Integer Complexity and Well-Ordering

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ABSTRACT. Define $\|n\|$ to be the *complexity* of n, the smallest number of ones needed to write n using an arbitrary combination of addition and multiplication. John Selfridge showed that $\|n\| \geq 3\log_3 n$ for all n. Define the *defect* of n, denoted $\delta(n)$, to be $\|n\| - 3\log_3 n$. In this paper, we consider the set $\mathscr{D} := \{\delta(n) : n \geq 1\}$ of all defects. We show that as a subset of the real numbers, the set $\mathscr{D}$ is well-ordered, of order type ω^ω. More specifically, for $k \geq 1$ an integer, $\mathscr{D} \cap [0, k)$ has order type ω^k. We also consider some other sets related to $\mathscr{D}$ and show that these too are well-ordered and have order type ω^ω.

1. Introduction

The *complexity* of a natural number n is the least number of 1s needed to write it using any combination of addition and multiplication, with the order of the operations specified using parentheses grouped in any legal nesting. For instance, $n = 11$ has a complexity of 8 since it can be written using eight ones as

$$(1+1+1)(1+1+1)+1+1,$$

but not with any fewer. This notion was implicitly introduced in 1953 by Mahler and Popken [18]; they actually considered the inverse function of the size of the largest number representable using k copies of the number 1. (More generally, they considered the same question for representations using k copies of a positive real number x.) Integer complexity was explicitly studied by John Selfridge and was later popularized by Guy [13; 14]. Following Arias de Reyna [3], we will denote the complexity of n by $\|n\|$.

Integer complexity is approximately logarithmic; it satisfies the bounds

$$3\log_3 n = \frac{3}{\log 3}\log n \leq \|n\| \leq \frac{3}{\log 2}\log n, \quad n > 1. \qquad (1.1)$$

The lower bound can be deduced from the result of Mahler and Popken and was explicitly proved by Selfridge [13]. It is attained with equality for $n = 3^k$ for all $k \geq 1$. The upper bound can be obtained by writing n in binary and finding a representation using Horner's algorithm. It is not sharp, and the constant $\frac{3}{\log 2}$ can be improved for large n [22].

The notion of integer complexity is similar in spirit but different in detail from the better known measure of *addition chain length,* which has application to computation of powers, and which is discussed in detail in Knuth [17, Sect. 4.6.3]. One important difference between the two notions is that integer complexity can

be computed by dynamic programming, whereas this does not seem to be the case for addition chain length. Specifically, integer complexity is computable via the dynamic programming recursion, for any $n > 1$,

$$\|n\| = \min_{\substack{a,b<n\in\mathbb{N} \\ a+b=n \text{ or } ab=n}} \|a\| + \|b\|.$$

There are many mysteries about $\|n\|$. For powers, we have

$$\|n^k\| \le k\|n\|,$$

and it is known that $\|3^k\| = 3k$ for all $k \ge 1$. However, other values have a more complicated behavior. For instance, powers of 5 do not work nicely since $\|5^6\| = 29 < 30 = 6\|5\|$. The behavior of powers of 2 remains unknown; it has been verified that

$$\|2^k\| = k\|2\| = 2k \quad \text{for } 1 \le k \le 39;$$

see [15].

We can also study arithmetic expressions for n that are not necessarily the shortest. For instance, we may see Gnang et al. [11] regarding the problem of counting the total number of arithmetic expressions for n, with multiplication by 1 disallowed. However, here we will be solely concerned with the lengths of arithmetic expressions for n and not their number.

1.1. Main Result

In an earlier paper, this author and Zelinsky [2] introduced the notion of the *defect* of an integer n, denoted $\delta(n)$, by

$$\delta(n) := \|n\| - 3\log_3 n.$$

This is a rescaled version of integer complexity, which, given n, contains equivalent information to $\|n\|$. In view of the lower bound (1.1), it satisfies $\delta(n) \ge 0$. The paper [2] exploited patterns in the dynamic programming structure of integer complexity to classify the structure of all integers with small values of the defect. In particular, it classifies all integers with $\delta(n) \le 1$.

The defect encodes interesting structure about integer complexity. In this paper, we will consider the image of this defect function in the general case:

DEFINITION 1.1. The *defect set* $\mathscr{D} \subseteq [0, \infty)$ is the set of all defect values $\{\delta(n) : n \in \mathbb{N}\}$.

Addition and multiplication tend to interact badly and unpredictably when placed on an equal footing. So we might not expect to find any particular sort of structure in the values of $\delta(n)$, even though its definition is based on powers of 3, which give the extremal case. In this paper, we will prove the following striking result:

THEOREM 1.2. *The set $\mathscr{D}$ is a well-ordered subset of $\mathbb{R}$, of order type ω^ω. Furthermore, for $k \ge 1$ an integer, the set $\mathscr{D} \cap [0, k)$ has order type ω^k.*

This well-ordering of the defect set $\mathscr{D}$ reveals new fundamental structure in the interaction between addition and multiplication. Some of the tangledness of that interaction may be reflected in how the set $\mathscr{D}$ grows more complicated as its elements get larger. In fact, the structure of $\mathscr{D}$ has even more regularity than what Theorem 1.2 describes, which we plan to discuss in a future paper.

In Section 7, we will also prove that the theorem still holds even if we replace $\mathscr{D}$ with any of several other closely related sets.

Theorem 1.2 is closely related to conjectures of Arias de Reyna [3] about integer complexity. We discuss these conjectures and use our results to prove modified versions of some of them in the Appendix.

In contrast to Theorem 1.2, little is known about the set of values of $\frac{\|n\|}{3\log_3 n}$, even though that might appear to be a more natural object of study. An open question is to determine the value

$$C_{\max} := \limsup_{n\to\infty} \frac{\|n\|}{3\log_3 n}.$$

The bounds (1.1) imply $1 \le C_{\max} \le \log_2 3$. It is an open problem to decide whether $C_{\max} = 1$ or $C_{\max} > 1$.

1.2. Low-Defect Polynomials

The strategy to prove the main theorem is to build up the set $\mathscr{D}$ by inductively building up the sets $\mathscr{D} \cap [0, s)$ for real numbers $s > 0$. The proof of Theorem 1.2 makes use of earlier work of this author with Zelinsky [2] classifying numbers of low defect. The paper [2] gave a method to list families of such integers and explicitly listed all integers of defect $\delta(n) < 1$. The innovation made here is that instead of treating the output of this method as an undifferentiated blob, we group it into tractable families.

We introduce a family of multilinear polynomials that we call *low-defect polynomials*. We show that for any $s > 0$, there exists a finite set of low-defect polynomials $\mathcal{S}_s$ such that any number of defect less than s can be written as $f(3^{n_1}, \ldots, 3^{n_k})3^{n_{k+1}}$ for some $f \in \mathcal{S}_s$ and nonnegative $n_1, \ldots, n_{k+1}$. Indeed, stronger statements are true; see Theorems 4.10 and 4.15. Note, however, that the low-defect polynomials may also produce extraneous numbers, with defect higher than intended; examples of these are given after Theorem 4.10.

To state this another way, these low-defect polynomials provide forms into which powers of 3 can be substituted to obtain all the numbers below the specified defect. As the defects get larger, the low-defect polynomials and the families of numbers we get this way become more complicated. And just as we can visualize expressions in $+$, $\times$, and 1 as trees, we can also visualize low-defect polynomials—or the expressions that generate them—as trees with open slots where powers of 3 can be plugged in. By attaching trees corresponding to powers of 3 we obtain trees for the numbers we get this way. This is illustrated in Figure 1 with the polynomial $(2x_1 + 1)x_2 + 1$. (Note, however, that this picture is not quite correct when we plug in 3^0; see Figure 2 in Section 4.)
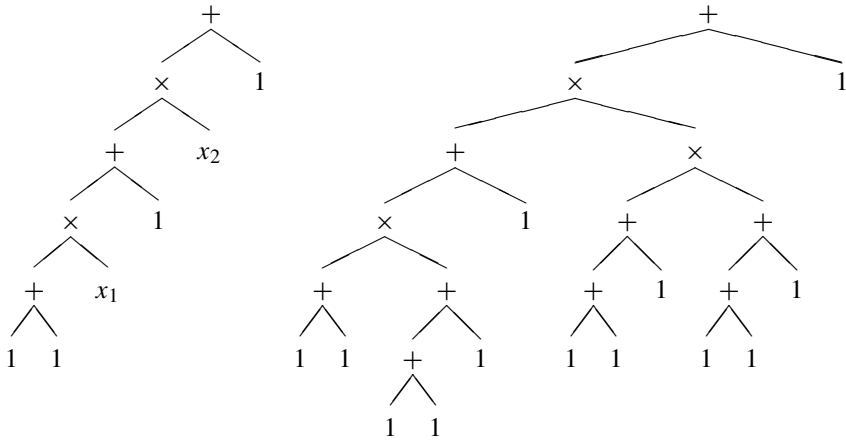
Figure 1　A tree corresponding to the polynomial $(2x_1 + 1)x_2 + 1$ and the same tree after making the substitution $x_1 = 3^1$, $x_2 = 3^2$

So with this approach, we can get at properties of the set of defects by examining properties of low-defect polynomials. For instance, as mentioned before, as the defects involved get larger, the low-defect polynomials required get more complicated; one way in which this occurs is that they require more variables. In fact, we will see (Theorem 4.10) that to cover defects up to a real number s, we need low-defect polynomials with up to $\lfloor s \rfloor$ variables. And it happens that if we have a low-defect polynomial f in k variables and consider the numbers $f(3^{n_1}, \ldots, 3^{n_k})$, then the defects of the numbers obtained this way form a well-ordered set of order type at least ω^k and less than ω^{k+1} (Proposition 6.3). It is this that leads us to Theorem 1.2, that for $k \geq 1$, the set $\mathscr{D} \cap [0, k)$ has order type precisely ω^k.

1.3. Variant Results

We also prove analogues of the main theorem for several other sets. The paper [2] showed that given the value of $\delta(n)$, the value of $\|n\|$ modulo 3 can be determined; see Theorem 2.1(6). It follows that we can split the set of defects $\mathscr{D}$ into sets $\mathscr{D}^0$, $\mathscr{D}^1$, and $\mathscr{D}^2$ according to these congruence classes modulo 3; see Definition 2.4. In Section 7, we prove analogues of the main theorem for each set $\mathscr{D}^a$ separately; see Theorem 7.4.

The paper [2] also introduced a notion of stable numbers; a number n is said to be *stable* if $\|3^k n\| = 3k + \|n\|$ for all $k \geq 0$ or, equivalently, if $\delta(3^k n) = \delta(n)$ for all $k \geq 0$. In Section 3, we show that given $\delta(n)$, we can determine whether or not a given number n is stable, and thus we can consider the set of "stable defects" $\mathscr{D}_{\mathrm{st}}$, which are the defect values for all stable numbers.

We can combine this notion with splitting based on the value of $\|n\|$ modulo 3 to define sets $\mathscr{D}_{\mathrm{st}}^0$, $\mathscr{D}_{\mathrm{st}}^1$, and $\mathscr{D}_{\mathrm{st}}^2$. In Section 7, we prove that each of these sets

is well-ordered of type ω^ω, as are the closures of all these sets. All these well-ordering results are collected in Theorem 7.4.

1.4. Computability Questions

Integer complexity captures part of the complicated interaction of addition and multiplication, where subtraction is not allowed; the underlying algebraic structure is that of a commutative semiring $(\mathbb{N}, +, \times)$. It is a very simple computational model but already exhibits difficult issues.

