Michigan Math. J. 64 (2015), 451–473

On Chow Quotients of Torus Actions

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Abstract. We consider torus actions on Mori dream spaces and ask whether the associated Chow quotient is again a Mori dream space and, if so, what does its Cox ring look like. We provide general tools for the study of these problems and give solutions for $\mathbb{K}^*$-actions on smooth quadrics.

1. Introduction

Consider an action $G \times X \to X$ of a connected linear algebraic group G on a projective variety X defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. The Chow quotient is an answer to the problem of associating in a canonical way a quotient to this action: it is defined as the closure of the set of general G-orbit closures viewed as points in the Chow variety; see Section 2 for more background. The Chow quotient always exists, but, in general, its geometry appears to be not easily accessible.

In the present paper, we consider algebraic torus actions $T \times X \to X$ and ask for the Mori dream property of the normalized Chow quotient Y, provided that X is a Mori dream space, that is, has finitely generated Cox ring [14]. A well understood example class is given by subtorus actions on toric varieties. There, the normalized Chow quotient is again toric and hence a Mori dream space. Moreover, the corresponding fan can be computed, and thus the Cox ring of the normalized Chow quotient is accessible as well [16; 6]. Note, however, that there is no hope for comparable statements in general. For example, Castravet and Tevelev [5] showed that the Chow quotient $\overline{M}_{0,n}$ of the maximal torus action on the Grassmannian $G(2,n)$ is not a Mori dream space for n sufficiently large.

Our aim is to provide tools for the treatment of nontoric examples and to open up the case of $\mathbb{K}^*$-actions on smooth projective quadrics as a new example class for positive results. The first main result is the following.

Theorem 1.1. *Let $\mathbb{K}^*$ act on a smooth projective quadric X. Then the associated normalized Chow quotient is a Mori dream space.*

The second result concerns the computation of the Cox ring; recall that the explicit knowledge of the Cox ring is an approach to the geometry of the underlying space [1]. We first prepare and state the result and then discuss the setting. After an equivariant embedding into a projective space and applying a suitable linear

Received August 23, 2013. Revision received May 19, 2015.

transformation, any smooth projective quadric X is of the following shape:

$$X = V(g_1) \subseteq \mathbb{P}_r, \quad g_1 = \begin{cases} T_0T_1 + \cdots + T_{r-1}T_r, & r \text{ odd}, \\ T_0T_1 + \cdots + T_{r-2}T_{r-1} + T_r^2, & r \text{ even}, \end{cases}$$

where the $\mathbb{K}^*$-action is diagonal with weights $\zeta_0, \dots, \zeta_r$, and the defining equation is of degree zero. In order to write down the Cox ring of the Chow quotient, consider the extended weight matrix

$$Q := \begin{bmatrix} \zeta_0 \cdots \zeta_r \\ 1 \ \dots \ 1 \end{bmatrix},$$

where we assume that the columns of Q generate $\mathbb{Z}^2$. Let P be an integral Gale dual, that is, an $r-1$ by $r+1$ matrix with the row space of Q as kernel. Determine the Gelfand–Kapranov–Zelevinsky decomposition Σ associated to P and put the primitive generators $b_1, \dots, b_l$ of Σ differing from the columns of P as columns into a matrix B. Then there is an integral matrix A such that $B = P \cdot A$. Define the shifted row sums

$$\eta_i := A_{i*} + A_{i+1*} + \mu \quad \text{for } i = 0, 2, \dots \quad \text{and} \quad \eta_r := 2A_{r*} + \mu \quad \text{if } r \text{ is even},$$

where μ is the componentwise minimal vector such that the entries of the η_i are all nonnegative. Then our result reads as follows.

Theorem 1.2. *In the previous setting, assume that any r columns of Q generate $\mathbb{Z}^2$, there remain at least two different weights ζ_i when removing two of maximal absolute value, and for odd (even) r, there are at least four (three) ζ_i of minimal absolute value. Then the normalized Chow quotient Y of the $\mathbb{K}^*$-action on X has the Cox ring*

$$\mathcal{R}(Y) = \mathbb{K}[T_0, \dots, T_r, S_1, \dots, S_l]/\langle g_2 \rangle$$

with

$$g_2 := \begin{cases} T_0T_1S^{\eta_0} + T_2T_3S^{\eta_2} + \cdots + T_{r-1}T_rS^{\eta_{r-1}}, & r \text{ odd}, \\ T_0T_1S^{\eta_0} + \cdots + T_{r-2}T_{r-1}S^{\eta_{r-2}} + T_r^2S^{\eta_r}, & r \text{ even}, \end{cases}$$

graded by $\mathbb{Z}^{l+2}$ via assigning to the ith variable the ith column of a Gale dual of the block matrix $[P, B]$.

Let us shed some light on this setting. The assumptions that any r columns of Q generate $\mathbb{Z}^2$ and there remain at least two different weights ζ_i when removing two of maximal absolute value mean exactly that Q defines a Cox ring of a projective "intrinsic quadric" Y' in the sense of [3] with divisor class group $\mathbb{Z}^2$; see [4] for other research on such varieties. The meaning of the Chow quotient Y computed in Theorem 1.2 is that it dominates in a minimal manner all normal projective varieties Y'' allowing a small quasi-modification $Y'' \to Y'$; see also Remark 4.6. A discussion of the assumption that, for odd (even) r, there are at least four (three) ζ_i of minimal absolute value is given in Remark 4.7.

The proof of Theorem 1.2 is performed in Section 4. Besides the explicit description of the rays of the Gelfand–Kapranov–Zelevinsky decomposition provided in Proposition 4.1, it requires controlling the behavior of the Cox ring under certain modifications. This technique is of independent interest and developed in full generality in Section 3. The proof of Theorem 1.1, given in Section 5, uses moreover methods from tropical geometry: we consider a "weak tropical resolution" of the Chow quotient (see Construction 5.3) and provide a reduction principle to divide out intrinsic torus symmetry (see Proposition 5.6).

2. Chow Quotients and Limit Quotients

We present the necessary background and a general result for the action of a torus T on a projective (irreducible) variety X. For a precise definition of the quotients, consider more generally the action $G \times X \to X$ of any connected linear algebraic group G on a projective variety X. The Chow quotient has been introduced by Kapranov, Sturmfels, and Zelevinsky [16]. Initially, the construction appears to depend on an embedding but finally turns out not to do so.

Construction 2.1. Suppose that X is a G-invariant closed subvariety of some projective space. For a suitable open invariant subset $U \subseteq X$, all orbit closures $c(x) := \overline{G \cdot x}$, where $x \in U$, have the same dimension k and degree d. Thus, each $x \in U$ defines a point $c(x) \in \mathrm{Ch}(X)$ in the Chow variety of k-cycles of degree d. The *Chow quotient* of the G-action on X is the closure

$$X /\!\!/_{\mathrm{CQ}} G := \overline{\{c(x);\, x \in U\}} \subseteq \mathrm{Ch}(X).$$

By the *normalized Chow quotient* we mean the normalization $X \,\tilde{/\!\!/}_{\mathrm{CQ}}\, G$ of $X /\!\!/_{\mathrm{CQ}} G$. With a suitably small chosen $U \subseteq X$, we obtain a commutative diagram of morphisms involving the normalization map:

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] & \\ X \,\tilde{/\!\!/}_{\mathrm{CQ}}\, G \arrow[rr] & & X /\!\!/_{\mathrm{CQ}} G \end{tikzcd}$$

The limit quotient arises from the variation of Mumford's GIT quotients [19]. Its construction relies on finiteness of the number of possible sets of semistable points [8; 21].

Construction 2.2. Suppose that G is reductive. Let $X_1, \dots, X_r \subseteq X$ be open sets of semistable points arising from G-linearized ample line bundles on X. Then, whenever $X_i \subseteq X_j$, we have a commutative diagram

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & X_j \arrow[d] \\ X_i /\!\!/ G \arrow[r, "\varphi_{ij}"'] & X_j /\!\!/ G \end{tikzcd}$$

where the induced map $\varphi_{ij}\colon X_i /\!\!/ G \to X_j /\!\!/ G$ of quotients is a dominant projective morphism. This turns the quotient spaces into a directed system, the *GIT system*. The associated *GIT limit* Y, that is, the inverse limit, comes with a canonical morphism

$$U := \bigcap_{i=1}^{r} X_i \to Y.$$

The closure of the image of this morphism is denoted by $X /_{\mathrm{LQ}} G$ and is called the *limit quotient*. There are canonical proper birational morphisms onto the GIT quotients:

$$\pi_i\colon X /_{\mathrm{LQ}} G \to X_i /\!\!/ G.$$

The *normalized limit quotient* is the normalization $X \tilde{/}_{\mathrm{LQ}} G$ of $X /_{\mathrm{LQ}} G$. Suitably shrinking the open set $U \subseteq X$, we obtain a commutative diagram involving the normalization map:

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] & \\ X \tilde{/}_{\mathrm{LQ}} G \arrow[rr] & & X /_{\mathrm{LQ}} G \end{tikzcd}$$

Note that, in the literature, $X /_{\mathrm{LQ}} G$ is also called the "canonical component" of the GIT limit, or even shortly the "GIT limit". Similar to the full inverse limit, the quotient $X /_{\mathrm{LQ}} G$ enjoys a universal property.

Remark 2.3. Given an irreducible variety W and a collection of dominant morphisms $\psi_i\colon W \to X_i /\!\!/ G$ with $\psi_j = \varphi_{ij} \circ \psi_i$ for all i, j, there is a unique morphism $\psi\colon W \to X /_{\mathrm{LQ}} G$ with $\psi_i = \pi_i \circ \psi$ for all i.

For a general reductive group action, the (normalized) Chow quotient and the (normalized) limit quotient need not coincide. For torus actions, however, they do. This statement seems to have folklore status; a proof under a certain hypothesis can be found in [13, Thm. 3.8]. Let us indicate how to deduce it from the corresponding statement in the case of subtorus actions on projective toric varieties obtained in [16; 6].

