlet form on the tiling space of a Pisot substitution tiling. These tilings share further properties that allow for the calculation of the Dirichlet form. It turns out that the corresponding Laplace operator is given by a standard elliptic second-order differential operator on the maximal equicontinuous factor (a torus or an inverse limit of tori). The somewhat technical details can be found in [14].

2. Preliminaries for Spectral Triples

A spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ for a unital C^*-algebra $\mathcal{A}$ is given by a Hilbert space $\mathfrak{H}$ carrying a faithful representation π of $\mathcal{A}$ by bounded operators and an unbounded self-adjoint operator D on $\mathfrak{H}$ with compact resolvent such that the set of $a \in \mathcal{A}$ for which the commutator $[D, \pi(a)]$ extends to a bounded operator on $\mathfrak{H}$ forms a dense subalgebra $\mathcal{A}_0 \subset \mathcal{A}$. The operator D is referred to as the Dirac operator. In all examples here, we will assume it to be invertible, with compact inverse. The spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ is termed *even* if there exists a $\mathbb{Z}/2$-grading operator χ on $\mathfrak{H}$ that commutes with $\pi(a)$, $a \in \mathcal{A}$, and anticommutes with D.

We consider the case of commutative C^*-algebras $\mathcal{A} = (C(X), \|\cdot\|_\infty)$ of continuous functions over a compact Hausdorff space X with the sup-norm, so we may speak about a spectral triple for the space X. The spaces we consider are far from being manifolds. Spectral triples for such spaces have been considered earlier for fractals in [7; 8; 16] and for ultrametric Cantor sets and tiling spaces in [17; 12; 15].

2.1. Zeta Function and Heat Kernel

Since the resolvent of D is supposed to be compact, $\mathrm{Tr}(|D|^{-s})$ can be expressed as a Dirichlet series in terms of the eigenvalues of $|D|$.[1] The spectral triple is called *finitely summable* if the Dirichlet series is summable for some $s \in \mathbb{R}$ and hence defines a function

$$\zeta(z) = \mathrm{Tr}(|D|^{-z})$$

on some half-plane $\{z \in \mathbb{C} : \Re(z) > s_0\}$, which is called the *zeta function* of the spectral triple. The smallest possible value for s_0 (the *abscissa of convergence* of the Dirichlet series) is called the *metric dimension* of the spectral triple. We call ζ *simple* if $\lim_{s\to s_0^+}(s-s_0)\zeta(s)$ exists. This is, for instance, the case if ζ can be meromorphically extended and then has a simple pole at s_0. We will then also refer to the meromorphic extension simply as the zeta function of the triple.

Another quantity to look at is the heat kernel e^{-tD^2} of the square of the Dirac operator. Thanks to the Mellin transform

$$\Gamma(s)\mu^{-2s} = \int_0^\infty e^{-t\mu^2} t^{s-1}\, dt,$$

where $\mu > 0$, and $\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1}\, dt$ is the gamma function, we can relate the zeta function to the heat kernel as follows:

$$\Gamma(s)\zeta(2s) = \int_0^\infty \mathrm{Tr}(e^{-tD^2}) t^{s-1}\, dt.$$

This of course makes sense only if e^{-tD^2} is trace class for all $t > 0$, which is anyway a necessary condition for finite summability. Notice that the trace class condition implies also that $s \mapsto \int_\delta^\infty \mathrm{Tr}(e^{-tD^2}) t^{s-1}\, dt$ is holomorphic for all $\delta > 0$.

The last formula is particularly useful if we know the asymptotic expansion of $\mathrm{Tr}(e^{-tD^2})$ at $t \to 0$ or only its leading term.[2] It is well known that the form of the asymptotic expansion is related to the singularites of the zeta-function [5; 11]. For instance, an expansion of the form

$$\mathrm{Tr}(e^{-tD^2}) = \sum_\alpha c_\alpha t^\alpha + h(t)$$

with $\Re(\alpha) < 0$, $c_\alpha \in \mathbb{C}$, and a function h bounded at 0 (in particular, without logarithmic terms like $\log t$) implies that the zeta function has a simple pole at -2α with residue equal to $2c_\alpha/\Gamma(-\alpha)$ and is regular at 0 [5]. We will see, however, that the situation is quite different in our case, where we have to replace c_α by

[1] For simplicity, we suppose (as will be the case in our applications) that $\ker(D)$ is trivial; otherwise, we would have to work with $\mathrm{Tr}_{\ker(D)^\perp}(|D|^{-s})$ or remove the kernel of D by adding a finite-rank perturbation.

[2] A function $f : \mathbb{R}^{>0} \to \mathbb{C}$ is asymptotically equivalent to $g : \mathbb{R}^{>0} \to \mathbb{R}$ as $t \to 0$, written $f \overset{t\to 0}{\sim} g$, if $|f-g| = o(|g|)$. The notation $f = O(g)$ means that $\exists M > 0$, $\exists \delta > 0$, $\forall 0 < t < \delta$: $|f(t)| \leq M|g(t)|$, and $f = o(g)$ means that $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall 0 < t < \delta$: $|f(t)| \leq \epsilon |g(t)|$.

functions that are periodic in $\log t$. Recall the Laplace transform of a function f at s:

$$\mathcal{L}[f](s) := \int_0^\infty f(x)e^{-sx}\,dx. \tag{3}$$

We assume therefore in the sequel that the asymptotic behavior of the trace of the heat kernel is given by

$$\mathrm{Tr}(e^{-tD^2}) \overset{t\to 0}{\sim} f(-\log t)t^\alpha, \tag{4}$$

where $\alpha < 0$, and $f : \mathbb{R}^{\geq 0} \to \mathbb{R}$ is a bounded, locally integrable function for which $\lim_{s\to 0} s\mathcal{L}[f](s)$ exists and is different from 0. This is the weakest assumption needed for ζ to be simple and have nonnegative abscissa of convergence, and to be able to compute its residues explicitly, as the following lemma shows (see also Remark 2.2 for a regular example of such f).

LEMMA 2.1. *If the trace of the heat kernel satisfies* (4), *then* ζ *is simple*, *has abscissa of convergence* f

$$s_0 = -2\alpha, \quad \textit{and} \quad \frac{1}{2}\Gamma\left(\frac{s_0}{2}\right)\lim_{s\to s_0^+}(s-s_0)\zeta(s) = \lim_{s\to 0} s\mathcal{L}[f](s).$$

If, *moreover*, $\mathcal{L}[f](s)$ *admits a meromorphic extension with simple pole at* 0 *and* $\mathrm{Tr}(e^{-tD^2}) - f(-\log t)t^\alpha = O(t^\beta)$ (*with* $\beta > \alpha$), *then* $\zeta(s)$ *has a simple pole at* $s_0 = -2\alpha$, *and hence* $\frac{1}{2}\Gamma(\frac{s_0}{2})\,\mathrm{Res}(\zeta, s_0) = \mathrm{Res}(\mathcal{L}[f], 0) = \lim_{s\to 0} s\mathcal{L}[f](s)$.

Proof. We adapt the arguments of [11]. Let $h(t) = \mathrm{Tr}(e^{-tD^2}) - f(-\log t)t^\alpha$ and $M = \sup_x |f(x)|$. Then, for all $\epsilon > 0$, there exists $\delta \leq 1$ such that $|h(t)| \leq \epsilon M t^\alpha$ if $t < \delta$. In particular, $H_\delta(s) := \int_0^\delta h(t)t^{s-1}\,dt$ satisfies $|H_\delta(s)| \leq \epsilon M \delta^{\alpha+s}/(\alpha+s)$, provided that $\alpha + s > 0$. Now, again for $\alpha + s > 0$,

$$\Gamma(s)\zeta(2s) = \int_0^\infty \mathrm{Tr}(e^{-tD^2})t^{s-1}\,dt = \int_0^\delta f(-\log t)t^{\alpha+s-1}\,dt + H_\delta(s) + g_\epsilon(s),$$

where $g_\epsilon(s) = \int_\delta^\infty \mathrm{Tr}(e^{-tD^2})t^{s-1}\,dt$ is holomorphic in s. This shows that $\zeta(2s)$ is finite for $s > -\alpha$. Furthermore,

$$\begin{aligned}\lim_{s\to-\alpha^+}(\alpha+s)\int_0^\delta f(-\log t)t^{\alpha+s-1}\,dt &= \lim_{s\to 0^+} s\int_{-\infty}^{\log\delta} f(-\tau)e^{\tau s}\,d\tau \\ &= \lim_{s\to 0^+} s\mathcal{L}[f](s),\end{aligned}$$

where we have used in the last equation that $\lim_{s\to 0^+} s\int_{\log\delta}^0 f(-\tau)e^{\tau s}\,d\tau = 0$. Since $\epsilon > 0$ is arbitrary, we conclude that

$$\lim_{s\to-\alpha^+}(\alpha+s)\Gamma(s)\zeta(2s) = \lim_{s\to 0^+} s\mathcal{L}[f](s).$$

Hence, $s_0 = -2\alpha$ is the abscissa of convergence.

Now if $h(t)$ is of order t^β, then we can find $M > 0$ and $\delta > 0$ such that $|h(t)t^{-\beta}| \leq M$ if $0 < t < \delta$. If $\beta > \alpha$, then the function $t \mapsto t^{s+\beta-1}$ is integrable on $(0, \delta)$ as long as s lies in a sufficiently small neighborhood of $-\alpha$. Since $s \mapsto$

$t^{s+\beta-1}$ is holomorphic for all $t > 0$, we find that $H_\delta(s) = \int_0^\delta (h(t)t^{-\beta})t^{s+\beta-1}\,dt$ is holomorphic near $s = -\alpha$, which shows that $(\alpha + s)\zeta(2s)$ is holomorphic there, too. Thus, ζ has a simple pole at -2α, and we have the stated formula for its residue. □

REMARK 2.2. If $f(\tau) = e^{ia\tau}$, then $\mathcal{L}[f](s) = \frac{1}{s-ia}$. Thus, if f is the restriction of a periodic function of class C^1, then upon using its representation as a Fourier series, we see that $s\mathcal{L}[f](s)$ extends to an analytic function around 0, and

$$\lim_{s\to 0} s\mathcal{L}[f](s) = \bar{f},$$

the mean of f.

2.2. *Spectral State*

Given a bounded operator A on $\mathfrak{H}$ such that $|D|^{-s_0}A$ is in the Dixmier ideal, we consider the expression

$$\mathcal{T}(A) = \mathrm{Tr}_\omega(|D|^{-s_0}A)/\,\mathrm{Tr}_\omega(|D|^{-s_0}),$$

which depends a priori on the choice of Dixmier trace Tr_ω. With a little luck, however, $\lim_{s\to s_0^+} \frac{1}{\zeta(s)}\,\mathrm{Tr}(|D|^{-s}A)$ exists, and then [4]

$$\mathcal{T}(A) = \lim_{s\to s_0^+} \frac{1}{\zeta(s)}\,\mathrm{Tr}(|D|^{-s}A).$$

We provide here a criterion for that. Note that the Mellin transform allows us to write

$$\mathrm{Tr}(|D|^{-s}A) = \frac{1}{\Gamma(s/2)}\int_0^\infty \mathrm{Tr}(e^{-tD^2}A)t^{s/2-1}\,dt.$$

We call $A \in \mathcal{B}(\mathfrak{H})$ *strongly regular* if there exists a number c_A such that

$$\mathrm{Tr}(e^{-tD^2}A) - c_A\,\mathrm{Tr}(e^{-tD^2}) = o(\mathrm{Tr}(e^{-tD^2})).$$

If $c_A \neq 0$, then we can thus say that $\mathrm{Tr}(e^{-tD^2}A) \overset{t\to 0}{\sim} c_A\,\mathrm{Tr}(e^{-tD^2})$. In the context in which the heat kernel satisfies (4), it is useful to consider the notion of *weakly regular* operators $A \in \mathcal{B}(\mathfrak{H})$. These are operators that satisfy

$$\mathrm{Tr}(e^{-tD^2}A) \overset{t\to 0}{\sim} f_A(-\log t)t^\alpha, \tag{5}$$

where α is the same as in (4), and $f_A : \mathbb{R}^{\geq 0} \to \mathbb{C}$ is a bounded, nonzero, locally integrable function for which $\lim_{s\to 0} s\mathcal{L}[f_A](s)$ exists. Clearly, strongly regular operators are weakly regular, and $f_A = c_A f$ in this case, where f is given in equation (4) (we actually have $c_A = \mathcal{T}(A)$; see Corollary 2.4).

LEMMA 2.3. *Assume that the trace of the heat kernel satisfies* (4) *and that* $A \in \mathcal{B}(\mathfrak{H})$ *is weakly regular, that is, satisfies* (5). *Then* $\lim_{s\to s_0^+} \frac{1}{\zeta(s)}\,\mathrm{Tr}(|D|^{-s}A)$ *exists and is equal to*

$$\mathcal{T}(A) = \lim_{s\to 0} \frac{\mathcal{L}[f_A](s)}{\mathcal{L}[f](s)}.$$

Proof. Under the hypothesis, for all $\epsilon > 0$, we can find $\delta > 0$ such that if $s > s_0$, then

$$\left|\int_0^\delta (\mathrm{Tr}(e^{-tD^2}A) - f_A(-\log t)t^\alpha)t^{s/2-1}\,dt\right| \le \epsilon \int_0^\delta |f_A(-\log t)|t^{\alpha+s/2-1}\,dt$$
$$\le \epsilon M_A \frac{\delta^{\alpha+s/2}}{\alpha+s/2},$$

where M_A is an upper bound for $|f_A(-\log t)|$. Since $\int_\delta^\infty \mathrm{Tr}(e^{-tD^2})t^{s/2-1}\,dt$ and hence also $\int_\delta^\infty \mathrm{Tr}(e^{-tD^2}A)t^{s/2-1}\,dt$ are finite for all $\delta > 0$, we get ($\alpha = -s_0/2$)

$$\lim_{s\to s_0^+} \frac{1}{\Gamma(s/2)\zeta(s)}\left|\int_0^\infty \mathrm{Tr}(e^{-tD^2}A)t^{s/2-1}\,dt - \int_0^1 f_A(-\log t)t^{(s-s_0)/2-1}\,dt\right|$$
$$\le \epsilon \widetilde{M}_A.$$

Notice that $\int_0^1 f_A(-\log t)t^{s-1}\,dt = \mathcal{L}[f_A](s)$. Since ϵ was arbitrary, we conclude that

$$\lim_{s\to s_0^+}\frac{1}{\zeta(s)}\mathrm{Tr}(|D|^{-s}A) = \lim_{s\to s_0^+}\frac{\mathcal{L}[f_A]((s-s_0)/2)}{\Gamma(s/2)\zeta(s)} = \lim_{s\to 0}\frac{\mathcal{L}[f_A](s)}{\mathcal{L}[f](s)}. \qquad \square$$

COROLLARY 2.4. *If $A \in \mathcal{B}(\mathfrak{H})$ is strongly regular, then $\mathrm{Tr}(e^{-tD^2}A) \overset{t\to 0}{\sim} \mathcal{T}(A)\,\mathrm{Tr}(e^{-tD^2}A)$. In other words, the functions in equations (4) and (5) satisfy $f_A = \mathcal{T}(A)f$.*

Proof. If A is strongly regular, then it is also weakly regular with $f_A = c_A f$. The Laplace transform is linear, so $\mathcal{L}[f_A](s) = c_A\mathcal{L}[f](s)$, and Lemma 2.3 then implies $c_A = \mathcal{T}(A)$. $\square$

Order the eigenvalues of $|D|$ increasingly (without counting multiplicity) and let F_n be the nth eigenspace of $|D|$.

