An Example Concerning Holomorphicity of Meromorphic Mappings Along Real Hypersurfaces

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ABSTRACT. We construct an example of a rational mapping F from $\mathbb{C}^2$ to $\mathbb{P}^2$ that has indeterminacies on the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ such that $F|_{\mathbb{S}^3}$ is continuous, the image $K := F(\mathbb{S}^3)$ is contained in the affine part $\mathbb{C}^2$ of $\mathbb{P}^2$, and K does not contain any germ of a nonconstant complex curve.

1. Introduction

Let $F : U \to \mathbb{P}^N$ be a meromorphic mapping from a domain $U \subset \mathbb{C}^n$ to the complex projective space. Here we always suppose that $n \geq 2$ and $N \geq 2$. Denote by I_F the set of points of indeterminacy of F, that is, $z_0 \in I_F$ if and only if F is not holomorphic in any neighborhood of z_0. As it is well known, I_F is an analytic subset of U of codimension at least two. Recall that the *full image* by F of a point z_0, denoted as $F[z_0]$, is the set of all cluster points of F at z_0, that is,

$$F[z_0] = \{x \in \mathbb{P}^N : \exists z_k \to z_0, z_k \notin I_F \text{ such that } F(z_k) \to x\}. \qquad (1.1)$$

The mapping F is holomorphic in a neighborhood of z_0 if and only if $F[z_0]$ is a singleton. Likewise, we can define the full image of z_0 by F along a closed subset M accumulating to z_0; for example, M can be a complex curve or a real hypersurface containing z_0:

$$F_M[z_0] = \{x \in \mathbb{P}^N : \exists m_k \in M \setminus I_F, m_k \to z_0 \text{ such that } F(m_k) \to x\}. \qquad (1.2)$$

If M is a complex curve, then $F_M[z_0]$ is always a singleton, regardless of whether z_0 is a regular or an indeterminacy point of F. As we shall see in our example, the same can happen if M occurs to be a real hypersurface with z_0 being an indeterminacy point of F. The proper image or transform of a closed subset $M \subset U$ under F is defined now to be the union of full images along M of its points. In other words,

$$F_M[M] = \bigcup_{m \in M} F_M[m]. \qquad (1.3)$$

If M is compact, for example, $M = \mathbb{S}^3 \subset \mathbb{C}^2 = U$, then

$$F_M[M] = \overline{F(M \setminus I_F)}. \qquad (1.4)$$

We denote as w_1, w_2 the standard coordinates in $\mathbb{C}^2$; $\mathbb{S}^3 = \{(w_1, w_2) : |w_1|^2 + |w_2|^2 = 1\}$ stands for the unit sphere in $\mathbb{C}^2$, and $\mathbb{B}^2$ for the *open* unit ball. Our goal in this note is to construct the following example.

EXAMPLE 1. There exists a rational mapping $F : \mathbb{C}^2 \to \mathbb{P}^2$ such that:

(i) Its indeterminacy set is $I_F = \{p_\pm := (0, \pm 1)\}$, its divisor of poles is $P_F = \{w_2 = \pm 1\}$, and $F_{\mathbb{S}^3}[p_\pm]$ are singletons, that is, $F|_{\mathbb{S}^3}$ is continuous.

(ii) The proper image $K := F_{\mathbb{S}^3}[\mathbb{S}^3]$ of the unit sphere $\mathbb{S}^3$ under F does not contain any germ of a nonconstant complex curve.

(iii) In addition, K is contained in an appropriate affine part $W \cong \mathbb{C}^2$ of $\mathbb{P}^2$.

Let us explain the interest in such an example. It was proved in Lemma 6.6, p. 128 of [P3], and later, independently in [C], that for a germ M of a real analytic hypersurface in $\mathbb{C}^n$ and a germ F of a meromorphic mapping near M such that $F_M[M] \subset \mathbb{S}^{2N-1}$, $2 \le n \le N$, F is holomorphic near M. It is of considerable interest and importance to replace in this result the unit sphere $\mathbb{S}^{2N-1} \subset \mathbb{C}^N$ by an arbitrary compact real analytic set $K \subset \mathbb{C}^N$ that does not contain nonconstant germs of complex curves, for instance, smooth and strictly pseudoconvex K. This problem has been under discussion among the experts, and our example shows that the before-mentioned conjecture cannot hold if one does not assume that K is smooth. See relevant Remarks 2.1 and 2.2 on these issues at the end of this note.