The model of computation treated in this paper could be considered as taking number inputs other than 1. Mahler and Popken [18] considered constructing numbers starting with copies of any fixed positive real number x. Note that as x varies the ordering of computed quantities on the positive real line will change. One feature of complexity for $x = 1$ (or for $x = k$, an integer) is that multiple ties can occur in doing the computations, which complicates determination of the structure of the minimal computation tree. For a generic (transcendental) x, the complexity issue simplifies to viewing the computation tree as computing a univariate polynomial with positive integer coefficients and a zero constant term. We can assign a complexity to the problem of computing such polynomials. Study of this simplified problem might be fruitful. Allowing multiple indeterminates as inputs, we can consider the complexity of computing multivariate polynomials, which is a much-studied topic. The model of computation allowing $+$ and $\times$ can compute all multivariate polynomials with nonnegative integer coefficients but is restricted in that it does not allow free reuse of polynomials already constructed. The complexity of computation in this restricted model can be compared to that in other computational models that allow additional operations beyond addition and multiplication, or allow free reuse of already computed polynomials (straight-line computation). It is much easier to compute polynomials in models with subtraction [21] or division [9] than with only addition and multiplication [5; 12; 16; 19]. Indeed, similar phenomena occur in the computation of integers as well as that of polynomials [4].

We can also ask about the computational complexity of integer complexity itself, or related notions, viewed in the polynomial hierarchy of complexity theory (see Garey and Johnson [10, Sect. 7.2]). An open question concerns the computational complexity of computing $\|n\|$. Consider the following problem:

INTEGER COMPLEXITY.

- *Instance:* Positive integers n and k, both encoded in binary.
- *Question:* Is $\|n\| \leq k$?

This problem is known to be in the complexity class *NP* (Arias de Reyna [3]), but it is not known to be either in *P* or in co-*NP*, nor is it known to be *NP*-complete. It is a very interesting candidate for a problem that is in *NP* but not *P*.

This paper introduces the ordering of defects as an object of investigation. Hence, we can also consider the following problem:

DEFECT ORDERING.

- *Instance:* Positive integers n_1 and n_2, both encoded in binary.
- *Question:* Is $\delta(n_1) \leq \delta(n_2)$?

This problem of computing the defect ordering is not known to be in the complexity class *NP*. If we could answer *Integer Complexity* in polynomial time, then we could also answer *Defect Ordering* in polynomial time. To show this, observe that the inequality $\delta(n_1) \leq \delta(n_2)$ is equivalent to

$$3^{\|n_1\|}(n_2)^3 \leq 3^{\|n_2\|}(n_1)^3,$$

and since $\|n\|$ is logarithmically small, this could be computed in polynomial time if we knew $\|n\|$. This argument shows that *Defect Ordering* belongs to the complexity class $P^{NP} = \Delta_2^P$.

   Another question related to the defect is that of computing a set $\mathcal{S}_s$ of low-defect polynomials sufficient to describe all integers of defect $\delta(n) < s$, that is, a set $\mathcal{S}_s$ satisfying the conditions of Theorem 4.10. What is the minimal cardinality of such a set, as a function of s? What is the complexity of computing one (say for s integral or rational)? The proof of Theorem 4.10 does give a construction of one such set $\mathcal{S}_s$; however, there exist other such sets $\mathcal{S}_s$, perhaps some smaller or computable more quickly than the one constructed.

2. Properties of the Defect

We begin by reviewing the relevant properties of integer complexity and the defect from [2]. They can be summed up in the following theorem.

THEOREM 2.1. *We have*:

(1) *For all n, $\delta(n) \geq 0$.*
(2) *For $k \geq 0$, $\delta(3^k n) \leq \delta(n)$, with equality if and only if $\|3^k n\| = 3k + \|n\|$. The difference $\delta(n) - \delta(3^k n)$ is a nonnegative integer.*
(3) *If the difference $\delta(n) - \delta(m)$ is rational, then $n = m3^k$ for some integer k (and so $\delta(n) - \delta(m) \in \mathbb{Z}$).*
(4) *Given any n, there exists L such that for all $k \geq L$, $\delta(3^k n) = \delta(3^L n)$. That is to say, $\|3^k n\| = \|3^L n\| + 3(k - L)$.*
(5) *For a given defect α, the set $\{m : \delta(m) = \alpha\}$ has either the form $\{n3^k : 0 \leq k \leq L\}$ for some n and L, or the form $\{n3^k : 0 \leq k\}$ for some n. This latter occurs if and only if α is the smallest defect among $\delta(3^k n)$ for $k \in \mathbb{Z}$.*
(6) *If $\delta(n) = \delta(m)$, then $\|n\| = \|m\| \pmod 3$.*
(7) *$\delta(1) = 1$, and for $k \geq 1$, $\delta(3^k) = 0$. No other integers occur as $\delta(n)$ for any n.*

Proof. Part (1) is just Selfridge's lower bound [13]. The first statement in part (2) is Proposition 9(3) from [2]; the second statement follows from the computation $\delta(n) - \delta(3^k n) = \|n\| - \|3^k n\| + 3k$. Part (3) is Proposition 14(1) from [2]. Parts (4) and (5) are Theorem 5 from [2]. Part (6) is part of Proposition 14(2) from [2]. For part (7), the fact that $\delta(1) = 1$ is immediate. The fact that $\delta(3^k) = 0$ for $k \geq 1$ is the same as the fact that $\|3^k\| = 3k$ for $k \geq 1$; that $\|3^k\| \leq 3k$ is obvious and

that $\|3^k\| \geq 3k$ follows from Selfridge's lower bound [13]. Finally, that no other integers occur as $\delta(n)$ for any n follows from part (3). □

We also recall the definitions made for discussing the statements of Theorem 2.1.

DEFINITION 2.2. A number m is called *stable* if $\|3^k m\| = 3k + \|m\|$ for every $k \geq 1$ or, equivalently, if $\delta(3^k m) = \delta(m)$ for every $k \geq 1$. Otherwise, it is called *unstable*.

DEFINITION 2.3. A natural number n is called a *leader* if it is the smallest number with a given defect. By part (5) of Theorem 2.1 this is equivalent to saying that either $3 \nmid n$, or, if $3 \mid n$, then $\delta(n) < \delta(n/3)$, that is, $\|n\| < 3 + \|n/3\|$.

Also, because of part (6) of Theorem 2.1, we can make the following definitions.

DEFINITION 2.4. For a a congruence class modulo 3, we define

$$\mathscr{D}^a = \{\delta(n) : \|n\| \equiv a \ (\mathrm{mod}\, 3), n \neq 1\}.$$

We explicitly exclude the number 1 here as it is dissimilar to other numbers whose complexity is congruent to 1 modulo 3. This is because, unlike other numbers that are 1 modulo 3, the number 1 cannot be written as $3j + 4$ for some j, and so the largest number that can be made with a single 1 is simply 1, rather than $4 \cdot 3^j$ (see the Appendix). For this reason, numbers of complexity 1 do not really go together with other numbers whose complexity is congruent to 1 modulo 3; however, the only such number is 1, so we simply explicitly exclude it. So $\mathscr{D}$ is the disjoint union of $\mathscr{D}^0$, $\mathscr{D}^1$, $\mathscr{D}^2$, and $\{1\}$.

Of course, we care not just about small defects, but about the numbers giving rise to those small defects; so we recall the following definitions.

DEFINITION 2.5. For any real $r \geq 0$, define the set of *r-defect numbers* A_r to be

$$A_r := \{n \in \mathbb{N} : \delta(n) < r\}.$$

Define the set of *r-defect leaders* B_r to be

$$B_r := \{n \in A_r : n \text{ is a leader}\}.$$

These sets are related by the following:

PROPOSITION 2.6. *For every $n \in A_r$, there exists a unique $m \in B_r$ and $k \geq 0$ such that $n = 3^k m$ and $\delta(n) = \delta(m)$; then $\|n\| = \|m\| + 3k$.*

Proof. The first part of this is Proposition 16(2) from [2]. The second part follows since then

$$\|n\| = \delta(n) + 3\log_3(3^k m) = 3k + \delta(m) + 3\log_3 m = \|m\| + 3k. \qquad \square$$

2.1. *Inductive covering of B_r and A_r*

In addition to the stated properties of the defect, there are two substantive theorems we will need from [2]. They allow us to inductively build up the sets A_r and B_r, or at least coverings of these. The first provides the base case:

THEOREM 2.7. *For every α with $0 < \alpha < 1$, the set of leaders B_α is a finite set.*

The other theorem provides the inductive step, telling us how to build up $B_{(k+1)\alpha}$ from previous $B_{i\alpha}$. In order to state it, we will first need some definitions.

DEFINITIONS 2.8. We say that n is *most-efficiently* represented as ab if $n = ab$ and $\|n\| = \|a\| + \|b\|$, or as $a + b$ if $n = a + b$ and $\|n\| = \|a\| + \|b\|$. In the former case, we will also say that $n = ab$ is a *good factorization* of n. We say that n is *solid* if it cannot be written most-efficiently as $a + b$ for any a and b. We say that n is *m-irreducible* if it cannot be written most-efficiently as ab for any a and b. And for a real number $\alpha \in (0, 1)$, we define the set T_α to consist of 1 together with those m-irreducible numbers n that satisfy

$$\frac{1}{n-1} > 3^{(1-\alpha)/3} - 1$$

and do not satisfy $\|n\| = \|n - b\| + \|b\|$ for any solid numbers b with $1 < b \leq n/2$.

Note that for any $0 < \alpha < 1$, the set T_α is a finite set due to the upper bound on the size of numbers $n \in T_\alpha$.

Now we can state the theorem. The theorem provides fives possibilities; three "generic cases" (1 through 3) and two "exceptional cases" (4 and 5).

THEOREM 2.9. *Suppose that $0 < \alpha < 1$ and that $k \geq 1$. Then any $n \in B_{(k+1)\alpha}$ can be most-efficiently represented in (at least) one of the following forms*:

(1) *For $k = 1$, there is either a good factorization $n = u \cdot v$ where $u, v \in B_\alpha$, or a good factorization $n = u \cdot v \cdot w$ with $u, v, w \in B_\alpha$;*
   For $k \geq 2$, there is a good factorization $n = u \cdot v$ where $u \in B_{i\alpha}$ and $v \in B_{j\alpha}$ with $i + j = k + 2$ and $2 \leq i, j \leq k$.
(2) *$n = a + b$ with $\|n\| = \|a\| + \|b\|$, $a \in A_{k\alpha}$, $b \leq a$ a solid number, and*

$$\delta(a) + \|b\| < (k+1)\alpha + 3\log_3 2.$$

(3) *There is a good factorization $n = (a + b)v$ with $v \in B_\alpha$, $a + b$ being a most-efficient representation, and a and b satisfying the conditions in the case (2).*
(4) *$n \in T_\alpha$ (and thus, in particular, either $n = 1$ or $\|n\| = \|n - 1\| + 1$).*
(5) *There is a good factorization $n = u \cdot v$ with $u \in T_\alpha$ and $v \in B_\alpha$.*

By applying these two theorems we can inductively build up the sets B_r and A_r; in a sense, they form the engine of our proof. However, without additional tools, it can be hard to say anything about just what these theorems output. In Section 4, we will show how to group the output of these theorems into tractable families, allowing us to go beyond the earlier work of this author and Zelinsky [2] and prove the main theorem.

3. Stable Defects and Stable Complexity

It will also be useful here to introduce the notions of "stable defect" and "stable complexity". First, let us discuss the defects of stable numbers.

