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Strange Duality for Height Zero Moduli Spaces of Sheaves on $\mathbb{P}^2$

Takeshi Abe

1. Introduction

Strange duality is a duality between vector spaces of global sections of line bundles on moduli spaces of sheaves. Originally it was studied for moduli spaces of vector bundles (or principal bundles) on curves (see [P]). In this paper we consider the strange duality for moduli spaces of sheaves on $\mathbb{P}^2$.

Result

For a coherent sheaf E of positive rank r on $\mathbb{P}^2$, we define the rational numbers $\mu(E)$ and $\Delta(E)$, called the slope and the discriminant, respectively, by

$$\mu(E) = \frac{c_1(E)}{r},$$
$$\Delta(E) = \frac{1}{r}\left(c_2(E) - \frac{r-1}{2r}c_1(E)^2\right).$$

For a positive integer r and rational numbers s, d, we denote by $M(r,s,d)$ the moduli space of rank r semistable sheaves E on $\mathbb{P}^2$ with $\mu(E)=s$ and $\Delta(E)=d$.

We recall the definition of a strange duality map. Fix positive integers r, r' and rational numbers s, s', d, d' such that $\chi(E\otimes E')=0$ for $E\in M := M(r,s,d)$ and $E'\in M' := M(r',s',d')$. Assume that $s+s'\geq 0$ (so that we have $\mathrm{H}^2(E\otimes E')=0$).

Consider the locus

$$\Theta := \{(E,E') \mid \mathrm{H}^0(E\otimes E')\neq 0\} \subset M\times M'.$$

If $\mathrm{H}^0(E\otimes E')\neq 0$ for all $E\in M$ and $E'\in M'$, then $\Theta = M\times M'$. Now we assume that for some $E\in M$ and $E'\in M'$, we have $\mathrm{H}^i(E\otimes E')=0$, $0\leq i\leq 2$. In this case, Θ is a divisor on $M\times M'$. The associated line bundle $\mathcal{O}(\Theta)$ is expressed as $\mathcal{D}\boxtimes\mathcal{D}'$ for line bundles $\mathcal{D}$ on M and $\mathcal{D}'$ on M'. By the Kunneth theorem, the section defining the divisor Θ gives rise to a duality map

$$\mathrm{H}^0(M',\mathcal{D}')^* \to \mathrm{H}^0(M,\mathcal{D}).$$

We call this the strange duality map. The purpose of this paper is to prove the following theorem.

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THEOREM 1.1. *Retain the notation and assumption in the previous paragraph. If either M or M' is of height zero, then the strange duality map is bijective.*

The height of a moduli space of sheaves on $\mathbb{P}^2$ is the notion introduced by Drezet [D87]. We recall its definition in Section 2.3.

Outline of the Proof of Theorem 1.1

The proof of Theorem 1.1 is a combination of the following three results

(i) Drezet's description of a moduli space of sheaves of height zero as a moduli space of Kronecker modules [D87],
(ii) Derksen and Weyman's duality theorem between spaces of global sections of line bundles on the moduli of Kronecker modules [DW],
(iii) Coskun, Huizenga, and Woolf's rational map of a moduli space of sheaves of positive height to a moduli of Kronecker modules [CHW].

Assume, for example, that M' is of height zero and M has positive height. By (i) and (iii) we have an isomorphism

$$\Phi' : M' \to Kr'$$

and a rational map

$$\Phi : M \cdots\to Kr,$$

where $Kr$and Kr' are moduli spaces of Kronecker modules with certain dimension vector. By (ii) we have an isomorphism

$$\mathrm{H}^0(Kr', \mathcal{L}')^* \to \mathrm{H}^0(Kr, \mathcal{L}),$$

where $\mathcal{L}$ and $\mathcal{L}'$ are certain line bundles on Kr and Kr', respectively. Consider a sequence of maps

$$\mathrm{H}^0(M', \Phi'^*\mathcal{L}')^* \xrightarrow{\sim} \mathrm{H}^0(Kr', \mathcal{L}')^* \xrightarrow{\sim} \mathrm{H}^0(Kr, \mathcal{L}) \xrightarrow{(*)} \mathrm{H}^0(M, \Phi^*\mathcal{L}), \tag{1.1}$$

where the first map is a dual of the pull-back by Φ', and the last map is a pull-back by Φ. We shall prove Theorem 1.1 by showing that

- $\Phi^*\mathcal{L} \simeq \mathcal{D}$ and $\Phi^*\mathcal{L}' \simeq \mathcal{D}'$,
- the composite of maps (1.1) is a strange duality map, and
- the map $(*)$ is an isomorphism.

Related Results

There are some other pairs (M, M') of moduli spaces of sheaves on $\mathbb{P}^2$ with certain numerical invariants for which the strange duality has been proved ([Da00; Da02; A]).

For abelian surfaces, Marian and Oprea [MO09] formulated three versions of strange duality and proved them on the numerical level. They proved two of them for product abelian surfaces in [MO14]. They also proved the strange duality for generic K3 surfaces under some numerical assumption in [MO13]. Bolognese, Marian, Oprea, and Yoshioka [BMOY] studied the remaining one of the three versions of strange duality for abelian surfaces. They first proved it for product abelian surfaces and then, using degeneration, for generic abelian surfaces.

Organization of the paper

In Section 2 we recall some results used in the proof of Theorem 1.1. In Section 3 we prove Theorem 1.1.

Notation and Convention

We work over an algebraically closed field k of characteristic zero. In this paper, $P(x)$ denotes the polynomial function$(x+1)(x+2)/2$.

2. Preliminaries

In Section 2.1 we fix the notation for moduli of sheaves and for G-linearized line bundles. In Section 2.2 we recall basic facts about exceptional bundles. In Section 2.3 we recall the definition of height introduced by Drezet. After fixing notation for moduli of Kronecker modules in Section 2.4, we recall in Section 2.5 the duality theorem proved by Derksen and Weyman between spaces of global sections of line bundles on the moduli of Kronecker modules. In Section 2.6 we recall Drezet's theorem saying that a moduli space of height zero is isomorphic to a moduli space of Kronecker modules. In Section 2.7 we recall a rational map constructed by Coskun, Huizenga, and Woolf of a moduli space of sheaves of positive height to a moduli space of Kronecker modules.

2.1. Notation

For $\xi = (r, s, d)$, where $r \in \mathbb{Z}_{\geq 0}$ and $s, d \in \mathbb{Q}$, $M(\xi)$ (resp. $\mathcal{M}(\xi)$) denotes the moduli space (resp. the moduli stack) of rank r semistable sheaves E with $\mu(E) = s$ and $\Delta(E) = d$. There is a natural morphism $\pi : \mathcal{M}(\xi) \to M(\xi)$. If $\mathcal{L}$ is a line bundle on $M(\xi)$, then we simply write $\mathcal{L}$ for $\pi^*\mathcal{L}$. Since $\mathrm{H}^0(\mathcal{M}(\xi), \pi^*\mathcal{L}) \simeq \mathrm{H}^0(M(\xi), \mathcal{L})$ (cf. the argument of [BL, Prop. 8.4]), this abuse of notation does not cause confusion.

Assume that an algebraic group G acts on a variety X. If $\chi : G \to k^\times$ is a character, then *the G-linearized line bundle associated to χ,* denoted by $\mathbb{L}_\chi$, is a trivial line bundle $X \times k$ with the G-action $g(x, a) := (gx, \chi(g)a)$ for $g \in G$ and $(x, a) \in X \times k$. *The line bundle $\mathcal{L}_\chi$ on the quotient stack $G\backslash X$ associated to* χ is the line bundle on $G\backslash X$ determined by $\mathbb{L}_\chi$ on X. We have $\mathrm{H}^0(G\backslash X, \mathcal{L}_\chi) \simeq \mathrm{H}^0(X, \mathbb{L}_\chi)^G$, that is, $\mathrm{H}^0(G\backslash X, \mathcal{L}_\chi)$ is isomorphic to the space of semiinvariant functions on X with weight χ. Here a function ϕ on X is semiinvariant with weight χ if $\phi(gx) = \chi(g)\phi(x)$ for $x \in X$ and $g \in G$.