We recall the necessary results and concepts from [16; 6]. Let Z be a projective toric variety with acting torus T_Z and consider the action of a subtorus $T \subseteq T_Z$. The toric variety Z arises from a fan Σ in some $\mathbb{Z}^r$, and $T \subseteq T_Z$ corresponds to an embedding $\mathbb{Z}^k \subseteq \mathbb{Z}^r$ of a sublattice. Let $P\colon \mathbb{Z}^r \to \mathbb{Z}^{r-k}$ be the projection. The *quotient fan* of Σ with respect to P is the fan in $\mathbb{Z}^{r-k}$ with the cones

$$\tau(v) := \bigcap_{\sigma \in \Sigma, v \in P(\sigma)} P(\sigma), \qquad v \in \mathbb{Q}^{r-k}.$$

Proposition 2.4. *See* [16; 6]. *Consider the toric variety Z arising from a fan Σ in $\mathbb{Z}^r$ and the action of a subtorus $T \subseteq T_Z$ corresponding to a sublattice $\mathbb{Z}^k \subseteq \mathbb{Z}^r$. Let Σ' be the quotient fan in $\mathbb{Z}^{r-k}$ with respect to $\mathbb{Z}^r \to \mathbb{Z}^{r-k}$, and Z' the* *associated toric variety. Then we have a commutative diagram*

$$\begin{tikzcd}
 & T_Z/T \arrow[dl] \arrow[d] \arrow[dr] & \\
Z \tilde{/}_{\mathrm{CQ}} T \arrow[r, leftrightarrow, "\cong"'] \arrow[d] & Z' \arrow[r, leftrightarrow, "\cong"'] & Z \tilde{/}_{\mathrm{LQ}} T \arrow[d] \\
Z /_{\mathrm{CQ}} \arrow[rr, leftrightarrow, "\cong"'] & & Z /_{\mathrm{LQ}} T
\end{tikzcd}$$

In particular, the (*normalized*) *Chow quotient and the* (*normalized*) *limit quotient of the T-action on Z are isomorphic to each other.*

We turn to the general case. The result is formulated for a projective variety X that is equivariantly embedded into a toric variety Z. Note that for a normal projective X, this can always be achieved, even with a projective space Z.

Proposition 2.5. *Let Z be a projective toric variety, $T \subseteq T_Z$ a subtorus of the big torus, and $X \subseteq Z$ a closed T-invariant subvariety intersecting T_Z. Then there is a commutative diagram*

$$\begin{tikzcd}
 & X /_{\mathrm{CQ}} T \arrow[r, "\text{embedding}"] \arrow[dddd, leftrightarrow, bend right=70, "\cong"'] & Z /_{\mathrm{CQ}} T \arrow[dddd, leftrightarrow, bend left=70, "\cong"] & \\
 & X \tilde{/}_{\mathrm{CQ}} T \arrow[u] \arrow[r, "\text{finite}"] \arrow[dd, equal] & Z \tilde{/}_{\mathrm{CQ}} T \arrow[u] \arrow[dd, equal] & \\
(X \cap T_Z)/T \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[rrr] & & & T_Z/T \arrow[uul] \arrow[ul] \arrow[dl] \arrow[ddl] \\
 & X \tilde{/}_{\mathrm{LQ}} T \arrow[d] \arrow[r, "\text{finite}"'] & Z \tilde{/}_{\mathrm{LQ}} T \arrow[d] & \\
 & X /_{\mathrm{LQ}} T \arrow[r, "\text{embedding}"'] & Z /_{\mathrm{LQ}} T &
\end{tikzcd}$$

where $X \tilde{/}_{\mathrm{CQ}} T \to Z \tilde{/}_{\mathrm{CQ}} T$ and $X \tilde{/}_{\mathrm{LQ}} T \to Z \tilde{/}_{\mathrm{LQ}} T$ normalize the closures of the images of $(X \cap T_Z)/T$ under the canonical open embeddings of T_Z/T.

Proof. The right part of the diagram is Proposition 2.4. The closed embedding $X /_{\mathrm{CQ}} T \to Z /_{\mathrm{CQ}} T$ exists by the construction of the Chow quotient; compare also [9, Thm. 3.2].

To obtain a morphism $X /_{\mathrm{LQ}} T \to Z /_{\mathrm{LQ}} T$, consider the sets of semistable points $V_1, \ldots, V_s \subseteq Z \tilde{/}_{\mathrm{CQ}} T$ defined by T-linearized ample line bundles on Z. Then the sets $U_i := X \cap V_i$ are sets of semistable points of the respective pullback bundles, see [19, Thm. 1.19], and we have induced morphisms $U_i /\!\!/ T \to V_i /\!\!/ T$. Since the $U_i /\!\!/ T$ form a subsystem of the full GIT-system of X, the universal property (Remark 2.3) yields a morphism of the limit quotients sending $X /_{\mathrm{LQ}} T$ birationally onto the closure of $(X \cap T_Z)/T$.

Now look at the canonical morphism $X /_{\mathrm{CQ}} T \to X /_{\mathrm{LQ}} T$ provided by [15; 21]. It fits into the diagram established so far, which in turn implies that $X /_{\mathrm{CQ}} T \to X /_{\mathrm{LQ}} T$

is an isomorphism and $X /\!\!/_{\mathrm{LQ}} T \to Z /\!\!/_{\mathrm{LQ}} T$ is an embedding. Finally, the respective normalizations fit into the diagram via their universal properties. □

Note that we will only use the part of Proposition 2.5 concerning the normalizations. This can be proved by similar arguments as before but without using the isomorphism $Z /\!\!/_{\mathrm{CQ}} T \to Z /\!\!/_{\mathrm{LQ}} T$ of Proposition 2.4.

Corollary 2.6. *Let $T \times X \to X$ be the action of a torus T on a normal projective variety X. Then the normalized Chow quotient $X \,\tilde{/\!\!/}_{\mathrm{CQ}}\, T$ and the normalized limit quotient $X \,\tilde{/\!\!/}_{\mathrm{LQ}}\, T$ are isomorphic to each other.*

The following corollary shows that for torus actions, the limit quotient is up to normalization already determined by the possible linearizations of a single ample bundle, a statement that fails in general for other reductive groups; compare also [15, Rem. 0.4.10].

Corollary 2.7. *Let $T \times X \to X$ be the action of a torus T on a normal projective variety X. Then the subsystem of GIT quotients arising from the possible T-linearizations of a given ample line bundle $\mathcal{L}$ has the same normalized limit quotient as the full system of GIT quotients.*

Proof. Fix a T-linearization of $\mathcal{L}$ and consider the T-equivariant embedding $X \to \mathbb{P}_r$ defined by a suitable power of $\mathcal{L}$. Then the subsystem of the GIT quotients on X arising from other linearizations of $\mathcal{L}$ is induced from the full GIT system on $\mathbb{P}_r$. Now apply Proposition 2.5. □

We now prove a reduction theorem, which says in particular that the Chow quotient of a torus action is birationally dominated by an iterated Chow quotient with respect to $\mathbb{K}^*$-actions.

Theorem 2.8. *Let $T \times X \to X$ be the action of a torus T on a normal projective variety X. Fix a subtorus $T_0 \subseteq T$ and set $T_1 := T/T_0$. Then we have canonical proper birational morphisms*

$$(X \,\tilde{/\!\!/}_{\mathrm{CQ}}\, T_0) \,\tilde{/\!\!/}_{\mathrm{CQ}}\, T_1 \to X \,\tilde{/\!\!/}_{\mathrm{CQ}}\, T, \qquad (X \,\tilde{/\!\!/}_{\mathrm{LQ}}\, T_0) \,\tilde{/\!\!/}_{\mathrm{LQ}}\, T_1 \to X \,\tilde{/\!\!/}_{\mathrm{LQ}}\, T.$$

Proof. First, consider the case that T is a subtorus of the big torus T_Z of a toric variety Z. Then the maps $T_Z \to T_Z/T_0 \to T_Z/T$ correspond to lattice homomorphisms $\mathbb{Z}^r \to \mathbb{Z}^{r-k_0} \to \mathbb{Z}^{r-k}$. The fan Σ of Z lives in $\mathbb{Z}^r$, and we have the quotient fan Σ_0 of Σ with respect to $\mathbb{Z}^r \to \mathbb{Z}^{r-k_0}$. The quotient fan of Σ_0 with respect to $\mathbb{Z}^{r-k_0} \to \mathbb{Z}^{r-k}$ refines the quotient fan of Σ with respect to $\mathbb{Z}^r \to \mathbb{Z}^{r-k}$. Translated to toric varieties, this means that we have the desired maps

$$(Z \,\tilde{/\!\!/}_{\mathrm{CQ}}\, T_0) \,\tilde{/\!\!/}_{\mathrm{CQ}}\, T_1 \to Z \,\tilde{/\!\!/}_{\mathrm{CQ}}\, T, \qquad (Z \,\tilde{/\!\!/}_{\mathrm{LQ}}\, T_0) \,\tilde{/\!\!/}_{\mathrm{LQ}}\, T_1 \to Z \,\tilde{/\!\!/}_{\mathrm{LQ}}\, T.$$

We turn to the general case. Suitably embedding X, we can arrange the setup of Proposition 2.5. Then we have a finite T_1-equivariant map $\nu\colon X \,\tilde{/\!\!/}_{\mathrm{CQ}}\, T_0 \to Z \,\tilde{/\!\!/}_{\mathrm{CQ}}\, T_0$. We consider the normalized limit quotient of the T_1-action on $X \,\tilde{/\!\!/}_{\mathrm{CQ}}\, T_0$. In a first step, we establish a commutative diagram

$$\begin{tikzcd}
 & (X \cap T_Z)/T_0 \arrow[dl] \arrow[dr] & \\
(X \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_0) /\!\!/_{\mathrm{LQ}} T_1 \arrow[rr] & & (Z \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_0) /\!\!/_{\mathrm{LQ}} T_1
\end{tikzcd}$$