PROPOSITION 3.1. *If $\delta(n) = \delta(m)$ and n is stable, then so is m.*

Proof. Suppose $\delta(n) = \delta(m)$ and n is stable. Then we can write $m = 3^k n$ for some $k \in \mathbb{Z}$. Now, a number a is stable if and only if $\delta(3^\ell a) = \delta(a)$ for all $\ell \geq 0$; so if $k \geq 0$, then m is stable. If, on the other hand, $k < 0$, then consider $\ell \geq 0$. If $\ell \geq -k$, then $\delta(3^\ell m) = \delta(3^{\ell+k}n) = \delta(n)$, whereas if $\ell \leq -k$, then $\delta(n) \leq \delta(3^\ell m) \leq \delta(m)$, so $\delta(3^\ell m) = \delta(m)$; hence, m is stable. □

Because of this proposition, it makes sense to make the following definition.

DEFINITION 3.2. We define a *stable defect* to be the defect of a stable number, and define $\mathscr{D}_{st}$ to be the set of all stable defects. Also, for a a congruence class modulo 3, we define $\mathscr{D}_{st}^a = \mathscr{D}^a \cap \mathscr{D}_{st}$.

Note that the integer 1 is not stable, and so its defect, which is also 1, would be excluded from $\mathscr{D}_{st}^1$ even if we had not explicitly excluded it in the definition of $\mathscr{D}^1$.

This double use of the word "stable" could potentially be ambiguous if we had a positive integer n that were also a defect. However, the only positive integer that is also a defect is 1, which is not stable in either sense.

PROPOSITION 3.3. *A defect α is stable if and only if it is the smallest $\beta \in \mathscr{D}$ such that $\beta \equiv \alpha \pmod{1}$.*

Proof. This follows from parts (2), (3), and (5) of Theorem 2.1. □

DEFINITION 3.4. For a positive integer n, define the *stable defect of n,* denoted $\delta_{st}(n)$, to be $\delta(3^k n)$ for any k such that $3^k n$ is stable. (This is well-defined as if $3^k n$ and $3^\ell n$ are stable, then $k \geq \ell$ implies $\delta(3^k n) = \delta(3^\ell n)$, and so does $\ell \geq k$.)

Here are two equivalent characterizations:

PROPOSITION 3.5. *The number $\delta_{st}(n)$ can be characterized by*:
(1) $\delta_{st}(n) = \min_{k \geq 0} \delta(3^k n)$,
(2) $\delta_{st}(n)$ *is the smallest $\alpha \in \mathscr{D}$ such that $\alpha \equiv \delta(n) \pmod{1}$.*

Proof. Part (1) follows from part (2) of Theorem 2.1 and the fact that m is stable if and only if $\delta(3^k m) = \delta(m)$ for all $k \geq 0$. To prove part (2), take k such that $3^k n$ is stable. Then $\delta(3^k n) \equiv \delta(n) \pmod{1}$, and it is the smallest such by Proposition 3.3. □

So we can think about $\mathscr{D}_{st}$ either as the subset of $\mathscr{D}$ consisting of the stable defects, or we can think about it as the image of δ_{st}. (This latter way of thinking does not work so well for the $\mathscr{D}_{st}^a$, however.)

Just as we can talk about the stable defect of a number n, we can also talk about its *stable complexity*—what the complexity would be "if n were stable".

DEFINITION 3.6. For a positive integer n, we define the *stable complexity of n,* denoted $\|n\|_{\mathrm{st}}$, to be $\|3^k n\| - 3k$ for any k such that $3^k n$ is stable. This is well-defined; if $3^k n$ and $3^\ell n$ are both stable, say with $k \le \ell$, then

$$\|3^k n\| - 3k = 3(k - \ell) + \|3^\ell n\| - 3k = \|3^\ell n\| - 3\ell.$$

PROPOSITION 3.7. *We have*:

(1) $\|n\|_{\mathrm{st}} = \min_{k \ge 0}(\|3^k n\| - 3k)$,
(2) $\delta_{\mathrm{st}}(n) = \|n\|_{\mathrm{st}} - 3 \log_3 n$.

Proof. To prove part (1), observe that $\|3^k n\| - 3k$ is nonincreasing in k since $\|3m\| \le 3 + \|m\|$. So a minimum is achieved if and only if for all ℓ,

$$\|3^{k+\ell} n\| - 3(k + \ell) = \|3^k n\| - 3k,$$

that is, for all ℓ, $\|3^{k+\ell} n\| = \|3^k n\| + 3\ell$, and thus $3^k n$ is stable.

To prove part (2), take k such that $3^k n$ is stable. Then

$$\delta_{\mathrm{st}}(n) = \delta(3^k n) = \|3^k n\| - 3 \log_3(3^k n) = \|3^k n\| - 3k - 3 \log_3 n$$
$$= \|n\|_{\mathrm{st}} - 3 \log_3 n. \qquad \square$$

PROPOSITION 3.8. *We have*:

(1) $\delta_{\mathrm{st}}(n) \le \delta(n)$, *with equality if and only if n is stable.*
(2) $\|n\|_{\mathrm{st}} \le \|n\|$, *with equality if and only if n is stable.*

Proof. The inequality in part (1) follows from Proposition 3.5. Also, if n is stable, then for any $k \ge 1$, we have $\delta(3^k n) = \delta(n)$, so $\delta_{\mathrm{st}}(n) = \delta(n)$. Conversely, if $\delta_{\mathrm{st}}(n) = \delta(n)$, then by Proposition 3.5, for any $k \ge 1$, we have $\delta(3^k n) \ge \delta(n)$. But also $\delta(3^k n) \le \delta(n)$ by part (2) of Theorem 2.1, and so $\delta(3^k n) = \delta(n)$, and n is stable.

Part (2) follows from part (1) along with part (2) of Proposition 3.7. $\square$

4. Low-Defect Polynomials

The primary tool we will use to prove the main theorem is to group the numbers produced by the main theorem of [2] into families. Each of these families will be expressed via a multilinear polynomial in $\mathbb{Z}[x_1, x_2, \ldots]$, which we will call a *low-defect polynomial*. We will associate these with a "base complexity" to form a *low-defect pair*. Formally:

DEFINITION 4.1. We define the set $\mathscr{P}$ of *low-defect pairs* as the smallest subset of $\mathbb{Z}[x_1, x_2, \ldots] \times \mathbb{N}$ such that:

(1) For any constant polynomial $k \in \mathbb{N} \subseteq \mathbb{Z}[x_1, x_2, \ldots]$ and any $C \ge \|k\|$, we have $(k, C) \in \mathscr{P}$.

(2) Given (f_1, C_1) and (f_2, C_2) in $\mathscr{P}$, we have $(f_1 \otimes f_2, C_1 + C_2) \in \mathscr{P}$, where, if f_1 is in r_1 variables and f_2 is in r_2 variables, then

$$(f_1 \otimes f_2)(x_1, \ldots, x_{r_1+r_2}) := f_1(x_1, \ldots, x_{r_1}) f_2(x_{r_1+1}, \ldots, x_{r_1+r_2}).$$

(3) Given $(f, C) \in \mathscr{P}$, $c \in \mathbb{N}$, and $D \geq \|c\|$, we have $(f \otimes x_1 + c, C + D) \in \mathscr{P}$ where $\otimes$ is as before.

The polynomials obtained this way will be referred to as *low-defect polynomials*. If (f, C) is a low-defect pair, then C will be called its *base complexity*. If f is a low-defect polynomial, then we will define its *absolute base complexity*, denoted $\|f\|$, to be the smallest C such that (f, C) is a low-defect pair.

Note that the degree of a low-defect polynomial is also equal to the number of variables it uses; see Proposition 4.2. We will often refer to the "degree" of a low-defect pair (f, C); this refers to the degree of f.

Note that we do not really care about what variables a low-defect polynomial (or pair) is in—if we permute the variables of a low-defect polynomial or replace them with others, then we will still regard the result as a low-defect polynomial. From this perspective, the meaning of $f \otimes g$ could be simply regarded as "relabel the variables of f and g so that they do not share any, then multiply f and g". Helpfully, the $\otimes$ operator is associative not only with this more abstract way of thinking about it, but also in the concrete way it was defined.

4.1. Properties of Low-Defect Polynomials

Let us begin by stating some structural properties of low-defect polynomials.

PROPOSITION 4.2. *Suppose f is a low-defect polynomial of degree r. Then f is a polynomial in the variables $x_1, \ldots, x_r$, and it is a multilinear polynomial, that is, it has degree 1 in each of its variables. The coefficients are nonnegative integers. The constant term is nonzero, and so is the coefficient of $x_1 \cdots x_r$, which we will call the* leading coefficient *of f.*

Proof. We prove the statement by structural induction.

If the low-defect polynomial f is just a constant n, then it has no variables, and the leading coefficient and constant term are both n, which is positive.

If $f = g \otimes h$, say $f(x_1, \ldots, x_r) = g(x_1, \ldots, x_s)h(x_{s+1}, \ldots, x_r)$, then by the inductive hypothesis, f is a product of two polynomials whose coefficients are nonnegative integers, and thus so is f. To see that f is multilinear, consider a variable x_i; if $1 \leq i \leq s$, then x_i has degree 1 in $g(x_1, \ldots, x_s)$ and degree 0 in $h(x_{s+1}, \ldots, x_r)$, whereas if $r + 1 \leq i \leq s$, then the reverse is true. Either way, x_i has degree 1 in f.

The coefficient of $x_1 \cdots x_r$ in f is the product of the coefficient of $x_1 \ldots x_s$ in g and the coefficient of $x_1 \cdots x_{r-s}$ in h and so does not vanish, and the constant term of f is the product of the constant terms of g and h and so does not vanish.

Finally, if $f = g \otimes x_1 + c$, say $f(x_1, \ldots, x_r) = g(x_1, \ldots, x_{r-1})x_r + c$, then since g has coefficients that are nonnegative integers, so does f. To see that f is

multilinear, consider a variable x_i; for $1 \leq i \leq r - 1$, the variable x_i has degree 1 in g and hence so does in f, whereas x_r has degree 0 in g and hence has degree 1 in f as well. Finally, the coefficient of $x_1 \cdots x_r$ in f is the same as the coefficient of $x_1 \cdots x_{r-1}$ in g and hence does not vanish, whereas the constant term of f is c, which is positive. $\qquad \square$

We will also need the following lemma in Section 6.

LEMMA 4.3. *For any low-defect polynomial f of degree $k > 0$, there exist low-defect polynomials g and h and a positive integer c such that $f = h \otimes (g \otimes x_1 + c)$.*

Proof. We apply structural induction. Since f has degree greater than zero, it is not a constant. Hence, it can be written either as $f_1 \otimes f_2$ (in which case at least one of these has degree greater than zero) or as $g \otimes x_1 + c$. In the latter case, we are done, writing $f = 1 \otimes (g \otimes x_1 + c)$.

In the former case, without loss of generality, say f_2 has degree $r > 0$. (Since if f_2 is a constant, then $f_1 \otimes f_2 = f_2 \otimes f_1$.) Then by the inductive hypothesis, there are low-defect polynomials g_2 and h_2 and a positive integer c_2 such that $f_2 = h_2 \otimes (g_2 \otimes x_1 + c)$, so $f = (f_1 \otimes h_2) \otimes (g_2 \otimes x_1 + c)$, as needed. $\qquad \square$

4.2. Numbers 3-Represented by Low-Defect Polynomials

We will obtain actual numbers from these polynomials by substituting in powers of 3 as mentioned in Section 1. Let us state here the following obvious but useful lemma.