COROLLARY 2.5. *Let $A \in \mathcal{B}(\mathfrak{H})$ and $\bar{A}_n = \mathrm{Tr}_{F_n}(A|_{F_n})/\dim F_n$. If the limit*

$$\bar{A} = \lim_{n\to\infty}\bar{A}_n$$

exists, then A is strongly regular, and $\mathcal{T}(A) = \bar{A}$.

Proof. Write $c_n = e^{-t\mu_n^2}\dim F_n$, where μ_n is the nth eigenvalue of $|D|$. We have

$$\mathrm{Tr}(e^{-tD^2}A) - \bar{A}\,\mathrm{Tr}(e^{-tD^2}) = \sum_{n\ge 1}(\bar{A}_n - \bar{A})c_n.$$

Now fix $\epsilon > 0$ and choose an integer N_ϵ such that $|\bar{A}_n - \bar{A}| \le \epsilon$ for all $n \ge N_\epsilon$. Then the series of the r.h.s. can be bound by $(\sup_n|\bar{A}_n - \bar{A}|)\sum_{n<N_\epsilon} c_n + \epsilon\sum_{n\ge N_\epsilon} c_n$. Using $\sum_{n\ge N_\epsilon} c_n \le \mathrm{Tr}(e^{-tD^2})$, we find that, for all $\epsilon > 0$, there exists C_ϵ such that

$$|\mathrm{Tr}(e^{-tD^2}A) - \bar{A}\,\mathrm{Tr}(e^{-tD^2})| \le C_\epsilon + \epsilon\,\mathrm{Tr}(e^{-tD^2}).$$

Since $\mathrm{Tr}(e^{-tD^2})$ tends to $+\infty$ as t tends to 0, this shows that $\mathrm{Tr}(e^{-tD^2}A) \overset{t\to 0}{\sim} f_A(-\log t)t^\alpha$ with $f_A = \bar{A} f$. Applying Lemma 2.3, we see that $\mathcal{T}(A) = \bar{A}$. $\square$

In the commutative case where $\mathcal{A} = C(X)$ for a compact Hausdorff space X, we are particularly concerned with operators of the form $A = \pi(f)$ for $f \in C(X)$ or for any Borel-measurable function f on X. By the Riesz representation theorem the functional $f \mapsto \mathcal{T}(\pi(f))$ gives a measure on X, which we call the *spectral measure.*

2.3. *Direct Sums*

The direct sum of two spectral triples $(\mathcal{A}_1, \mathfrak{H}_1, D_1)$ and $(\mathcal{A}_2, \mathfrak{H}_2, D_2)$ is the spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ given by

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2, \qquad \mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2, \qquad D = D_1 \oplus D_2,$$

with direct sum representation. The zeta function ζ of the direct sum is clearly the sum of the zeta functions ζ_i of the summands, and thus its abscissa of convergence s_0 is equal to the largest abscissa of the two zeta functions ζ_i. Let $\mathcal{T}$ denote the spectral state of the direct sum triple, and $\mathcal{T}_i$ those of the summands, and assume that all zeta functions are simple. Then, for regular operators A_1 and A_2, we have

$$\mathcal{T}(A) = \frac{1}{c_1 + c_2}(c_1 \mathcal{T}(A_1) + c_2 \mathcal{T}(A_2)), \tag{6}$$

where $c_i = \lim_{s\to s_0^+}(s - s_0)\zeta_i(s)$. Notice that $c_1 = 0$ if the abscissa of convergence of ζ_1 is smaller than that of ζ_2, in which case $\mathcal{T}(A) = \mathcal{T}(A_2)$ (and similarly with 1 and 2 exchanged: $\mathcal{T}(A) = \mathcal{T}(A_1)$). Notice also that $c_1 + c_2 = \lim_{s\to s_0^+}(s - s_0)\zeta(s)$.

2.4. *Tensor Products*

The tensor product of an even spectral triple $(\mathcal{A}_1, \mathfrak{H}_1, D_1)$ with grading operator χ and another spectral triple $(\mathcal{A}_2, \mathfrak{H}_2, D_2)$ is the spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ given by

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \qquad \mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2, \qquad D = D_1 \otimes 1 + \chi \otimes D_2.$$

Notice that $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$. It follows that the trace of the heat kernel is multiplicative: $\mathrm{Tr}(e^{-tD^2}) = \mathrm{Tr}_{\mathfrak{H}_1}(e^{-tD_1^2})\,\mathrm{Tr}_{\mathfrak{H}_2}(e^{-tD_2^2})$. This allows us to obtain information on the spectral state of the zeta function.

LEMMA 2.6. *Suppose that* $\mathrm{Tr}_{\mathfrak{H}_1}(e^{-tD_1^2})$ *and* $\mathrm{Tr}_{\mathfrak{H}_2}(e^{-tD_2^2})$ *satisfy* (4) *with* $f = f_1$ *and* $f = f_2$, *respectively. Suppose that* $\lim_{s\to 0} s\mathcal{L}[f_1 f_2](s)$ *exists and is nonzero. Then the metric dimension* s_0 *of the tensor product spectral triple is the sum of the metric dimensions of the factors, and the zeta function* ζ *of the tensor product is simple with*

$$\frac{1}{2}\Gamma\left(\frac{s_0}{2}\right)\lim_{s\to s_0^+}(s - s_0)\zeta(s) = \lim_{s\to 0} s\mathcal{L}[f_1 f_2](s).$$

Proof. Due to the multiplicativity of the trace of the heat kernel, we have

$$\mathrm{Tr}(e^{-tD^2}) \overset{t\to 0}{\sim} f_1(-\log t) f_2(-\log t) t^{\alpha_1+\alpha_2},$$

and hence the result follows from Lemma 2.1. □

LEMMA 2.7. *Assume the conditions of Lemma 2.6. Let $A_1 \in \mathcal{B}(\mathfrak{H}_1)$ and $A_2 \in \mathcal{B}(\mathfrak{H}_2)$ be weakly regular with functions f_{A_1} and f_{A_2}. Then*

$$\mathcal{T}(A_1 \otimes A_2) = \lim_{s\to 0^+} \frac{\mathcal{L}[f_{A_1} f_{A_2}](s)}{\mathcal{L}[f_1 f_2](s)}.$$

Proof. Let $\epsilon > 0$ and choose $\delta > 0$ such that $|\mathrm{Tr}_{\mathfrak{H}_i}(e^{-tD_i^2} A_i) - f_{A_i}(-\log t)t^{\alpha_i}| \le \epsilon t^{\alpha_i}$ for $0 < t < \delta$. Then

$$\begin{aligned}
&\int_0^\delta \mathrm{Tr}_{\mathfrak{H}}(e^{-tD^2} A_1 \otimes A_2) t^{s-1}\, dt \\
&\quad = \int_0^\delta \mathrm{Tr}_{\mathfrak{H}_1}(e^{-tD_1^2} A_1)\,\mathrm{Tr}_{\mathfrak{H}_2}(e^{-tD_2^2} A_2) t^{s-1}\, dt \\
&\quad = \int_0^\delta (f_{A_1}(-\log t) f_{A_2}(-\log t) + O(\epsilon)) t^{\alpha_1+\alpha_2+s-1}\, dt \\
&\quad = \mathcal{L}[f_{A_1} f_{A_2}](\alpha_1+\alpha_2+s) + O(\epsilon)\frac{\delta^{\alpha_1+\alpha_2+s}}{\alpha_1+\alpha_2+s},
\end{aligned}$$

from which the result follows by similar arguments as before. □

COROLLARY 2.8. *Let A_1 and A_2 be weakly regular operators.*

(i) *If A_1 is strongly regular, then $\mathcal{T}(A_1 \otimes A_2) = \mathcal{T}_1(A_1)\mathcal{T}(\mathbf{1} \otimes A_2)$.*
(ii) *If both A_1 and A_2 are strongly regular, then $\mathcal{T}(A_1 \otimes A_2) = \mathcal{T}_1(A_1)\mathcal{T}_2(A_2)$.*

REMARK 2.9. The result of the corollary says that the spectral state factorizes for tensor products of strongly regular operators. This corresponds to the formula on page 563 in [4]. It should be noticed, however, that this factorization is in general not valid for tensor products of weakly regular operators since the Laplace transform of a product is not the product of the Laplace transforms. We consider examples of this type at the end of Section 3.3.

3. The Spectral Triple Associated with a Stationary Bratteli Diagram

An oriented graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the data of a set of vertices $\mathcal{V}$ and a set of edges $\mathcal{E}$ with two maps $\mathcal{E} \overset{r}{\underset{s}{\rightrightarrows}} \mathcal{V}$, one assigning to an edge ε its source vertex $s(\varepsilon)$ and the second assigning its range $r(\varepsilon)$. The *graph matrix* of $\mathcal{G}$ is the matrix A with coefficients A_{vw} equal to the numbers of edges that have source v and range w.

We construct a spectral triple from the following data (see Figure 1 for an illustration of the construction):

1. A finite oriented graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a distinguished one-edge-loop l^*. We suppose that the graph is strongly connected: for any two vertices v_1, v_2, there

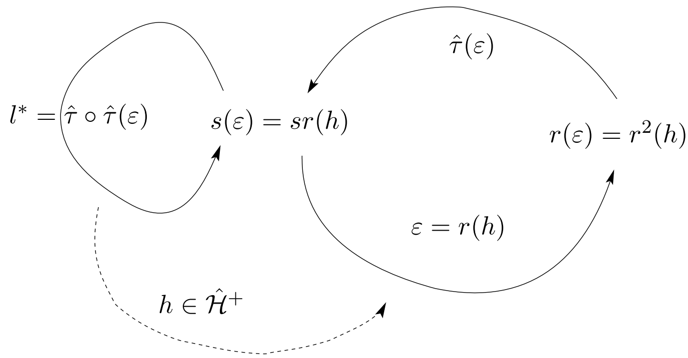

Figure 1 The graph $\mathcal{G}$ associated with the Fibonacci matrix and horizontal edges $\hat{\mathcal{H}} = \{h, h^{\mathrm{op}}\}$

exists an oriented path from v_1 to v_2 and an oriented path from v_2 to v_1. This is equivalent to saying that the graph matrix A is irreducible. We will further assume that A is *primitive* (i.e., $\exists N \in \mathbb{N}$, $\forall v, w$, $A^N_{vw} > 0$).

Alternatively, we can pick a distinguished loop made of $p > 1$ edges and resume the case described before by replacing the matrix A by A^p and considering its associated graph $\mathcal{G}^p$ instead of $\mathcal{G}$.

2. A function $\hat{\tau} : \mathcal{E} \to \mathcal{E}$ satisfying for all $\varepsilon \in \mathcal{E}$:
   (a) if $r(\varepsilon)$ is the vertex of l^*, then $\hat{\tau}(\varepsilon) = l^*$,
   (b) if $r(\varepsilon)$ is not the vertex of l^*, then $\hat{\tau}(\varepsilon)$ is an edge starting at $r(\varepsilon)$ and such that $r(\hat{\tau}(\varepsilon))$ is closer[3] to the vertex of l^* in $\mathcal{G}$.
3. A symmetric subset $\hat{\mathcal{H}} = \mathcal{H}(\mathcal{G})$ of $\mathcal{E} \times \mathcal{E}$:

$$\hat{\mathcal{H}} \subseteq \{(\varepsilon, \varepsilon') \in \mathcal{E} \times \mathcal{E} : \varepsilon \neq \varepsilon', s(\varepsilon) = s(\varepsilon')\}.$$

   This can be understood as a graph with vertices $\mathcal{E}$ and edges $\hat{\mathcal{H}}$ that has no loops. We fix an orientation of the edges in $\hat{\mathcal{H}}$ and write $\hat{\mathcal{H}} = \hat{\mathcal{H}}^+ \cup \hat{\mathcal{H}}^-$ for the decomposition into positively and negatively oriented edges.
4. A real number $\rho \in (0, 1)$.

NOTATION. We still denote the range and source maps by r, s on $\hat{\mathcal{H}}$: $\hat{\mathcal{H}} \overset{r}{\underset{s}{\rightrightarrows}} \mathcal{E}$. We allow compositions with the source and range maps from $\mathcal{E}$ to $\mathcal{V}$, which we denote by s^2, r^2, sr, rs: $\hat{\mathcal{H}} \overset{r^2,s^2}{\underset{rs,sr}{\rightrightarrows\!\!\rightrightarrows}} \mathcal{V}$. See Figure 1 for an illustration with the Fibonacci matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We denote by $\Pi_n(\mathcal{G})$, or simply by Π_n if the graph is understood, the set of paths of length n over $\mathcal{G}$, that is, sequences of n edges $\varepsilon_1 \cdots \varepsilon_n$ such that $r(\varepsilon_i) = s(\varepsilon_{i+1})$. We also set $\Pi_0(\mathcal{G}) = \mathcal{V}$. We extend the range and source maps to paths: if $\gamma =$

[3] For the combinatorial graph metric where nonloop edges have length 1 and loop edges have length 0.