For this, let $V_1, \ldots, V_s \subseteq Z \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_0$ be the sets of semistable points arising from T_1-linearized ample line bundles. Then the inverse images $\nu^{-1}(V_i) \subseteq X \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_0$ are sets of semistable points of the respective pullback bundles, see [19, Thm. 1.19]. Note that we have canonical induced maps

$$\nu^{-1}(V_i) /\!\!/ T_1 \to V_i /\!\!/ T_1.$$

Consequently, the limit quotient of the system of the quotients $\nu^{-1}(V_i) /\!\!/ T_1$ maps to the limit quotient $(Z \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_0) /\!\!/_{\mathrm{LQ}} T_1$. Since the $\nu^{-1}(V_i) /\!\!/ T_1$ form a subsystem of the full GIT system of $X \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_0$, this gives rise to a morphism

$$(X \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_0) /\!\!/_{\mathrm{LQ}} T_1 \to (Z \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_0) /\!\!/_{\mathrm{LQ}} T_1$$

as needed for the previous commutative diagram. As in the proof of Proposition 2.5, we may pass to the normalizations and thus obtain a morphism

$$(X \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_0) \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_1 \to (Z \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_0) \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T_1.$$

Now, by the toric case, we have a proper birational morphism from the toric variety on the right-hand side onto $Z \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} T$. Using once more Proposition 2.5, the assertion follows. □

3. Toric Ambient Modifications

In this section, we provide a general machinery to study the effect of modifications on the Cox ring. Similar to [17], we use toric embeddings. In contrast to the geometric criteria given there, our approach here is purely algebraic, based on the results of [2]. The heart is a construction of factorially graded rings out of given ones.

We begin with recalling the necessary algebraic concepts. Let K be a finitely generated Abelian group, and R a finitely generated integral K-graded $\mathbb{K}$-algebra. A homogeneous nonzero nonunit $f \in R$ is called *K-prime* if $f \mid gh$ with homogeneous $g, h \in R$ always implies $f \mid g$ or $f \mid h$. The algebra R is called *factorially K-graded* if every homogeneous nonzero nonunit $f \in R$ is a product of K-primes.

We enter the construction of factorially graded rings. Consider a grading of the polynomial ring $\mathbb{K}[T_1, \ldots, T_{r_1}]$ by a finitely generated Abelian group K_1 such that the variables T_i are homogeneous. Then we have a pair of exact sequences

$$0 \longrightarrow \mathbb{Z}^{k_1} \xrightarrow{Q_1^*} \mathbb{Z}^{r_1} \xrightarrow{P_1} \mathbb{Z}^n$$

$$0 \longleftarrow K_1 \xleftarrow[Q_1]{} \mathbb{Z}^{r_1} \xleftarrow[P_1^*]{} \mathbb{Z}^n \longleftarrow 0$$

where $Q_1 \colon \mathbb{Z}^{r_1} \to K_1$ is the degree map sending the ith canonical basis vector e_i to $\deg(T_i) \in K_1$. We enlarge P_1 to an $n \times r_2$ matrix P_2 by concatenating further $r_2 - r_1$ columns. This gives a new pair of exact sequences

$$0 \longrightarrow \mathbb{Z}^{k_2} \xrightarrow{Q_2^*} \mathbb{Z}^{r_2} \xrightarrow{P_2} \mathbb{Z}^n$$

$$0 \longleftarrow K_2 \xleftarrow[Q_2]{} \mathbb{Z}^{r_2} \xleftarrow[P_2^*]{} \mathbb{Z}^n \longleftarrow 0$$

Construction 3.1. Given a K_1-homogeneous ideal $I_1 \subseteq \mathbb{K}[T_1, \ldots, T_{r_1}]$, we transfer it to a K_2-homogeneous ideal $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{r_2}]$ by taking extensions and contractions according to the scheme

$$\begin{tikzcd}
\mathbb{K}[T_1, \ldots, T_{r_2}] \arrow[d, "\imath_2"'] & & \mathbb{K}[T_1, \ldots, T_{r_1}] \arrow[d, "\imath_1"] \\
\mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}] & \mathbb{K}[S_1^{\pm 1}, \ldots, S_n^{\pm 1}] \arrow[l, "\pi_2^*"] \arrow[r, "\pi_1^*"'] & \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}]
\end{tikzcd}$$

where $\imath_1, \imath_2$ are the canonical embeddings, and π_i^* are the homomorphisms of group algebras defined by $P_i^* \colon \mathbb{Z}^n \to \mathbb{Z}^{r_i}$.

Now let $I_1 \subseteq \mathbb{K}[T_1, \ldots, T_{r_1}]$ be a K_1-homogeneous ideal, and $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{r_2}]$ the transferred K_2-homogeneous ideal. Our result relates factoriality properties of the algebras $R_1 := \mathbb{K}[T_1, \ldots, T_{r_1}]/I_1$ and $R_2 := \mathbb{K}[T_1, \ldots, T_{r_2}]/I_2$ to each other.

Theorem 3.2. *Assume that R_1, R_2 are integral, $T_1, \ldots, T_{r_1}$ define K_1-primes in R_1, and $T_1, \ldots, T_{r_2}$ define K_2-primes in R_2. Then the following statements are equivalent*:

(i) *The algebra R_1 is factorially K_1-graded.*
(ii) *The algebra R_2 is factorially K_2-graded.*

Proof. First, observe that the homomorphisms π_j^* embed $\mathbb{K}[S_1^{\pm 1}, \dots, S_n^{\pm 1}]$ as the degree zero part of the respective K_j-grading and fit into a commutative diagram

$$\begin{tikzcd}
I_2 \arrow[d] & \subseteq & \mathbb{K}[T_1, \dots, T_{r_2}] \arrow[d, "\iota_2"'] & & \mathbb{K}[T_1, \dots, T_{r_1}] \arrow[d, "\iota_1"] & \supseteq & I_1 \arrow[d] \\
I_2' & \subseteq & \mathbb{K}[T_1^{\pm 1}, \dots, T_{r_2}^{\pm 1}] \arrow[rr, "\psi : T_i \mapsto \begin{cases} T_i, & 1 \le i \le r_1, \\ 1, & r_1+1 \le i \le r_2 \end{cases}"] & & \mathbb{K}[T_1^{\pm 1}, \dots, T_{r_1}^{\pm 1}] & \supseteq & I_1' \\
& & I_2'' \arrow[ull] \arrow[u] & \mathbb{K}[S_1^{\pm 1}, \dots, S_n^{\pm 1}] \arrow[ul, "\pi_2^*"'] \arrow[ur, "\pi_1^*"] & I_1'' \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

The factor ring R_1' of the extension $I_1' := \langle \iota_1(I_1) \rangle$ is obtained from R_1 by localization with respect to K_1-primes $T_1, \dots, T_{r_1}$:

$$R_1' := \mathbb{K}[T_1^{\pm 1}, \dots, T_{r_1}^{\pm 1}]/I_1' \cong (R_1)_{T_1 \cdots T_{r_1}}.$$

The ideal I_1'' is the degree zero part of I_1'. Thus, its factor algebra is the degree zero part of R_1':

$$R_1'' := \mathbb{K}[T_1^{\pm 1}, \dots, T_{r_1}^{\pm 1}]_0 / I_1'' \cong (R_1')_0.$$

Note that $\mathbb{K}[T_1^{\pm 1}, \dots, T_{r_1}^{\pm 1}]$ and hence R_1' admit units in every degree. Thus, [2, Thm. 1.1] yields that R_1 is a factorially K_1-graded if and only if R_1'' is a UFD.

The homomorphism ψ restricts to an isomorphism ψ_0 of the respective degree zero parts. Thus, the shifted ideal $I_2'' := \psi_0^{-1}(I_1'')$ defines an algebra R_2'' isomorphic to R_1'':

$$R_2'' := \mathbb{K}[T_1^{\pm 1}, \dots, T_{r_2}^{\pm 1}]_0 / I_2'' \cong R_1''.$$

The ideal $I_2' := \langle \pi_2^*((\pi_1^*)^{-1}(I_1')) \rangle$ has I_2'' as its degree zero part, and $\mathbb{K}[T_1^{\pm 1}, \dots, T_{r_2}^{\pm 1}]$ admits units in every degree. The associated K_2-graded algebra

$$R_2' := \mathbb{K}[T_1^{\pm 1}, \dots, T_{r_2}^{\pm 1}]/I_2'$$

is the localization of R_2 by the K_2-primes $T_1, \dots, T_{r_2}$. Again by [2, Thm. 1.1] we obtain that R_2'' is a UFD if and only if R_2 is factorially K_2-graded. $\square$

The following observation is intended for practical purposes; it reduces, for example, the number of necessary primality tests.