LEMMA 4.4. *For any a, b, and n, $\|ab^n\| \leq \|a\| + n\|b\|$.*

Proof. If $n \geq 1$, then $\|ab^n\| \leq \|a\| + \|b^n\| \leq \|a\| + n\|b\|$, whereas if $n = 0$, then $\|ab^n\| = \|a\| = \|a\| + n\|b\|$. $\qquad \square$

This provides an upper bound on the complexities of the outputs of these polynomials:

PROPOSITION 4.5. *If (f, C) is a low-defect pair of degree r, then*

$$\|f(3^{n_1}, \ldots, 3^{n_r})\| \leq C + 3(n_1 + \cdots + n_r).$$

Proof. We prove the statement by structural induction. If f is a constant k, then $C \geq \|k\|$, and we are done.

If there are low-defect pairs (g_1, D_1) and (g_2, D_2) (say of degrees s_1 and s_2) such that $f = g_1 \otimes g_2$ and $C = D_1 + D_2$, then

$$\|f(3^{n_1}, \ldots, 3^{n_r})\| \leq \|g_1(3^{n_1}, \ldots, 3^{n_{s_1}})\| + \|g_2(3^{n_{s_1+1}}, \ldots, 3^{n_r})\|$$
$$\leq D_1 + D_2 + 3(n_1 + \cdots + n_r)$$
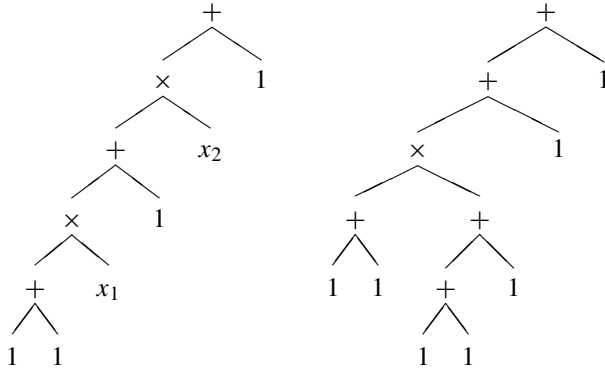$$= C + 3(n_1 + \cdots + n_r).$$

Figure 2   A tree corresponding to the polynomial $(2x_1 + 1)x_2 + 1$ and the same tree after making the substitution $x_1 = 3^1$, $x_2 = 3^0$; observe how the top multiplication node disappears

In the last case, if there is a low-defect pair (g, D) and a constant c with $C \geq D + \|c\|$ such that $f = g \otimes x_1 + c$, then we apply Lemma 4.4:

$$\|f(3^{n_1}, \ldots, 3^{n_r})\| \leq \|g(3^{n_1}, \ldots, 3^{n_{r-1}})\| + 3n_r + \|c\|$$
$$\leq D + \|c\| + 3(n_1 + \cdots + n_r) \leq C + 3(n_1 + \cdots + n_r). \quad \square$$

Note that because of the two cases in the proof of Lemma 4.4, the picture in Figure 1 is slightly inaccurate; this is only the picture when 3^k is plugged in for $k \geq 1$. See Figure 2 for an illustration of what happens when we plug in 3^0.

Because of Proposition 4.5, we define:

DEFINITION 4.6. Given a low-defect pair (f, C) (say of degree r) and a number N, we will say that (f, C) *efficiently 3-represents* N if there exist nonnegative integers $n_1, \ldots, n_r$ such that $N = f(3^{n_1}, \ldots, 3^{n_r})$ and $\|N\| = C + 3(n_1 + \cdots + n_r)$. More generally, we will also say that f 3-represents N if there exist nonnegative integers $n_1, \ldots, n_r$ such that $N = f(3^{n_1}, \ldots, 3^{n_r})$.

Note that if (f, C) efficiently 3-represents N, then $(f, \|f\|)$ efficiently 3-represents N, which means that in order for (f, C) to 3-represent anything efficiently at all, we must have $C = \|f\|$. However, it is still worth using low-defect pairs rather than just low-defect polynomials since we may not always know $\|f\|$. This paper will not be concerned with these sorts of computational issues, but in a future paper [1], we will discuss how to refine the theorems here to allow for computation.

For this reason, it makes sense to use "f efficiently 3-represents N" to mean "some (f, C) efficiently 3-represents N" or, equivalently, "$(f, \|f\|)$ efficiently 3-represents N".

In keeping with the name, the numbers 3-represented by a low-defect polynomial have bounded defect. First, let us make two definitions.

DEFINITION 4.7. Given a low-defect pair (f, C), we define $\delta(f, C)$, the defect of (f, C), to be $C - 3 \log_3 a$, where a is the leading coefficient of f. When we are not concerned with keeping track of base complexities, we will use $\delta(f)$ to mean $\delta(f, \|f\|)$.

DEFINITION 4.8. Given a low-defect pair (f, C) of degree r, we define

$$\delta_{f,C}(n_1, \ldots, n_r) = C + 3(n_1 + \cdots + n_r) - 3 \log_3 f(3^{n_1}, \ldots, 3^{n_r}).$$

We will also define δ_f to mean $\delta_{f,\|f\|}$ when we are not concerned with keeping track of base complexities.

Then we have the following:

PROPOSITION 4.9. *Let (f, C) be a low-defect pair of degree r, and let $n_1, \ldots, n_r$ be nonnegative integers.*

(1) *We have*
$$\delta(f(3^{n_1}, \ldots, 3^{n_r})) \leq \delta_{f,C}(n_1, \ldots, n_r),$$
and the difference is an integer.

(2) *We have*
$$\delta_{f,C}(n_1, \ldots, n_r) \leq \delta(f, C),$$
and if $r \geq 1$, then this inequality is strict.

Proof. For part (1), observe that this inequality is just Proposition 4.5 with the quantity $3 \log_3(f(3^{n_1}, \ldots, 3^{n_r}))$ subtracted off both sides. And since Proposition 4.5 is an inequality of integers, the difference is an integer.

For part (2), let a denote the leading coefficient of f. Then by Proposition 4.2,

$$f(3^{n_1}, \ldots, 3^{n_r}) \geq a \cdot 3^{n_1 + \cdots + n_r},$$

and this inequality is strict if $r \geq 1$ (since the constant term of f does not vanish). So

$$\begin{aligned}
\delta_{f,C}(n_1, \ldots, n_r) &= C + 3(n_1 + \cdots + n_r) - 3 \log_3 f(3^{n_1}, \ldots, 3^{n_r}) \\
&\leq C + 3(n_1 + \cdots + n_r) - 3 \log_3(a) - 3(n_1 + \cdots + n_r) \\
&= C - 3 \log_3(a) = \delta(f, C),
\end{aligned}$$

and this inequality is strict if $r \geq 1$.   □

4.3. *Low-Defect Polynomials Give All Leaders of Small Defect*

The reason these polynomials are relevant is as follows.

THEOREM 4.10. *For any real $r \geq 0$, there exists a finite set $\mathcal{S}_r$ of low-defect pairs satisfying the following conditions:*

(1) *Each $(f, C) \in \mathcal{S}_r$ has degree at most $\lfloor r \rfloor$;*

(2) *For every $N \in B_r$, there exists some $(f, C) \in \mathcal{S}_r$ that efficiently 3-represents N.*

Proof. We prove this statement in the following form: For any real $\alpha \in (0, 1)$ and any integer $k \geq 1$, there exists a finite set $\mathcal{S}_{k,\alpha}$ of low-defect pairs, each of degree at most $k - 1$, such that for every $N \in B_{k\alpha}$, there exists some $(f, C) \in \mathcal{S}_{k,\alpha}$ that efficiently 3-represents N. Once we have this, the result will follow for $r > 0$ by taking $\mathcal{S}_r = \mathcal{S}_{k,\alpha}$ for $k = \lfloor r \rfloor + 1$ and $\alpha = \frac{r}{\lfloor r \rfloor + 1}$. For $r = 0$, we can take $\mathcal{S}_r = \emptyset$.

We prove this by induction on k. If $k = 1$, then B_α is finite by Theorem 2.7, so we can take $\mathcal{S}_{1,\alpha} = \{(N, \|N\|) : N \in B_\alpha\}$. Now suppose the statement is true for k, and we want to prove it for $k + 1$, so we have already constructed sets $\mathcal{S}_{i,\alpha}$ for $i \leq k$.

We will define the set $\mathcal{S}_{k+1,\alpha}$ to consist of the following:

(1) If $k + 1 > 2$, then for $(f, C) \in \mathcal{S}_{i,\alpha}$ and $(g, D) \in \mathcal{S}_{j,\alpha}$ with $2 \leq i, j \leq k$ and $i + j = k + 2$, we include $(f \otimes g, C + D)$ in $\mathcal{S}_{k+1,\alpha}$;
Whereas if $k + 1 = 2$, then for $(f_1, C_1), (f_2, C_2), (f_3, C_3) \in \mathcal{S}_{1,\alpha}$, we include $(f_1 \otimes f_2, C_1 + C_2)$ and $(f_1 \otimes f_2 \otimes f_3, C_1 + C_2 + C_3)$ in $\mathcal{S}_{2,\alpha}$.
(2) For $(f, C) \in \mathcal{S}_{k,\alpha}$ and any solid number b with $\|b\| < (k + 1)\alpha + 3 \log_3 2$, we include $(f \otimes x_1 + b, C + \|b\|)$ in $\mathcal{S}_{k+1,\alpha}$.
(3) For $(f, C) \in \mathcal{S}_{k,\alpha}$, any solid number b with $\|b\| < (k + 1)\alpha + 3 \log_3 2$, and any $v \in B_\alpha$, we include $(v(f \otimes x_1 + b), C + \|b\| + \|v\|)$ in $\mathcal{S}_{k+1,\alpha}$.
(4) For all $n \in T_\alpha$, we include $(n, \|n\|)$ in $\mathcal{S}_{k+1,\alpha}$.
(5) For all $n \in T_\alpha$ and $v \in B_\alpha$, we include $(vn, \|vn\|)$ in $\mathcal{S}_{k+1,\alpha}$.

This is a finite set since the $\mathcal{S}_{i,\alpha}$ for $i \leq k$ are all finite, B_α is finite, T_α is finite, and there are only finitely many b satisfying $\|b\| < (k + 1)\alpha + 3 \log_3 2$ since this implies that

$$3 \log_3 b < (k + 1)\alpha + 3 \log_3 2.$$

Also, all elements of $\mathcal{S}_{k+1,\alpha}$ have degree at most k: In case (1), if $k + 1 > 2$, then f and g have degrees at most $i - 1$ and $j - 1$ respectively, so $f \otimes g$ has degree at most $i + j - 2 = k$, whereas if $k + 1 = 2$, then f_1, f_2, and f_3 all have degree 0, so $f_1 \otimes f_2$ and $f_1 \otimes f_2 \otimes f_3$ also have degree 0. In cases (2) and (3), f has degree at most $k - 1$, so $f \otimes x_1 + b$ has degree at most k. Finally, in cases (4) and (5), we are adding low-defect pairs of degree 0.

So suppose that $N \in B_{(k+1)\alpha}$; we apply Theorem 2.9.

In case (1) of Theorem 2.9, if $k + 1 > 2$, then there is a good factorization $N = uv$ where $u \in B_{i\alpha}$, $v \in B_{j\alpha}$ with $i + j = k + 2$ and $2 \leq i, j \leq k$. So by the inductive hypothesis we can take $(f, C) \in \mathcal{S}_{i,\alpha}$ and $(g, D) \in \mathcal{S}_{j,\alpha}$ such that (f, C) efficiently 3-represents u and (g, D) efficiently 3-represents v. Since the factorization $N = uv$ is good, it follows that $(f \otimes g, C + D)$ efficiently represents N. If $k + 1 = 2$, then there is either a good factorization $n = u_1 u_2$ or a good factorization $n = u_1 u_2 u_3$ with all $u_\ell \in B_\alpha$. So take $(f_\ell, C_\ell) \in \mathcal{S}_{1,\alpha}$ such that (f_ℓ, C_ℓ) efficiently 3-represents u_l; then either $(f_1 \otimes f_2, C_1 + C_2)$ or $(f_1 \otimes f_2 \otimes f_3, C_1 + C_2 + C_3)$ efficiently 3-represents N, as appropriate.