2. Construction of the Example

Step 1. *Construction of the mapping.* As it is well known, the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2 \subset \mathbb{P}^2$, after an appropriate projective transformation of $\mathbb{P}^2$ (see (2.3) below) in an appropriate affine chart $U_0 = \mathbb{C}^2$, can be represented in the form

$$M := \{\text{Im } z_2 = |z_1|^2\}. \tag{2.1}$$

As a rational map F in question we shall take

$$F : (z_1, z_2) \to \left(\frac{z_1^3}{z_2}, z_2 \right). \tag{2.2}$$

For the moment, it is better to consider this map as a mapping from $U_0 = \mathbb{C}^2$ to $\mathbb{P}^1 \times \mathbb{C}$. Therefore, F is holomorphic on $M \setminus \{0\}$. To see that F is continuous on the whole of M, we should remark that $F_M[0] = 0$, that is, the full image of 0 along M is a singleton. This is nearly obvious because along M we have $|z_2| \ge |z_1|^2$, and therefore along M the first function in (2.2) is $O(|z_1|)$.

Set $l := M \cap \{z_1 = 0\}$, which is a real line. We easily check that F is the identity when restricted to l and that it is locally injective on $M \setminus l$. The latter can be seen by computing the Jacobian of F, and therefore $F|_{M \setminus l} : M \setminus l \to F(M \setminus l)$ is a three-to-one cover. This means that $F(M \setminus l)$ is a strictly pseudoconvex hypersurface. Therefore, it cannot contain any germ of a complex curve. As a result, a germ of a complex curve in $F(M)$ should be contained in $F(l) = l$, which is impossible. Thus, $F(M)$ does not contain germs of complex curves.

Step 2. *Globalization.* It is easy to check that the projective transformation

$$R = \begin{cases} z_1 = \frac{w_1}{1 - w_2}, \\ z_2 = i \frac{1 + w_2}{1 - w_2} \end{cases} \tag{2.3}$$

sends the unit sphere $\mathbb{S}^3 = \{|w_1|^2 + |w_2|^2 = 1\}$ to our model M. Writing F as a mapping to $\mathbb{P}^2$, that is, as

$$F : (z_1, z_2) \to [z_1^3 : z_2^2 : z_2], \tag{2.4}$$

and composing it with R, we get the following expression of our map in coordinates w_1, w_2:

$$F : (w_1, w_2) \to [w_1^3 : -(1 + w_2)^2(1 - w_2) : i(1 + w_2)(1 - w_2)^2]. \tag{2.5}$$

We refer to (2.5) as to the form of the mapping F in coordinates (w_1, w_2). This is the mapping of our example. Now let us prove the assertions (i)–(iii) as they are stated in the Introduction.

(i) It is easy to see that it has two points of indeterminacy $p_\pm = (0, \pm 1)$ and that it behaves at these points similarly. The behavior at p_- was already studied, and the behavior at p_+ is the same. The divisor of poles is $\{w_2 = \pm 1\}$, as required.

(ii) $K := F(\mathbb{S}^2)$ cannot contain any germ of a nonconstant complex curve. Indeed, let $C \subset K$ be such a germ. Then C should intersect an affine neighborhood either of p_+ or of p_-, and this was proved to be not the case.

(iii) To prove that K is situated in an appropriate affine part of $\mathbb{P}^2$, consider in homogeneous coordinates $[\zeta_0 : \zeta_1 : \zeta_2]$ the line

$$L := \{\zeta_2 - i\zeta_1 = 0\}. \tag{2.6}$$

Putting $F(w)$ into this equation, we get

$$\zeta_2 - i\zeta_1 = i(1 + w_2)(1 - w_2)^2 + i(1 + w_2)^2(1 - w_2) = 2i(1 - w_2^2). \tag{2.7}$$