PROPOSITION 3.3. *Assume that R_1 is integral and the canonical map $K_2 \to K_1$ admits a section (e.g., K_1 is free).*

(i) *Let $T_1, \dots, T_{r_1}$ define K_1-primes in R_1 and $T_{r_1+1}, \dots, T_{r_2}$ define K_2-primes in R_2. If no T_j with $j \ge r_1 + 1$ divides a T_i with $i \le r_1$, then also $T_1, \dots, T_{r_1}$ define K_2-primes in R_2.*

(ii) *The ring R_2 is integral. Moreover, if R_1 is normal and $T_{r_1+1}, \dots, T_{r_2}$ define primes in R_2 (e.g., they are K_2-prime, and K_2 is free), then R_2 is normal.*

Proof. The exact sequences involving the grading groups K_1 and K_2 fit into a commutative diagram where the upwards sequences are exact and $\mathbb{Z}^{r_2-r_1} \to K_2'$ is an isomorphism:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 & K_1 \arrow[l] \arrow[u] & \mathbb{Z}^{r_1} \arrow[l, "Q_1"'] \arrow[u] & \mathbb{Z}^{n} \arrow[l, "P_1^*"'] \arrow[u] & 0 \arrow[l] \\
0 & K_2 \arrow[l] \arrow[u] & \mathbb{Z}^{r_2} \arrow[l, "Q_2"] \arrow[u] & \mathbb{Z}^{n} \arrow[l, "P_2^*"] \arrow[u] & 0 \arrow[l] \\
 & K_2' \arrow[u] & \mathbb{Z}^{r_2-r_1} \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Moreover, denoting by $K_1' \subseteq K_2$ the image of the section $K_1 \to K_2$, there is a splitting $K_2 = K_2' \oplus K_1'$. Since $K_2' \subseteq K_2$ is the subgroup generated by the degrees of $T_{r_1+1}, \ldots, T_{r_2}$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{K}[T_1, \ldots, T_{r_2}] \arrow[d, "\imath_2"'] & \\
\mathbb{K}[T_1, \ldots, T_{r_1}, T_{r_1+1}^{\pm 1}, \ldots, T_{r_2}^{\pm 1}] \arrow[r, "{\psi\colon T_i \mapsto \begin{cases} T_i, & 1 \le i \le r_1, \\ 1, & r_1+1 \le i \le r_2 \end{cases}}"] & \mathbb{K}[T_1, \ldots, T_{r_1}] \\
\mathbb{K}[T_1, \ldots, T_{r_1}, T_{r_1+1}^{\pm 1}, \ldots, T_{r_2}^{\pm 1}]_0 \arrow[u, "\mu"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

where the map μ denotes the embedding of the degree zero part with respect to the K_2'-grading. By the splitting $K_2 = K_2' \oplus K_1'$ the image of μ is precisely the Veronese subalgebra associated with the subgroup $K_1' \subseteq K_2$. For the factor rings R_2 and R_1 by the ideals I_2 and I_1, the previous diagram leads to the following situation:

$$\begin{tikzcd}
R_2 \arrow[d, "\imath_2"'] & \\
(R_2)_{T_{r_1+1} \cdots T_{r_2}} \arrow[r, "\psi"] & R_1 \\
((R_2)_{T_{r_1+1} \cdots T_{r_2}})_0 \arrow[u, "\mu"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

To prove (i), consider a variable T_i with $1 \le i \le r_1$. We have to show that T_i defines a K_2-prime element in R_2. By the previous diagram, T_i defines a K_1'-prime element in $((R_2)_{T_{r_1+1} \cdots T_{r_2}})_0$, the Veronese subalgebra of R_2 defined by $K_1' \subseteq K_2$. Since every K_2-homogeneous element of $(R_2)_{T_{r_1+1} \cdots T_{r_2}}$ can be shifted by a homogeneous unit into $((R_2)_{T_{r_1+1}\cdots T_{r_2}})_0$, we see that T_i defines a K_2-prime in $(R_2)_{T_{r_1+1}\cdots T_{r_2}}$. By assumption, $T_{r_1+1}, \dots, T_{r_2}$ define K_2-primes in R_2 and are all coprime to T_i. It follows that T_i defines a K_2-prime in R_2.

We turn to assertion (ii). As just observed, the degree zero part $((R_2)_{T_{r_1+1}\cdots T_{r_2}})_0$ of the K_2'-grading is isomorphic to R_1 and thus integral (normal if R_1 is). Moreover, the K_2'-grading is free in the sense that the associated torus $\operatorname{Spec}\mathbb{K}[K_2']$ acts freely on $\operatorname{Spec}(R_2)_{T_{r_1+1}\cdots T_{r_2}}$. It follows that $(R_2)_{T_{r_1+1}\cdots T_{r_2}}$ is integral (normal if R_1 is). Construction 3.1 gives that R_2 is integral. Moreover, if $T_{r_1+1}, \dots, T_{r_2}$ define primes in R_2, then we can conclude that R_2 is normal. □

Let us apply the results to Cox rings. We first briefly recall the basic definitions and facts; for details, we refer to [1]. For a normal variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ and $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$, its Cox ring is defined as the graded ring

$$\mathcal{R}(X) := \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

This ring is normal and factorially $\mathrm{Cl}(X)$-graded. Moreover, if $\mathcal{R}(X)$ is finitely generated, then we can reconstruct X from $\mathcal{R}(X)$ as a good quotient of an open subset of $\operatorname{Spec}\mathcal{R}(X)$ by the action of $\operatorname{Spec}\mathbb{K}[\mathrm{Cl}(X)]$.

Now return to the setting fixed at the beginning of the section and assume in addition that the columns of P_2 are pairwise different primitive vectors in $\mathbb{Z}^n$ and those of P_1 generate $\mathbb{Q}^n$ as a convex cone. Suppose that we have toric Cox constructions $\pi_i\colon \hat{Z}_i \to Z_i$, where $\hat{Z}_i \subseteq \mathbb{K}^{r_i}$ are open toric subvarieties, and π_i are toric morphisms defined by P_i; see [7]. Then the canonical map $Z_2 \to Z_1$ is a toric modification. Consider the ideal I_1 as discussed before and the geometric data

$$\bar{X}_1 := V(I_1) \subseteq \mathbb{K}^{r_1}, \qquad \hat{X}_1 := \bar{X}_1 \cap \hat{Z}_1, \qquad X_1 := \pi_1(\hat{X}_1) \subseteq Z_1.$$

Assume that R_1 is normal and factorially K_1-graded and $T_1, \dots, T_{r_1}$ define pairwise nonassociated prime elements in R_1. Then R_1 is the Cox ring of X_1 by [1, Thm. 3.2.1.4]. Our statement concerns the Cox ring of the proper transform $X_2 \subseteq Z_2$ of $X_1 \subseteq Z_1$ with respect to $Z_2 \to Z_1$.

Corollary 3.4. *In this setting, assume that R_2 is normal and the variables $T_1, \dots, T_{r_2}$ define pairwise nonassociated K_2-prime elements in R_2. Then the K_2-graded ring R_2 is the Cox ring of X_2.*

Proof. According to Theorem 3.2, the ring R_2 is factorially K_2-graded. Moreover, with the toric Cox construction $\pi_2\colon \hat{Z}_2 \to Z_2$, we obtain that R_2 is the algebra of functions of the closure $\hat{X}_2 \subseteq \hat{Z}_2$ of $\pi_2^{-1}(X_2 \cap \mathbb{T}^{r_2})$. Thus, [1, Thm. 3.2.1.4] yields that R_2 is the Cox ring of X_2. □

Example 3.5. We start with the UFD $R_1 = \mathbb{K}[T_1, \dots, T_8]/I_1$, where the ideal I_1 is defined as

$$I_1 = \langle T_1T_2 + T_3T_4 + T_5T_6 + T_7T_8 \rangle.$$

The ideal I_1 is homogeneous with respect to the standard grading given by $Q_1 = [1, \dots, 1]$. A Gale dual is $P_1 = [e_0, e_1, \dots, e_7]$, where $e_0 = -e_1 - \cdots - e_7$ and e_i are the canonical basis vectors. Concatenating $e_1 + e_3$ gives a matrix P_2. The resulting UFD is $R_2 = \mathbb{K}[T_1, \dots, T_9]/I_2$ with

$$I_2 = \langle T_1T_2T_9 + T_3T_4T_9 + T_5T_6 + T_7T_8 \rangle.$$

4. Proof of Theorem 1.2

We approach the Chow quotient via toric embedding. The idea then is to obtain the Cox ring via toric ambient modifications. An essential step for this is an explicit description of the rays of certain Gelfand–Kapranov–Zelevinsky decompositions given in Proposition 4.1; note that in the setting of polytopes, related statements implicitly occur in literature (e.g., [11; 12]).

Recall that the Gelfand–Kapranov–Zelevinsky decomposition associated with a matrix $P \in \mathrm{Mat}(n, r+1; \mathbb{Z})$ is the fan Σ in $\mathbb{Q}^n$ with the cones $\sigma(v) = \bigcap_{v \in \tau^\circ} \tau$, where $v \in \mathbb{Q}^n$, and τ runs through the P-cones, that is, the cones generated by some of the columns $p_0, \dots, p_r$ of P. Fix a Gale dual matrix $Q \in \mathrm{Mat}(k, r+1; \mathbb{Z})$, where $r+1 = k+n$, and denote the columns of Q by $q_0, \dots, q_r$. Then we have mutually dual exact sequences of rational vector spaces

$$0 \longrightarrow \mathbb{Q}^k \xrightarrow{Q^*} \mathbb{Q}^{r+1} \xrightarrow{P} \mathbb{Q}^n \longrightarrow 0$$

$$0 \longleftarrow \mathbb{Q}^k \xleftarrow[Q]{} \mathbb{Q}^{r+1} \xleftarrow[P^*]{} \mathbb{Q}^n \longleftarrow 0$$

By a *Q-hyperplane* we mean a linear hyperplane in $\mathbb{Q}^k$ generated by some of the columns $q_0, \dots, q_r$. Given a Q-hyperplane, we write it as the kernel $u^\perp$ of a linear form u and associate with it a ray in $\mathbb{Q}^n$ as follows:

$$\varrho(u) := \mathrm{cone}\Bigg(\sum_{u(q_i)>0} u(q_i)p_i\Bigg).$$

It turns out that $\varrho(u) = \varrho(-u)$ and thus the ray is well defined. We say that a column q_i of Q is *extremal* if it does not belong to the relative interior of the "movable cone" $\bigcap_i \mathrm{cone}(q_j; j \neq i)$.