$\varepsilon_1 \cdots \varepsilon_n$, then $r(\gamma) := r(\varepsilon_n)$, $s(\gamma) := s(\varepsilon_1)$. Also, given $\gamma = \varepsilon_1 \cdots \varepsilon_i \cdots \varepsilon_n$, we denote by $\gamma_i = \varepsilon_i$ the ith edge along the path.

The number of paths of length n starting from v and ending in w then is ${A^n}_{vw}$. Recall that we require that A is *primitive:* $\exists N \in \mathbb{N}, \forall v, w, A^N_{vw} > 0$. (For the graph $\mathcal{G}$, this means that for any two vertices v, w, there is at least one oriented path of length N from v to w; for the graph $\mathcal{G}^N$, this means that for any two vertices v, w, there is at least one oriented edge from v to w.) Under this assumption, A has a nondegenerate positive eigenvalue λ_{PF} that is strictly larger than the modulus of any other eigenvalue. This is the Perron–Frobenius eigenvalue of A. Let us denote by L and R the left and right Perron–Frobenius eigenvectors of A (i.e., associated with λ_{PF}) normalized so that

$$\sum_j R_j = 1, \qquad \sum_j R_j L_j = 1. \tag{7}$$

Let us also write the minimal polynomial of A as $\mu_A(\lambda) = \prod_{k=1}^p (\lambda - \lambda_k)^{m_k}$ with $\lambda_1 = \lambda_{\mathrm{PF}}$ and $m_1 = 1$. Then from the Jordan decomposition of A we can compute the asymptotics of the powers of A as follows [10]:

$$A^n_{ij} = R_i L_j \lambda^n_{\mathrm{PF}} + \sum_{k=2}^p P_k^{(ij)}(n)\lambda_k^n, \tag{8}$$

where $P_k^{(ij)}$ is a polynomial of degree m_k if $n \geq m_k$ and of degree less than m_k if $n < m_k$.

Let M_j be the algebraic multiplicity of the jth eigenvalue of A (hence, $M_1 = 1$). Let $R^{j,l}$ for $1 \leq j \leq p$ and $1 \leq l \leq M_j$ be a basis of (right) eigenvectors of A: $AR^{j,l} = \lambda_j R^{j,l}$. Let also $L^{j,l}$ be a basis of left eigenvectors of A normalized so that $R^{j,l} \cdot L^{j',l'} = \delta_{jj'}\delta_{ll'}$. So $R^{1,1} = R$ and $L^{1,1} = L$ as defined in equation (8). For $1 \leq j \leq p$, let us define

$$C^j_{\hat{\mathcal{H}}} = \frac{1}{\lambda_j} \sum_{l=1}^{M_j} \sum_{\substack{v \in \mathcal{V} \\ h \in \hat{\mathcal{H}}}} R_v^{j,l} L^{j,l}_{s^2(h)}. \tag{9}$$

Given $\mathcal{G}$, we consider the topological space Π_∞ of all (one-sided) infinite paths over $\mathcal{G}$ with the standard topology. It is compact and metrizable. The set $\Pi_{\infty *}$ of infinite paths that eventually become l^* forms a dense set.

REMARK 3.1. This construction is equivalent to that of a *stationary Bratteli diagram* [2]: this is an infinite directed graph with a copy of the vertices $\mathcal{V}$ at each level $n \geq 0$ and a copy of the edges $\mathcal{E}$ linking the vertices at level n to those at level $n+1$ for all n (there are also a root and edges linking it down to the vertices at level 0). So, for instance, the set Π_n of paths of length n is viewed here as the set of paths from level 0 down to level n in the diagram. See Figure 2 for an illustration (the root is represented by the hollow circle to the left).

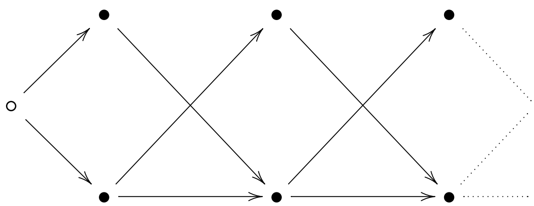

Figure 2 The stationary Bratteli diagram associated with the Fibonacci matrix

Given $\hat{\tau}$, we obtain an embedding of Π_n into Π_{n+1} by $\varepsilon_1 \cdots \varepsilon_n \mapsto \varepsilon_1 \cdots \varepsilon_n \hat{\tau}(\varepsilon_n)$ and hence, by iteration, into $\Pi_{\infty *} \subset \Pi_\infty$. We denote the corresponding inclusion $\Pi_n \hookrightarrow \Pi_\infty$ by τ.

Given $\hat{\mathcal{H}}$, we define horizontal edges $\mathcal{H}_n$ between paths of Π_n, namely $(\gamma, \gamma') \in \mathcal{H}_n$ if γ and γ' differ only on their last edges ε and ε', and with $(\varepsilon, \varepsilon') \in \hat{\mathcal{H}}$. For all n, we carry the orientation of $\hat{\mathcal{H}}$ over to $\mathcal{H}_n$.

The *approximation graph* $G_\tau = (V, E)$ is given by:

$$V = \bigcup_{n \geq 0} V_n, \quad V_n = \tau(\Pi_n) \subset \Pi_{\infty *},$$
$$E = \bigcup_{n \geq 1} E_n, \quad E_n = \tau \times \tau(\mathcal{H}_n),$$

together with the orientation inherited from $\mathcal{H}_n$: so we write $E_n = E_n^+ \cup E_n^-$ for all n and $E = E^+ \cup E^-$. Given $h \in \hat{\mathcal{H}}$, we denote by $\mathcal{H}_n(h)$, $E_n(h)$, and $E(h)$ the corresponding sets of edges of type h.

We can think of an element in $\mathcal{H}_n$ as a pair of paths of length n that agree on their first $n-1$ edges and disagree on their nth edges and of an element in E_n as a pair of infinite paths that agree on their first $n-1$ edges, disagree on their nth edges, and merge further down (according to the definition of $\hat{\tau}$) to the tail made of the infinite repetition of l^*. With this in mind, it is clear that $E_n \cap E_m = \emptyset$ if $n \neq m$.

LEMMA 3.2. *The approximation graph $G_\tau = (V, E)$ is connected if and only if, for all $\varepsilon, \varepsilon' \in \mathcal{E}$ with $s(\varepsilon) = s(\varepsilon')$, there is a path in $\hat{\mathcal{H}}$ linking ε to ε'. Its set of vertices V is dense in Π_∞.*

Proof. Let $x, y \in V$, $x \neq y$, and let n be the largest integer such that $x_i = y_i$, $i = 1, \ldots, n-1$, and $x_n \neq y_n$ (so $s(x_n) = s(y_n)$). Any path in G_τ linking x to y must contain a subpath linking x_n to y_n. Hence, G_τ is connected iff the given condition on $\hat{\mathcal{H}}$ is satisfied.

The density of V is clear since any base clopen set for the topology of Π_∞: $[\gamma] = \{x \in \Pi_\infty : x_i = \eta_i, i \leq n\}$, $\gamma \in \Pi_n$, $n \in \mathbb{N}$, contains a point of V, namely $\tau(\gamma)$. $\square$

Given an edge $e \in E$, we write e^{op} for the edge with the opposite orientation: $s(e^{\mathrm{op}}) = r(e)$, $r(e^{\mathrm{op}}) = s(e)$. Now our earlier construction [15] yields a spectral triple from the data of the approximation graph G_τ. The C^*-algebra is $C(\Pi_\infty)$, and it is represented on the Hilbert space $\ell^2(E)$ by

$$\pi(f)\psi(e) = f(s(e))\psi(e). \tag{10}$$

The Dirac operator is given by

$$D\varphi(e) = \rho^{-n}\varphi(e^{\mathrm{op}}), \quad e \in E_n. \tag{11}$$

The orientation yields a decomposition of $\ell^2(E)$ into $\ell^2(E^+) \oplus \ell^2(E^-)$.

THEOREM 3.3. *The triple $(C(\Pi_\infty(\mathcal{G})), \ell^2(E), D)$ is an even spectral triple with $\mathbb{Z}/2$-grading χ that flips the orientation. Its representation is faithful. If $\hat{\mathcal{H}}$ is sufficiently large (i.e., satisfies the condition in Lemma 3.2), then its spectral distance d_s is compatible with the topology on $\Pi_\infty(\mathcal{G})$, and we have*

$$d_s(x, y) = c_{xy}\rho^{n_{xy}} + \sum_{n > n_{xy}} (b_n(x) + b_n(y))\rho^n \quad \textit{for } x \neq y, \tag{12}$$

where n_{xy} is the largest integer such that $x_i = y_i$ for $i < n_{xy}$, and $b_n(z) = 1$ if $\hat{\tau}(z_n) \neq z_{n+1}$ and $b_n(z) = 0$ otherwise for any $z \in \Pi_\infty$. The coefficient c_{xy} is the number of edges in a shortest path in $\hat{\mathcal{H}}$ linking $x_{n_{xy}}$ to $y_{n_{xy}}$. If $\hat{\mathcal{H}}$ is maximal, that is, $\hat{\mathcal{H}} = \{(\varepsilon, \varepsilon') \in \mathcal{E} \times \mathcal{E} : \varepsilon \neq \varepsilon', s(\varepsilon) = s(\varepsilon')\}$, then $c_{xy} = 1$ for all $x, y \in \Pi_\infty$.

Proof. The first statements are clear. In particular, the commutator $[D, \pi(f)]$ is bounded if f is a locally constant function. The representation is faithful by the denseness of V in Π_∞.

If $\hat{\mathcal{H}}$ satisfies the condition in Lemma 3.2, then the graph G_τ is connected. It is also a metric graph with lengths given by ρ^n for all edges $e \in E_n$. By a previous result (Lemma 2.5 in [15]) d_s is an extension to Π_∞ of this graph metric, and since $\sum \rho^n < +\infty$, it is continuous and given by equation (12) (by straightforward generalizations of Lemma 4.1 and Corollary 4.2 in [15]). □

3.1. Zeta Function

We determine the zeta function for the triple $(C(\Pi_\infty(\mathcal{G})), \ell^2(E), D)$ associated with the above data.

THEOREM 3.4. *Suppose that the graph matrix is diagonalizable with eigenvalues λ_j, $j = 1, \ldots, p$. The zeta function ζ extends to a meromorphic function on $\mathbb{C}$ that is invariant under the translation $z \mapsto z + \frac{2\pi\imath}{\log\rho}$. It is given by*

$$\zeta(z) = \sum_{j=1}^{p} \frac{C^j_{\hat{\mathcal{H}}}}{1 - \lambda_j\rho^z} + h(z),$$

where h is an entire function, and $C^j_{\hat{\mathcal{H}}}$ is given in equation (9). In particular, ζ has only simple poles, which are located at $\{(\log\lambda_j + 2\pi\imath k)/(-\log\rho) : k \in \mathbb{Z}, j =$

$1, \ldots, p\}$ *with residues given by*

$$\mathrm{Res}\left(\zeta, \frac{\log \lambda_j + 2\pi \imath k}{-\log \rho}\right) = \frac{C^j_{\hat{\mathcal{H}}} \lambda_j}{-\log \rho}. \tag{13}$$

In particular, the metric dimension is equal to $s_0 = \log \lambda_{\mathrm{PF}}/(-\log \rho)$.

Proof. Clearly,

$$\zeta(z) = \sum_{n \geq 1} \rho^{nz} \# E_n.$$

The cardinality $\# E_n$ of E_n can be computed by summing, over all vertices $v \in \mathcal{V}$ and all edges $h \in \hat{\mathcal{H}}$ with $s^2(h) = v$, the number of paths of length $n-1$:

$$\# E_n = \sum_{v \in \mathcal{V}} \sum_{h \in \hat{\mathcal{H}}} A^{n-1}_{v s^2(h)}. \tag{14}$$

Now since A is diagonalizable, the polynomials P^{ij}_k in equation (8) are all constant and can be expressed in terms of its (right and left) eigenvectors $R^{k,l}$, $L^{k,l}$, $1 \leq l \leq M_k$. (These vectors were normalized so that $R^{k,l}$ are the columns of the matrix of change of basis that diagonalizes A and the vectors $L^{k,l}$ are the rows of its inverse.) So from equation (8) we get $\# E_n = \sum_{k=1}^p C^k_{\hat{\mathcal{H}}} \lambda_k^n$ for all n, and hence

$$\zeta(z) = \sum_{k=1}^p C^k_{\hat{\mathcal{H}}} \sum_{n \geq 1} \lambda_k^n \rho^{nz} = \sum_{k=1}^p C^k_{\hat{\mathcal{H}}} \frac{\lambda_k \rho^z}{1 - \lambda_k \rho^z}.$$

Hence, ζ has a simple pole at values z for which $\rho^z \lambda_k = 1$, $k = 1, \ldots, p$. The calculation of the residues is direct. □

The periodicity of the zeta function with purely imaginary period whose length is only determined by the factor ρ is a feature that distinguishes our self-similar spectral triples from the known triples for manifolds. Note also that ζ may have a (simple) pole at 0, namely if 1 is an eigenvalue of the graph matrix A.

REMARK 3.5. In the general case, when A is not diagonalizable, it is no longer true that the zeta function has only simple poles. Here the polynomials $P^{ij}_k(n)$ in equation (8) are nonconstant (of degree $m_k - 1 > 0$ for $k = 2, \ldots, p$) and give power terms in the sum for $\zeta(z)$ written in the proof of Theorem 3.4 (we get sums of the form $\sum_{n \geq 1} n^a \lambda_k^n \rho^{nz}$ for integers $a \leq m_k - 1$). In this case, $\zeta(z)$ has poles of order m_j at $z = (\log \lambda_j + 2\pi \imath k)/(-\log \rho)$.

3.2. *Heat Kernel*

We derive here the asymptotic behavior of the trace of the heat kernel $\mathrm{Tr}(e^{-tD^2})$ around $t = 0$.