In case (2) of Theorem 2.9, there are a and b with $N = a + b$, $\|N\| = \|a\| + \|b\|$, $a \in A_{k\alpha}$, $b \leq a$ a solid number, and

$$\delta(a) + \|b\| < (k + 1)\alpha + 3 \log_3 2.$$

In particular, we have $\|b\| < (k+1)\alpha + 3\log_3 2$. Write $a = a'3^\ell$ with a' a leader and $\|a\| = \|a'\| + 3\ell$, so $a' \in B_{k\alpha}$, and pick $(f,C) \in \mathcal{S}_{k,\alpha}$ that efficiently 3-represents a'. Then $(f \otimes x_1 + b, C + \|b\|)$ is in $\mathcal{S}_{k+1,\alpha}$ and efficiently 3-represents N. In case (3) of Theorem 2.9, there is a good factorization $n = (a+b)v$ with $v \in B_\alpha$ and a and b satisfying the conditions in the case (2) of Theorem 2.9, so the proof is similar; if we write $a = a'3^\ell$ with a' a leader and $\|a\| = \|a'\| + 3\ell$ and pick $(f,C) \in \mathcal{S}_{k,\alpha}$ efficiently 3-representing a', then $(v(f \otimes x_1 + b), C + \|b\| + \|v\|)$ efficiently 3-represents N.

Finally, in cases (4) and (5) of Theorem 2.9, the pair $(N, \|N\|)$ is itself in $\mathcal{S}_{k+1,\alpha}$, by cases (4) and (5). This proves the theorem. □

Note that while this theorem produces a covering of B_r, there is no guarantee that for $f \in \mathcal{S}_r$, all the numbers 3-represented by f will have defect less than r; and in general this will not be the case. For instance, if we use the method of the proof of Theorem 4.10 to produce the set $\mathcal{S}_1$, it will contain the polynomial $16x_1 + 1$, which 3-represents the number 17, which has defect greater than 1. This deficiency will be remedied in a sequel paper [1], where it will be shown how to choose the $\mathcal{S}_r$ to get this additional property. There is also no guarantee that the numbers 3-represented by f will be leaders; for instance, if we use this method to produce the set $\mathcal{S}_1$, then it will also contain the constant polynomials 9 and 27.

4.4. Augmented Low-Defect Polynomials

Theorem 4.10 gives us a representation of the leaders with defect less than a fixed r, but we want to consider all numbers with defect less than r. However, by Proposition 2.6, any number can be written most-efficiently as $3^k m$ for some $k \geq 0$ and some leader m. To account for this, we introduce the notion of an augmented low-defect polynomial.

DEFINITION 4.11. For any low-defect polynomial f, we define $\hat{f} = f \otimes x$. The polynomial $\hat{f}$ will be called an *augmented low-defect polynomial*. For a low-defect pair (f, C), the pair $(\hat{f}, C)$ will be called an *augmented low-defect pair*.

Note that augmented low-defect polynomials are never low-defect polynomials; by Proposition 4.2, low-defect polynomials always have a nonzero constant term, whereas an augmented low-defect polynomial always has a zero constant term.

We can then make the following observations and definitions, parallel to the contents of Sections 4.2 and 4.3.

COROLLARY 4.12. *If (f, C) is a low-defect pair of degree r, then*

$$\|\hat{f}(3^{n_1}, \ldots, 3^{n_{r+1}})\| \leq C + 3(n_1 + \cdots + n_{r+1}).$$

Proof. This is immediate from Proposition 4.5 and Lemma 4.4. □

DEFINITION 4.13. Given a low-defect pair (f, C) (say of degree r) and a number N, we will say that $(\hat{f}, C)$ efficiently 3-represents N if there exist

$n_1, \ldots, n_{r+1}$ such that $N = \hat{f}(3^{n_1}, \ldots, 3^{n_{r+1}})$ and $\|N\| = C + 3(n_1 + \cdots + n_{r+1})$. More generally, we will also say that $\hat{f}$ 3-represents N if there exist $n_1, \ldots, n_{r+1}$ such that $N = \hat{f}(3^{n_1}, \ldots, 3^{n_{r+1}})$.

COROLLARY 4.14. *Let (f, C) be a low-defect pair of degree r, and let $n_1, \ldots, n_r$ be nonnegative integers. Then*

$$\delta(\hat{f}(3^{n_1}, \ldots, 3^{n_{r+1}})) \leq \delta_{f,C}(n_1, \ldots, n_r),$$

and the difference is an integer.

Proof. This inequality is just Corollary 4.12 with $3\log_3 \hat{f}(3^{n_1}, \ldots, 3^{n_{r+1}})$ subtracted off both sides. And since Corollary 4.12 is an inequality of integers, the difference is an integer. □

THEOREM 4.15. *For any real $r \geq 0$, there exists a finite set $\mathcal{S}_r$ of low-defect pairs satisfying the following conditions*:
(1) *Each $(f, C) \in \mathcal{S}_r$ has degree at most $\lfloor r \rfloor$*;
(2) *For every $N \in A_r$, there exists some $(f, C) \in \mathcal{S}_r$ such that $(\hat{f}, C)$ efficiently 3-represents N.*

Proof. This is immediate from Theorem 4.10 and Proposition 2.6. □

5. Facts from Order Theory and Topology

This section collects facts about well orderings and partial orderings needed to prove the main result. Recall that a *well partial order* is a partial order that is well-founded (has no infinite descending chains) and has no infinite antichains. Any totally ordered extension of a well partial order is well-ordered. Given a well partial order X, we can consider the set of order types of well-orders obtained by extending the ordering on X. It was proved by De Jongh and Parikh [8, Theorem 2.13] that for any well partial order X, the set of ordinals obtained this way has a maximum; this maximum is denoted $o(X)$. They further proved [8, Theorem 3.4, Theorem 3.5] the following:

THEOREM 5.1. *Let X and Y be two well partial orders. Then $X \amalg Y$ and $X \times Y$ are well partial orders, $o(X \amalg Y) = o(X) \oplus o(Y)$, and $o(X \times Y) = o(X) \otimes o(Y)$, where $\oplus$ and $\otimes$ are the operations of natural sum and natural product (also known as the Hessenberg sum and Hessenberg product).*

The natural sum and natural product are defined as follows [8].

DEFINITION 5.2. The *natural sum* (also known as the *Hessenberg sum*) of two ordinals α and β, here denoted $\alpha \oplus \beta$, is defined by simply adding up their Cantor normal forms as if they were "polynomials in ω". That is to say, if there are ordinals $\gamma_0 < \cdots < \gamma_n$ and whole numbers $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ such that $\alpha = \omega^{\gamma_n} a_n + \cdots + \omega^{\gamma_0} a_0$ and $\beta = \omega^{\gamma_n} b_n + \cdots + \omega^{\gamma_0} b_0$, then

$$\alpha \oplus \beta = \omega^{\gamma_n}(a_n + b_n) + \cdots + \omega^{\gamma_0}(a_0 + b_0).$$

Similarly, the *natural product* (also known as the *Hessenberg product*) of α and β, here denoted $\alpha \otimes \beta$, is defined by multiplying their Cantor normal forms as if they were "polynomials in ω", using the natural sum to add the exponents. That is to say, if we write $\alpha = \omega^{\gamma_n} a_n + \cdots + \omega^{\gamma_0} a_0$ and $\beta = \omega^{\delta_m} b_m + \cdots + \omega^{\delta_0} b_0$ with $\gamma_0 < \cdots < \gamma_n$ and $\delta_0 < \cdots < \delta_m$ ordinals and the a_i and b_i whole numbers, then

$$\alpha \otimes \beta = \bigoplus_{\substack{0 \le i \le n \\ 0 \le j \le m}} \omega^{\gamma_i \oplus \delta_j} a_i b_j.$$

These operations are commutative and associative, and $\otimes$ distributes over $\oplus$. The expression $\alpha \oplus \beta$ is strictly increasing in α and β; and $\alpha \otimes \beta$ is strictly increasing in β so long as $\alpha \ne 0$, and vice versa [6].

There are other definitions of these operations. Given ordinals α and β, $\alpha \oplus \beta$ is sometimes defined as $o(\alpha \sqcup \beta)$, and $\alpha \otimes \beta$ as $o(\alpha \times \beta)$, where for this definition, we consider α and β as partial orders. As noted before, De Jongh and Parikh showed the stronger statement Theorem 5.1, from which it follows that

$$o(\alpha_1 \sqcup \cdots \sqcup \alpha_n) = \alpha_1 \oplus \cdots \oplus \alpha_n,$$
$$o(\alpha_1 \times \cdots \times \alpha_n) = \alpha_1 \otimes \cdots \otimes \alpha_n.$$

There is also a recursive definition [7].

Note also the following statements about well partial orderings.

PROPOSITION 5.3. *Suppose that X is a well partially ordered set, S a totally ordered set, and $f : X \to S$ is monotonic. Then $f(X)$ is well-ordered and has order type at most $o(X)$.*

Proof. Pick a well-ordering extending the ordering $\le$ on X; call it $\preceq$. Define another total ordering on X, call it $\le'$, by $a <' b$ if either $f(a) < f(b)$ or $f(a) = f(b)$ and $a \prec b$. Observe that $\le'$ is an extension of $\le$ as f is monotonic, so it is a well-ordering and has order type at most $o(X)$. Since f is clearly also monotonic when we instead use the ordering $\le'$ on the domain, its image is therefore also well-ordered and of order type at most $o(X)$.   □

Note in particular that if X is the union of $X_1, \ldots, X_n$, then $o(X) \le o(X_1) \oplus \cdots \oplus o(X_n)$ since X is a monotonic image of $X_1 \sqcup \cdots \sqcup X_n$. So we have the following:

PROPOSITION 5.4. *We have*:

(1) *If S is a well-ordered set and $S = S_1 \cup \cdots \cup S_n$, and S_1 through S_n all have order type less than ω^k, then so does S.*

(2) *If S is a well-ordered set of order type ω^k and $S = S_1 \cup \cdots \cup S_n$, then at least one of S_1 through S_n also has order type ω^k.*

Proof. For (1), observe that the order type of S is at most the natural sum of those of $S_1, \ldots, S_n$, and the natural sum of ordinals less than ω^k is again less than ω^k.

For (2), by (1), if $S_1, \ldots, S_k$ all had order type less than ω^k, so would S; so at least one has order type at least ω^k, and it necessarily also has order type at most ω^k, being a subset of S. □

For the proof of the main result we will also need some facts about well-ordered sets sitting inside the real numbers. In particular, we need results about closures and limit points of such sets, with the ambient space carrying the order topology. Since we have not found all the following results in the literature, we supply proofs.

PROPOSITION 5.5. *Let X be a totally ordered set, and let S be a well-ordered subset of order type α. Then $\overline{S}$ is also well-ordered and has order type either α or $\alpha + 1$. If $\alpha = \gamma + k$ where γ is a limit ordinal and k is finite, then $\overline{S}$ has order type $\alpha + 1$ if and only if the initial segment of S of order type γ has a supremum in X that is not in S.*

Proof. We induct on α. If $\alpha = 0$, then S is empty, and thus so is $\overline{S}$.