PROPOSITION 4.1. *Let Q and P be Gale dual matrices as before, assume that the columns of P are pairwise linearly independent nonzero vectors generating $\mathbb{Q}^n$ as a cone, and let Σ be the Gelfand–Kapranov–Zelevinsky decomposition associated with P.*

- (i) *If a ray $\varrho \in \Sigma$ is the intersection of two P-cones, then $\varrho = \varrho(u)$ with a Q-hyperplane $u^\perp$.*
- (ii) *If $k = 2$, then every ray of Σ can be obtained as an intersection of two P-cones.*

(iii) *Assume that* $k = 2$ *and fix nonzero linear forms* u_i *with* $u_i \perp q_i$. *Then the rays of* Σ *are* $\mathrm{cone}(p_0), \ldots, \mathrm{cone}(p_r)$ *and the* $\varrho(u_i)$ *with* q_i *not extremal.*

The proof relies on the fact that Σ describes the lifts of regular Q-subdivisions. We adapt the precise formulation of this statement to our needs. Let $\gamma \subseteq \mathbb{Q}^{r+1}$ be the positive orthant and define a *γ-collection* to be a set $\mathfrak{B}$ of faces of γ such that any two $\gamma_1, \gamma_2 \in \mathfrak{B}$ admit an *invariant separating linear form* f in the sense that

$$P^*(\mathbb{Q}^n) \subseteq f^\perp, \qquad f_{|\gamma_1} \geq 0, \qquad f_{|\gamma_2} \leq 0, \qquad f^\perp \cap \gamma_i = \gamma_1 \cap \gamma_2.$$

Write $\mathfrak{B}_1 \leq \mathfrak{B}_2$ if for every $\gamma_1 \in \mathfrak{B}_1$, there is a $\gamma_2 \in \mathfrak{B}_2$ with $\gamma_1 \subseteq \gamma_2$. Moreover, call a γ-collection $\mathfrak{B}$ *normal* if it cannot be enlarged as a γ-collection and the images $Q(\gamma_0)$, where $\gamma_0 \in \mathfrak{B}$, form the normal fan of a polyhedron. For a face $\gamma_0 \preceq \gamma$, we denote by $\gamma_0^* = \gamma_0^\perp \cap \gamma^\vee$ the corresponding face of the dual cone $\gamma^\vee$.

Now assume that the columns of P are pairwise different nonzero vectors. Then [1, Sect. 2.2] provides us with an order-reversing bijection

$$\{\text{normal } \gamma\text{-collections}\} \to \Sigma, \qquad \mathfrak{B} \mapsto \bigcap_{\gamma_0 \in \mathfrak{B}} P(\gamma_0^*).$$

Proof of Proposition 4.1. We prove (i). Let $\varrho = P(\gamma_1^*) \cap P(\gamma_2^*)$ with $\gamma_1, \gamma_2 \preceq \gamma$. We may assume that the relative interiors $P(\gamma_1^*)^\circ$ and $P(\gamma_2^*)^\circ$ intersect nontrivially. Then γ_1 and γ_2 admit an invariant separating linear form $f = Q^*(u)$ with a linear form u on $\mathbb{Q}^k$. In terms of the components of $f_i = u(q_i)$ of f, we have

$$\gamma_1 = \mathrm{cone}(e_i;\ f_i \geq 0), \qquad \gamma_2 = \mathrm{cone}(e_i;\ f_i \leq 0).$$

Write $f = f^+ - f^-$ with the unique vectors $f^+, f^- \in \mathbb{Q}^{r+1}$ having only nonnegative components. Then $P(f) = 0$ gives $P(f^+) = P(f^-)$. We conclude that $\varrho = \mathrm{cone}(P(f^+))$, and the assertion follows.

We prove (ii) and (iii). The rays of Σ arise from normal γ-collections that are submaximal with respect to "$\leq$" in the sense that the only dominating γ-collection is the trivial collection $\langle\gamma\rangle$ consisting of all faces $\gamma_0 \preceq \gamma$ that are invariantly separable from γ. There are precisely two types of such submaximal collections:

- the normal γ-collections $\mathfrak{B} = \langle\gamma_0\rangle$, where $\gamma_0 \precneqq \gamma$ is a facet satisfying $Q(\gamma_0) = Q(\gamma)$,
- the normal γ-collections $\mathfrak{B} = \langle\gamma_1, \gamma_2\rangle$, where $\gamma_1, \gamma_2 \precneqq \gamma$ are invariantly separable from each other and satisfy

$$\gamma_i = Q^{-1}(Q(\gamma_i)) \cap \gamma, \qquad Q(\gamma) = Q(\gamma_1) \cup Q(\gamma_2).$$

The submaximal γ-collections of the first type give the rays $\mathrm{cone}(p_i) \in \Sigma$ with q_i not extremal. If q_i is extremal, then the (unique) γ-collection of the second type with $Q(\gamma_1) = \mathrm{cone}(q_j;\ j \neq i)$ defines the ray $\mathrm{cone}(p_i)$. The remaining rays of Σ are of the form $\varrho = P(\gamma_1^*) \cap P(\gamma_2)^*$ with the remaining collections of the second type. $\square$

REMARK 4.2. Statements (ii) and (iii) of Proposition 4.1 hold as well for pairs P, Q, where the columns of Q generate the cone over a so-called *totally-2-splittable* polytope; these have been studied in [11; 12].

As a further preparation of the proof of Theorem 1.2, we have to specialize the discussion of Section 3 to the case of a single defining equation. The following notion will be used for an explicit description of the transferred ideal.

Definition 4.3. Consider an $n \times (r+1)$ matrix P and an $n \times l$ matrix B, both integral. A *weak B-lifting* (*with respect to P*) is an integral $(r+1) \times l$ matrix A allowing a commutative diagram

$$\begin{tikzcd}
 & \mathbb{Z}^{r+1+l} \arrow[dl, "{\substack{e_i \mapsto e_i \\ e_j \mapsto m_j e_j}}"'] \arrow[dr, "{[E_{r+1}, A]}"] & \\
\mathbb{Z}^{r+1+l} \arrow[d, "{[P, B]}"'] & & \mathbb{Z}^{r+1} \arrow[d, "P"] \\
\mathbb{Z}^n \arrow[rr, "E_n"'] & & \mathbb{Z}^n
\end{tikzcd}$$

where e_i are the first $r+1$ and e_j the last l canonical basis vectors of $\mathbb{Z}^{r+1+l}$, m_j are positive integers, and E_n and E_{r+1} denote the unit matrices of sizes n and $r+1$, respectively.

Note that weak B-liftings A always exist. Given such A, consider the following homomorphism of Laurent polynomial rings:

$$\psi_A \colon \mathbb{K}[T_0^{\pm 1}, \dots, T_r^{\pm 1}] \to \mathbb{K}[T_0^{\pm 1}, \dots, T_r^{\pm 1}, S_1^{\pm 1}, \dots, S_l^{\pm 1}],$$
$$\sum \alpha_\nu T^\nu \mapsto \sum \alpha_\nu T^\nu S^{A^t \cdot \nu}.$$

Set $K_1 := \mathbb{Z}^{r+1}/P^*(\mathbb{Z}^n)$. Then the left-hand side algebra is K_1-graded by assigning to the ith variable the class of e_i in K_1.

Lemma 4.4. *In the previous notation, let $g_1 \in \mathbb{K}[T_0^{\pm 1}, \dots, T_r^{\pm 1}]$ be a K_1-homogeneous polynomial.*

(i) *We have $T^\nu S^\mu \psi_A(g_1) = g_2'$ with $\nu \in \mathbb{Z}^{r+1}$, $\mu \in \mathbb{Z}^l$ and a unique monomial-free $g_2' \in \mathbb{K}[T_0, \dots, T_r, S_1, \dots, S_l]$.*

(ii) *The polynomial g_2' is of the form $g_2' = g_2(T_0, \dots, T_{r+1}, S_1^{m_1}, \dots, S_l^{m_1})$ with a $g_2 \in \mathbb{K}[T_0, \dots, T_r, S_1, \dots, S_l]$ not depending on the choice of A.*

(iii) *If, in the setting of Construction 3.1, we have $I_1 = \langle g_1 \rangle$, then the transferred ideal is given by $I_2 = \langle g_2 \rangle$.*

(iv) *The variable T_i defines a prime element in $\mathbb{K}[T_0, \dots, T_{r+l+1}]/\langle g_2 \rangle$ if and only if the polynomial $g_2(T_1, \dots, T_{i-1}, 0, T_{i+1}, \dots, T_{r+l+1})$ is irreducible.*

Proof. Consider the commutative diagram of group algebras corresponding to the dualized diagram (Definition 4.3). There, ψ_A occurs as the homomorphism of group algebras defined by the transpose $[E_{r+1}, A]^*$. Let T^κ be any monomial of g_1. Then $g_1' := T^{-\kappa} g_1$ gives rise to the same g_2, but g_1' is of K_1-degree zero and hence a pullback $g_1' = \psi_{P^*}(h)$. The latter allows us to use commutativity of the diagram, which gives (i) and (ii). Assertions (iii) and (iv) are clear. □

Proof of Theorem 1.2. Recall that we consider the quadric $X = V(g_1) \subseteq \mathbb{P}_r$ with $g_1 = T_0T_1 + \cdots + T_{r-1}T_r$, where we replace the last term with T_r^2 in the case of an even r, and a $\mathbb{K}^*$-action on $\mathbb{P}_r$, given by weights $\zeta_0, \ldots, \zeta_r$ such that g_0 is of degree zero and, in particular, X is invariant.