For $r > 0$, $\Re(\alpha) > 0$, and $s \in \mathbb{R}$, we define

$$\widetilde{\mathfrak{p}}(r, \alpha, s) = \sum_{k=-\infty}^{\infty} \Gamma\left(\alpha + \frac{2\pi \imath k}{r}\right) e^{2\pi \imath k s}. \tag{15}$$

In particular,

$$\mathfrak{p}_{r,a}(\sigma) := \frac{1}{r}\widetilde{\mathfrak{p}}\left(r, \frac{a}{r}, \frac{\sigma}{r}\right) \tag{16}$$

is a periodic function of period r with average $\bar{\mathfrak{p}}_{r,a} = \frac{1}{r}\Gamma(\frac{a}{r})$ over a period.

THEOREM 3.6. *Consider the above spectral triple* $(C(\Pi_\infty(\mathcal{G})), \ell^2(E), D)$ *with graph matrix* A *and parameter* $\rho \in (0,1)$. *We assume that* A *has no eigenvalue of modulus one. Write its eigenvalues* λ_j, $j = 1, \dots, p$. *Let* m_j *be the size of the largest Jordan block of* A *to eigenvalue* λ_j, *and* P_j *the polynomial of degree* m_j *as in equation* (8). *Then the trace of the heat kernel has the following expansion as* $t \to 0^+$:

$$\begin{aligned}\operatorname{Tr}(e^{-tD^2}) = \sum_{j:|\lambda_j|>1} & P_j\left(\frac{1}{-\log\rho}\frac{d}{ds_j}\right)\mathfrak{p}_{-2\log\rho, -s_j\log\rho}(-\log t)t^{-s_j/2} \\ & + h(t),\end{aligned} \tag{17}$$

where $s_j = \log\lambda_j/(-\log\rho)$, *and* h *is a smooth function around* 0. *The leading term of the expansion comes from the Perron–Frobenius eigenvalue, and we have the asymptotic behavior*

$$\operatorname{Tr}(e^{-tD^2}) \overset{t\to 0}{\sim} C^1_{\hat{\mathcal{H}}}\mathfrak{p}_{-2\log\rho, \log\lambda_{\mathrm{PF}}}(-\log t)t^{-s_0/2}, \tag{18}$$

where $s_0 = \log\lambda_{\mathrm{PF}}/(-\log\rho)$ *is the spectral dimension as given in Theorem 3.4, and* $C^1_{\hat{\mathcal{H}}}$ *is given in equation* (9).

Proof. From equations (14) and (8) we have $\#E_n = \sum_{j=1}^p P_j(n)\lambda_j^n$, where P_j is a polynomial of degree m_j, for all n greater than or equal to $\max\{m_j : 0 \le j \le p\}$ (and if $n < \max\{m_j : 0 \le j \le p\}$, then P_j has to be replaced by a polynomial of degree less than or equal to m_j, depending on n). Setting $v = \rho^{-2}$, the trace of the heat kernel reads

$$\sum_n \#E_n e^{-t\rho^{-2n}} = \sum_{j=1}^{p}\sum_{n=0}^{+\infty} P_j(n)\lambda_j^n \exp(-tv^n) + g_1(t), \tag{19}$$

where $g_1(t)$ is a smooth function around zero (the finite sum over $n < \max\{m_k : 0 \le k \le p\}$ of terms correcting the formula for $\#E_n$).

First, consider an eigenvalue λ_j with $|\lambda_j| < 1$. We have

$$|P_j(n)\lambda_j^n e^{-tv^n}| \le |P_j(n)\lambda_j^{n/2}||\lambda_j|^{n/2}|e^{-tv^n}|| \le c_j|\lambda_j|^{n/2},$$

where c_j is a constant. So the corresponding series in equation (19) is absolutely summable, and therefore eigenvalues of modulus less than 1 do not contribute to the singular behavior of the trace.

Hence, the trace of equation (19) can be rewritten as

$$\sum_n \#E_n e^{-t\rho^{-2n}} = \sum_{j:|\lambda_j|>1} \sum_{n=0}^{+\infty} P_j(n) v^{ns_j/2} \exp(-tv^n) + g_2(t)$$
$$= \sum_{j:|\lambda_j|>1} P_j\left(\frac{1}{-\log\rho}\frac{d}{ds_j}\right) \sum_{n=0}^{+\infty} v^{ns_j/2} \exp(-tv^n) + g_2(t),$$

where g_2 is a smooth function around 0.

Consider now an eigenvalue λ_j with $|\lambda_j| > 1$. We thus have $\Re(s_j) > 0$, and we can write

$$\sum_n v^{ns_j/2} e^{-t\rho^{-2n}} = \sum_{n=-\infty}^{+\infty} v^{ns_j/2} \exp(-tv^n) - \sum_{n=1}^{+\infty} v^{-ns_j/2} \exp(-tv^{-n}).$$

The term $\sum_{n=1}^{+\infty} v^{-ns_j/2} \exp(-tv^{-n})$ being bounded at 0, we only need to concentrate on the first sum. Clearly, $t^{s_j/2} \sum_{n=-\infty}^{+\infty} v^{ns_j/2} \exp(-tv^n) = f(\frac{\log t}{\log v})$, where

$$f(s) = \sum_{n=-\infty}^{\infty} v^{(n+s)s_j/2} \exp(-v^{n+s}).$$

By standard arguments this series is uniformly convergent and defines a smooth 1-periodic function f. It follows that the singular behavior of $\sum_{n=0}^{+\infty} v^{ns_j/2} \times \exp(-tv^n)$ as $t \to 0^+$ is given by $f(\frac{\log t}{\log v})t^{-s_j/2}$. So we get that the trace reads

$$\sum_n \#E_n e^{-t\rho^{-2n}} = \sum_{j:|\lambda_j|>1} P_j\left(\frac{1}{-\log\rho}\frac{d}{ds_j}\right) f\left(\frac{\log t}{\log v}\right) t^{-s_j/2} + h(t), \tag{20}$$

where h is a smooth function around 0. So we are left with identifying the function f. Its Fourier coefficients are given by

$$\hat{f}_k = \int_{-\infty}^{\infty} e^{\log(v)s_j/2x} \exp(-v^x) e^{2\pi\imath kx}\,dx = \frac{\Gamma(s_j/2 + (2\pi\imath k)/\log v)}{\log v},$$

so we see from equation (15) and (16) that $f(s) = \frac{1}{-2\log\rho}\widetilde{\mathfrak{p}}(-2\log\rho, s_j/2, -s)$. Hence, the singular term associated with λ_j reads $f(\frac{\log t}{\log v})t^{-s_j/2} = \mathfrak{p}_{-2\log\rho, -s_j\log\rho}(-\log t)t^{-s_j/2}$. We substitute this back into equation (20) to complete the proof of equation (17).

Clearly, $s_0 = \log\lambda_{\mathrm{PF}}/(-\log\rho)$ has the greatest modulus among all the other s_j. Hence, the leading term in the expansion comes from the Perron–Frobenius eigenvalue. Since $\lambda_1 = \lambda_{\mathrm{PF}}$ is an eigenvalue, $P_1 = C^1_{\hat{\mathcal{H}}}$ is the constant polynomial given in equation (9), which proves equation (18). □

We could determine an asymptotic expansion of the trace of the heat kernel using the inverse Mellin transform of the function $\zeta(2s)\Gamma(s)$. This is, of course, a lot more complicated than the direct computations. When using the inverse Mellin transform of $\zeta(2s)\Gamma(s)$, we can see that the origin of the periodic function $\mathfrak{p}$ is directly related to the periodicity of the zeta function and that the appearance of the term $\log t$ in the trace of the heat kernel expansion arises from a simple pole of $\zeta(s)$ at $s=0$, which amounts to a *double* pole of $\zeta(2s)\Gamma(s)$ at $s=0$.

REMARK 3.7. If A is not diagonalizable, then we do not know how to compute contributions of eigenvalues of modulus one, so we assumed that A did not have any in Theorem 3.6. But if A is diagonalizable, contributions of such eigenvalues are easily computed. Eigenvalues of modulus one, not equal to one, do not contribute to the singular behavior of the trace. Only the eigenvalue $\lambda_{j_0}=1$, if present in the spectrum of A, gives an extra term. Eigenvalues of modulus greater than one contribute as in equation (17), but with $P_j=1$ since A is diagonalizable. The trace of the heat kernel in this case has the following expansion as $t\to 0^+$:

$$\begin{aligned}\operatorname{Tr}(e^{-tD^2}) = &\sum_{j:|\lambda_j|>1} C^j_{\hat{\mathcal{H}}}\mathfrak{p}_{-2\log\rho,-s_j\log\rho}(-\log t)t^{-s_j/2}\\ &+ C^{j_0}_{\hat{\mathcal{H}}}\frac{-\log t}{-2\log\rho}+h(t),\end{aligned}\tag{21}$$

where $s_j=\lambda_j/(-\log\rho)$, $C^j_{\hat{\mathcal{H}}}$ is given in equation (9), j_0 is the index for the eigenvalue $\lambda_{j_0}=1$ (setting $C^{j_0}_{\hat{\mathcal{H}}}=0$ if A has no eigenvalue equal to 1), and h is a smooth function around 0. The trace of the heat kernel has therefore the same asymptotic behavior as in equation (18).

3.3. Spectral State

There is a natural Borel probability measure on Π_∞. Indeed, due to the primitivity of the graph matrix, there is a unique Borel probability measure that is invariant under the action of the groupoid given by tail equivalence. We explain that.

If we denote by $[\gamma]$ the cylinder set of infinite paths beginning with γ, then invariance under the above-mentioned groupoid means that $\mu([\gamma])$ depends only on the length $|\gamma|$ of γ and its range, that is, $\mu([\gamma])=\mu(|\gamma|,r(\gamma))$. By additivity we have

$$\mu([\gamma])=\sum_{\varepsilon:s(\varepsilon)=r(\gamma)}\mu([\gamma\varepsilon]),$$

which translates into

$$\mu(n,v)=\sum_w A_{vw}\mu(n+1,w).$$

The unique solution to that equation is

$$\mu(n,v)=\lambda_{\mathrm{PF}}^{-n}R_v,$$

where R is the *right* Perron–Frobenius eigenvector of the adjacency matrix A, normalized as in equation (7). So if $\gamma\in\Pi_n$ is a path of length n, then $\mu([\gamma])=\lambda_{\mathrm{PF}}^{-n}R_{r(\gamma)}$.

THEOREM 3.8. *All operators of the form $\pi(f)$, $f \in C(\Pi_\infty)$, are strongly regular. Moreover, the measure μ defined on $f \in C(\Pi_\infty)$ by*

$$\mu(f) := \mathcal{T}(\pi(f))$$

is the unique measure that is invariant under the groupoid of tail equivalence.

Proof. Let f be a measurable function on Π_∞ and set

$$\mu_n(f) = \frac{\mathrm{Tr}_{E_n}(\pi(f)|_{E_n})}{\# E_n} = \frac{\sum_{e \in E_n} f(s(e))}{\# E_n}.$$

To check that the sequence $(\mu_n(f))_{\mathbb{N}}$ converges, it suffices to consider f to be a characteristic function of a base clopen set for the (Cantor) topology of Π_∞. Let γ be a finite path of length $|\gamma| < n$ and denote by χ_γ the characteristic function on $[\gamma]$. Then $\chi_\gamma(s(e))$ is nonzero if the path $s(e)$ starts with γ. Given that the tail of the path $s(e)$ is determined by the choice function τ, the number of $e \in E_n$ for which $\chi_\gamma(s(e))$ is nonzero coincides with the number of paths of length $n - |\gamma| - 1$ that start at $r(\gamma)$ and end at $s(s(e)_{n-1}) = s^2(h)$ for some $h \in \hat{\mathcal{H}}$. Hence,

$$\sum_{e \in E_n} \chi_\gamma(s(e)) = \sum_{h \in \hat{\mathcal{H}}} A^{n-|\gamma|-1}_{r(\gamma) s^2(h)}.$$

As noted before in the proof of Theorem 3.4, the cardinality of E_n is asymptotically $C^1_{\hat{\mathcal{H}}} \lambda^n_{\mathrm{PF}}$, so we have

$$\mu_n(\chi_\gamma) = \frac{\sum_{e \in E_n} \chi_\gamma(s(e))}{\# E_n} = \lambda_{\mathrm{PF}}^{-|\gamma|-1} \frac{1}{C^1_{\hat{\mathcal{H}}}} \sum_{h \in \hat{\mathcal{H}}} R_{r(\gamma)} L_{s^2(h)} (1 + o(1)).$$

Set $U_v = (\lambda_{\mathrm{PF}}^{-1} / C^1_{\hat{\mathcal{H}}}) \sum_{h \in \hat{\mathcal{H}}} R_v L_{s^2(h)}$. We readily check that U is a (right) eigenvector of A with eigenvalue λ_{PF}, and since its coordinates add up to 1, we have $U = R$. So we get

$$\mu_n(\chi_\gamma) = \lambda_{\mathrm{PF}}^{-|\gamma|} R_{r(\gamma)} (1 + o(1)) \xrightarrow{n \to +\infty} \lambda_{\mathrm{PF}}^{-|\gamma|} R_{r(\gamma)} = \mu([\gamma]).$$

Now Corollary 2.5 implies that $\pi(f)$ is strongly regular and $\mathcal{T}(\pi(f)) = \mu(f)$. □

We now consider weakly regular operators showing, among other, that they do not necessarily satisfy the product decomposition of Corollary 2.8.

DEFINITION 3.9. Let $\rho, \rho' \in (0, 1)$. We call $\varphi \in (0, 2\pi)$ a *nonresonant phase* for (ρ, ρ') if

$$\varphi + 2\pi k + 2\pi \frac{\log \rho}{\log \rho'} k' \neq 0, \quad \forall k, k' \in \mathbb{Z}. \tag{22}$$

Consider an operator $A \in \mathcal{B}(\mathfrak{H})$ defined on the Hilbert space of our spectral triple. Recall the definition of $\bar{A}_n$ from Corollary 2.5. Given any bounded sequence f_n of complex numbers, there is an operator $A \in \mathcal{B}(\mathfrak{H})$ such that $\bar{A}_n = f_n$, so we have just to define A on the subspace F_n to be f_n times the identity.