If $\alpha = \beta + 1$, say x is the maximum element of S and $T = S \setminus \{x\}$. Then $\overline{S} = \overline{T} \cup \{x\}$, and x is the maximum element of $\overline{S}$. If $x \in \overline{T}$, then $\overline{S} = \overline{T}$; otherwise, its order type is 1 greater. So since $\overline{T}$ has order type either β or $\beta + 1$ by the inductive hypothesis, $\overline{S}$ has order type β, $\beta + 1 = \alpha$, or $\beta + 2 = \alpha + 1$. Of course, the first of these is impossible since its order type must be at least α and it contains S, so the order type is either α or $\alpha + 1$.

Furthermore, if $\beta = \gamma + k$ where γ is a limit ordinal, then we can let R be the initial segment of T (equivalently, of S) of order type γ. Then by the inductive hypothesis, $\overline{T}$ has order type $\beta + 1$ if and only if R has a supremum in X that is not in T. In the case where $x \notin \overline{T}$, then $x \notin \overline{R}$, and so x cannot be a supremum of R in X. Hence, in this case, $\overline{T}$ has order type $\beta + 1$ if and only if R has a supremum in X that is not in S, and so $\overline{S}$ has order type $\beta + 2 = \alpha + 1$ if and only if R has a supremum in X that is not in S.

In the case where $x \in \overline{T}$, it must be that x is a supremum of T in X. Since x is not itself in T, this requires that β be a limit ordinal and hence that $\beta = \gamma$, that is, $T = R$, since γ is the largest limit ordinal smaller than S. So R has a supremum that is not in T, namely, x; and so by the inductive hypothesis, $\overline{T}$ has order type $\beta + 1$. Since $\overline{S} = \overline{T}$ in this case, it also has order type $\beta + 1 = \alpha$. Furthermore, R has a supremum x, but this supremum is in S; thus, the theorem is true in this case.

Finally, we have the case where α is a limit ordinal. If $x \in \overline{S}$, then either x is an upper bound of S, or it is not; we will first consider R, the subset of $\overline{S}$ consisting of those elements that are not upper bounds of S. For any $x \in R$, there is some $y \in S$ with $y > x$, and so $x \in (-\infty, y) \cap \overline{S}$. Since the former is an open set, this means $x \in \overline{S \cap (-\infty, y)}$. Since $S \cap (-\infty, y)$ is a proper initial segment of S, by the inductive hypothesis, its closure is well-ordered. Note that for varying y, the sets $\overline{S \cap (-\infty, y)}$ form a chain under inclusion of well-ordered sets, with smaller ones being initial segments of larger ones. So since R is the union of these, it is well-ordered, and its order type is equal to their supremum. Now clearly the order

type of R is at least α since R includes S; and by the inductive hypothesis, it is at most $\lim_{\beta < \alpha}(\beta + 1) = \alpha$. So R has order type α.

This leaves the question of elements of $\overline{S}$ that are upper bounds of S (and hence R). The only way such an element can exist is if it is the supremum of S. Hence, if S has a supremum in X, and this supremum is not already in S, then $\overline{S}$ has order type $\alpha + 1$, and otherwise it has order type α. $\square$

PROPOSITION 5.6. *Suppose X is a totally ordered set, S a subset of X, and T an initial segment of S. Then $\overline{T}$ is an initial segment of $\overline{S}$.*

Proof. Suppose $x \in \overline{T}$, $y \in \overline{S}$, and $y < x$; we want to show that $y \in \overline{T}$. The set (y, ∞) is an open subset of X and contains $x \in \overline{T}$; thus, it also contains some $t \in \overline{T}$. That is to say, there is some $t \in T$ with $t > y$.

Now let U be any open neighborhood of y; then $U \cap (-\infty, t)$ is again an open neighborhood of y, and since $y \in \overline{S}$, there must exist some $s \in S \cap U \cap (-\infty, t)$. But then $s \in S$, $s < t$, and $t \in T$, so $s \in T$ as well since we assumed that T was an initial segment of S. Thus, each neighborhood U of y contains some element of T, that is to say, $y \in \overline{T}$. $\square$

COROLLARY 5.7. *Let X be a totally ordered set with the least upper bound property, and S a well-ordered subset of X of order type α. Then if $\beta < \alpha$ is a limit ordinal, then the βth element of $\overline{S}$ is the supremum (limit) of the initial β elements of S.*

Proof. Let T be the initial segment of S of order type β. Since $\beta < \alpha$, T is bounded above in S and thus in X, and thus it has a supremum s. This supremum s is not in T since T has order type β, a limit ordinal, and thus has no maximum. So $\overline{T}$, by Proposition 5.5, has order type $\beta + 1$, and s is clearly its final element. So by Proposition 5.6 it is the βth element of $\overline{S}$ as well, and by definition it is the supremum of the initial β elements of S. $\square$

PROPOSITION 5.8. *If S is a well-ordered set of order type $\alpha < \omega^{n+1}$ with n finite, then S', the set of limit points of S (in the order topology) has order type strictly less than ω^n.*

Proof. Since we are considering S purely as a totally ordered set and not embedded in anything else, we may assume that it is an ordinal. Let β be the order type of S'. The elements of S' consist of the limit ordinals less than α. If $n = 0$, then α is finite, and so $\beta = 0 < \omega^0$.

Otherwise, $\alpha < \omega^{n+1}$, so say $\alpha \leq \omega^n k$. An ordinal γ is a limit ordinal if and only if it can be written as $\omega \gamma'$ for some $\gamma' > 0$. Since, assuming that $n > 0$, $\omega \gamma' < \omega^n k$ if and only if $\gamma' < \omega^{n-1} k$, the order type of the set of limit ordinals less than $\omega^n k$ is easily seen to be $\omega^{n-1} k - 1$ (where the 1 is subtracted off the beginning; this only makes a difference if $n = 1$). So the order type of β is at most $\omega^{n-1} k - 1 < \omega^n$. $\square$

It is not too hard to write down a general formula for the order type of S' in terms of the order type of S (even without the restriction that $\alpha < \omega^\omega$), but we will not need such detail here. See [20, Theorem 8.6.6] for more on this.

PROPOSITION 5.9. *Let T be a totally ordered set, and S a well-ordered subset. If S' (in the order topology on T) has order type at least ω^n with n finite, then S has order type at least ω^{n+1}.*

Proof. Suppose S has order type less than ω^{n+1}. Then by Proposition 5.5 so does $\overline{S}$. Since $\overline{S}' = S'$, we can just consider $\overline{S}$. And we can consider the order topology on $\overline{S}$ instead of the subspace topology since the former is coarser and thus $\overline{S}$ has more limit points under it. But by Proposition 5.8 the order type of $\overline{S}'$ in the order topology on $\overline{S}$ is less than ω^n. Hence, $\overline{S}'$ under the subspace topology also has order type less than ω^n, and hence S' has order type less than ω^n. So if S' has order type at least ω^n, then S has order type at least ω^{n+1}. $\square$

6. Well-Ordering of Defects

We now begin proving well-ordering theorems about defects.

PROPOSITION 6.1. *Let (f, C) be a low-defect pair. Then the function $\delta_{f,C}$ is strictly increasing in each variable.*

Proof. Suppose f has degree r. We can define g, the reverse polynomial of f:

$$g(x_1, \ldots, x_r) = x_1 \cdots x_r f(x_1^{-1}, \ldots, x_r^{-1}).$$

So g is a multilinear polynomial in $x_1, \ldots, x_r$ with the coefficient of $\prod_{i \in S} x_i$ in g being the coefficient of $\prod_{i \notin S} x_i$ in f. By Proposition 4.2, f has nonnegative coefficients, so does g; since the constant term of f does not vanish, the $x_1 \ldots x_r$ term of g does not vanish. Hence, g is strictly increasing in each variable.

Then

$$\delta_{f,C}(n_1, \ldots, n_r) = C + 3(n_1 + \cdots + n_r) - 3 \log_3 f(3^{n_1}, \ldots, 3^{n_r})$$

$$= C - 3 \log_3 \frac{f(3^{n_1}, \ldots, 3^{n_r})}{3^{n_1 + \cdots + n_r}} = C - 3 \log_3 g(3^{-n_1}, \ldots, 3^{-n_r})$$

which is strictly increasing in each variable, as claimed. $\square$

PROPOSITION 6.2. *Let (f, C) be a low-defect pair of degree r. Then the image of $\delta_{f,C}$ is a well-ordered subset of $\mathbb{R}$ with order type ω^r.*

Proof. By Proposition 6.1, $\delta_{f,C}$ is a monotonic function from $\mathbb{Z}_{\geq 0}^r$ to $\mathbb{R}$, and $\mathbb{R}$ is totally ordered, so by Proposition 5.3 and Theorem 5.1 its image is a well-ordered set of order type at most ω^r.

For the lower bound, we induct on r. Let S denote the image of $\delta_{f,C}$. If $r = 0$, then $\delta_{f,C}$ is a constant, and so S has order type $1 = \omega^0$. Now suppose $r \geq 1$ and that this is true for $r - 1$. By Lemma 4.3 we can write $f = h \otimes (g \otimes x_1 + c)$

where c is a positive integer and g and h are low-defect polynomials. Un-
packing this statement, if s is the degree of h, then we have $f(x_1, \ldots, x_r) =$
$h(x_1, \ldots, x_s)(g(x_{s+1}, \ldots, x_{r-1})x_r + c)$. Then

$$\delta_{f,C}(n_1, \ldots, n_r) = (C - \|h\|) + \delta_h(n_1, \ldots, n_s) + 3(n_{s+1} + \cdots + n_{r-1})$$
$$- 3\log_3(g(3^{n_{s+1}}, \ldots, 3^{n_{r-1}}) + c3^{-n_r}).$$

Thus,

$$\lim_{n_r \to \infty} \delta_{f,C}(n_1, \ldots, n_r) = C - \|h\| + \delta_h(n_1, \ldots, n_s)$$
$$+ 3(n_{s+1} + \cdots + n_{r-1}) - 3\log_3(g(3^{n_{s+1}}, \ldots, 3^{n_{r-1}}))$$
$$= C - \|h\| - \|g\| + \delta_h(n_1, \ldots, n_s) + \delta_g(n_{s+1}, \ldots, n_{r-1})$$
$$= C + 3(n_1 + \cdots + n_{r-1})$$
$$- 3\log_3(h(n_1, \ldots, n_s)g(n_{s+1}, \ldots, n_{r-1}))$$
$$= C - \|g \otimes h\| + \delta_{g \otimes h}(n_1, \ldots, n_{r-1}).$$

Since $\delta_{f,C}$ is increasing in n_r, this means that this is in fact a limit point of S.
So we see that S' contains a translate of the image of $\delta_{g \otimes h}$. The degree of $g \otimes h$
is $r - 1$, so by the inductive hypothesis this image has order type at least ω^{r-1}.
Thus, S' has order type at least ω^{r-1}, and so by Proposition 5.9 this means that S
has order type at least ω^r.   □

PROPOSITION 6.3. *Let (f, C) be a low-defect pair of degree r. Then the set of*
$\delta(n)$ *for all n 3-represented by the augmented low-defect polynomial $\hat{f}$ is a well-*
ordered subset of $\mathbb{R}$ with order type at least ω^r and at most $\omega^r(\lfloor\delta(f,C)\rfloor + 1) <$
ω^{r+1}. *The same is true if f is used instead of the augmented version $\hat{f}$.*

Proof. Let S be the set of all $\delta(n)$ for all n that are 3-represented by $\hat{f}$, and let T
be the image of $\delta_{f,C}$. By Proposition 6.2, T is a well-ordered subset of $\mathbb{R}$ of order
type ω^r. Suppose $n = \hat{f}(3^{m_1}, \ldots, 3^{m_{r+1}})$. Then by Corollary 4.14,

$$\delta(n) = \delta_{f,C}(m_1, \ldots, m_{r+1}) - k$$

for some $k \geq 0$. But $\delta_{f,C}(m_1, \ldots, m_{r+1}) \leq \delta(f, C)$ by Proposition 4.9, and since
$\delta(n) \geq 0$, this implies $k \leq \delta(f, C)$. Since k is an integer, this implies

$$k \in \{0, \ldots, \lfloor\delta(f,C)\rfloor\},$$

which is a finite set. Let ℓ refer to the number $\lfloor\delta(f,C)\rfloor$.