In a first step, we construct a suitable GIT quotient X_1 of the $\mathbb{K}^*$-action on X. Lifting the data to $\mathbb{K}^{r+1}$ gives $\bar{X} := V(g_1) \subseteq \mathbb{K}^{r+1}$, which is invariant under the action of $\mathbb{T}^2 = \mathbb{K}^* \times \mathbb{K}^*$ on $\mathbb{K}^{r+1}$ given by the weight matrix

$$Q := \begin{bmatrix} \zeta_0 \cdots \zeta_r \\ 1 \; \ldots \; 1 \end{bmatrix}.$$

Consider the weight $w = (0, 1)$ of $\mathbb{T}^2$ and the associated set of semistable points $\hat{Z}_1 \subseteq \mathbb{K}^{r+1}$, that is, the union of all localizations $\mathbb{K}^{r+1}_f$, where f is homogeneous with respect to some positive multiple of w. Then $\hat{Z}_1$ is a toric open subset, and with $\hat{X}_1 := \bar{X} \cap \hat{Z}_1$, we obtain a commutative diagram

$$\begin{tikzcd} \hat{X}_1 \arrow[r, phantom, "\subseteq"] \arrow[d, "/\!\!/\mathbb{T}^2"'] & \hat{Z}_1 \arrow[d, "/\!\!/\mathbb{T}^2"] \\ X_1 \arrow[r] & Z_1 \end{tikzcd}$$

where the induced map $X_1 \to Z_1$ of quotients is a closed embedding. We are in the setting presented before Corollary 3.4. In particular, $\hat{Z}_1 \to Z_1$ is a toric Cox construction with a Gale dual P of Q as describing matrix; note that the columns of P generate $\mathbb{Z}^{r-1}$ as a lattice. Moreover, the Cox ring of X_1 is the $\mathbb{Z}^2$-graded ring

$$R_1 = \mathbb{K}[T_0, \ldots, T_r]/\langle g_1 \rangle.$$

Observe that X_1 is as well the $\mathbb{K}^*$-quotient of the image of $\hat{X}_1$ in X, which in turn is the set of semistable points of a suitable linearization of $\mathcal{O}(1)$.

Set $n := r - 1$ and consider the Gelfand–Kapranov–Zelevinsky decomposition Σ associated to P. Then, according to Proposition 2.4, the toric variety Z_2 determined by Σ is the normalized Chow quotient of the $\mathbb{K}^*$-action on $\mathbb{P}_r$. Moreover, let $X_2 \subseteq Z_2$ denote the proper transform of $X_1 \subseteq Z_1$ under the toric morphism $Z_2 \to Z_1$. Then Proposition 2.5 tells us that X_2 and the Chow quotient $X /\!_{\mathrm{CQ}}\, \mathbb{K}^*$ share the same normalization.

We will now show that X_2 is in fact normal and that its Cox ring is as claimed in the theorem. As before, put the primitive generators $b_1, \ldots, b_l$ of rays of Σ differing from columns of P into a matrix B and choose a weak B-lifting A with respect to P; using the fact that the columns of P generate $\mathbb{Z}^n$, we can choose the numbers m_j all equal to one. With the shifted row sums $\eta_0, \eta_2, \ldots, \eta_{r-1}$, we set

$$g_2 := \begin{cases} T_0T_1S^{\eta_0} + T_2T_3S^{\eta_2} + \cdots + T_{r-1}T_rS^{\eta_{r-1}}, & r \text{ odd}, \\ T_0T_1S^{\eta_0} + \cdots + T_{r-2}T_{r-1}S^{\eta_{r-2}} + T_r^2S^{\eta_r}, & r \text{ even}. \end{cases}$$

Lemma 4.4 then ensures that $I_2 := \langle g_2 \rangle$ is the transferred ideal of $I_1 := \langle g_1 \rangle$ in the sense of Construction 3.1; define $P_1 := P$ and $P_2 := [P, B]$ to adapt the settings.

Consider the ring

$$R_2 = \mathbb{K}[T_0, \dots, T_r, S_1, \dots, S_l]/\langle g_2 \rangle.$$

Our task is to show that the variables $S_1, \dots, S_l$ define prime elements in R_2. Then Proposition 3.3 tells us that R_2 and thus X_2 are normal, and Corollary 3.4 yields that the Cox ring of X_2 is R_2 together with the $\mathbb{Z}^{2+l}$-grading defined by a Gale dual Q_2 of $P_2 = [P, B]$.

Suitably renumbering the variables T_i, we achieve that $|\zeta_{r-3}|, \dots, |\zeta_r|$ are minimal among all $|\zeta_i|$ in the case of odd r and, similarly, in the case of even r, we have $\zeta_{r-3} = \zeta_{r-2} = \zeta_{r-1} = 0$. In order to see that the S_j define primes, it suffices to show that, according to odd and even r,

$$g_2 = T_{r-3}T_{r-2} + T_{r-1}T_r + h, \quad \text{or} \quad g_2 = T_{r-2}T_{r-1} + T_r^2 + h$$

with a polynomial $h \in \mathbb{K}[T_0, \dots, T_r, S_1, \dots, S_l]$ not depending on the last four (three) T_i; see Lemma 4.4(iv). This in turn is seen by constructing a suitable weak B-lifting via the description of the rays through $b_1, \dots, b_l$ provided by Proposition 4.1. Each b_j (or a suitable integral multiple) stems from a Q-hyperplane, and the u_j can be chosen to be nonpositive on the last four (three) q_i. Putting $\max(0, u_j(q_i))$ into a matrix A' gives a weak B-lifting A' with $A'_{i*} = 0$ for the last four (three) rows. By Lemma 4.4 the weak B-lifting A' yields the same g_2, which now has the desired form. □

EXAMPLE 4.5. Consider the quadric $X = V(T_0T_1 + T_2T_3 + T_4T_5 + T_6^2) \subseteq \mathbb{P}_6$ and the action of $\mathbb{K}^*$ on $\mathbb{P}_6$ given by

$$t \cdot [x_0, \dots, x_6] := [t^{-2}x_0, t^2x_1, t^{-1}x_2, t^1x_3, x_4, x_5, x_6].$$

An integral Gale dual P of the extended weight matrix Q is of size 5×7 and explicitly given as

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

Computing the associated Gelfand–Kapranov–Zelevinsky decomposition, we see that it comes with one new ray, namely

$$b_1 = (-1, 0, -1, 1, 0) = 2p_0 + p_2,$$

where $p_0, \dots, p_6$ are the columns of P. The Cox ring of the normalized Chow quotient $X \mathbin{\tilde{/}_{\mathrm{CQ}}} \mathbb{K}^*$ is the ring

$$\mathcal{R}(X \mathbin{\tilde{/}_{\mathrm{CQ}}} \mathbb{K}^*) = \mathbb{K}[T_0, \dots, T_6, S_1]/ \langle T_0T_1S_1^2 + T_2T_3S_1 + T_4T_5 + T_6^2 \rangle$$

together with the grading by $\mathrm{Cl}(X) = \mathbb{Z}^3$ via a Gale dual of $[p_0, \dots, p_6, b_1]$, that is, the degrees of the variable are the columns of

$$\begin{bmatrix} -2 & 2 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

REMARK 4.6. The setting of Theorem 1.2 can also be interpreted in terms of Mori theory, especially in the sense of [14]. There are (up to isomorphism) finitely many normal projective varieties $Y_1, \dots, Y_s$ sharing as their Cox ring a given $R_1 = \mathbb{K}[T_0, \dots, T_n]/\langle g_1 \rangle$ with its $\mathbb{Z}^2$-grading coming from the extended weight matrix Q. Each Y_i is a GIT-quotient of the induced $\mathbb{K}^*$-action on the quadric $X = V(g_1) \subseteq \mathbb{P}_r$ and thus dominated in universal manner by the normalized Chow quotient $Y = X \widetilde{/\!\!/}_{\mathrm{CQ}} \mathbb{K}^*$. Thus, Y is the “Mori master space” controlling the whole class of small birational relatives Y_i. This picture obviously extends to all Mori dream spaces, and it is a natural desire to study the geometry of such Mori master spaces.

REMARK 4.7. The assumption in Theorem 1.2 that, for odd (even) r, there are at least four (three) weights ζ_i of minimal absolute value is used for verifying the primality conditions on the variables T_i in Corollary 3.4. It would be interesting to see what happens beyond this assumption. We expect that the Cox ring then has further generators, in addition to the variables S_j; note that in the setting of Remark 4.6, the S_j correspond to the canonical sections of exceptional divisors of $Y \to Y_i$ for some fixed i.

5. Proof of Theorem 1.1

The main idea of the proof is to consider, instead of the Chow quotient, its “weak tropical resolution” and to use intrinsic symmetry of the latter space. This approach applies also to problems beyond $\mathbb{K}^*$-actions on quadrics; we therefore develop it in sufficient generality. We begin with recalling the necessary concepts from tropical geometry.

Let f be a Laurent polynomial in n variables. The Newton polytope $B_f \subseteq \mathbb{Q}^n$ is the convex hull over the exponent vectors of f. The tropical variety $\mathrm{trop}(V(f))$ of the zero set $V(f) \subseteq \mathbb{T}^n$ lives in $\mathbb{Q}^n$ and is defined to be the union of all $(n-1)$-dimensional cones of the normal fan of B_f. The tropical variety of an arbitrary closed subset $Y \subseteq \mathbb{T}_n$ is the intersection $\mathrm{trop}(Y)$ over all $\mathrm{trop}(V(f))$, where f runs through the ideal of Y. It turns out that $\mathrm{trop}(Y)$ is the support of an (in general not unique and not pointed) fan in $\mathbb{Q}^n$.

DEFINITION 5.1. Consider a toric variety Z defined by a fan Σ in $\mathbb{Q}^n$ and an irreducible subvariety $Y \subseteq Z$ intersecting the big torus $\mathbb{T}^n \subseteq Z$ nontrivially. We call the embedding $Y \subseteq Z$ *weakly tropical* if the support $|\Sigma| \subseteq \mathbb{Q}^n$ equals the tropical variety $\mathrm{trop}(Y \cap \mathbb{T}^n) \subseteq \mathbb{Q}^n$.