2.2. *Exceptional Bundle*

An exceptional bundle E on $\mathbb{P}^2$ is a stable vector bundle with $\mathrm{Ext}^1(E, E) = 0$. A rational number α is called an exceptional slope if α is the slope of an exceptional bundle. For an exceptional slope α, there is only one (up to isomorphism) exceptional bundle with slope α, which we denote by E_α ([DL, Lemma 4.3]). This implies that

$$E_\alpha^* \simeq E_{-\alpha} \quad \text{and} \quad E_\alpha \otimes \mathcal{O}(n) \simeq E_{\alpha+n}.$$

The rank r_α of a rational number α is defined to be the smallest positive integer such that $\alpha r_\alpha \in \mathbb{Z}$. Put $\Delta_\alpha = \frac{1}{2}(1 - 1/r_\alpha^2)$. Then for an exceptional slope α, the rank and discriminant of E_α are r_α and Δ_α, respectively (cf. the proof of [DL, Lemma 4.3]). In particular, we know that $\Delta(E_\alpha) < 1/2$. This is another characterization of exceptional bundles, that is, a stable sheaf E is exceptional if and only if $\Delta(E) < 1/2$.

The subset $\mathfrak{E} \subset \mathbb{Q}$ of all exceptional slopes is described as follows. For rational numbers α, β with $3 + \alpha - \beta \neq 0$, we define

$$\alpha.\beta := \frac{\alpha+\beta}{2} + \frac{\Delta_\beta - \Delta_\alpha}{3 + \alpha - \beta}.$$

Let $\mathfrak{D} := \{(p/2^q) \mid q \in \mathbb{Z}_{\geq 0}, p \in \mathbb{Z}\}$. We define a function $\varepsilon : \mathfrak{D} \to \mathbb{Q}$ inductively as follows:

- $\varepsilon(n) = n$ for $n \in \mathbb{Z}$,
- $\varepsilon((2p+1)/2^{q+1}) = \varepsilon(p/2^q).\varepsilon((p+1)/2^q)$.

Then ε is a strictly increasing function, and $\mathfrak{E} = \varepsilon(\mathfrak{D})$ ([DL, Thm. A]).

The following cohomological properties are used later. For $\alpha < \beta$,

$$\mathrm{Hom}(E_\beta, E_\alpha) = 0, \tag{2.1}$$

$$\mathrm{Ext}^i(E_\alpha, E_\beta) = 0 \quad \text{for } i > 0. \tag{2.2}$$

Property (2.1) is a consequence of stability, and (2.2) is due to [D86, Thm. 6]. (Note that the vanishing (2.2) holds also for $\alpha = \beta$.)

By [D86, p. 30], for $\alpha = \varepsilon(p/2^q)$ and $\beta = \varepsilon((p+1)/2^q)$, we have

$$\chi(E_\alpha, E_\beta) = 3 r_\alpha r_\beta (\beta - \alpha). \tag{2.3}$$

2.3. *Height of Moduli Space*

For $\xi = (r, s, d)$ with $r \in \mathbb{Z}_{\geq 0}$ and $s, d \in \mathbb{Q}$, we denote by Q_ξ the parabola in the (μ, Δ)-plane defined by $\Delta = P(\mu + s) - d$. By the Riemann–Roch theorem, we have

$$\chi(E, F) = r(E)r(F)(P(\mu(F) - \mu(E)) - \Delta(E) - \Delta(F))$$

for sheaves E and F on $\mathbb{P}^2$ having positive ranks $r(E)$ and $r(F)$, respectively. So, for a sheaf E, the point $(\mu(E), \Delta(E))$ in the (μ, Δ)-plane lies on the parabola Q_ξ if and only if $\chi(E \otimes F) = 0$ for F with $\mu(F) = s$ and $\Delta(F) = d$.

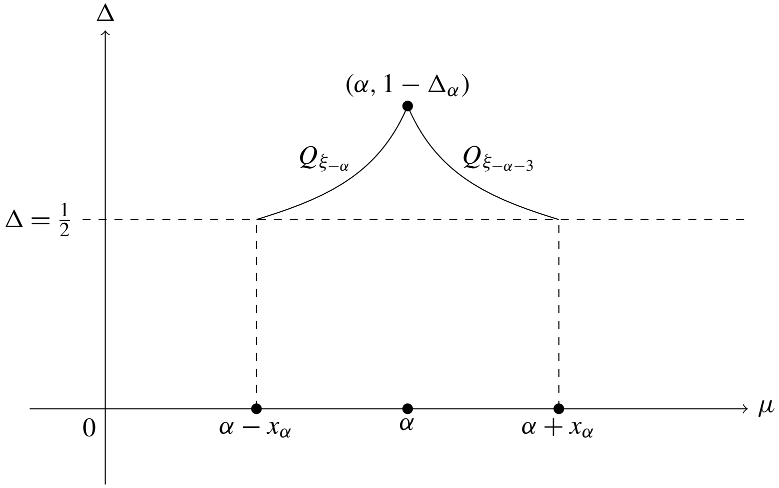

Figure 1 The graph of $\Delta = \delta(\mu)$ over $(\alpha - x_\alpha, \alpha + x_\alpha)$

For an exceptional slope α, put

$$x_\alpha := \frac{3 - \sqrt{9 - 4/r_\alpha^2}}{2}.$$

Let $I_\alpha := \{\mu \in \mathbb{R} \mid |\mu - \alpha| < x_\alpha\}$. Then $\mathbb{Q}$ is the disjoint union $\bigsqcup_{\alpha \in \mathfrak{E}} (I_\alpha \cap \mathbb{Q})$ [D87, Thm. 1]. Let $\delta : \bigsqcup_{\alpha \in \mathfrak{E}} I_\alpha \to \mathbb{R}$ be the function such that $\delta(\mu) = P(-|\mu - \alpha|) - \Delta_\alpha$ for $\mu \in I_\alpha$. Since x_α is the smaller solution of the equation

$$P(-x) - \Delta_\alpha = \frac{1}{2},$$

the graph $\Delta = \delta(\mu)$ in the (μ, Δ)-plane is above the line $\Delta = \frac{1}{2}$. For $\alpha - x_\alpha < \mu < \alpha$, the graph $\Delta = \delta(\mu)$ is part of $Q_{\xi_{-\alpha}}$, and for $\alpha < \mu < \alpha + x_\alpha$, it is part of $Q_{\xi_{-\alpha-3}}$, where $\xi_\alpha = (1, \alpha, \Delta_\alpha)$. See Figure 1.

For a sheaf F with $\alpha - x_\alpha < \mu(F) \le \alpha$, the point $(\mu(F), \Delta(F))$ in the (μ, Δ)-plane lies on the graph of $\Delta = \delta(\mu)$ if and only if $\chi(E_\alpha, F) = 0$. If such a sheaf F is semistable, then $\operatorname{Ext}^i(E_\alpha, F) = 0$ for all i because we have $\operatorname{Hom}(E_\alpha, F) = 0$ by semistability and $\operatorname{Ext}^2(E_\alpha, F) \simeq \operatorname{Hom}(F, E_{\alpha-3})^* = 0$ by duality and semistability. Analogously, for a sheaf F with $\alpha < \mu < \alpha + x_\alpha$, the point $(\mu(F), \Delta(F))$ lies on the graph of $\Delta = \delta(\mu)$ if and only if $\chi(F, E_\alpha) = 0$. If such a sheaf F is semistable, then $\operatorname{Ext}^i(F, E_\alpha) = 0$ for all i.

Let $r \in \mathbb{Z}_{>0}$ and $s, d \in \mathbb{Q}$. If $\dim M(r, s, d) = 0$, then $s \in \mathfrak{E}$ and $M(r, s, d) = \{E_s^{\oplus r/r_s}\}$ (see [DL, Prop. (4.4)]). Now assume that $\dim M(r, s, d) > 0$. The exceptional slope α such that $s \in I_\alpha$ is called *the associated exceptional slope* to $M(r, s, d)$. We have $d \ge \delta(s)$ [D87, Thm. 1]. The height $h(M(r, s, d))$ of the moduli space $M(r, s, d)$ is defined to be $r r_\alpha (d - \delta(s))$. It is a nonnegative integer.