Thus, S is covered by finitely many translates of T; more specifically, we can
partition T into T_0 through T_ℓ such that

$$S = T_0 \cup (T_1 - 1) \cup \cdots \cup (T_\ell - \ell).$$

Then the T_i all have order type at most ω^r, and by Proposition 5.4 at least one has
order type ω^r. Hence, S is well-ordered of order type at most $\omega^r(\lfloor\delta(f,C)\rfloor + 1) <$
ω^{r+1} by Propositions 5.1 and 5.3. By the previous reasoning it also has order type
at least ω^r.

The proof for f instead of $\hat{f}$ is similar.   □

PROPOSITION 6.4. *For any $s > 0$, the set $\mathscr{D} \cap [0, s)$ is a well-ordered subset of $\mathbb{R}$ with order type at least $\omega^{\lfloor s \rfloor}$ and less than $\omega^{\lfloor s \rfloor + 1}$.*

Proof. By Theorem 4.15 there exists a finite set $\mathcal{S}_s$ of low-defect polynomials of degree at most $\lfloor s \rfloor$ such that each $n \in A_s$ can be 3-represented by $\hat{f}$ for some $f \in \mathcal{S}_s$. By Proposition 6.3, for each $f \in \mathcal{S}$, the set of defects of numbers 3-represented by $\hat{f}$ is a well-ordered set of order type less than $\omega^{\lfloor s \rfloor + 1}$. Since $\mathscr{D} \cap [0, s)$ is covered by a finite union of these, it is also well-ordered of order type less than $\omega^{\lfloor s \rfloor + 1}$ by Proposition 5.4.

For the lower bound on the order type, if $0 < s < 1$, then observe that $0 \in \mathscr{D} \cap [0, s)$. Otherwise, let $k = \lfloor s \rfloor$ and consider the low-defect polynomial

$$f = (\cdots(((3x_1 + 1)x_2 + 1)x_3 + 1)\cdots)x_k + 1.$$

We have $\|f\| \leq 3 + k$, so $\delta(f) \leq k \leq s$. Since $k \geq 1$, by Proposition 4.9 the set of $\delta(n)$ for n that are 3-represented by f is contained in $\mathscr{D} \cap [0, s)$, whereas by Proposition 6.3 it has order type at least ω^k, proving the claim. $\square$

We can thus conclude.

THEOREM 6.5. *The set $\mathscr{D}$ is a well-ordered subset of $\mathbb{R}$ of order type ω^ω.*

Proof. By Proposition 6.4 we see that each initial segment of $\mathscr{D}$ is well-ordered and with order type less than ω^ω; hence, $\mathscr{D}$ is well-ordered and has order type at most ω^ω. Also by Proposition 6.4 we can find initial segments of $\mathscr{D}$ with order type at least ω^n for any $n \in \mathbb{N}$, so $\mathscr{D}$ has order type at least ω^ω. $\square$

We have now determined the order type of $\mathscr{D}$. However, we have not fully determined the order types of $\mathscr{D} \cap [0, s]$ for real numbers s. Of course, in general, determining this is complicated, but we can answer the question when s is an integer.

THEOREM 6.6. *For any whole number $k \neq 1$, $\mathscr{D} \cap [0, k]$ is a well-ordered subset of $\mathbb{R}$ with order type ω^k, whereas $\mathscr{D} \cap [0, 1]$ has order type $\omega + 1$.*

Proof. The order type of $\mathscr{D} \cap [0, k]$ is either the same as that of $\mathscr{D} \cap [0, k)$ or that same order type plus 1, depending on whether or not $k \in \mathscr{D}$. By Theorem 2.1 the only integral elements of $\mathscr{D}$ are 0 and 1, so it remains to determine the order type of $\mathscr{D} \cap [0, k)$. For $k = 0$, the set $\mathscr{D} \cap [0, 0] = \{0\}$ clearly has order type $1 = \omega^0$, making the statement true for this case, so assume that $k \geq 1$.

By Proposition 6.4, $\mathscr{D} \cap [0, k)$ is well-ordered and has order type at least ω^k. However, its order type is also equal to the supremum of the order types of $\mathscr{D} \cap [0, r)$ for $r < k$, and by Proposition 6.4, since k is an integer, these are all less than ω^k. Hence, its order type is also at most ω^k and thus exactly ω^k. Thus, for $k \geq 1$, the order type of $\mathscr{D} \cap [0, k]$ is exactly ω^k, unless $k = 1$, in which case it is $\omega + 1$. $\square$

Putting these together, we have the main theorem:

Proof of Theorem 1.2. The first part is Theorem 6.5. The second part follows from the proof of Theorem 6.6 or from Theorem 6.6 and the fact that 1 is the only nonzero defect that is also an integer.   □

7. Variants of the Main Theorem

In this section, we prove several variants of the main theorem, all showing ω^ω well-ordering for various related sets.

We begin with proving that the well ordering holds for the closure $\overline{\mathscr{D}}$ of the defect set in $\mathbb{R}$.

PROPOSITION 7.1. *The set $\overline{\mathscr{D}}$, the closure of the defect set, is well-ordered with order type ω^ω. Furthermore, for an integer $k \geq 1$, the order type of $\overline{\mathscr{D}} \cap [0, k]$ is $\omega^k + 1$. (And $k \in \overline{\mathscr{D}}$, so k is the ω^kth element of $\overline{\mathscr{D}}$.)*

Proof. By Proposition 5.5 the set $\overline{\mathscr{D}}$ is well-ordered, and its order type is ω^ω since $\mathscr{D}$ is unbounded in $\mathbb{R}$. For the set $\overline{\mathscr{D}} \cap [0, k]$, observe that this set is the same as the closure of $\mathscr{D} \cap [0, k]$ within $[0, k]$, so Proposition 5.5 implies that this has order type $\omega^k + 1$ since $[0, k]$ has the least-upper-bound property. Since by Proposition 5.5, for $r < k$, the set $\overline{\mathscr{D}} \cap [0, r]$ has order type less than ω^k, the ω^kth element must be k itself.   □

The other variants of the main result include considering defect sets for integers n whose complexity $\|n\|$ falls in individual congruence classes modulo 3 and, in a separate direction, restricting to stable defects. Furthermore, results in both directions can be combined. These defect sets are all well-ordered by virtue of being contained in $\overline{\mathscr{D}}$, and the issue is to show that they have the appropriate order type.

To prove the main theorem, we needed to know that given a low-defect pair (f, C) of degree k, we have $\|f(3^{n_1}, \ldots, 3^{n_k})\| \leq C + 3(n_1 + \cdots + n_k)$. In order to prove these more detailed versions, as a preliminary result, we demonstrate that for certain low-defect pairs (f, C), equality holds for "most" choices of $(n_1, \ldots, n_k)$. Indeed, we will need an even stronger statement: Since $\|f(3^{n_1}, \ldots, 3^{n_k})\| \leq C + 3(n_1 + \cdots + n_k)$, it follows that also

$$\|f(3^{n_1}, \ldots, 3^{n_k})\|_{\mathrm{st}} \leq C + 3(n_1 + \cdots + n_k),$$

and it is equality in this form that we will need for "most" $(n_1, \ldots, n_k)$.

PROPOSITION 7.2. *Let (f, C) be a low-defect pair of degree k with $\delta(f, C) < k+1$. Define its "exceptional set" to be*

$$S := \{(n_1, \ldots, n_k) : \|f(3^{n_1}, \ldots, 3^{n_k})\|_{\mathrm{st}} < C + 3(n_1 + \cdots + n_k)\}.$$

Then the set $\{\delta(f(3^{n_1}, \ldots, 3^{n_k})) : (n_1, \ldots, n_k) \in S\}$ has order type less than ω^k. In particular, the set $\{\delta(f(3^{n_1}, \ldots, 3^{n_k})) : (n_1, \ldots, n_k) \notin S\}$ has order type at least ω^k, and thus so does the set

$$\{\delta(f(3^{n_1}, \ldots, 3^{n_k})) : (n_1, \ldots, n_k) \in \mathbb{Z}_{\geq 0}^k\} \cap \mathscr{D}_{\mathrm{st}}^C.$$

Proof. The set S can be equivalently written as

$$\{(n_1, \ldots, n_k) : \|f(3^{n_1}, \ldots, 3^{n_k})\|_{\text{st}} \le C + 3(n_1 + \cdots + n_k) - 1\}$$

and hence as

$$\{(n_1, \ldots, n_k) : \delta_{\text{st}}(f(3^{n_1}, \ldots, 3^{n_k})) \le \delta_{f,C}(n_1, \ldots, n_k) - 1\}.$$

Hence, for $(n_1, \ldots, n_k) \in S$, we have

$$\delta_{\text{st}}(f(3^{n_1}, \ldots, 3^{n_k})) \le \delta(f, C) - 1 < k,$$

and thus by Proposition 6.4, the set of these stable defects has order type less than ω^k.

Equivalently, by Proposition 5.4, the set

$$\{\delta(f(3^{n_1}, \ldots, 3^{n_k})) : (n_1, \ldots, n_k) \in S\}$$

and the set $\delta_{f,C}(S)$ have order type less than ω^k since each is a finite union of translates of subsets of the set $\{\delta_{\text{st}}(f(3^{n_1}, \ldots, 3^{n_k})) : (n_1, \ldots, n_k) \in S\}$.

So consider the set

$$\{\delta(f(3^{n_1}, \ldots, 3^{n_k})) : (n_1, \ldots, n_k) \notin S\},$$

which can equivalently be written as

$$\{\delta_{\text{st}}(f(3^{n_1}, \ldots, 3^{n_k})) : (n_1, \ldots, n_k) \notin S\}$$

since for $(n_1, \ldots, n_k) \notin S$, the number $f(3^{n_1}, \ldots, 3^{n_k})$ is stable. This set must have order type at least ω^k by Propositions 6.3 and 5.4. Since for $(n_1, \ldots, n_k) \notin S$, we have that $f(3^{n_1}, \ldots, 3^{n_k})$ is stable and

$$\|f(3^{n_1}, \ldots, 3^{n_k})\| = C + 3(n_1 + \cdots + n_k) \equiv C \pmod{3},$$

this implies that the set

$$\{\delta(f(3^{n_1}, \ldots, 3^{n_k})) : (n_1, \ldots, n_k) \in \mathbb{Z}_{\ge 0}^k\} \cap \mathscr{D}_{\text{st}}^C,$$

being a superset of the above, has order type at least ω^k. $\qquad\square$

Recall that $\mathscr{D}_{\text{st}}^a$ denotes the set of defect values $\delta(n)$ taken by stable numbers n having complexity $\|n\| \equiv a \pmod{3}$. Using Proposition 7.2, we can now prove the following:

THEOREM 7.3. *For $a = 0, 1, 2$, the stable defect sets $\mathscr{D}_{\text{st}}^a$ are well-ordered with order type ω^ω. Furthermore, if $k \equiv a \pmod{3}$, then the set $\mathscr{D}_{\text{st}}^a \cap [0, k]$ has order type ω^k.*

Proof. Each of these sets is a subset of $\mathscr{D}$, and so they are well-ordered with order type at most ω^ω. To check that it is in fact exactly ω^ω, consider the following low-defect polynomial:

$$f_{a,k} := (\cdots(((ax_1 + 1)x_2 + 1)x_3 + 1)\cdots)x_k + 1.$$

Specifically, consider the low-defect pair $(f_{a,k}, \|a\| + k)$ for $a = 2, 3, 4$. Observe that $\delta(f_{a,k}, \|a\| + k) = \delta(a) + k$, and for these choices of a, we have $\delta(a) < 1$. Thus, for $a = 2, 3, 4$, $f_{a,k}$ satisfies the conditions of Proposition 7.2. Thus, for

$a = 2, 3, 4$ and $k \geq 0$, $\mathscr{D}_{\text{st}}^{a+k}$ has order type at least ω^k. Since regardless of k, the set $\{2+k, 3+k, 4+k\}$ is a complete system of residues modulo 3, it follows that for $a = 0, 1, 2$ and any k, the set $\mathscr{D}_{\text{st}}^a$ has order type at least ω^k. Hence, $\mathscr{D}_{\text{st}}^a$ has order type at least ω^ω and hence exactly ω^ω.