Remark 5.2. Any tropical embedding in the sense of Tevelev [22] is weakly tropical. If $Y \subseteq Z$ is a weakly tropical subvariety of a toric variety Z, then, by [10, Sect. 14], for any toric orbit $\mathbb{T}^n \cdot z \subseteq Z$ intersecting Y nontrivially, we have

$$\dim(Z) - \dim(\mathbb{T}^n \cdot z) = \dim(Y) - \dim(\mathbb{T}^n \cdot z \cap Y).$$

Construction 5.3 (Weak tropical resolution). Let Z be a complete toric variety arising from a fan Σ in $\mathbb{Q}^n$, and $Y \subseteq Z$ an irreducible subvariety intersecting the big torus $\mathbb{T}^n \subseteq Z$ nontrivially. Fix a fan structure Σ_Y carried on the tropical variety $\mathrm{trop}(Y \cap \mathbb{T}^n) \subseteq \mathbb{Q}^n$ for $Y \cap \mathbb{T}^n$ and consider the coarsest common refinement

$$\Sigma' := \Sigma \sqcap \ \Sigma_Y = \{\tau \cap \sigma;\, \sigma \in \Sigma,\ \tau \in \Sigma_Y\}$$

of the fans Σ and Σ_Y. Then the canonical map of fans $\Sigma' \to \Sigma$ defines a birational toric morphism $Z' \to Z$ of the associated toric varieties. With the proper transform $Y' \subseteq Z'$ of $Y \subseteq Z$, we obtain a proper birational map $Y' \to Y$, which we call a *weak tropical resolution* of $Y \subseteq Z$.

Proof. The only thing to show is the properness of the morphism $Y' \to Y$. But this follows directly from Tevelev's criterion [22, Prop. 2.3]. □

The use of passing to the weak tropical resolution in our context is that it enables us to divide out torus symmetries in a controlled manner. This leads to an explicit version of [18, Thm. 1.2] relating the Mori dream space property of a variety to the Mori dream space property of a certain quotient.

Construction 5.4. Consider a toric variety Z arising from a fan Σ in $\mathbb{Q}^r$ and a weakly tropical embedded subvariety $Y \subseteq Z$. Suppose that Y is invariant under the action of a subtorus $T \subseteq \mathbb{T}^r$. Set

$$Z_0 := \{z \in Z;\, \dim(\mathbb{T}^r \cdot z) \geq r-1,\, T_z \text{ finite}\}, \qquad Y_0 := Y \cap Z_0.$$

Then $Z_0 \subseteq Z$ is an open toric subset corresponding to a subfan $\Sigma_0 \preceq \Sigma$ with certain rays $\varrho_1, \dots, \varrho_s$ of Σ as its maximal cones. Let the matrix $P \in \mathrm{Mat}(n, r; \mathbb{Z})$ describe an epimorphism $\pi : \mathbb{T}^r \to \mathbb{T}^n$ with $\ker(\pi) = T$ and consider the following fan in $\mathbb{Z}^n$:

$$\Delta_0 := \{0, P(\varrho_1), \dots, P(\varrho_s)\}.$$

Note that $\varrho_1, \dots, \varrho_s$ are precisely the rays of Σ that are not contained in $\ker(P)$. The matrix P determines a toric morphism $Z_0 \to Z /\!_0\, T$ onto the toric variety associated to Δ_0. We define $Y /\!_0\, T \subseteq Z /\!_0\, T$ to be the closure of the image $\pi(Y \cap \mathbb{T}^r)$.

Remark 5.5. The tropical variety $\mathrm{trop}(Y /\!_0\, T \cap \mathbb{T}^n)$ contains all rays $P(\varrho_1), \dots, P(\varrho_s)$ of the fan Δ_0. If there is a fan Δ in $\mathbb{Z}^n$ having $\mathrm{trop}(Y /\!_0\, T \cap \mathbb{T}^n)$ as its support and $P(\varrho_1), \dots, P(\varrho_s)$ as its rays, then $Y /\!_0\, T$ admits a weakly tropical completion with boundary of codimension at least two.

PROPOSITION 5.6. *Consider a toric variety Z and a weakly tropical subvariety $Y \subseteq Z$. Suppose that Y is invariant under the action of a subtorus $T \subseteq \mathbb{T}^r$. Then the following statements are equivalent*:

(i) *The normalization of Y has finitely generated Cox ring.*
(ii) *The normalization of $Y /\!\!/_0 T$ has finitely generated Cox ring.*

Proof. Let $\nu \colon \tilde{Y} \to Y$ be the normalization map. By $W \subseteq Y$ we denote the open T-invariant subset consisting of all points $y \in Y$ having a finite isotropy group T_y. The fact that $Y \subseteq Z$ is tropically embedded ensures that $Y_0 \subseteq W$ has a complement of codimension at least two in W. This property is preserved when passing to the respective normalizations $\tilde{W} := \nu^{-1}(W)$ and $\tilde{Y}_0 := \nu^{-1}(Y_0)$. In particular, the separations in the sense of [18, p. 978] of the corresponding quotients $\tilde{W}/T$ and $\tilde{Y}_0/T$ have the same Cox rings. Since normalizing commutes with taking quotients and separating, the latter space is isomorphic to the normalization of $Y /\!\!/_0 T$. Thus, the assertion follows from [18, Thm. 1.2]. □

PROPOSITION 5.7. *Let Z be a toric variety, $Y \subseteq Z$ a complete subvariety invariant under a subtorus T of the big torus of Z, and $Y' \to Y$ be a weak tropical resolution. If the normalization of $Y' /\!\!/_0 T$ has finitely generated Cox ring, then the normalization $\tilde{Y}$ of Y is a Mori dream space.*

Proof. Since the normalization of $Y' /\!\!/_0 T$ has finitely generated Cox ring, Proposition 5.6 shows that the normalization Y'' of Y' has finitely generated Cox ring and thus is a Mori dream space. The canonical morphism $\pi \colon Y'' \to \tilde{Y}$ is proper and birational. In order to see that $\tilde{Y}$ is a Mori dream space, we may apply the general [20, Thm. 10.4] or look at a suitable sheaf $\mathcal{S} = \bigoplus_K \mathcal{O}_{\tilde{Y}}(D)$ of divisorial algebras on $\tilde{Y}$ mapping onto the Cox sheaf $\mathcal{R}$ of $\tilde{Y}$. By the properness of π we obtain $\mathcal{S} = \pi_* \mathcal{S}''$ over the set $W \subseteq \tilde{Y}$ of regular points for $\mathcal{S}'' = \bigoplus_K \mathcal{O}_{Y''}(\pi^*(D))$. Since Y'' is a Mori dream space, $\Gamma(\pi^{-1}(W), \mathcal{S}'')$ is finitely generated. This implies finite generation of the Cox ring $\mathcal{R}(\tilde{Y}) = \Gamma(W, \mathcal{R})$. □

A second preparation of the proof of Theorem 1.1 concerns toric ambient modification. We will always write $e_1, \ldots, e_n \in \mathbb{Z}^n$ for the canonical basis vectors and set $e_0 := -e_1 - \cdots - e_n$. Moreover, we denote by $\Delta(n)$ the fan in $\mathbb{Z}^n$ consisting of all cones spanned by at most n of the vectors $e_0, \ldots, e_n$ and by $\Delta'(n) \subseteq \Delta(n)$ the subfan consisting of all cones of dimension at most $n-1$.

LEMMA 5.8. *Consider nonzero vectors $v_1, \ldots, v_l \in \mathbb{Q}^n$ contained in a maximal cone $\tau \in \Delta(n)$, a cone $\sigma \subseteq \mathbb{Q}^n$ generated by some of the vectors $e_0, \ldots, e_n$, $v_1, \ldots, v_l$, and a cone $\delta \in \Delta'(n)$. Suppose that $\varrho := \delta \cap \sigma$ is one-dimensional and $\varrho \notin \Delta'(n)$. Then ϱ is contained in some facet of τ.*

Proof. We may assume that $\tau = \operatorname{cone}(e_1, \ldots, e_n)$. Replacing δ and σ with suitable faces, we may assume that $\varrho^\circ = \delta^\circ \cap \sigma^\circ$. The proof uses Gale duality, and we work in the notation of Section 4. Consider the matrix $P :=$

$[e_0, \dots, e_n, v_1, \dots, v_l]$ and its Gale dual

$$Q := [q_0, \dots, q_{n+l}] := \begin{bmatrix} 0 & v_{11} & \cdots & v_{1n} & -1 & & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \\ 0 & v_{l1} & \cdots & v_{ln} & 0 & & -1 \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Set $r := n + l$ and let $e'_0, \dots, e'_r$ denote the canonical basis vectors of $\mathbb{Z}^{r+1}$ and $\gamma := \mathbb{Q}^{r+1}_{\geq 0}$ the positive orthant. Then there are faces $\gamma_1, \gamma_2 \preceq \gamma$ such that for the corresponding dual faces γ_i^*, we have

$$P(\gamma_1^*) = \delta, \qquad P(\gamma_2^*) = \sigma, \qquad P(\gamma_1^*)^\circ \cap P(\gamma_2^*)^\circ \neq \emptyset.$$

For some $n+1 \leq j \leq r$, we have $e'_j \in \gamma_2^*$, and we may assume that γ_1^* is generated by at most $n-1$ of the vectors $e'_0, \dots, e'_n$. The latter implies $e'_{n+1}, \dots, e'_{n+l} \in \gamma_1$. Let $f = Q^*(u)$ be a separating linear form for γ_1 and γ_2. Then $f_{|\gamma_1} \geq 0$ implies

$$u(q_{n+1}), \dots, u(q_{n+l}) \geq 0, \qquad u(q_0) \geq u(q_1), \dots, u(q_n).$$