2.4. Kronecker Module

Let V be a finite-dimensional vector space. A Kronecker V-module is a linear map $e : A \otimes V \to B$ with A, B finite-dimensional vector spaces. The pair $(\dim A, \dim B)$ is called the dimension vector of the Kronecker module. A morphism from a Kronecker V-module $e : A \otimes V \to B$ to $e' : A' \otimes V \to B'$ is a pair (f, g) of linear maps $f : A \to A'$ and $g : B \to B'$ such that $g \circ e = e' \circ (f \otimes \mathrm{id}_V)$.

A submodule of a Kronecker V-module $e : A \otimes V \to B$ is a Kronecker V-module $e : A' \otimes V \to B'$ such that A' and B' are subspaces of A and B, respectively, and $e' = e|_{A' \otimes V}$. A Kronecker V-module $e : A \otimes V \to B$ is semistable (resp. stable) if for any nonzero proper submodule $e : A' \otimes V \to B'$ of e, the inequality

$$-\dim A \cdot \dim B' + \dim B \cdot \dim A' \leq 0 \quad (\text{resp.} <)$$

holds.

We can also consider a family of Kronecker V-modules. For a scheme S, a Kronecker V-module over S is a morphism $e : \mathcal{A} \otimes_k V \to \mathcal{B}$ of $\mathcal{O}_S$-modules with $\mathcal{A}$, $\mathcal{B}$ locally free $\mathcal{O}_S$-modules of finite rank. The moduli stack $\mathcal{K}r_V(a, b)$ of Kronecker V-modules with dimension vector (a, b) is defined as follows. $\mathcal{K}r_V(a, b)$ associates to a scheme S the groupoid consisting of Kronecker V-modules $e : \mathcal{A} \otimes_k V \to \mathcal{B}$ over S such that $\operatorname{rank} \mathcal{A} = a$ and $\operatorname{rank} \mathcal{B} = b$. We denote by $\mathcal{K}r_V(a, b)^{ss}$ the open substack of $\mathcal{K}r_V(a, b)$ consisting of semistable Kronecker V-modules. We denote by $\mathbf{Kr}_V(a, b)^{ss}$ the coarse moduli space of (S-equivalence of) semistable Kronecker V-modules with dimension vector (a, b). We have a natural morphism $\mathcal{K}r_V(a, b)^{ss} \to \mathbf{Kr}(a, b)^{ss}$. The stack $\mathcal{K}r_V(a, b)$ is described as a quotient stack of an affine space as follows. For a Kronecker V-module $e : k^a \otimes V \to k^b$ and $(f, g) \in \mathrm{GL}(a, b) := \mathrm{GL}(a) \times \mathrm{GL}(b)$, let $(f, g)e$ be the Kronecker V-module $g \circ e \circ (f \otimes \mathrm{id}_V)^{-1}$. This defines a left action of $\mathrm{GL}(a, b)$ on $\mathbf{H} := \mathrm{Hom}(k^a \otimes V, k^b)$. We have an isomorphism $\mathcal{K}r_V(a, b) \simeq \mathrm{GL}(a, b) \backslash \mathbf{H}$ of stacks.

For integers l, m, the line bundle $\mathcal{L}_{l,m}$ on $\mathcal{K}r_V(a, b)$ is defined by associating to a Kronecker V-module $e : \mathcal{A} \otimes_k V \to \mathcal{B}$ over a scheme S the line bundle $(\det \mathcal{A})^{\otimes l} \otimes (\det \mathcal{B})^{\otimes m}$ on S. Under the isomorphism $\mathcal{K}r_V(a, b) \simeq \mathrm{GL}(a, b) \backslash \mathbf{H}$, the line bundle $\mathcal{L}_{l,m}$ is isomorphic to $\mathcal{L}_{\chi_{l,m}}$, where $\chi_{l,m} : \mathrm{GL}(a, b) \to k^*$ is the character defined by $(f, g) \mapsto (\det f)^l (\det g)^m$. By this we find that the space $\mathrm{H}^0(\mathcal{K}r_V(a, b), \mathcal{L}_{l,m})$ of global sections of $\mathcal{L}_{l,m}$ is isomorphic to the space of semiinvariant functions on $\mathbf{H}$ with weight $\chi_{l,m}$.

LEMMA 2.1. *Let l, m be integers with m positive such that $la + mb = 0$. Then the natural map*

$$\mathrm{H}^0(\mathcal{K}r_V(a, b), \mathcal{L}_{l,m}) \xrightarrow{\text{restr.}} \mathrm{H}^0(\mathcal{K}r_V(a, b)^{ss}, \mathcal{L}_{l,m})$$

is bijective.

Proof. We need to show that if ϕ is a semiinvariant function with weight $\chi_{l,m}$ on $\mathbf{H}^{ss}$ (= the open subset consisting of semistable Kronecker V-modules $k^a \otimes V \to k^b$), then ϕ extends to $\mathbf{H}$. By [K, Prop. 3.1] $\mathbf{H}^{ss}$ is the set of GIT-semistable points with respect to the $\mathrm{GL}(a,b)$-linearized trivial line bundle $\mathbb{L}_{\chi_{l,m}}$ on $\mathbf{H}$. Then the lemma is a consequence of (the poof of) [NR, Lemma 4.15]. □

2.5. Derksen–Weyman's Theorem

Let a, a', b, b' be positive integers, and put $v = \dim V$. Let $e : \mathcal{A} \otimes_k V \to \mathcal{B}$ and $e' : \mathcal{A}' \otimes_k V \to \mathcal{B}'$ be universal Kronecker V-modules over $\mathcal{K}r_V(a,b)$ and $\mathcal{K}r_V(a',b')$, respectively. The pull-backs of these Kronecker V-modules to the product $\mathcal{K}r_V(a,b) \times \mathcal{K}r_V(a',b')$ are denoted by the same letters. Over the product stack $\mathcal{K}r_V(a,b) \times \mathcal{K}r_V(a',b')$, we consider a morphism

$$\lambda : \mathcal{H}om(\mathcal{A}',\mathcal{A}) \oplus \mathcal{H}om(\mathcal{B}',\mathcal{B}) \to \mathcal{H}om(\mathcal{A}' \otimes_k V, \mathcal{B})$$

of locally free sheaves of the same rank, defined by $\lambda(f,g) = e \circ (f \otimes \mathrm{id}_V) - g \circ e'$. Now assume that $a'bv = aa' + bb'$. This assumption means that the source and target of the morphism λ have the same rank. If $\mathcal{V}$ and $\mathcal{W}$ denote the source and target of λ, respectively, then $\det(\lambda)$ defines a section of the line bundle $\det \mathcal{W} \otimes (\det \mathcal{V})^*$, which is isomorphic, by direct calculation, to $\mathcal{L}_{-a',a'v-b'} \boxtimes \mathcal{L}_{a-vb,b}$. Therefore, $\det(\lambda)$ gives rise to a duality map (up to scalar)

$$\mathrm{H}^0(\mathcal{K}r_V(a,b), \mathcal{L}_{-a',a'v-b'})^* \to \mathrm{H}^0(\mathcal{K}r_V(a',b'), \mathcal{L}_{a-vb,b}). \tag{2.4}$$

The following is a particular case of [DW, Thm. 1 and Cor. 1].

THEOREM 2.2 (Derksen–Weyman). *The map* (2.4) *is an isomorphism.*

2.6. Moduli of Height Zero

A triad is a triple (E,G,F) of exceptional bundles such that their slopes are of the form $(\alpha, \alpha.\beta, \beta)$, $(\beta - 3, \alpha, \alpha.\beta)$, or $(\alpha.\beta, \beta, \alpha + 3)$, where $\alpha = \varepsilon(p/2^n)$ and $\beta = \varepsilon((p+1)/2^n)$ for some $n \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}$.

Let (E,G,F) be a triad. Let M be the cokernel of the natural morphism

$$G \to F \otimes_k \mathrm{Hom}(G,F)^*.$$

Then M is also an exceptional bundle, and (E,F,M) is a triad. Put $G_{-2} = E$, $G_{-1} = G$, $G_0 = F$, $F_{-2} = E^*(-3)$, $F_{-1} = M^*$, $F_0 = F^*$, and $G_i = F_i = 0$ for $i \notin \{0,-1,-2\}$. The following spectral sequence proved by Drezet is called the generalized Beilinson spectral sequence.