LEMMA 3.10. *Consider the above spectral triple $(C(\Pi_\infty(\mathcal{G})), \ell^2(E), D)$. Let $A \in \mathcal{B}(\mathfrak{H})$ be such that $\bar{A}_n \overset{n\to\infty}{\sim} e^{\imath n\varphi}$ for some $\varphi \in (0, 2\pi)$. Then*

$$\mathrm{Tr}(e^{-tD^2} A) \overset{t\to 0}{\sim} e^{\imath(\varphi \log t)/(2\log\rho)} C^1_{\hat{\mathcal{H}}} \mathfrak{p}_{-2\log\rho, \log\lambda_{\mathrm{PF}}+\imath\varphi}(-\log t) t^{(\log\lambda_{\mathrm{PF}})/(2\log\rho)}.$$

In particular, A is weakly regular, and $\mathcal{T}(A) = 0$.

Proof. As before we set $\sigma = \frac{\log t}{\log v}$, $v = \rho^{-2}$, $\alpha = \log\lambda_{\mathrm{PF}}/(-2\log\rho)$, so that we have to determine the asymptotic behavior of $\sum_{n=1}^\infty e^{\imath n\varphi} v^{\alpha(n+\sigma)} e^{-v^{n+\sigma}}$ as $\sigma \to -\infty$. Since $\sum_{n=-\infty}^{-1} e^{\imath n\varphi} v^{\alpha(n+\sigma)} e^{-v^{n+\sigma}}$ is absolutely convergent, the sum over $\mathbb{N}$ has the same asymptotic behavior as the sum over $\mathbb{Z}$. Now we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} e^{\imath n\varphi} v^{\alpha(n+\sigma)} e^{-v^{n+\sigma}} &= e^{-\imath\sigma\varphi} \sum_{n=-\infty}^{\infty} v^{(\alpha+(\imath\varphi)/\log v)(n+\sigma)} e^{-v^{n+\sigma}} \\ &= e^{-\imath\sigma\varphi} \frac{1}{\log v} \widetilde{\mathfrak{p}}\left(\log v, \frac{\log\lambda_{\mathrm{PF}} + \imath\varphi}{\log v}, -\sigma\right),\end{aligned}$$

from which the first statement follows. Now $\mathcal{T}(A) = \lim_{s\to 0^+}(\mathcal{L}[f_A](s))/(\mathcal{L}[f](s))$ is equal to the mean of the function $\sigma \mapsto e^{-\imath\sigma\varphi} \frac{1}{\log v} \widetilde{\mathfrak{p}}(\log v, (\log\lambda_{\mathrm{PF}} + \imath\varphi)/\log v, -\sigma)$. Developing this function into a Fourier series (see equation (15)), we can compute this mean term by term (the gamma function is rapidly decreasing as k tends to infinity). Since for all integer k, $\varphi + 2\pi k \neq 0$, the mean of each term vanishes. □

We now consider the tensor product of two spectral triples associated with two possibly different graphs with Perron–Frobenius eigenvalue λ_{PF} and λ'_{PF} and parameters $\rho < 1$ and $\rho' < 1$.

LEMMA 3.11. *Consider two spectral triples of the type discussed before with Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ and parameters ρ and ρ'. Let $\mathcal{T}$ be the tensor product state of the two spectral states $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $A \in \mathcal{B}(\mathfrak{H}_1)$ be as in the last lemma with nonresonant φ. Then $\mathcal{T}(A \otimes \mathbf{1}) = 0$.*

Proof. By Lemma 2.7 we have $\mathcal{T}(A\otimes\mathbf{1}) = \lim_{s\to 0^+}(\mathcal{L}[f_A f_2](s))/(\mathcal{L}[f_1 f_2](s))$, and this quantity is equal to the mean of the function $s \mapsto F(\varphi, s) := e^{\imath(s\varphi)/\log v} \times \widetilde{\mathfrak{p}}(\log v, (\log\lambda_{\mathrm{PF}} + \imath\varphi)/\log v, (-s)/\log v)\widetilde{\mathfrak{p}}(\log v', (\log\lambda'_{\mathrm{PF}})/\log v', (-s)/\log v')$ divided by the mean of the function $s \mapsto F(0, s)$. Now $F(0, s)$ is a product of two positive functions, always strictly greater than zero. By developing the two $\widetilde{\mathfrak{p}}$-functions whose product is $F(\varphi, s)$ into Fourier series we can determine the mean of $F(\varphi, s)$ as a sum over the means of the functions $s \mapsto e^{\imath(s\varphi)/\log v} \times e^{-\imath(2\pi ks)/\log v} e^{-\imath(2\pi k's)/\log v'}$ (times the corresponding Fourier coefficients). If φ is nonresonant, that is, $\varphi + 2\pi k + 2\pi(\log\rho/\log\rho')k' \neq 0$ for all integer k, k', then the phases above are all nonzero, and hence the means are all equal to 0. □

This proof yields an easy method to construct weakly regular operators A for which $\mathcal{T}(A \otimes \mathbf{1}) \neq \mathcal{T}_1(A)\mathcal{T}_2(\mathbf{1})$. In fact, if $\log\rho/\log\rho'$ is irrational and, for instance, $\varphi = 2\pi \log\rho/\log\rho'$, then the phase in $s \mapsto e^{\imath(s\varphi)/\log v} e^{-\imath(2\pi k s)/\log v} \times e^{-\imath(2\pi k' s)/\log v'}$ is 0 precisely for $k = 0$ and $k' = -1$. It follows that the mean of $F(\varphi, s)$ is $\Gamma((\log\lambda'_{\mathrm{PF}} - 2\pi\imath)/\log v')$, which does not vanish. Thus, $\mathcal{T}(A \otimes \mathbf{1}) \neq 0$, whereas, as we saw, $\mathcal{T}_1(A) = 0$.

3.4. *Telescoping*

There is a standard equivalence relation among Bratteli diagrams, which is generated by isomorphisms and so-called telescoping. Since we are looking at stationary diagrams, we consider stationary telescopings only. Then the following operations generate the equivalence relation we consider:

1. *Telescoping:* Given the above data built from a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a positive integer p, we consider a new graph $\mathcal{G}^p := (\mathcal{V}^p, \mathcal{E}^p)$ with the same vertices $\mathcal{V}^p = \mathcal{V}$ and the paths of length p as edges, $\mathcal{E}^p = \Pi_p(\mathcal{G})$. The corresponding parameter is taken to be $\rho_p = \rho^p$.
2. *Isomorphism:* Given two graphs as before, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, we say that the corresponding stationary Bratteli diagrams are isomorphic if there are two bijections $\mathcal{V} \to \mathcal{V}'$ and $\mathcal{E} \to \mathcal{E}'$ that intertwine the range and source maps. We need in this case the associated parameters to be equal and the sets of horizontal edges to be isomorphic (through a map that intertwines the range and source maps).

We show now that this equivalence relation leaves the properties of the associated spectral triple invariant:

(i) The zeta functions are equivalent and thus have the same spectral dimension: $s_0 = \log\lambda_{\mathrm{PF}}/(-\log\rho)$;
(ii) The spectral measures are both equal to the invariant probability measure μ on Π_∞;
(iii) Both spectral distances generate the topology of Π_∞ (provided that $\hat{\mathcal{H}}$ is large enough as in Lemma 3.2) and are furthermore Lipschitz equivalent.

The invariance under isomorphism is trivial. We explain briefly how things work under telescoping. The horizontal edges for $\mathcal{G}^p$ are given as for $\mathcal{G}$ by the corresponding subset

$$\hat{\mathcal{H}}^p \subseteq \{(\varepsilon, \varepsilon') \in \mathcal{E}^p \times \mathcal{E}^p : \varepsilon \neq \varepsilon', s(\varepsilon) = s(\varepsilon')\},$$

and so we have the identifications

$$\mathcal{H}_n^p \cong \bigcup_{i=0}^{p-1} \mathcal{H}_{np+i}, \qquad E_n^p \cong \bigcup_{i=0}^{p-1} E_{np+i},$$

which allows us to determine the approximation graph $G_\tau^p = (V^p, E^p)$ and yields a unitary equivalence $\ell^2(E) \cong \ell^2(E^p)$. We identify the two Hilbert spaces $\ell^2(E) \cong \ell^2(E^p)$ and the representations $\pi \cong \pi_p$, whereas the Dirac operators satisfy

$$D_p = W^\dagger D W \quad \text{with } W : \begin{cases} \ell^2(E) \to \ell^2(E^p), \\ \delta_e \mapsto \rho^{-k/2}\delta_e \qquad \text{for } e \in E_n \text{ with } n = k \mod p. \end{cases}$$

From the inequalities $\mathbf{1} \le W \le \rho^{-p}\mathbf{1}$ we deduce that the zeta functions are equivalent and that both spectral triples have the same spectral dimension s_0 and give rise to the same spectral measure μ. By Theorem 3.3 both Connes distances generate the topology of $\Pi_\infty = \Pi_\infty^p$, provided that $\hat{\mathcal{H}}$ is large enough. Let us denote d_s^p the spectral metric associated with $\mathcal{G}^p$, with the corresponding coefficients $n_{xy}^p, c_{xy}^p, b_n^p$ as in equation (12). Writing $n_{xy} = pn_{xy}^p + k_{xy}$ for some $k_{xy} \le p-1$, we have

$$d_s(x,y) = c_{xy}\rho^{k_{xy}}(\rho^p)^{n_{xy}^p} + \sum_{n > n_{xy}^p} (\rho^p)^n \sum_{k=0}^{p-1} (b_{np+k}(x) + b_{np+k}(y))\rho^k.$$

Now we see that $b_{np+k}(z) = 1 \Rightarrow b_n^p(z) = 1$, whereas if $b_{np+k}(z) = 0$ for all $k = 0, \dots, p-1$, then $b_n^p(z) = 0$ too, so that we have $b_n^p(z) \le \sum_{k=0}^{p-1} b_{np+k}(z) \le pb_n^p(z)$. We substitute this back into the previous equation to get the Lipschitz equivalence:

$$c_p d_s^p(x,y) \le d_s(x,y) \le p\rho^p C_p d_s^p(x,y)$$

with c_p, C_p, the respective min and max of c_{xy}/c_{xy}^p (which only depends on $\mathcal{H}$ and p).

4. Substitution Tiling Spaces

Bratteli diagrams occur naturally in the description of substitution tilings. The path space of the Bratteli diagram defined by the substitution graph has been used to describe the transversal of such a tiling [6; 13]. As we will first show, an extended version can also be used to describe a dense set of the continuous hull Ω_Φ of the tiling, and therefore we will employ it and the construction of the previous section to construct a spectral triple for Ω_Φ.

4.1. Preliminaries

We recall the basic notions of tiling theory, namely tiles, patches, tilings of the Euclidean space $\mathbb{R}^d$, and substitutions. For a more detailed presentation in particular of substitution tilings, we refer the reader to [9]. A *tile* is a compact subset of $\mathbb{R}^d$ that is homeomorphic to a ball. It possibly carries a decoration (for instance, its collar). A *tiling* of $\mathbb{R}^d$ is a countable set of tiles $(t_i)_{i\in I}$ whose union covers $\mathbb{R}^d$ and with pairwise disjoint interiors. Given a tiling T, we call a *patch* of T any set of tiles in T that covers a bounded and simply connected set. A *prototile* (resp. *protopatch*) is an equivalence class of tiles (resp. patches) modulo translations. We will only consider tilings with finitely many prototiles and for which there are only finitely many protopatches containing two tiles (such tilings have finite local complexity, FLC).

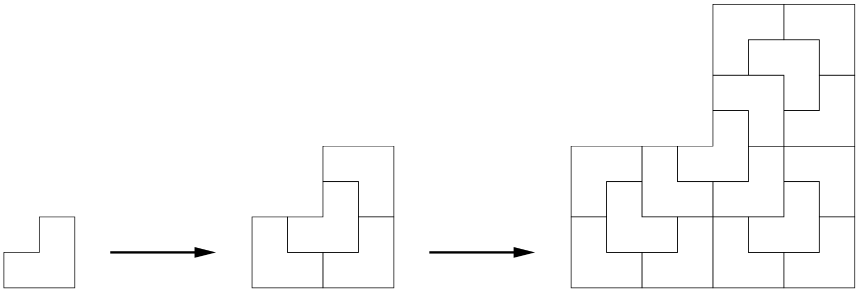

Figure 3 A process of inflation and substitution (chair tiling). A whole tiling of $\mathbb{R}^2$ can be obtained as a fixed point of this map

The tilings we are interested in are constructed from a (finite) prototile set $\mathcal{A}$ and a substitution rule on the prototiles. A substitution rule is a decomposition rule followed by a rescaling, that is, each prototile is decomposed into smaller tiles, which, when stretched by a common factor $\theta > 1$, are congruent to some prototiles. We call θ the *dilation factor* of the substitution. The decomposition rule can be applied to patches and whole tilings by simply decomposing each tile and so can be the substitution rule when the result of the decomposition is stretched by a factor of θ. We denote the decomposition rule by δ and the substitution rule by Φ. In particular, we have, for a tile t, $\delta(t+a) = \delta(t) + a$ and $\Phi(t+a) = \Phi(t) + \theta a$ for all $a \in \mathbb{R}^d$. See Figure 3 for an example in $\mathbb{R}^2$.

A patch of the form $\Phi^n(t)$ for some tile t is called an *n-supertile,* or nth-order supertile. A rescaled tile $\theta^n t$ will be called a *level n tile,* but also, if $n = -m < 0$, an *m-microtile,* or an mth-order microtile.

A substitution defines a tiling space Ω_Φ, the set of all tilings T with the property that any patch of T occurs in a supertile of sufficiently high order.

We will assume that the substitution is *primitive* and *aperiodic:* there exists an integer n such that any n-supertile contains tiles of each type and all tilings of Ω_Φ are aperiodic. This implies that by inspection of a large enough but finite patch around them the tiles of Ω_Φ can be grouped into supertiles (Φ is said to be recognizable) so that δ and Φ are invertible. In particular, Φ is a homeomorphism of Ω_Φ if the latter is equipped with the standard tiling metric [1].

We may suppose that the substitution *forces the border* [13]. The condition says that given any tile t, its nth substitute does not only determine the n-supertile $\Phi^n(t)$, but also all tiles that can be adjacent to it. This condition can be realized, for instance, by considering decorations of each types of tiles and replacing $\mathcal{A}$ by the larger set of collared prototiles.