The latter is not zero on $\mathbb{S}^3$ if and only if $w_2 \neq \pm 1$, that is, if w is not $p_\pm$. But at p_- along $\mathbb{S}^3$ our map takes the value $[0 : 0 : 1] \notin L$ and at p_+ the value $[0 : 1 : 0] \notin L$. Therefore, $F_{\mathbb{S}^3}[\mathbb{S}^3] \subset \mathbb{P}^2 \setminus L =: W \cong \mathbb{C}^2$. Now we can make the coordinate change

$$\begin{cases} \xi_0 = \zeta_0, \\ \xi_1 = \zeta_1, \\ \xi_2 = \zeta_2 - i\zeta_1 \end{cases} \tag{2.8}$$

in $\mathbb{P}^2$ and write our map in the form

$$F : (w_1, w_2) \to [w_1^3 : -(1 + w_2)(1 - w_2^2) : 2i(1 - w_2^2)] \tag{2.9}$$

or, in the affine charts,

$$F : (w_1, w_2) \to \left(-\frac{i}{2} \frac{w_1^3}{1 - w_2^2}, \frac{i}{2}(1 + w_2) \right). \tag{2.10}$$

This is the final form of F as a mapping from $\mathbb{C}^2 \subset \mathbb{P}^2$ to $\mathbb{C}^2 \subset \mathbb{P}^2$, that is, $K \subset \mathbb{C}^2$ if F is given in the form (2.10).

Let us make a few additional remarks about our example.

REMARK 2.1.

1. The image $K = F(\mathbb{S}^2)$ in our example is not smooth. Indeed, it is given by the equation

$$(\text{Im } z_2)^3 = |z_1 z_2|^2. \tag{2.11}$$

It is easy to see that the set of singular points of (2.11) is exactly our line $l = F(M) \cap \{z_1 = 0\}$ considered before. Outside of l, our K is a spherical hypersurface.

2. The nonsmoothness of K can be overcome at the cost of increasing the dimension of the image. Our example written as a map $\mathbb{C}^2 \to \mathbb{C}^3$, that is, as

$$F : (z_1, z_2) \to \left(\frac{z_1^3}{z_2}, z_2, 0 \right), \tag{2.12}$$

takes its values in the smooth algebraic hypersurface

$$K = \{y_3 + y_2^3 - |z_1 z_2|^2 = 0\}. \tag{2.13}$$

However, this hypersurface contains the complex curve $C = \{z_2 = z_3 = 0\}$.

REMARK 2.2. In equal dimensions with strictly pseudoconvex both M and K, such an example does not exist. Namely, the following is true.

PROPOSITION 2.1. *Let $(M, 0)$ be a germ of strictly pseudoconvex hypersurface in $\mathbb{C}^n$, $n \geq 2$, and let K be a compact strictly pseudoconvex hypersurface in $\mathbb{C}^n$. Let $F : V \to \mathbb{C}^n$ be a meromorphic mapping from some neighborhood of the origin $V \subset \mathbb{C}^n$ to $\mathbb{C}^n$ such that $F(M) \subset K$. Then F is holomorphic in a neighborhood of the origin.*

Proof. If F is constant, then there is nothing to prove. If not, then it is locally biholomorphic at every point of $M \setminus I_F$. Now we have two cases.

Case 1. M is spherical. Then K is spherical as well. Set $p := F(0)$ and let $j : (K, p) \to (\mathbb{S}^{2n-1}, 0)$ be a germ of a biholomorphic mapping. By the result of [P1] germ j extends locally biholomorphically along any path in K. Take some point $a \in M \setminus I_F$ and set $b := F(a)$. Take a path γ in M from 0 and a. Set $\beta := F(\gamma)$; it is a path from p to b in K. The composition $j \circ F$ extends along γ^{-1} locally biholomorphically from a to 0 by the mentioned result of [P1]. Since j extends along β and therefore j^{-1} extends along $j(\beta^{-1})$, we can take the composition $j^{-1} \circ (j \circ F)$ and extend it from a to 0 to obtain the holomorphicity of F at the origin.

Case 2. M is not spherical. Then K is not spherical as well. We can then apply another theorem of Pinchuk [P2]. Take any Moser chain through 0 on M that is not contained entirely in $M \cap I_F$, say a path $\gamma : [0, 1] \to M$ such that $\gamma(0) \notin I_F$ and $\gamma(1) = 0$. Extend F along γ by [P2] to conclude that F is holomorphic at zero. $\square$

References

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