THEOREM 2.3 [D86]. *If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^2$, then there exists a spectral sequence*

$$E_1^{p,q} = G_p \otimes \mathrm{H}^q(F_p \otimes \mathcal{F})$$

that converges to $\mathcal{F}$ in degree 0 *and to* 0 *in other degrees.*

Let $M(r,s,d)$ be a moduli space of height zero, and let γ be the associated exceptional slope to it. Write as $\gamma = \alpha.\beta$ with $\alpha = \varepsilon(p/2^n)$, $\beta = \varepsilon((p+1)/2^n)$. By the description of the set $\mathfrak{E}$ of exceptional slopes in Section 2.2, such α and β are determined uniquely. There are two cases:

(a) $\gamma - x_\gamma < s \leq \gamma$,
(b) $\gamma < s < \gamma + x_\gamma$.

In the case (a), by applying Theorem 2.3 to $\mathcal{F} \in M(r,s,d)$ with $(E,G,F) = (E_{\beta-3}, E_\alpha, E_{\alpha.\beta})$ we obtain a short exact sequence

$$0 \to E_{\beta-3}^m \xrightarrow{A_\mathcal{F}} E_\alpha^n \to \mathcal{F} \to 0, \tag{2.5}$$

where

$$m = -\chi(E_{-\beta} \otimes \mathcal{F}) \quad \text{and} \quad n = -\chi(E_{-(\alpha.\beta).\beta} \otimes \mathcal{F}). \tag{2.6}$$

From the morphism $A_\mathcal{F}$ we obtain a Kronecker V-module

$$t_\mathcal{F} : k^m \otimes V \to k^n \tag{2.7}$$

with $V = \mathrm{Hom}(E_{\beta-3}, E_\alpha)^*$. (Conversely, we can recover $A_\mathcal{F}$ from $t_\mathcal{F}$. So giving $A_\mathcal{F}$ is equivalent to giving $t_\mathcal{F}$.)

THEOREM 2.4 [D87]. *By the correspondence $\mathcal{F} \mapsto t_\mathcal{F}$ we obtain an isomorphism*

$$\tau : M(r,s,d) \to \mathbf{Kr}_V(m,n)^{ss}.$$

In the case (b), all sheaves in $M(r,s,d)$ are locally free ([D87, p. 40, Remarks]). By taking dual we have an isomorphism $M(r,s,d) \simeq M(r,-s,d)$. Then $-\gamma - x_{-\gamma} < -s < -\gamma$, so we are in the case (a). Let $\mathcal{F} \in M(r,s,d)$. By applying Theorem 2.3 to $\mathcal{F}^*$ with $(E,G,F) = (E_{-\alpha-3}, E_{-\beta}, E_{-\alpha.\beta})$ we obtain a short exact sequence

$$0 \to E_{-\alpha-3}^m \xrightarrow{\mathcal{A}'_\mathcal{F}} E_{-\beta}^n \to \mathcal{F}^* \to 0, \tag{2.8}$$

where

$$m = -\chi(E_\alpha \otimes \mathcal{F}^*) \quad \text{and} \quad n = -\chi(E_{\alpha.(\alpha.\beta)} \otimes \mathcal{F}^*). \tag{2.9}$$

Giving the morphism $\mathcal{A}'_\mathcal{F}$ is equivalent to giving a Kronecker V-module

$$t'_\mathcal{F} : k^m \otimes V \to k^n \tag{2.10}$$

with $V = \mathrm{Hom}(E_{-\alpha-3}, E_{-\beta})^*$.

THEOREM 2.5 [D87]. *By the correspondence $\mathcal{F} \mapsto t'_\mathcal{F}$ we have an isomorphism*

$$\tau' : M(r,s,d) \to \mathbf{Kr}_V(m,n)^{ss}.$$

2.7. Moduli of Positive Height

Consider the moduli space $M(\xi)$ where $\xi = (r,s,d)$. Assume that the height of $M(\xi)$ is positive. As in Section 2.3, we let Q_ξ be the parabola in the (μ, Δ)-plane defined by the equation $\Delta = P(\mu + s) - d$. Coskun, Huizenga, and Woolf proved the following [CHW, Thm. 3.1].

THEOREM 2.6. *The parabola Q_ξ intersects the line $\Delta = \frac{1}{2}$ at two points. If $\mu_0 \in \mathbb{R}$ is the larger of the two slopes such that $(\mu_0, \frac{1}{2}) \in Q_\xi$, then there is an exceptional slope $\gamma \in \mathfrak{E}$ such that $\mu_0 \in I_\gamma$.*

The exceptional slope γ in the theorem is called *the corresponding exceptional slope to* ξ. Express γ as $\alpha.\beta$ with $\alpha = \varepsilon(p/2^n)$, $\beta = \varepsilon((p+1)/2^n)$.

2.7.1. The case $\chi(U \otimes E_{\alpha.\beta}) > 0$ *for* $U \in M(\xi)$. By [CHW, Prop. 5.3], a general $U \in M(\xi)$ has a resolution of the form

$$0 \to E_{-\alpha-3}^{m_1} \to E_{-\beta}^{m_2} \oplus E_{-(\alpha.\beta)}^{m_3} \to U \to 0, \tag{2.11}$$

where $m_1 = -\chi(U \otimes E_\alpha)$, $m_2 = -\chi(U \otimes E_{\alpha.(\alpha.\beta)})$ and $m_3 = \chi(U \otimes E_{\alpha.\beta})$. Put $\mathcal{A} := E_{-\alpha-3}^{m_1}$ and $\mathcal{B} := E_{-\beta}^{m_2} \oplus E_{-(\alpha.\beta)}^{m_3}$. Let $\mathbb{H}$ be the affine space $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Let V_1 be the open subset of $\mathbb{H}$ consisting of $f : \mathcal{A} \to \mathcal{B}$ such that f is injective with torsion-free cokernel. Let V_2 be the open subset of V_1 consisting of f such that $\mathrm{Coker} f$ is semistable.

LEMMA 2.7. *The following inequalities hold*:

(1) $\mathrm{codim}(\mathbb{H} \setminus V_1, \mathbb{H}) \geq 2$,
(2) $\mathrm{codim}(V_1 \setminus V_2, V_1) \geq 2$.

Proof. (1) Consider the subset $Z := \{(x, f) \mid f|_x \text{ is not injective}\}$ of $\mathbb{P}^2 \times \mathbb{H}$. By [H, Lemma 5.4] $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ is globally generated, and by a standard dimension counting argument we find that Z is an irreducible closed subvariety of codimension $r + 1$ of $\mathbb{P}^2 \times \mathbb{H}$. We need to show that the subset $\{f \in \mathbb{H} \mid \dim(\mathbb{H} \times \{f\} \cap Z) \geq 1\}$ has codimension ≥ 2 in $\mathbb{H}$. This is true either if $r \geq 2$ or if $r = 1$ and the projection $Z \to \mathbb{H}$ is surjective. If $r = 1$ and $Z \to \mathbb{H}$ is not surjective, then $M(\xi)$ parameterizes a line bundle, which contradicts the assumption that the height of $M(\xi)$ is positive.

(2) In the proof of [CHW, Prop. 5.3], it is proved that the family $\mathcal{U}/V_1$ of quotients parameterized by V_1 is a complete family of prioritary sheaves. Then the lemma is a consequence of [LP, Lemma 18.3.1]. □

Let $G := \mathrm{Aut}(\mathcal{A}) \times \mathrm{Aut}(\mathcal{B})$ act on $\mathbb{H}$ by $(g_1, g_2)f := g_2 \circ f \circ g_1^{-1}$ for $(g_1, g_2) \in G$ and $f \in \mathbb{H}$. If $U \in M(\xi)$ fits in exact sequences, $i = 1, 2$,

$$0 \to \mathcal{A} \xrightarrow{f_i} \mathcal{B} \xrightarrow{\pi_i} U \to 0,$$

then there exists a unique $(g_1, g_2) \in G$ such that $f_2 \circ g_1 = g_2 \circ f_1$ and $\pi_2 \circ g_2 = \pi_1$. It follows from this that we have an isomorphism

$$\alpha : \mathcal{M}(\xi)^\circ \to G \backslash V_2$$

of stacks, where $\mathcal{M}(\xi)^\circ$ is the open substack of $\mathcal{M}(\xi)$ consisting of sheaves U having a resolution of the form (2.11).