There is a canonical action of $\mathbb{R}^d$ on the tiling space Ω_Φ by translation that makes it a topological dynamical system. Under the above assumptions, the dynamical system $(\Omega_\Phi, \mathbb{R}^d)$ is minimal and uniquely ergodic. The unique invariant and ergodic probability measure on Ω_Φ will be denoted μ.

A particularity of tiling dynamical system is that they admit particular transversals to the $\mathbb{R}^d$-action. To define such a transversal Ξ, we associate with each prototile a particular point, called its *puncture.* Each level n tile being similar to a unique prototile, we may then associate with the level n tile the puncture that is the image of the puncture of the prototile under the similarity. The transversal[4] Ξ is the subset of tilings $T \in \Omega$ that has the puncture of one of its tiles at the origin of $\mathbb{R}^d$. The measure μ induces an invariant probability measure on Ξ, which gives the frequencies of the tiles and patches.

4.2. Substitution Graph and the Robinson Map

The *substitution matrix* of the substitution Φ is the matrix with coefficients A_{ij} equal to the number of tiles of type t_i in $\Phi(t_j)$. The graph $\mathcal{G}$ of Section 3 underlying our constructions will be here the *substitution graph:* the graph whose graph matrix is the substitution matrix. More precisely, its vertices $v \in \mathcal{V}$ are in one-to-one correspondence with the prototiles, and we denote by t_v the prototile corresponding to $v \in \mathcal{V}$, that is, the prototile set reads $\mathcal{A} = \{t_v : v \in \mathcal{V}\}$. Between the vertices u and v, there are A_{uv} edges corresponding to the A_{uv} different occurrences of tiles of type t_u in $\Phi(t_v)$. Here we call u (or t_u) the source and v (or t_v) the range of these edges. Notice that the Perron–Frobenius eigenvalue of A is the dth-power of the dilation factor θ: $\lambda_{\mathrm{PF}} = \theta^d$. The asymptotics of the powers of A are given by equation (8) as before. The coordinates of the left and right Perron–Frobenius eigenvectors L, R are now related to the volumes and the frequencies of the prototiles as follows: for all $v \in \mathcal{V}$, we have

$$\mathrm{freq}(t_v) = R_v, \qquad \mathrm{vol}(t_v) = L_v, \tag{23}$$

where $\mathrm{freq}(t_v)$ is the frequency, and $\mathrm{vol}(t_v)$ is the volume of t_v, the volume being normalized as in equation (7) so that the average volume of a tile is 1.

Given a choice of punctures to define the transversal Ξ of Ω_Φ, there is a map

$$\mathcal{R} : \Xi \to \Pi_\infty(\mathcal{G})$$

onto the set of half-infinite paths in $\mathcal{G}$. Indeed, given a tiling $T \in \Xi$ (so with a puncture at the origin) and an integer $n \in \mathbb{N}$, we define:

- $v_n(T) \in \mathcal{V}$ to be the vertex corresponding to the prototile type of the tile in $\Phi^{-n}(T)$ that contains the origin;
- $\varepsilon_n(T) \in \mathcal{E}$ to be the edge corresponding to the occurrence of $v_{n-1}(T)$ in $\Phi(v_n(T))$.

Then $\mathcal{R}(T)$ is the sequence $(\varepsilon_n(T))_{n\geq 1}$. We call $\mathcal{R}$ the Robinson map as it was first defined for the Penrose tilings by Robinson; see [9].

THEOREM 4.1 ([13]). *$\mathcal{R}$ is a homeomorphism.*

We extend the map $\mathcal{R}$ to the continuous hull Ω_Φ. The idea is simple: the definition of $\mathcal{R}$ makes sense, provided that the origin lies in a single tile but becomes

[4]Sometimes, Ξ is referred to as the *canonical transversal* or the *discrete hull.*

ambiguous as soon as it lies in the common boundary of several tiles. We will therefore always assign that boundary to a unique tile in the following way.

We suppose that the boundaries of the tiles are sufficiently regular so that there exists a vector $\vec{v} \in \mathbb{R}^d$ such that, for all points x of a tile t, either $\exists \eta > 0$, $\forall \epsilon \in (0, \eta)$: $x + \epsilon\vec{v} \in t$ or $\exists \eta > 0$, $\forall \epsilon \in (0, \eta)$: $x + \epsilon\vec{v} \notin t$. This is clearly the case for polyhedral tilings. We fix such a vector v. Given a prototile t (a closed set), we define the half-open prototile $[t)$ as follows:

$$[t) := \{x \in t : \exists \eta > 0\ \forall \epsilon \in [0, \eta) : x + \epsilon\vec{v} \in t\}.$$

It follows that any tiling T gives rise to a partition of $\mathbb{R}^d$ by half-open tiles. We extend the Robinson map to

$$\mathcal{R} : \Omega_\Phi \to \Pi_{-\infty,+\infty},$$

where $\Pi_{-\infty,+\infty}$ is the space of bi-infinite sequences over $\mathcal{G}$ using half-open prototiles as follows. For $n \in \mathbb{Z}$, we define:

- $v_n(T) \in \mathcal{V}$ to be the vertex corresponding to the prototile type of the half-open tiles in $\Phi^{-n}(T)$ that contains the origin. So $v_n(T)$ corresponds to
  - the nth-order (half-open) supertile in T containing the origin for $n > 0$,
  - nth-order (half-open) microtile in $\delta^{-n}(T)$ containing the origin for $n \le 0$;
- $\varepsilon_n(T) \in \mathcal{E}$ to be the edge corresponding to the occurrence of $v_{n-1}(T)$ in $\Phi(v_n(T))$.

Also, we set $\mathcal{R}(T)$ to be the bi-infinite sequence $\mathcal{R}(T) = (\varepsilon_n(T))_{n\in\mathbb{Z}}$.

REMARK 4.2. As in Remark 3.1, we can see this construction as a Bratteli diagram, but the diagram is bi-infinite this time. There is a copy of $\mathcal{V}$ at each level $n \in \mathbb{Z}$ and edges of $\mathcal{E}$ between levels n and $n+1$. Level 0 corresponds to prototiles, level 1 to supertiles, and level $n > 1$ to nth-order supertiles, whereas level -1 corresponds to microtiles and level $n < -1$ to nth-order microtiles. For the "negative" part of the diagram, we can alternatively consider the reversed substitution graph $\widetilde{\mathcal{G}} = (\mathcal{V}, \widetilde{\mathcal{E}})$ that is $\mathcal{G}$ with all orientations of the edges reversed. The graph matrix of $\widetilde{\mathcal{G}}$ is then the transpose of the substitution matrix: $\widetilde{A} = A^T$. So for $n \le 0$, there are edges of $\widetilde{\mathcal{E}}$ between levels n and $n-1$: there are $\widetilde{A}_{uv} = A_{vu}$ such edges linking u to v.

As for Theorem 4.1, we prove, using the border forcing condition, that $\mathcal{R}$ is injective.

Given a path $\xi \in \Pi_{-\infty,+\infty}$ and $m < n \in \mathbb{Z} \cup \{\pm\infty\}$, we denote by $\xi_{[m,n]}$, $\xi_{(m,n]}$, $\xi_{[m,n)}$ and $\xi_{(m,n)}$ its restrictions from level m to n (with end points included or not). Also, ξ_n will denote its nth edge from level n to level $n+1$ for $n \in \mathbb{Z}$. We similarly define $\Pi_{m,n}$ (with end points included). For instance, $\Pi_{0,+\infty}$ is simply $\Pi_\infty = \Pi_\infty(\mathcal{G})$.

We say that an edge $e \in \mathcal{E}$ is *inner* if it encodes the position of a tile t in the supertile p such that $\exists \eta > 0$, $\forall \epsilon \in [0, \eta)$: $t + \epsilon\vec{v} \in p$. This says that the occurrence of t in p does not intersect the open part of the border of p.

It is not true that $\mathcal{R}$ is bijective, but we have the following.

LEMMA 4.3. *$X := \operatorname{im}\mathcal{R}$ contains the set of paths $\xi \in \Pi_{-\infty,+\infty}$ such that $\xi_{(-\infty,0]}$ has infinitely many inner edges.*

Proof. Recall the following: If $(A_n)_n$ is a sequence of subsets of $\mathbb{R}^d$ such that $A_{n+1} \subset A_n$ and $\operatorname{diam}(A_n) \to 0$, then there exists a unique point $x \in \mathbb{R}^d$ such that $x \in \bigcap_n \overline{A_n}$.

By construction, for any tiling T, we have $0 \in \bigcap_{n\leq 0}[v_n(T))$. Hence, $\xi = \mathcal{R}(T)$ whenever $\bigcap_{n\leq 0}[s(\xi_n)) \neq \emptyset$, where $[s(\xi_0))$ is the standard representative for the half-open prototile of type $s(\xi_0)$, and $[s(\xi_n))$ is the half-open nth-order microtile of type $s(\xi_n)$ in $[s(\xi_0))$ that is encoded by the path $\xi_{[n,0]}$.

Suppose that ξ_n is inner. Then $[s(\xi_n))$ does not lie at the open border of $[r(\xi_n) = s(\xi_{n+1}))$. Hence, $[s(\xi_n)) \cap [s(\xi_{n+1})) = [s(\xi_n)] \cap [s(\xi_{n+1}))$, where $[s(\xi_n)]$ is the closure of $[s(\xi_n))$. Suppose that infinitely many edges of $\xi_{(-\infty,0]}$ are inner. Then

$$\bigcap_{n<0:\xi_n \text{ inner}} [s(\xi_n)] \subset \bigcap_{n<0}[s(\xi_{n+1})),$$

showing that the r.h.s. contains an element, and hence $\xi \in \operatorname{im}\mathcal{R}$. □

COROLLARY 4.4. *The set X is a dense and shift-invariant subset of $\Pi_{-\infty,+\infty}$.*

Proof. The shift invariance is clear. The denseness follows immediately from Lemma 4.3. □

In particular, for $n \in \mathbb{N}$, each element of $\Pi_{-n,n}$ can be the middle part of a sequence in $\mathcal{R}(\Omega_\Phi)$, that is, for all $\gamma \in \Pi_{-n,n}$, there exists $T \in \Omega_\Phi$ such that $\mathcal{R}(T)_{[-n,n]} = \gamma$.

REMARK 4.5. For $v \in \mathcal{V}$, let $\Pi^v_{-\infty,\infty}$ be the set of bi-infinite paths that pass through v at level 0, and set $X^v = X \cap \Pi^v_{-\infty,\infty}$. Then $\mathcal{R}$ yields a bijection between $\Xi_{t_v} \times [t_v)$ and X^v, where t_v is the prototile corresponding to v, and Ξ_{t_v} its acceptance domain (the set of all tilings in Ξ that have t_v at the origin).

Notice that $\Pi_{-\infty,0}$ can be identified with $\Pi_\infty(\widetilde{\mathcal{G}})$, where $\widetilde{\mathcal{G}}$ is the graph obtained from $\mathcal{G}$ by reversing the orientation of its edges: we simply read paths backward and so follow the edges along their opposite orientations. We then see that the Robinson map yields a homeomorphism between Ξ_{t_v} and $\Pi^v_{0,+\infty} = \Pi^v_\infty$ and a map with dense image from $[t_v)$ into $\Pi^v_{-\infty,0} = \Pi^v_\infty(\widetilde{\mathcal{G}})$.

4.3. The Transversal Triple for a Substitution Tiling

Our aim here is to construct a spectral triple for the transversal Ξ. We apply the general construction of Section 3 to the substitution graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We may suppose[5] that the substitution has a fixed point T^* such that $\mathbb{R}^d$ is covered by the union over n of the nth-order supertiles of T^* containing the origin. It follows that $\mathcal{R}(T^*)$ is a constant path in $\Pi_{-\infty,+\infty}(\mathcal{G})$, that is, the infinite repetition of a loop

[5]This can always be achieved by going over to a power of the substitution.

edge of $\mathcal{G}$, which we choose to be ε^*. We then fix τ, take $\rho = \rho_{\text{tr}}$ as a parameter, and choose a subset

$$\hat{\mathcal{H}}_{\text{tr}} \subset \mathcal{H}(\mathcal{G}) = \{(\varepsilon, \varepsilon') \in \mathcal{E} \times \mathcal{E} : \varepsilon \neq \varepsilon', s(\varepsilon) = s(\varepsilon')\},$$

which we suppose to satisfy the conditions of Lemma 3.2: if $s(\varepsilon) = s(\varepsilon')$, then there is a path of edges in $\hat{\mathcal{H}}_{\text{tr}}$ linking ε' with ε'. The horizontal edges of level $n \in \mathbb{N}$ are then given by

$$\mathcal{H}_{\text{tr},n} = \{(\eta\varepsilon, \eta\varepsilon') : \eta \in \Pi_{n-1}(\mathcal{G}), (\varepsilon, \varepsilon') \in \hat{\mathcal{H}}_{\text{tr}}\} \subset \Pi_n(\mathcal{G}) \times \Pi_n(\mathcal{G}).$$

They define the transverse approximation graph $G_{\text{tr},\tau} = (V_{\text{tr}}, E_{\text{tr}})$ as in Section 3:

$$V_{\text{tr}} = \bigcup_n V_{\text{tr},n}, \quad V_{\text{tr},n} = \tau(\Pi_n(\mathcal{G})) \subset \Pi_\infty^*(\mathcal{G}),$$
$$E_{\text{tr}} = \bigcup_n E_{\text{tr},n}, \quad E_{\text{tr},n} = \tau \times \tau(\mathcal{H}_{\text{tr},n}),$$

together with the orientation inherited from $\hat{\mathcal{H}}_{\text{tr}}$, so that $E_{\text{tr},n} = E_{\text{tr},n}^+ \cup E_{\text{tr},n}^-$ for all $n \in \mathbb{N}$, and $E_{\text{tr}} = E_{\text{tr}}^+ \cup E_{\text{tr}}^-$. We also write $E_n(h) = \tau \times \tau(\mathcal{H}_{\text{tr},n}(h))$, where if $h = (\varepsilon, \varepsilon')$, then $\mathcal{H}_{\text{tr},n}(h) = \{(\eta\varepsilon, \eta\varepsilon') : \eta \in \Pi_{n-1}(\mathcal{G})\}$. By our assumption on $\hat{\mathcal{H}}$ the approximation graph $G_{\text{tr},\tau}$ is connected, and its vertices are dense in $\Pi_\infty(\mathcal{G})$.