Now suppose we take $k \equiv a \pmod 3$. We know that if $k \neq 1$, then $\mathscr{D}_{\text{st}}^a \cap [0, k]$ has order type at most ω^k by Theorem 6.6. (If $k = 1$, then we know this because $1 \notin \mathscr{D}_{\text{st}}$.) To see that it is at least ω^k, we consider the low-defect pair $(f_{3,k}, 3+k)$. Observe that $\delta(f_{3,k}, 3+k) = k$, and so (by Proposition 7.2) the set $\mathscr{D}_{\text{st}}^{3+k} \cap [0, k]$ has order type at least ω^k. Since $3 + k \equiv a \pmod 3$, this is the same as the set $\mathscr{D}_{\text{st}}^a \cap [0, k]$, proving the claim. $\qquad\square$

With this result in hand, we can now prove the following:

THEOREM 7.4. *We have*:
(1) *The defect set $\mathscr{D}$ and stable defect set $\mathscr{D}_{\text{st}}$ are both well-ordered, both with order type ω^ω. Furthermore, the set $\mathscr{D}_{\text{st}} \cap [0, k]$ has order type ω^k, and for $k \neq 1$, so does $\mathscr{D} \cap [0, k]$.*
(2) *The sets $\overline{\mathscr{D}_{\text{st}}}$ and $\overline{\mathscr{D}}$ are well-ordered, both with order type ω^ω. Furthermore, for $k \geq 1$, the sets $\overline{\mathscr{D}_{\text{st}}} \cap [0, k]$ and $\overline{\mathscr{D}_{\text{st}}} \cap [0, k]$ have order type $\omega^k + 1$ (and both contain k, so k is the ω^kth element of both).*
(3) *For $a = 0, 1, 2$, the sets $\mathscr{D}^a$ and $\mathscr{D}_{\text{st}}^a$ are all well-ordered, each with order type ω^ω. Furthermore, if $a \equiv k \pmod 3$, then $\mathscr{D}^a \cap [0, k]$ and $\mathscr{D}_{\text{st}}^a \cap [0, k]$ have order type ω^k.*
(4) *For $a = 0, 1, 2$, the sets $\overline{\mathscr{D}^a}$ and $\overline{\mathscr{D}_{\text{st}}^a}$ are well-ordered with order type ω^ω. Furthermore, if $k \geq 1$ and $a \equiv k \pmod 3$, then $\overline{\mathscr{D}^a} \cap [0, k]$ and $\overline{\mathscr{D}_{\text{st}}^a} \cap [0, k]$ have order type $\omega^k + 1$ (and each contains k, so k is the ω^kth element).*

Proof. The part of (1) for $\mathscr{D}$ is just Theorem 6.6. To prove the rest, observe that the order type of $\mathscr{D}_{\text{st}}$ is ω^ω because it is contained in $\mathscr{D}$ and contains, for example, $\mathscr{D}_{\text{st}}^0$. For $k \neq 1$, we can see that the order type of $\mathscr{D}_{\text{st}} \cap [0, k]$ is at most ω^k because it is contained in $\mathscr{D} \cap [0, k]$. For $k = 1$, we need to additionally note that $1 \notin \mathscr{D}_{\text{st}}$. Finally, the order type of $\mathscr{D}_{\text{st}} \cap [0, k]$ is at least ω^k because it contains $\mathscr{D}_{\text{st}}^k \cap [0, k]$.

The part of (2) for $\overline{\mathscr{D}}$ is Proposition 7.1. To prove the rest, note that by (1), $\mathscr{D}_{\text{st}}$ is unbounded in $\mathbb{R}$, and so Proposition 5.5 implies that $\overline{\mathscr{D}_{\text{st}}}$ is well-ordered with order type ω^ω. For $\overline{\mathscr{D}_{\text{st}}} \cap [0, k]$, (1) together with Proposition 5.5 implies that this has order $\omega^k + 1$. Since by Proposition 5.5, for $r < k$, the set $\overline{\mathscr{D}_{\text{st}}} \cap [0, r]$ has order type less than ω^k, the ω^kth element must be k itself.

The part of (3) for $\overline{\mathscr{D}^a}$ is just Theorem 7.3. To prove the rest, observe that the sets $\mathscr{D}^a$ are well-ordered with order type ω^ω because they contain $\mathscr{D}_{\text{st}}^a$ and are contained in $\mathscr{D}$. Furthermore, if $a \equiv k \pmod 3$, then $\mathscr{D}^a \cap [0, k]$ has order type at least ω^k by Theorem 7.3. If $k \neq 1$, then Theorem 6.6 shows that it has order type at most ω^k; for $k = 1$, we need to additionally note that $1 \notin \mathscr{D}^a$.

Finally, to prove (4), note that by Theorem 7.3 and (3), $\mathscr{D}^a$ and $\mathscr{D}_{\text{st}}^a$ are unbounded in $\mathbb{R}$, and so Proposition 5.5 implies that $\overline{\mathscr{D}_{\text{st}}^a}$ and $\overline{\mathscr{D}^a}$ are well-ordered with order type ω^ω. For $\overline{\mathscr{D}_{\text{st}}^a} \cap [0, k]$ and $\overline{\mathscr{D}^a} \cap [0, k]$, Theorem 7.3 and (3) together

with Proposition 5.5 imply that these have order type $\omega^k + 1$. Since by Proposition 5.5, for $r < k$, the sets $\overline{\mathscr{D}_{\mathrm{st}}^a} \cap [0, r]$ and $\overline{\mathscr{D}^a} \cap [0, r]$ have order type less than ω^k, the ω^kth element must be k itself. $\qquad\square$

We can also restate this result in the following way.

COROLLARY 7.5. *We have*:

(1) *For $k \geq 1$, the ω^kth elements of $\overline{\mathscr{D}}$ and $\overline{\mathscr{D}_{\mathrm{st}}}$ are both k. If $a \equiv k \pmod 3$, then this is also true for $\overline{\mathscr{D}^a}$ and $\overline{\mathscr{D}_{\mathrm{st}}^a}$.*
(2) *For $k \geq 0$, the supremum of the initial ω^k elements of $\mathscr{D}$ is k, and so is that of the initial ω^k elements of $\mathscr{D}_{\mathrm{st}}$. If $a \equiv k \pmod 3$, then this is also true for $\mathscr{D}^a$ and $\mathscr{D}_{\mathrm{st}}^a$.*

Proof. Part (1) is just Theorem 7.4. Part (2), for $k \geq 1$, is Theorem 7.4 and Corollary 5.7. For $k = 0$, this is just the observation that 0 is the initial element of $\mathscr{D}$ and so also of $\mathscr{D}_{\mathrm{st}}$, $\mathscr{D}^0$, and $\mathscr{D}_{\mathrm{st}}^0$ (since these all contain 0). $\qquad\square$

So we have now exhibited sixteen particular sets of defects that are well-ordered with order type ω^ω: $\mathscr{D}$, $\mathscr{D}_{\mathrm{st}}$, the closures of these sets, and for $a = 0, 1, 2$, the sets $\mathscr{D}^a$, $\mathscr{D}_{\mathrm{st}}^a$, and their closures. We leave it for future work to resolve which of these sets are distinct.

Appendix: Conjectures of Juan Arias de Reyna

In his paper "Complejidad de los números naturales" [3], Arias de Reyna proposed a series of conjectures about integer complexity. These conjectures also proposed a structure to integer complexity described by ordinal numbers, but using a different language. These conjectures make assertions similar in spirit to some of the stated results. We prove modified versions of his conjectures 5 through 7.

The conjectures deal with the quantity $n3^{-\lfloor \|n\|/3 \rfloor}$, which is related to (in fact, determined by) the quantity $\delta(n)$. We recall first the formula for the largest number writable with k ones, which was proved by Selfridge (see [13]).

DEFINITION A.1. Let $E(k)$ denote the largest number writable with k ones, that is, the largest number with complexity at most k.

THEOREM A.2 (Selfridge). *The number $E(k)$ is given by the following formulae*:

$$E(1) = 1,$$
$$E(3j) = 3^j,$$
$$E(3j + 2) = 2 \cdot 3^j,$$
$$E(3j + 4) = 4 \cdot 3^j.$$

Based on this, in [2], this author and Zelinsky noted the following:

PROPOSITION A.3. *We have* $\delta(1) = 1$ *and*

$$\delta(n) = \begin{cases} 3\log_3 \frac{E(\|n\|)}{n} & \text{if } \|n\| \equiv 0 \pmod 3, \\ 3\log_3 \frac{E(\|n\|)}{n} + 2\delta(2) & \text{if } \|n\| \equiv 1 \pmod 3 \text{ with } n > 1, \\ 3\log_3 \frac{E(\|n\|)}{n} + \delta(2) & \text{if } \|n\| \equiv 2 \pmod 3. \end{cases}$$

That is to say, for $n > 1$, given the congruence class of $\|n\|$ modulo 3, the quantity $nE(\|n\|)^{-1}$ is a one-to-one and order-reversing function of $\delta(n)$.

As noted before, whereas this author and Zelinsky considered $nE(\|n\|)^{-1}$, Arias de Reyna considered $n3^{-\lfloor \|n\|/3 \rfloor}$. However, this is much the same thing:

PROPOSITION A.4. *For* $k > 1$,

$$E(k) = c3^{\lfloor k/3 \rfloor},$$

where

$$c = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 3, \\ 4/3 & \text{if } k \equiv 1 \pmod 3, \\ 2 & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

So for $n > 1$, within each congruence class of $\|n\|$ modulo 3, the quantity $n3^{-\lfloor \|n\|/3 \rfloor}$ is also a one-to-one and order-reversing function of $\delta(n)$, being the same as $nE(\|n\|)^{-1}$ up to a constant factor.

This allows us to conclude the following result, which is a modified version of what one gets if one combines Arias de Reyna's Conjectures 5, 6, and 7 with his Conjectures 3 and 4.

THEOREM A.5 (Modified Arias de Reyna Conjectures 5, 6, 7). *For* $a = 0, 1, 2$, *the sets*

$$\left\{ \frac{n}{3^{\lfloor \|n\|/3 \rfloor}} : \|n\| \equiv a \pmod 3, n \text{ stable} \right\}$$

are reverse well-ordered with reverse order type ω^ω.

Equivalently, for $a = 0, 1, 2$, *so are the sets*

$$\left\{ \frac{n}{E(\|n\|)} : \|n\| \equiv a \pmod 3, n \text{ stable} \right\}.$$

Proof. By Propositions A.3 and A.4, each of these is the image of some $\mathscr{D}_{\text{st}}^a$ under an order-reversing function.   □

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