Put $\bar{\mathbb{H}} = \mathrm{Hom}(\mathcal{A}, E_{-\beta}^{m_2})$ and $\bar{G} = \mathrm{Aut}(\mathcal{A}) \times \mathrm{Aut}(E_{-\beta}^{m_2})$. We let $\bar{G}$ act on $\bar{\mathbb{H}}$ naturally. Since $\mathrm{Hom}(E_{-(\alpha.\beta)}, E_{-\beta}) = 0$ by (2.1), we have a natural homomorphism $h : G \to \bar{G}$. We have a morphism $\mathbb{H} \to \bar{\mathbb{H}}$ by associating to a morphism

$$\mathcal{A} \xrightarrow{(f,g)} \mathcal{B} = E_{-\beta}^{m_2} \oplus E_{-\alpha.\beta}^{m_3} \tag{2.12}$$

a morphism $f : \mathcal{A} \to E_{-\beta}^{m_2}$. This morphism is compatible with the actions of G and $\bar{G}$, so it induces a morphism

$$\beta : G\backslash\mathbb{H} \to \bar{G}\backslash\bar{\mathbb{H}}$$

of stacks. Let Ψ be the composite of morphisms

$$\mathcal{M}(\xi)^\circ \xrightarrow{\alpha} G\backslash V_2 \subset G\backslash\mathbb{H} \xrightarrow{\beta} \bar{G}\backslash\bar{\mathbb{H}}. \tag{2.13}$$

LEMMA 2.8. *Let $\bar{\chi} : \bar{G} \to k^\times$ be a character. The natural map*

$$\mathrm{H}^0(\bar{G}\backslash\bar{\mathbb{H}}, \mathcal{L}_{\bar{\chi}}) \to \mathrm{H}^0(\mathcal{M}(\xi)^\circ, \Psi^*\mathcal{L}_{\bar{\chi}})$$

is an isomorphism.

Proof. By Lemma 2.7 the natural map

$$\mathrm{H}^0(G\backslash\mathbb{H}, \beta^*\mathcal{L}_{\bar{\chi}}) \to \mathrm{H}^0(\mathcal{M}(\xi)^\circ, \Psi^*\mathcal{L}_{\bar{\chi}})$$

is an isomorphism, so we have only to show that the map

$$\mathrm{H}^0(\bar{G}\backslash\bar{\mathbb{H}}, \mathcal{L}_{\bar{\chi}}) \to \mathrm{H}^0(G\backslash\mathbb{H}, \beta^*\mathcal{L}_{\bar{\chi}})$$

is an isomorphism. Put $\chi := \bar{\chi} \circ h$. Then $\beta^*\mathcal{L}_{\bar{\chi}} \simeq \mathcal{L}_\chi$. Let ϕ be a semiinvariant function on $\mathbb{H}$ with weight χ. We shall show that ϕ is a pull-back of a function on $\bar{\mathbb{H}}$. Since the element

$$\left(\mathrm{id}_{\mathcal{A}}, \begin{pmatrix} \mathrm{id}_{E_{-\beta}^{m_2}} & 0 \\ 0 & a\,\mathrm{id}_{E_{-\alpha.\beta}^{m_3}} \end{pmatrix}\right) \in G$$

maps to the identity in $\bar{G}$ by h, where $a \in k^\times$, ϕ is invariant by the action of this element. From this we see that for $(f, g) \in \mathbb{H} = \mathrm{Hom}(\mathcal{A}, E_{-\beta}^{m_2}) \oplus \mathrm{Hom}(\mathcal{A}, E_{-\alpha.\beta}^{m_3})$, we have $\phi(f, g) = \phi(f, 0)$. Hence, ϕ is a pull-back of a function on $\bar{\mathbb{H}}$. $\square$

2.7.2. The case $\chi(U \otimes E_{\alpha.\beta}) \le 0$ for $U \in M(\xi)$. In this case, a general $U \in M(\xi)$ fits in an exact sequence of the form

$$0 \to E_{-\alpha-3}^{m_1} \oplus E_{-\alpha.\beta-3}^{m_3} \to E_{-\beta}^{m_2} \to U \to 0, \tag{2.14}$$

where $m_1 = \chi(U \otimes E_{(\alpha.\beta)\beta})$, $m_2 = \chi(U \otimes E_\beta)$, and $m_3 = -\chi(U \otimes E_{\alpha.\beta})$ (cf. [CHW, Section 5.4]). Put $\bar{\mathbb{H}} := \mathrm{Hom}(E_{-\alpha-3}^{m_1}, E_{-\beta}^{m_3})$ and $\bar{G} := \mathrm{Aut}(E_{-\alpha-3}^{m_1}) \times \mathrm{Aut}(E_{-\beta}^{m_2})$, and let $\bar{G}$ act on $\bar{\mathbb{H}}$ naturally. Let $\mathcal{M}(\xi)^\circ$ be the open substack of $\mathcal{M}(\xi)$ consisting of sheaves U having a resolution of the form (2.14). As in the previous case, we define a morphism

$$\Psi : \mathcal{M}(\xi)^\circ \to \bar{G}\backslash\bar{\mathbb{H}} \tag{2.15}$$

by associating to a sheaf having a resolution

$$0 \to E_{-\alpha-3}^{m_1} \oplus E_{-\alpha.\beta-3}^{m_3} \xrightarrow{f+g} E_{-\beta}^{m_2} \to U \to 0 \tag{2.16}$$

the (equivalence class of) morphism $[f] \in \bar{G}\backslash\bar{\mathbb{H}}$. Then the same statement as in Lemma 2.8 holds.

3. Proof of Theorem 1.1

We prove Theorem 1.1. Put $\xi = (r, s, d)$ and $\xi' = (r', s', d')$. We assume that $M' = M(\xi')$ is of height zero. There are three cases:

Case (1) The height of M is positive.

Case (2) The height of M is zero.

Case (3) $\dim M = 0$.

Case (1) is a "generic" case, and the others are special cases. We will divide Case (1) into subcases Case (1-a) and Case (1-b) and give a complete proof for Case (1-a). For the other cases, we omit some arguments if they are similar to Case (1-a).

Case (1). Let γ be the corresponding exceptional slope to ξ. In the (μ, Δ)-plane, the subset $\{(\mu, \Delta) \mid \mu \in I_\gamma, \Delta = \delta(\mu)\}$ and the parabola Q_ξ intersect at one point. First, we note the following.

Claim 3.0.1. *The intersection point of the subset* $\{(\mu, \Delta) \mid \mu \in I_\gamma, \Delta = \delta(\mu)\}$ *and the parabola* Q_ξ *is* (s', d').

Proof. By assumption, the point (s', d') is both on the parabola Q_ξ and on the graph of the function $\delta(\mu)$. The slope of the tangent line of Q_ξ at (s', d') is $s' + s + 3/2$, which is positive by the assumption $s' + s \geq 0$. Let us show that for an exceptional slope $\zeta > \gamma$, Q_ξ does not intersect $\{(\mu, \Delta) \mid \mu \in I_\zeta, \Delta = \delta(\mu)\}$ (then the claim follows). The slope of the tangent line of the parabola $\Delta = P(\mu - \zeta) - \Delta_\zeta$ at $(\zeta - x_\zeta, \frac{1}{2})$ is $-x_\zeta + \frac{3}{2}$, and the slope of the tangent line of Q_ξ at $(\zeta - x_\zeta, P(\zeta - x_\zeta + s) - d)$ is $\zeta - x_\zeta + s + \frac{3}{2}$. It suffices to check that $\zeta - x_\zeta + s + \frac{3}{2} > -x_\zeta + \frac{3}{2}$ or, equivalently, $\zeta + s > 0$. But if $(\bar{\mu}, \frac{1}{2})$ is one of the intersection points of the parabola Q_ξ and the line $\Delta = \frac{1}{2}$ with bigger μ-coordinate, then $\bar{\mu} + s > 0$ since the parabola Q_ξ is increasing for $\mu \geq -s - 3/2$ and $P(-s + s) - d = 1 - d < 1/2$ (because $d > 1/2$). By the definition of γ we have $\bar{\mu} \in I_\gamma$. Since $\zeta > \bar{\mu}$, we have $\zeta + s > 0$. □

There are two cases:

Case (1-a) $\gamma - x_\gamma < s' \leq \gamma$.