An edge $h \in \hat{\mathcal{H}}_{\text{tr}}$ has the following interpretation: The two paths $\tau(s(h))$ and $\tau(r(h))$ have the same source vertex, say v_0, they differ on their first edge, and then, at some minimal $n_h > 0$, they come back together coinciding for all further edges. This is a consequence of the property of τ. Let us denote the vertex at which the two edges come back together with v_h. Neglecting the part after that vertex, we obtain a pair (γ, γ') of paths of length n_h that both start at v_0 and end at v_h. Reading the definition of the Robinson map $\mathcal{R}$ backward, we see that the pair (γ, γ') describes a pair of tiles (t, t') of type v_0 in an n_hth-order supertile of type v_h. Of importance further will be the vector $r_h \in \mathbb{R}^d$ of translation from t to t'.

The interpretation of an edge $e \in E_{\text{tr},n}(h)$ (so an edge of type h) is similar, except that the paths $\tau(s(e))$ and $\tau(r(e))$ coincide up to level n and meet again at level $n + n_h$. In particular, e describes a pair of nth-order supertiles (t, t') of type v_0 in an $(n + n_h)$th-order supertile of type v_h. If we denote by $r_e \in \mathbb{R}^d$ the translation vector between t and t', then, due to the selfsimilarity, we have

$$r_e = \theta^n r_h. \tag{24}$$

See Figure 4 for an illustration.

Theorem 3.3 provides us with a spectral triple for the algebra $C(\Pi_\infty(\mathcal{G}))$. We adapt this slightly to get a spectral triple for $C(\Xi)$. Since the nth-order supertiles of T^* on 0 eventually cover $\mathbb{R}^d$, $\mathcal{R}$ identifies $\Pi_\infty^*(\mathcal{G})$ with the translates of T^* that belong to Ξ. We may thus consider the spectral triple $(C(\Xi), \mathfrak{H}_{\text{tr}}, D_{\text{tr}})$ (which depends on ρ_{tr} and on the choices for τ and $\mathcal{H}$) with representation and Dirac

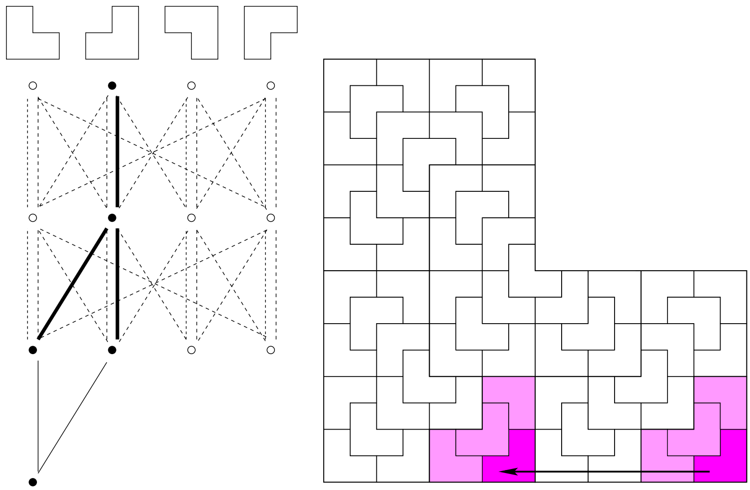

Figure 4 A doubly pointed pattern associated with a horizontal arrow $e \in E_{\mathrm{tr},3}(h)$. The arrow represents the vector r_e. Here $n = 2$ (the paths have lengths 2), and $n_h = 1$ (the paths join further down at level $n + n_h = 3$)

operator defined as in equations (10) and (11) by

$$\mathfrak{H}_{\mathrm{tr}} = \ell^2(E_{\mathrm{tr}}), \qquad \pi_{\mathrm{tr}}(f)\varphi(e) = f(\mathcal{R}^{-1}(s(e)))\varphi(e),$$
$$D_{\mathrm{tr}}\varphi(e) = \rho_{\mathrm{tr}}^{-n}\varphi(e^{\mathrm{op}}), \quad e \in E_{\mathrm{tr},n}.$$

We call it the *transverse spectral triple* of the substitution tiling. By Theorem 3.3 it is an even spectral triple with grading χ (which flips the orientation). Also, since $\mathcal{H}_{\mathrm{tr}}$ satisfies the hypothesis of Lemma 3.2 as noted before, the Connes distance induces the topology of Ξ. By Theorems 3.4 and 3.8 the transversal spectral triple has metric dimension $s_{\mathrm{tr}} = (d\log(\theta))/(-\log(\rho_{\mathrm{tr}}))$, and its spectral measure is the unique ergodic measure on Ξ, which is invariant under the tiling groupoid action.

For $v \in \mathcal{V}$, we will also consider the spectral triple $(C(\Xi_{t_v}), \mathfrak{H}_{\mathrm{tr}}^v, D_{\mathrm{tr}})$ for $\Xi_{t_v} = \mathcal{R}^{-1}(\Pi_\infty^v(\mathcal{G}))$: the acceptance domain of t_v (see Remark 4.5). We call it the *transverse spectral triple for the prototile* t_v. It is obtained from the transverse spectral triple by restriction to the Hilbert space $\mathfrak{H}_{\mathrm{tr}}^v = \ell^2(E_{\mathrm{tr}}^v)$, where E_{tr}^v are the horizontal edges between paths that start on v. This restriction has the effect that

$$\zeta_v = R_v\zeta + reg.,$$

that is, up to a perturbation that is regular at s_{tr}, the new zeta function is $R_v = \mathrm{freq}(t_v)$ times the old one. It hence has the same abscissa of convergence, $s_{\mathrm{tr}}^v = s_{\mathrm{tr}}$, but its residue at s_{tr} is $\mathrm{freq}(t_v)$ times the old one.

Like for Ξ, the Connes distance induces the topology of Ξ_{t_v}. Finally, the spectral measure μ_{tr}^v is the restriction to Ξ_{t_v} of the invariant measure on Ξ, normalized so that the total measure of Ξ_{t_v} is 1. The spectral triple for Ξ is in fact the direct sum over $v \in \mathcal{V}$ of the spectral triples for Ξ_{t_v}.

4.4. *The Longitudinal Triple for a Substitution Tiling*

We now aim at constructing what we call the longitudinal spectral triple for the substitution tiling, which is based on the reversed substitution graph $\widetilde{\mathcal{G}} = (\mathcal{V}, \widetilde{\mathcal{E}})$ ($\mathcal{G}$ with all orientations of the edges reversed, so with adjacency matrix $\widetilde{A} = A^T$). Set $\widetilde{\varepsilon}^* = \varepsilon^*$ and choose $\widetilde{\tau}$. We take $\rho = \rho_{\text{lg}}$ as a parameter and choose a subset

$$\begin{aligned}\hat{\mathcal{H}}_{\text{lg}} \subset \mathcal{H}(\widetilde{\mathcal{G}}) &= \{(\widetilde{\varepsilon}, \widetilde{\varepsilon}') \in \widetilde{\mathcal{E}} \times \widetilde{\mathcal{E}} : \widetilde{\varepsilon} \neq \widetilde{\varepsilon}', s(\widetilde{\varepsilon}) = s(\widetilde{\varepsilon}')\} \\ &= \{(\varepsilon, \varepsilon') \in \mathcal{E} \times \mathcal{E} : \varepsilon \neq \varepsilon', r(\varepsilon') = r(\varepsilon')\}\end{aligned}$$

again satisfying the condition of Lemma 3.2. We denote the horizontal edges of level $n \in \mathbb{N}$ by

$$\mathcal{H}_{\text{lg},n} = \{(\eta\widetilde{\varepsilon}, \eta\widetilde{\varepsilon}') : \eta \in \Pi_{n-1}(\widetilde{\mathcal{G}}), (\widetilde{\varepsilon}, \widetilde{\varepsilon}') \in \hat{\mathcal{H}}_{\text{lg}}\} \subset \Pi_n(\widetilde{\mathcal{G}}) \times \Pi_n(\widetilde{\mathcal{G}})$$

and define the longitudinal approximation graph $G_{\text{lg},\tau} = (V_{\text{lg}}, E_{\text{lg}})$ as in Section 3 by

$$\begin{aligned} V_{\text{lg}} &= \bigcup_n V_{\text{lg},n}, \quad V_{\text{lg},n} = \tau(\Pi_n(\widetilde{\mathcal{G}})) \subset \Pi^*_\infty(\widetilde{\mathcal{G}}), \\ E_{\text{lg}} &= \bigcup_n E_{\text{lg},n}, \quad E_{\text{lg},n} = \tau \times \tau(\mathcal{H}_{\text{lg},n}), \end{aligned}$$

together with the orientation inherited from $\hat{\mathcal{H}}_{\text{lg}}$: so $E_{\text{lg},n} = E^+_{\text{lg},n} \cup E^-_{\text{lg},n}$ for all $n \in \mathbb{N}$, and $E_{\text{lg}} = E^+_{\text{lg}} \cup E^-_{\text{lg}}$.

With these choices made, Theorem 3.3 provides us with a spectral triple for the algebra $C(\Pi_\infty(\widetilde{\mathcal{G}}))$.

A longitudinal horizontal edge $h \in \hat{\mathcal{H}}_{\text{lg}}$ has the following interpretation: As for the transversal horizontal edges, $\tau(s(h))$ and $\tau(r(h))$ start on a common vertex v_0, differ on their first edge, and then come back to finish equally. To obtain their interpretation, it is more useful, however, to reverse their orientation since this is the way the Robinson map $\mathcal{R}$ was defined. Then $h = (\widetilde{\varepsilon}, \widetilde{\varepsilon}')$ with $r(\widetilde{\varepsilon}) = r(\widetilde{\varepsilon}')$ determines a pair of microtiles (t, t') of type $s(\widetilde{\varepsilon})$ and $s(\widetilde{\varepsilon}')$, respectively, in a tile of type $r(\varepsilon)$. The remaining part of the double path $(\tau(\widetilde{\varepsilon}), \tau(\widetilde{\varepsilon}'))$ serves to fix a point in the two microtiles. Of importance is now the vector of translation a_h between the two points of the microtiles.

Similarly, an edge in $E_{\text{lg},n}$ will describe a pair of $(n+1)$th-order microtiles in an nth-order microtile. By self-similarity again, the corresponding translation vector $a_e \in \mathbb{R}^d$ between the two $(n+1)$th-order microtiles will satisfy

$$a_e = \theta^{-n} a_h \tag{25}$$

if $e \in E_{\text{lg},n}(h)$. See Figure 5 for an illustration.

Recall from Remark 4.5 that we can identify $\Pi_\infty(\widetilde{\mathcal{G}}) = \Pi_{-\infty,0}(\mathcal{G})$. The inverse of the Robinson map $\mathcal{R}$ also induces a dense map $\Pi^v_{-\infty,0} \to t_v$ that is one-to-one on the preimage of $\Pi^*_{-\infty,0}(\mathcal{G})$; we still denote this map by $\mathcal{R}^{-1}$. Hence, the approximation graph for $\Pi^v_{-\infty,0}$ is also an approximation graph for t_v. Let E^v_{lg} denote the set of edges whose corresponding paths pass through v at level 0. We may thus adapt the above spectral triple to get the spectral triple $(C(t_v), \mathfrak{H}^v_{\text{lg}}, D_{\text{lg}})$

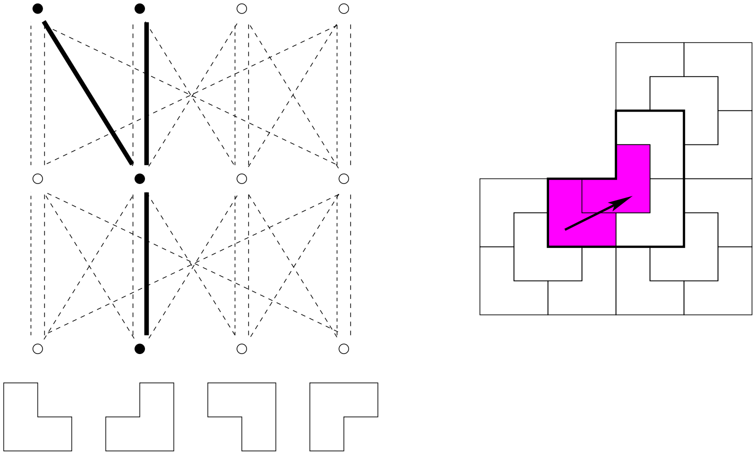

Figure 5 A microtile pattern associated with a horizontal arrow $e \in E_{\mathrm{lg},2}(h)$ (the pattern shown has the size of a single tile). The arrow represents the vector r_e

(which depends on ρ_{lg}) with representation and Dirac defined as in equations (10) and (11) by

$$\mathfrak{H}^v_{\mathrm{lg}} = \ell^2(E^v_{\mathrm{lg}}), \qquad \pi_{\mathrm{lg}}(f)\varphi(e) = f(\mathcal{R}^{-1}(s(e)))\varphi(e),$$
$$D_{\mathrm{lg}}\varphi(e) = \rho_{\mathrm{lg}}^{-n}\varphi(e^{\mathrm{op}}), \quad e \in E^v_{\mathrm{lg},n}.$$

The bounded commutator axiom is satisfied by the following lemma and the fact that the Hölder-continuous functions are dense in $C(t_v)$.