Note that we must have $f(e'_j) = u(q_j) > 0$ because e'_j does not lie in γ_2. Let $\tau_1, \tau_2 \preceq \gamma$ be the maximal faces with $f_{|\tau_1} \geq 0$ and $f_{|\tau_2} \leq 0$. Then f separates τ_1 and τ_2, and $\tau_i^* \subseteq \gamma_i^*$. We conclude

$$\emptyset \neq P(\tau_1^*)^\circ \cap P(\tau_2^*)^\circ \subseteq P(\tau_1^*) \cap P(\tau_2^*) \subseteq P(\gamma_1^*) \cap P(\gamma_2^*) = \varrho.$$

Since $e'_j \notin \tau_2$, we obtain $\tau_2^* \neq \{0\}$ and thus $0 \notin P(\tau_2^*)^\circ$. Together with the displayed line, this gives $P(\tau_1^*) \cap P(\tau_2^*) = \varrho$. Since at least two of $e'_0, \dots, e'_n$ lie in γ_1, we obtain $e'_0 \in \tau_1$, and thus

$$\varrho \subseteq P(\tau_1^*) \subseteq \operatorname{cone}(e_1, \dots, e_n). \qquad \square$$

LEMMA 5.9. *For $n \in \mathbb{Z}_{\geq 1}$, consider $\Delta'(n)$ and let $b_1, \dots, b_l \in \mathbb{Q}^n$ be pairwise different primitive vectors lying on the support of $\Delta'(n)$ but not on its rays. Denote by $\sigma_j \in \Delta'(n)$ the minimal cone with $b_j \in \sigma_j$ and write*

$$b_j = a_{0j} e_0 + \cdots + a_{nj} e_n, \quad \text{where } a_{ij} > 0 \text{ if } e_i \in \sigma_j, a_{ij} = 0 \text{ if } e_i \notin \sigma_j.$$

Then, for $P := [e_0, \dots, e_n]$ and $B := [b_1, \dots, b_l]$, the matrix $A := (a_{ij})$ is a weak B-lifting with respect to P. The lift of $h_1 = T_0 + \cdots + T_n$ in the sense of Lemma 4.4 is given by

$$h_2 = T_0 S_1^{a_{01}} \cdots S_l^{a_{0l}} + \cdots + T_n S_1^{a_{n1}} \cdots S_l^{a_{nl}}.$$

Moreover, the variables $T_0, \dots T_n, S_1, \dots, S_l$ define pairwise nonassociated prime elements in $\mathbb{K}[T_0, \dots T_n, S_1, \dots, S_l]/\langle h_2 \rangle$ if and only if the vectors $b_1, \dots, b_l$ lie in a common cone of $\Delta(n)$.

Proof. Only the last sentence needs some explanation. The fact that $b_1, \dots, b_l$ lie in a common cone of $\Delta(n)$ is equivalent to the fact that there is a term of h_2 not depending on $S_1, \dots, S_l$, and, moreover, for every k, there is a further term of h_2 not depending on S_k. Now, Lemma 4.4(iv) gives the desired characterization. $\square$

Proof of Theorem 1.1. We may assume that $X = V(g_1) \subseteq \mathbb{P}_r$ with a polynomial $g_1 = T_0T_1 + \cdots + T_{r-1}T_r$, where we replace the last term with T_r^2 in the case of an even r, and $\mathbb{K}^*$ acts linearly with weights $\zeta_0, \ldots, \zeta_r$, where $|\zeta_r|$ is minimal among all $|\zeta_i|$; see [1, Prop. 3.2.4.7].

The first step is to determine the normalized Chow quotient of the $\mathbb{K}^*$-action on X. As observed in Proposition 2.5, the Chow quotient $X /\!\!/_{\mathrm{cq}} \mathbb{K}^*$ is canonically embedded into the Chow quotient of $\mathbb{P}_r$ by the $\mathbb{K}^*$-action. To determine the latter, consider the extended weight matrix

$$Q := \begin{bmatrix} \zeta_0 \ldots \zeta_r \\ 1 \ldots \ 1 \end{bmatrix}$$

and let P be a Gale dual matrix. Then, according to Proposition 2.4, the normalized Chow quotient of the $\mathbb{K}^*$-action on $\mathbb{P}_r$ is the toric variety Z having the Gelfand–Kapranov–Zelevinsky decomposition Σ defined by the columns of P as its fan. Moreover, by Proposition 2.5 the Chow quotient of the $\mathbb{K}^*$-action on X has the same normalization as the closure

$$Y = \overline{(X \cap \mathbb{T}^r)/\mathbb{K}^*} \subseteq Z.$$

The second step is to determine a weak tropical resolution of $Y \subseteq Z$. For this, we first need $\mathrm{trop}(Y \cap T_Z)$. Let $\mu_0, \ldots, \mu_n \in \mathbb{Z}^{r+1}$ be the vertices of the Newton polytope g_1 and consider the matrix P_{gr} with the rows $\mu_i - \mu_0$, $i = 1, \ldots, n$. Then we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Q}^2 \arrow[r] \arrow[d] & \mathbb{Q}^{r+1} \arrow[r, "P"] \arrow[d, equal] & \mathbb{Q}^{r-1} \arrow[r] \arrow[d, "\Pi"] & 0 \\
0 \arrow[r] & \mathbb{Q}^{r+1-n} \arrow[r] & \mathbb{Q}^{r+1} \arrow[r, "P_{\mathrm{gr}}"'] & \mathbb{Q}^n \arrow[r] & 0
\end{tikzcd}$$

Note that g_1 equals T^{μ_0} times the pullback of the polynomial $h_1 := 1 + S_1 + \cdots + S_n$ under the homomorphism of tori $\mathbb{T}^r \to \mathbb{T}^n$ defined by P_{gr}. The tropical variety of $V(h_1) \subseteq \mathbb{T}^n$ is the support of the fan $\Delta'(n)$, and thus we have

$$\mathrm{trop}(Y \cap T_Z) = \Pi^{-1}(\mathrm{trop}(V(h_1))) = \Pi^{-1}(|\Delta'(n)|).$$

We endow $\mathrm{trop}(Y \cap T_Z)$ with the natural fan structure lifting $\Delta'(n)$; note that the cones are in general not pointed. By definition the weak tropical resolution Y' of Y is the closure of $Y \cap T_Z$ in the toric variety Z' with the coarsest common refinement $\Sigma' := \Sigma \sqcap \mathrm{trop}(Y \cap T_Z)$ as its fan.

In the third step, we pass to $Y' /\!\!/_0 T_{Y'}$, where $T_{Y'}$ is the kernel of the homomorphism of tori $T_Z \to \mathbb{T}^n$ defined by Π. By Construction 5.4 the quotient $Y' /\!\!/_0 T_{Y'}$ is the closure of the image of $Y \cap T_Z$ under $T_Z \to \mathbb{T}^n$ in the toric variety $Z' /\!\!/_0 T_{Y'}$ associated to the describing fan in $\mathbb{Z}^n$ having as maximal cones the rays $\Pi(\varrho)$, where ϱ runs through the rays of Σ'.

Claim. *For every ray $\varrho \in \Sigma'$, there is a facet of* $\mathrm{cone}(e_0, \ldots, e_{n-1})$ *containing the image $b := \Pi(\varrho) \in \mathbb{Q}^n$.*

Indeed, since every cone of $\mathrm{trop}(Y \cap T_Z)$ is saturated with respect to Π, we have $\Pi(\varrho) = \Pi(\sigma) \cap \delta$ for some $\sigma \in \Sigma$ and $\delta \in \Delta'(n)$. The image $\Pi(\sigma)$ is a cone spanned by some e_i and some images $v_j := \Pi(\nu_j)$, where ν_j are the primitive generators of the rays of Σ different from columns p_i of P. Proposition 4.1 yields the presentations

$$\nu_j = \sum_{i=0}^{r-1} \alpha_{ij} p_i \quad \text{with certain } \alpha_{ij} \geq 0.$$

Hence, we obtain $v_j \in \mathrm{cone}(e_0, \dots, e_{n-1})$. Lemma 5.8 then shows that $\Pi(\varrho)$ lies in some facet of $\mathrm{cone}(e_0, \dots, e_{n-1})$, and the claim is verified.

Finally, in the fourth step, we show that $Y' /\!\!/_{0} T_{Y'}$ is normal and has finitely generated Cox ring; by Proposition 5.7 this will complete the proof. First, note that we have the toric modification $Z' /\!\!/_{0} T_{Y'} \to W$, where $W \subseteq \mathbb{P}_n$ is the open toric subset corresponding to the subfan $\Delta'(n)$ of $\Delta(n)$. Moreover, $Y' /\!\!/_{0} T_{Y'}$ is the proper transform under $Z' /\!\!/_{0} T_{Y'} \to W$ of the closure of $V(h_1) \subseteq \mathbb{T}^n$ in W. The claim just verified and Lemma 5.9 ensure that we may apply Proposition 3.3 and Corollary 3.4. In particular, we see that $Y' /\!\!/_{0} T_{Y'}$ is normal with finitely generated Cox ring. □

EXAMPLE 5.10. Consider the quadric $X = V(T_0T_1 + \cdots + T_6T_7) \subseteq \mathbb{P}_7$ and the action of $\mathbb{K}^*$ on $\mathbb{P}_7$ given by

$$t \cdot [x_0, \dots, x_7] := [t^{-3}x_0, t^3x_1, t^{-3}x_2, t^3x_3, t^{-2}x_4, t^2x_5, t^{-1}x_6, tx_7].$$

Theorem 1.2 and its proof do not apply to this case because only two weights ζ_i have minimal absolute value. The way through the weak toric resolution Y' as gone in the proof of Theorem 1.1 produces a quotient $Y' /\!\!/_{0} T_{Y'}$ embedded into the toric variety with fan obtained by subdividing $\Delta(3)$ at $(0, -1, -1)$.

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