Case (1-b) $\gamma < s' < \gamma + x_\gamma$.

Express γ as $\alpha.\beta$ with $\alpha = \varepsilon(p/2^n)$, $\beta = \varepsilon((p + 1)/2^n)$.

Case (1-a). We define a morphism

$$\Phi' : \mathcal{M}(\xi') \to \mathcal{K}r' := \mathcal{K}r_V(m, n)$$

by associating to $\mathcal{F} \in \mathcal{M}(\xi')$ fitting in the exact sequence (2.5) the Kronecker V-module in (2.7), where $V = \mathrm{Hom}(E_{\beta-3}, E_\alpha)^*$, and m, n are given by (2.6).

Next, we shall define a morphism

$$\Phi : \mathcal{M}(\xi)^\circ \to \mathcal{K}r := \mathcal{K}r_V(m_2, m_1)$$

as follows. Recall that in (2.13) we defined a morphism

$$\Psi : \mathcal{M}(\xi)^\circ \to \bar{G}\backslash\bar{\mathbb{H}}$$

such that if $U \in \mathcal{M}(\xi)^\circ$ is the cokernel of the morphism (2.12), then $\Psi(U)$ is (represented by) the morphism $f : E_{-\alpha-3}^{m_1} \to E_{-\beta}^{m_2}$. If we apply $\mathcal{H}om(-, \mathcal{O}(-3))$ to f, then we obtain a morphism

$$f^\dagger : E_{\beta-3}^{m_2} \to E_\alpha^{m_1},$$

which is equivalent to giving a Kronecker V-module

$$t_U^\dagger : k^{m_2} \otimes V \to k^{m_1}.$$

Conversely, we can recover f from $t_U^\dagger$. So we have an isomorphism

$$\bar{G} \backslash \bar{\mathbb{H}} \to \mathcal{K}r$$

of stacks. The morphism $\Phi : \mathcal{M}(\xi)^\circ \to \mathcal{K}r$ is defined to be the composite of Ψ and this isomorphism.

Let $e : \mathcal{A} \otimes V \to \mathcal{B}$ and $e' : \mathcal{A}' \otimes V \to \mathcal{B}'$ be universal families over $\mathcal{K}r$ and $\mathcal{K}r'$, respectively. As explained in Section 2.5, we define a morphism

$$\mathcal{H}om(\mathcal{A}', \mathcal{A}) \oplus \mathcal{H}om(\mathcal{B}', \mathcal{B}) \xrightarrow{\lambda} \mathcal{H}om(\mathcal{A}' \otimes_k V, \mathcal{B}) \tag{3.1}$$

over $\mathcal{K}r \times \mathcal{K}r'$. The source and target of the morphism λ have the same rank (see Remark 3.1). Define the divisor $\Xi \subset \mathcal{K}r \times \mathcal{K}r'$ to be the degenerate locus of the morphism $\det \lambda$.

Let $\mathcal{U}$ and $\mathcal{U}'$ be universal families for the moduli stacks $\mathcal{M}(\xi)$ and $\mathcal{M}(\xi')$, respectively. The pull-backs of $\mathcal{U}$ and $\mathcal{U}'$ to $\mathbb{P}^2 \times \mathcal{M}(\xi) \times \mathcal{M}(\xi')$ are denoted by the same letters. Let

$$p : \mathbb{P}^2 \times \mathcal{M}(\xi) \times \mathcal{M}(\xi') \to \mathcal{M}(\xi) \times \mathcal{M}(\xi')$$

be the projection. Define Θ to be the divisor on $\mathcal{M}(\xi) \times \mathcal{M}(\xi')$ defined by the Fitting ideal of $R^1 p_*(\mathcal{U} \otimes \mathcal{U}')$. This is the divisor used to define the strange duality map. Put $\Theta^\circ := \Theta|_{\mathcal{M}(\xi)^\circ \times \mathcal{M}(\xi')}$.

CLAIM 3.0.2. $\Theta^\circ = (\Phi \times \Phi')^* \Xi$ *as divisors on* $\mathcal{M}(\xi)^\circ \times \mathcal{M}(\xi')$.

Proof. Over $\mathbb{P}^2 \times (\mathcal{M}(\xi)^\circ \times \mathcal{M}(\xi'))$, express $\mathcal{U}$ and $\mathcal{U}'$ as

$$0 \to E_{\beta-3} \boxtimes \mathcal{W}_1' \xrightarrow{h} E_\alpha \boxtimes \mathcal{W}_2' \to \mathcal{U}' \to 0,$$

$$0 \to E_{-\alpha-3} \boxtimes \mathcal{W}_2 \xrightarrow{(f,g)} E_{-\beta} \boxtimes \mathcal{W}_1 \oplus E_{-(\alpha.\beta)} \boxtimes \mathcal{S} \to \mathcal{U} \to 0,$$

where $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{S}$, $\mathcal{W}_1'$, and $\mathcal{W}_2'$ are locally free sheaves over $\mathcal{M}(\xi)^\circ \times \mathcal{M}(\xi)$ of ranks m_2, m_1, m_3, m, n, respectively.

Let $G^\bullet$ and $F^\bullet$ be respectively the two-term complexes

$$E_{-\alpha-3} \boxtimes \mathcal{W}_2 \xrightarrow{f} E_{-\beta} \boxtimes \mathcal{W}_1 \quad \text{and} \quad E_{\beta-3} \boxtimes \mathcal{W}_1' \xrightarrow{h} E_\alpha \boxtimes \mathcal{W}_2'$$

with terms only in degree -1, 0. These are related to the universal families e, e' of Kronecker V-modules as follows. Applying $\mathcal{H}om(-, E_{-\beta})$ to f, we obtain a morphism

$$\mathcal{E}nd(E_{-\beta}) \boxtimes \mathcal{W}_1^* \to \mathcal{H}om(E_{-\alpha-3}, E_{-\beta}) \boxtimes \mathcal{W}_2^* \simeq \mathcal{H}om(E_{\beta-3}, E_\alpha) \boxtimes \mathcal{W}_2^*.$$

Taking its push-forward by p, we obtain a morphism

$$\mathcal{W}_1^* \to V^* \otimes \mathcal{W}_2^* \quad \text{or equivalently} \quad \mathcal{W}_1^* \otimes V \to \mathcal{W}_2^*$$

over $\mathcal{M}(\xi)^\circ \times \mathcal{M}(\xi')$. By the definition of Φ, this is the pull-back by $\Phi \times \Phi'$ of the universal Kronecker V-module e. Analogously, applying $\mathcal{H}(E_{\beta-3}, -)$ to h and taking the push-forward of the resulting morphism by p, we obtain a morphism

$$\mathcal{W}_1' \to V^* \otimes \mathcal{W}_2' \quad \text{or, equivalently,} \quad \mathcal{W}_1' \otimes V \to \mathcal{W}_2'.$$

This is the pull-back of the universal Kronecker V-module e' by $\Phi \times \Phi'$.