LEMMA 4.6. *If $f \in C(t_v)$ is Hölder continuous (w.r.t. the Euclidean metric d) with exponent $\alpha = -\log(\rho_{\mathrm{lg}})/\log(\theta)$, then $[D_{\mathrm{lg}}, \pi_{\mathrm{lg}}(f)]$ is bounded.*

Proof. Suppose that $f \in C(t_v)$ is Hölder continuous with exponent $\alpha = -\log(\rho_{\mathrm{lg}})/\log(\theta)$, that is, $|(f(x) - f(y))/d(x,y)| \le C$ for some $C > 0$ and all $x, y \in t_v$. Then

$$\|[D, \pi(f)]\| = \sup_n \sup_{e \in E_{\mathrm{lg},n}} \left| \frac{f(\mathcal{R}^{-1}(r(e))) - f(\mathcal{R}^{-1}(s(e)))}{d(\mathcal{R}^{-1}(r(e)), \mathcal{R}^{-1}(s(e)))^\alpha} \right| \times \frac{d(\mathcal{R}^{-1}(r(e)), \mathcal{R}^{-1}(s(e)))^\alpha}{\rho_{\mathrm{lg}}^n}.$$

This expression is finite since the first factor is bounded by C. By self-similarity there exists $C' > 0$ such that $d(\mathcal{R}^{-1}(r(e)), \mathcal{R}^{-1}(s(e))) \le C'\theta^{-n}$. Also, α has been chosen so that $\theta^{-n\alpha}\rho_{\mathrm{lg}}^{-n} = 1$. $\square$

We refer to this spectral triple $(C(t_v), \mathfrak{H}^v_{\mathrm{lg}}, D_{\mathrm{lg}})$ as the *longitudinal spectral triple for the prototile t_v*. It should be noted that although the map $\mathcal{R}^{-1}$ is continuous, the topologies of t_v and $\Pi^v_{-\infty,0}$ are quite different, and so the Connes distance of this spectral triple does not induce the topology of t_v. By Theorems 3.4 and 3.8 the longitudinal spectral triple has metric dimension $s_{\mathrm{lg}} = (d\log(\theta))/(-\log(\rho_{\mathrm{lg}}))$ for all v, but what depends on v is the residue of the zeta function. In fact, as compared to the zeta function of the full triple, it has to be rescaled: $\zeta^v_{\mathrm{tr}} = (L_v/\sum_u L_u)\zeta_{\mathrm{tr}}$.

The spectral measure μ^v_{lg} is easily seen to be the normalized Lebesgue measure on t_v since the groupoid of tail equivalence acts by partial translations.

4.5. *The Spectral Triple for* Ω_Φ

We now combine the above triples to get a spectral triple $(C(\Omega_\Phi), \mathfrak{H}, D)$ for the whole tiling space Ω_Φ. The graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ have the same set of vertices $\mathcal{V}$, so we notice from Remark 4.5 that the identification

$$\Pi_{-\infty,+\infty}(\mathcal{G}) = \bigcup_{v\in\mathcal{V}} \Pi^v_{-\infty,0}(\mathcal{G}) \times \Pi^v_{0,+\infty}(\mathcal{G}) = \bigcup_{v\in\mathcal{V}} \Pi^v_{\infty}(\widetilde{\mathcal{G}}) \times \Pi^v_{\infty}(\mathcal{G})$$

suggests to construct the triple for Ω_Φ as a direct sum of tensor product spectral triples related to the transversal and the longitudinal parts. In fact, $\Pi^v_{-\infty,0}(\mathcal{G}) \times \Pi^v_{0,+\infty}(\mathcal{G})$ is dense in $t_v \times \Xi_{t_v}$ (see Remark 4.5), and so we can use the tensor product construction for spectral triples to obtain a spectral triple for $C(t_v \times \Xi_{t_v}) \cong C(t_v) \otimes C(\Xi_{t_v})$ from the two spectral triples considered before. Furthermore, the C^*-algebra $C(\Omega_\Phi)$ is a subalgebra of $\bigoplus_{v\in\mathcal{V}} C(t_v \times \Xi_{t_v})$, and so the direct sum of the tensor product spectral triples for the different tiles t_v provides us with a spectral triple for $C(\Omega)$:

$$\begin{aligned} \mathfrak{H} &= \bigoplus_{v\in\mathcal{V}} \mathfrak{H}^v_{\mathrm{tr}} \otimes \mathfrak{H}^v_{\mathrm{lg}}, \qquad \pi = \bigoplus_{v\in\mathcal{V}} \pi^v_{\mathrm{tr}} \otimes \pi^v_{\mathrm{lg}}, \\ D &= \bigoplus_{v\in\mathcal{V}} (D^v_{\mathrm{tr}} \otimes \mathbf{1} + \chi \otimes D^v_{\mathrm{lg}}), \end{aligned} \tag{26}$$

where χ is the grading of the transversal triple (which flips the orientations in E_{tr}). The representation of a function $f \in C(\Omega)$ then reads

$$\pi(f) = \sum_{v\in\mathcal{V}} f^v_{\mathrm{tr}} \otimes f^v_{\mathrm{lg}}, \quad \text{with } f^v_{\mathrm{tr}} = \pi^v_{\mathrm{tr}}(f) \in C(\Xi_{t_v}),\ f^v_{\mathrm{lg}} = \pi^v_{\mathrm{lg}}(f) \in C(t_v). \tag{27}$$

From the results in Section 2.4 we now get all the spectral information of $(C(\Omega_\Phi), \mathfrak{H}, D)$. To formulate our results more concisely, let us call $A \in \mathcal{B}(\mathfrak{H})$ *nonresonant* if $\bar{A}_n \overset{n\to\infty}{\sim} c_A e^{\imath n\varphi}$ for some $c_A > 0$ (see Corollary 2.5 for the definition of $\bar{A}_n$), and $\varphi \in (0, 2\pi)$ is nonresonant (Definition 3.9).

Theorem 4.7. *The above is a spectral triple for* $C(\Omega_\Phi)$. *Its spectral dimension is*

$$s_0 = s_{\mathrm{tr}} + s_{\mathrm{lg}} = \frac{d\log\theta}{-\log\rho_{\mathrm{tr}}} + \frac{d\log\theta}{-\log\rho_{\mathrm{lg}}},$$

and its zeta function $\zeta(z)$ *has a simple pole at* s_0 *with strictly positive residue. Moreover, suppose that* $A = \bigoplus_v A^v_{\mathrm{tr}} \otimes A^v_{\mathrm{lg}}$ *with either both* A^v_{tr} *and* A^v_{lg} *strongly* *regular or that one is strongly regular and the other is the sum of a strongly regular and a nonresonant part. Then we have*

$$\mathcal{T}(A) = \sum_{v \in \mathcal{V}} \operatorname{freq}(t_v) \operatorname{vol}(t_v) \mathcal{T}^v_{\mathrm{tr}}(A^v_{\mathrm{tr}}) \mathcal{T}^v_{\mathrm{lg}}(A^v_{\mathrm{lg}}). \tag{28}$$

In particular, the spectral measure is the unique invariant ergodic probability measure μ on Ω_Φ.

Proof. Consider first the triple for the matchbox $t_v \times \Xi_{t_v}$, that is, the tensor product spectral triple for $C(t_v) \otimes C(\Xi_{t_v})$. Applying Lemma 2.6, we obtain the value $s_0^v = (d \log \theta)/(-\log \rho_{\mathrm{tr}}) + (d \log \theta)/(-\log \rho_{\mathrm{lg}})$ for the abscissa of convergence of its zeta function ζ^v. In particular, this value does not depend on v. Furthermore,

$$\begin{aligned}\lim_{s \to s_0^+} (s - s_0)\zeta^v(s) = {} & \frac{\operatorname{freq}(t_v) \operatorname{vol}(t_v)}{\sum_u \operatorname{vol}(t_u)} \\ & \times \left(\sum_{k=-\infty}^{\infty} \Gamma\left(\frac{d \log \theta + 2\pi i k}{-2 \log(\rho_{\mathrm{tr}})} \right) \Gamma\left(\frac{d \log \theta - 2\pi i k}{-2 \log(\rho_{\mathrm{lg}})} \right) \right) \\ & \Big/ \left(2\Gamma\left(\frac{s_0}{2}\right) \log(\rho_{\mathrm{tr}}) \log(\rho_{\mathrm{lg}}) \right).\end{aligned}$$

This number is in fact a strictly positive real number since it is up to a positive factor the mean of two strictly positive periodic functions. It follows that the abscissa of convergence for the zeta function of the direct sum of the above triples ζ is equal to the common value $s_0 = s_0^v$. From this, with the help of Lemma 2.7 and (6), we can now determine the spectral state.

If A^v_{tr} and A^v_{lg} are both strongly regular, then, by Corollary 2.8, $\mathcal{T}(A^v_{\mathrm{tr}} \otimes A^v_{\mathrm{lg}}) = n_v \mathcal{T}^v_{\mathrm{tr}}(A^v_{\mathrm{tr}}) \mathcal{T}^v_{\mathrm{lg}}(A^v_{\mathrm{lg}})$ with the factor $n_v = \operatorname{freq}(t_v) \operatorname{vol}(t_v)$ because the states are normalized. If, say, A^v_{tr} is regular and $A^v_{\mathrm{lg}} = A^v_{\mathrm{lg,reg}} + A^v_{\mathrm{lg,nres}}$ is the sum of a strongly regular and a nonresonant part, then

$$\begin{aligned}\mathcal{T}(A^v_{\mathrm{tr}} \otimes A^v_{\mathrm{lg}}) &= \mathcal{T}(A^v_{\mathrm{tr}} \otimes A^v_{\mathrm{lg,sreg}}) + \mathcal{T}(A^v_{\mathrm{tr}} \otimes A^v_{\mathrm{lg,nres}}) \\ &= n_v \mathcal{T}^v_{\mathrm{tr}}(A^v_{\mathrm{tr}}) \mathcal{T}^v_{\mathrm{lg}}(A^v_{\mathrm{lg,sreg}}) + \operatorname{freq}(t_v) \mathcal{T}^v_{\mathrm{tr}}(A^v_{\mathrm{tr}}) \mathcal{T}(\mathbf{1} \otimes A^v_{\mathrm{lg,nres}}) \\ &= n_v \mathcal{T}^v_{\mathrm{tr}}(A^v_{\mathrm{tr}}) \mathcal{T}_2(A^v_{\mathrm{lg,sreg}}) + 0 \\ &= n_v \mathcal{T}^v_{\mathrm{tr}}(A^v_{\mathrm{tr}}) \mathcal{T}^v_{\mathrm{lg}}(A^v_{\mathrm{lg,sreg}}) + n_v \mathcal{T}^v_{\mathrm{tr}}(A^v_{\mathrm{tr}}) \mathcal{T}^v_{\mathrm{lg}}(A^v_{\mathrm{lg,nres}}) \\ &= n_v \mathcal{T}^v_{\mathrm{tr}}(A^v_{\mathrm{tr}}) \mathcal{T}^v_{\mathrm{lg}}(A^v_{\mathrm{lg}}),\end{aligned}$$

where the second line follows by Corollary 2.8, the third by Lemma 3.11, and the fourth by Lemma 3.10 (the state of a nonresonant operator vanishes: $\mathcal{T}^v_{\mathrm{lg}}(A_{\mathrm{lg,nres}}) = 0$). The argument is the same if A^v_{lg} is strongly regular and A^v_{tr} is the sum of a strongly regular and a nonresonant part. Hence, in both cases, we get

$$\mathcal{T}(A) = \mathcal{T}\left(\sum_v A^v_{\mathrm{tr}} \otimes A^v_{\mathrm{lg}} \right) = \sum_v n_v \mathcal{T}^v_{\mathrm{tr}}(A^v_{\mathrm{tr}}) \mathcal{T}^v_{\mathrm{lg}}(A^v_{\mathrm{lg}}).$$

Since $n_v = \operatorname{freq}(t_v) \operatorname{vol}(t_v)$ is the μ-measure of the matchbox $t_v \times \Xi_{t_v}$, we see that the spectral measure coincides with μ. □

References

[1] J. E. Anderson and I. F. Putnam, *Topological invariants for substitution tilings and their associated C^*-algebras,* Ergodic Theory Dynam. Systems 18 (1998), 509–537.

[2] O. Bratteli, *Inductive limits of finite dimensional C^*-algebra,* Trans. Amer. Math. Soc. 171 (1972), 195–234.

[3] E. Christensen and C. Ivan, *Sums of two-dimensional spectral triples,* Math. Scand. 100 (2007), 35–60.

[4] A. Connes, *Noncommutative geometry,* Academic Press, San Diego, CA, 1994.

[5] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives,* Amer. Math. Soc. Colloq. Publ., 55, AMS, Providence, RI, 2008.

[6] A. Forrest, *K-groups associated with substitution minimal systems,* Israel J. Math. 98 (1997), 101–139.

[7] D. Guido and T. Isola, *Dimensions and singular traces for spectral triples, with applications for fractals,* J. Funct. Anal. 203 (2003), 362–400.

[8] ———, *Dimension and spectral triples for fractals in $\mathbb{R}^n$,* Advances in operator algebras and mathematical physics, Theta Ser. Adv. Math., 5, pp. 89–108, Theta, Bucarest, 2005.

[9] B. Grünbaum and G. C. Shephard, *Tilings and patterns,* Freeman and Company, New York, 1987.

[10] R. Horn and C. R. Johnson, *Topics in matrix analysis,* Cambridge University Press, Cambridge, 1994.

[11] B. Iochum, *Spectral geometry,* Lecture notes for the summer school geometric, algebraic an topological methods for quantum field theory, Chile, 2011.

[12] A. Julien and J. Savinien, *Transverse Laplacians for substitution tilings,* Comm. Math. Phys. 301 (2010), 285–318.

[13] J. Kellendonk, *Non commutative geometry of tilings and gap labelling,* Rev. Math. Phys. 7 (1995), 1133–1180.

[14] J. Kellendonk and J. Savinien, *Spectral triples from stationary Bratteli diagrams,* arXiv:1210.7310v2 [math.OA].

[15] ———, *Spectral triples and characterization of aperiodic order,* Proc. Lond. Math. Soc. (3) 104 (2012), 123–157.

[16] M. Lapidus, *Towards a noncommutative fractal geometry? Laplacians and volume measures on fractals,* Harmonic analysis and nonlinear differential equations, Contemp. Math., 208, pp. 211–252, Amer. Math. Soc., Providence, RI, 1997.

[17] J. Pearson and J. Bellissard, *Noncommutative Riemannian geometry and diffusion on ultrametric Cantor sets,* J. Noncommut. Geom. 3 (2009), 447–480.

J. Kellendonk
Univ. Lyon
Université Claude Bernard Lyon 1
CNRS UMR 5208
Institut Camille Jordan
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne cedex
France

J. Savinien
IECL
Université de Lorraine—Metz
France