Since $\mathbf{R}p_*(E_{-(\alpha.\beta)} \otimes \mathcal{U}') = 0$ by the explanation in the penultimate paragraph of Section 2.3, we have $\mathbf{R}p_*(\mathcal{U} \otimes \mathcal{U}') \simeq \mathbf{R}p_*(G^\bullet \otimes \mathcal{U}')$. In the derived category we have the isomorphisms

$$\begin{aligned} G^\bullet \otimes \mathcal{U}' &\simeq \mathcal{H}om(G^{\bullet *}, \mathcal{U}') \\ &\simeq \mathcal{H}om(G^{\bullet *}, F^\bullet) \\ &\simeq \mathcal{H}om(\mathcal{H}om(F^\bullet[-1], G^{\bullet *}), \mathcal{O}[1]) \\ &\simeq \mathcal{H}om(K^\bullet, \mathcal{O}(-3)[1]), \end{aligned}$$

where $K^\bullet := \mathcal{H}om(F^\bullet[-1], G^{\bullet *} \otimes \mathcal{O}(-3))$. By this we have the isomorphisms

$$\begin{aligned} \mathrm{H}^1(\mathbf{R}p_*(\mathcal{U} \otimes \mathcal{U}')) &\simeq \mathrm{H}^0(\mathbf{R}p_* \mathcal{H}om(K^\bullet, \mathcal{O}(-3)[2])) \\ &\simeq \mathrm{H}^0(\mathbf{R}\mathcal{H}om(\mathbf{R}p_*(K^\bullet), \mathcal{O})) \\ &\simeq \mathrm{H}^0(\mathcal{H}om((\Phi \times \Phi')^*(\text{the complex (3.1)}), \mathcal{O})), \end{aligned}$$

where the second isomorphism is a duality isomorphism. The last isomorphism is a result of the isomorphism

$$\mathbf{R}p_*(K^\bullet) \simeq (\Phi \times \Phi')^*(\text{the complex (3.1)}), \qquad (*)$$

which is explained in the next paragraph. Since for a morphism $\varphi : \mathcal{C} \to \mathcal{D}$ of locally free sheaves of the same rank, $\mathrm{Div}(\det \varphi) = \mathrm{Div}(\det \varphi^*)$, the claim follows from the above isomorphisms.

Finally, we explain the isomorhism $(*)$. The sheaves appearing in the complex $K^\bullet$ are of the form

$$\mathcal{E}nd(E_{\beta-3}) \boxtimes ?, \qquad \mathcal{E}nd(E_\alpha) \boxtimes ? \quad \text{or} \quad \mathcal{H}om(E_{\beta-3}, E_\alpha) \boxtimes ?,$$

where ? are locally free sheaves on $\mathcal{M}^\circ(\xi) \times \mathcal{M}(\xi')$. Since the higher cohomologies of $\mathcal{E}nd(E_{\beta-3})$, $\mathcal{E}nd(E_\alpha)$ and $\mathcal{H}om(E_{\beta-3}, E_\alpha)$ vanish by (2.2), we have $\mathbf{R}p_*(K^\bullet) \simeq p_*(K^\bullet)$. The degree 0 term of $p_*(K^\bullet)$ is $\mathcal{H}om(\mathcal{W}_1', \mathcal{W}_1^*) \oplus \mathcal{H}om(\mathcal{W}_2', \mathcal{W}_2^*)$, and the degree 1 term is $V^* \otimes \mathcal{H}om(\mathcal{W}_1', \mathcal{W}_2^*)$. From the description of the relation of the complexes $G^\bullet$ and $F^\bullet$ with the universal Kronecker V-modules e and e' we see that these are respectively the pull-back by $\Phi \times \Phi'$ of the source and target of the morphism (3.1). By comparing the definition of the morphism λ and the definition of a Hom complex we can see that $p_*(K^\bullet)$ is the pull-back of the complex (3.1) by $\Phi \times \Phi'$. $\square$

REMARK 3.1. The proof of the claim shows that the source and target of the morphism λ have the same rank. In fact, we have rank $\mathbf{R}p_*(\mathcal{U}\otimes\mathcal{U}')=0$ by the assumption of Theorem 1.1, and by the proof of the claim we have

$$\mathbf{R}p_*(\mathcal{U}\otimes\mathcal{U}')\simeq \mathcal{H}om((\Phi\times\Phi')^*(\text{the complex (3.1)}),\mathcal{O})[-1].$$

Hence, the rank of the complex (3.1) is zero.

Recall that $\mathcal{D}$ and $\mathcal{D}'$ are line bundles on $\mathcal{M}(\xi)$ and $\mathcal{M}(\xi')$, respectively, such that $\mathcal{D}\boxtimes\mathcal{D}'\simeq\mathcal{O}(\Theta)$. Let $\mathcal{L}$ and $\mathcal{L}'$ be line bundles on $\mathcal{K}r$ and $\mathcal{K}r'$, respectively, such that $\mathcal{L}\boxtimes\mathcal{L}'\simeq\mathcal{O}(\Xi)$. Then by the claim we have the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{H}^0(\mathcal{M}(\xi),\mathcal{D}) \arrow[d, "\text{restriction}"] \\
\mathrm{H}^0(\mathcal{M}(\xi'),\mathcal{D}')^* \arrow[ur, "\text{(a)}"] \arrow[r, "\text{(b)}"'] \arrow[d, "\text{(c)}"] & \mathrm{H}^0(\mathcal{M}(\xi)^\circ,\mathcal{D}|_{\mathcal{M}(\xi)^\circ}) \\
\mathrm{H}^0(\mathcal{K}r',\mathcal{L}')^* \arrow[r, "\text{(d)}"] & \mathrm{H}^0(\mathcal{K}r,\mathcal{L}). \arrow[u, "\text{(e)}"']
\end{tikzcd}$$

Here (a) is the strange duality map, (c) is the dual of the pull-back map by Φ', (d) is the map induced by the divisor Ξ, and (e) is the pull-back map by Φ.

By Theorem 2.4 and Lemma 2.1 the map (c) is an isomorphism. By Theorem 2.2 the map (d) is an isomorphism. By Lemma 2.8 the map (e) is an isomorphism. From these we see that the map (b) is an isomorphism. So the map (a) is an isomorphism because the vertical restriction map is injective. This finishes the proof in Case (1-a).

CASE (1-b). First, note that in this case we have $\chi(U\otimes E_{\alpha.\beta})\le 0$ for $U\in\mathcal{M}(\xi)$. In fact, if $\chi(U\otimes E_{\alpha.\beta})>0$, then the assumption that $\mathrm{H}^i(E\otimes E')=0$ holds for some $E\in\mathcal{M}(\xi)$ and $E'\in\mathcal{M}(\xi')$ is not satisfied because of [CHW, Thm. 3.4].

We define a morphism

$$\Phi':\mathcal{M}(\xi')\to\mathcal{K}r':=\mathcal{K}r_V(m,n)$$

by associating to $\mathcal{F}\in\mathcal{M}(\xi')$ fitting in the exact sequence (2.8) the Kronecker V-module (2.10), where $V=\mathrm{Hom}(E_{-\alpha-3},E_{-\beta})^*$, and n, m are given by (2.9).

Recall that in (2.15) we defined a morphism

$$\Psi:\mathcal{M}(\xi)^\circ\to\bar{G}\backslash\bar{\mathbb{H}}$$

such that if $U\in\mathcal{M}(\xi)^\circ$ fits in an exact sequence (2.16), then $\Psi(U)$ is the morphism $f:E_{-\alpha-3}^{m_1}\to E_{-\beta}^{m_2}$. Giving f is equivalent to giving a Kronecker V-module

$$t_U:k^{m_1}\otimes V\to k^{m_2}.$$

We define a morphism $\Phi:\mathcal{M}(\xi)^\circ\to\mathcal{K}r:=\mathcal{K}r_V(m_1,m_2)$ by $\Phi(U)=t_U$.

As in Case (1-a), we can define divisors Ξ, Θ, and Θ° on $\mathcal{K}r\times\mathcal{K}r'$, $\mathcal{M}(\xi)\times\mathcal{M}(\xi')$, and $\mathcal{M}(\xi)^\circ\times\mathcal{M}(\xi')$, respectively. Let us see that the same statement as

Claim 3.0.2 holds. Suppose that $U \in \mathcal{M}(\xi)^{\circ}$ fits in the exact sequence (2.16) and that $\mathcal{F} \in \mathcal{M}(\xi')$ fits in the exact sequence (2.8). Let $G^{\bullet}$ and $F^{\bullet}$ be respectively the two-term complexes

$$E_{-\alpha-3}^{m_1} \xrightarrow{f} E_{-\beta}^{m_2} \quad \text{and} \quad E_{-\alpha-3}^{m} \xrightarrow{A'_{\mathcal{F}}} E_{-\beta}^{n}$$

with terms only in degree -1, 0. Then we have

$$\begin{aligned} \mathrm{H}^0(\mathbb{P}^2, U \otimes \mathcal{F}) \neq 0 &\Leftrightarrow \mathrm{H}^0(\mathbb{P}^2, G^{\bullet} \otimes \mathcal{F}) \neq 0 \\ &\Leftrightarrow \mathrm{Hom}_{D(\mathbb{P}^2)}(\mathcal{F}^*, G^{\bullet}) \neq 0 \\ &\Leftrightarrow \mathrm{Hom}_{D(\mathbb{P}^2)}(F^{\bullet}, G^{\bullet}) \neq 0 \\ &\Leftrightarrow \mathrm{Hom}(\Phi'(\mathcal{F}), \Phi(U)) \neq 0, \end{aligned}$$

where the first equivalence follows from $\mathrm{H}^i(\mathbb{P}^2, \mathcal{F} \otimes E_{-\alpha.\beta-3}) = 0$ for any i (this is explained in the penultimate paragraph of Section 2.3), and the last equivalence follows from the vanishing of higher cohomologies of each term of the complex $\mathcal{H}om(F^{\bullet}, G^{\bullet})$. This shows that the same statement as Claim 3.0.2 holds.

Now the rest of the proof goes as in Case (1-a).

CASE (2). Let γ and γ' be the associated exceptional slopes to $M(\xi)$ and $M(\xi')$, respectively. We divide into two cases:

Case (2-a) $\gamma - x_{\gamma} < s \leq \gamma$ or $\gamma' - x_{\gamma'} < s' \leq \gamma'$.

Case (2-b) $\gamma < s < \gamma + x_{\gamma}$ and $\gamma' < s' < \gamma' + x_{\gamma'}$.

CASE (2-a). By symmetry we may assume that $\gamma - x_{\gamma} < s \leq \gamma$. Consider the parabola Q_{ξ} defined by $\Delta = P(\mu + s) - d$ in the (μ, Δ)-plane. We have $P(-s + s) - d = 1 - d < \frac{1}{2}$, and by easy calculation we can see that $P(-\gamma + x_{\gamma} + s) - d > \frac{1}{2}$. So the graph of $\Delta = \delta(\mu)$ and the parabola Q_{ξ} intersect at one point in the area $\{(\mu, \Delta) \mid -\gamma < \mu < -\gamma + x_{\gamma}\}$. By the same argument as Claim 3.0.1 the intersection point is (s', d'). Write γ as $\alpha.\beta$ with $\alpha = \varepsilon(p/2^n)$, $\beta = \varepsilon((p+1)/2^n)$. Define a morphism

$$\Phi : \mathcal{M}(\xi) \to \mathcal{K}r_V(m, n)$$

by associating to $\mathcal{F} \in \mathcal{M}(\xi)$ fitting in the exact sequence (2.5) the Kronecker V-module $t_{\mathcal{F}}$ in (2.7), where $V = \mathrm{Hom}(E_{\beta-3}, E_{\alpha})^*$, and m, n are given by (2.6).

Every $\mathcal{G} \in \mathcal{M}(\xi')$ fits in an exact sequence

$$0 \to E_{\beta-3}^{m'} \xrightarrow{g} E_{\alpha}^{n'} \to \mathcal{G}^* \to 0, \tag{3.3}$$

where $m' = -\chi(\mathcal{G}^* \otimes E_{-\beta})$ and $n' = -\chi(\mathcal{G}^* \otimes E_{-(\alpha.\beta).\beta})$. Giving g is equivalent to giving a Kronecker V-module

$$t_{\mathcal{G}} : k^{m'} \otimes V \to k^{n'}.$$

We have a morphism

$$\Phi' : \mathcal{M}(\xi') \to \mathcal{K}r_V(m', n'),$$

which maps $\mathcal{G}$ to $t_{\mathcal{G}}$. For $\mathcal{F} \in \mathcal{M}(\xi)$ fitting in the exact sequence (2.5) and $\mathcal{G} \in \mathcal{M}(\xi')$ fitting in the exact sequence (3.3), we have

$$\begin{aligned} \mathrm{H}^0(\mathcal{F} \otimes \mathcal{G}) \neq 0 &\Leftrightarrow \mathrm{Hom}(\mathcal{G}^*, \mathcal{F}) \neq 0 \\ &\Leftrightarrow \mathrm{Hom}(E_{\beta-3}^{m'} \to E_\alpha^{n'}, E_{\beta-3}^{m} \to E_\alpha^{n}) \neq 0 \\ &\Leftrightarrow \mathrm{Hom}(\Phi'(\mathcal{G}), \Phi(\mathcal{F})) \neq 0, \end{aligned}$$

where the two-term complexes in the second line have terms in degree -1, 0.

Now the rest of the proof goes as in the previous cases.

CASE (2-b). We shall show that this case does not occur. Consider the parabola Q_ξ in the (μ, Δ)-plane in Case (2-a). The argument in Case (2-a) shows that the parabola Q_ξ and the graph of $\Delta = \delta(\mu)$ intersect at one point in the area $\{(\mu, \Delta) \mid -s < \mu < -\gamma + x_\gamma\}$. Again by the same argument as Claim 3.0.1 the intersection point is (s', d'). Write γ as $\alpha.\beta$ with $\alpha = \varepsilon(p/2^n)$, $\beta = \varepsilon((p+1)/2^n)$.

Since we are in Case (2-b), we have $-\gamma < s' < -\gamma + x_\gamma$. So every $\mathcal{G} \in \mathcal{M}(\xi')$ fits in the exact sequence (3.3). Every $\mathcal{F} \in \mathcal{M}(\xi)$ fits in the exact sequence

$$0 \to E_{-\alpha-3}^m \xrightarrow{f} E_{-\beta}^n \to \mathcal{F}^* \to 0$$

with m, n positive integers. Let $G^\bullet$ and $F^\bullet$ be respectively the two-term complexes

$$E_{\beta-3}^{m'} \xrightarrow{g} E_\alpha^{n'} \quad \text{and} \quad E_\beta^{n} \xrightarrow{f^*} E_{\alpha+3}^{m}$$

having terms in degree -1, 0. Then we have

$$\begin{aligned} \mathrm{H}^0(\mathcal{F} \otimes \mathcal{G}) \neq 0 &\Leftrightarrow \mathrm{Hom}(\mathcal{G}^*, \mathcal{F}) \neq 0 \\ &\Leftrightarrow \mathrm{Hom}(G^\bullet, F^\bullet[-1]) \neq 0 \\ &\Leftrightarrow \mathrm{Hom}(E_\alpha^{n'}, E_\beta^{n}) \neq 0. \end{aligned}$$

Since $\chi(E_\alpha, E_\beta) > 0$ by (2.3) and $\mathrm{Ext}^i(E_\alpha, E_\beta) = 0$ for $i > 0$ by (2.2), we have $\mathrm{Hom}(E_\alpha, E_\beta) \neq 0$. Therefore, we have $\mathrm{H}^0(\mathcal{F} \otimes \mathcal{G}) \neq 0$ for every $\mathcal{F} \in \mathcal{M}(\xi)$ and $\mathcal{G} \in \mathcal{M}(\xi')$, which contradicts the assumption of the theorem.

CASE (3). In this case we have $s \in \mathfrak{E}$, $d = \Delta_s$ and $M(\xi) = \{E_s^{r/r_s}\}$. We shall show that the line bundle on $M(\xi')$ determined by E_s is the trivial line bundle. To see that, we show that for every $\mathcal{G} \in M(\xi')$, we have $\mathrm{H}^i(E_s \otimes \mathcal{G}) = 0$ for any i.

In the (μ, Δ)-plane, the only point with μ-coordinate greater than or equal to $-s$ that line on both the parabola defined by $\Delta = P(\mu + s) - d$ and the graph $\Delta = \delta(\mu)$ is $(-s, 1-d)$. Hence, $(s', d') = (-s, 1-d)$. Since $\mu(E_s^*) = \mu(\mathcal{G})$ and $\Delta(E_s^*) = d < 1 - d = \Delta(\mathcal{G})$, we have $\mathrm{H}^0(E_s \otimes \mathcal{G}) = 0$ by semistability. We also have $\mathrm{H}^2(E_s \otimes \mathcal{G}) = \mathrm{H}^0(E_s^* \otimes \mathcal{G}^*(-3)) = 0$ again by semistability. Therefore, we have $\mathrm{H}^i(E_s \otimes \mathcal{G}) = 0$ for any i.

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Graduate School of Science
and Technology
Kumamoto University
2-39-1 Kurokami
Kumamoto 860-8555
Japan

abeken@sci.kumamoto-u.ac.jp