Michigan Math. J. **64 (2015), 293–318**

On Stable Conjugacy of Finite Subgroups of the Plane Cremona Group, II

Yuri Prokhorov

Abstract. We prove that, except for a few cases, stable linearizability of finite subgroups of the plane Cremona group implies linearizability.

1. Introduction

This is a follow-up paper to [BP13]. Let $\Bbbk$ be an algebraically closed field of characteristic 0. Recall that the *Cremona group* $\mathrm{Cr}_n(\Bbbk)$ is the group of birational automorphisms $\mathrm{Bir}(\mathbb{P}^n)$ of the projective space $\mathbb{P}^n$ over $\Bbbk$. Subgroups $G \subset \mathrm{Cr}_n(\Bbbk)$ and $G' \subset \mathrm{Cr}_m(\Bbbk)$ are said to be *stably conjugate* if, for some $N \geq n, m$, they are conjugate in $\mathrm{Cr}_N(\Bbbk)$, where the embeddings $\mathrm{Cr}_n(\Bbbk), \mathrm{Cr}_m(\Bbbk) \subset \mathrm{Cr}_N(\Bbbk)$ are induced by birational isomorphisms $\mathbb{P}^N \dashrightarrow \mathbb{P}^n \times \mathbb{P}^{N-n} \dashrightarrow \mathbb{P}^m \times \mathbb{P}^{N-m}$.

Any embedding of a finite subgroup $G \subset \mathrm{Cr}_n(\Bbbk)$ is induced by a biregular action on a rational variety X. A subgroup $G \subset \mathrm{Cr}_n(\Bbbk)$ is said to be *linearizable* if one can take $X = \mathbb{P}^n$. A subgroup $G \subset \mathrm{Cr}_n(\Bbbk)$ is said to be *stably linearizable* if it is stably conjugate to a linear action of G on a vector space $\Bbbk^m$.

The following question is a natural extension of the famous Zariski cancellation problem [BCSD85] to the geometric situation.

Question 1.1. Let $G \subset \mathrm{Cr}_2(\Bbbk)$ be a stably linearizable finite subgroup. Is it true that G is linearizable?

In this paper, we give a partial answer by finding a (very restrictive) list of all subgroups $G \subset \mathrm{Cr}_2(\Bbbk)$ that potentially can give counterexamples to the question.

It is easy to show (see [BP13]) that the group $H^1(G, \mathrm{Pic}(X))$ is a stable birational invariant. In particular, if $G \subset \mathrm{Cr}_n(\Bbbk)$ is stably linearizable, then $H^1(G_1, \mathrm{Pic}(X)) = 0$ for any subgroup $G_1 \subset G$ (then we say that $G \subset \mathrm{Cr}_n(\Bbbk)$ is *H^1-trivial*). Any finite subgroup $G \subset \mathrm{Cr}_2(\Bbbk)$ is induced by an action on either a del Pezzo surface or a conic bundle [Isk80]. In the first case, our main result is the following theorem, which is based on a computation of $H^1(G, \mathrm{Pic}(X))$ in [BP13] (see Theorem 2.9).

Theorem 1.2. *Let X be a del Pezzo surface, and let $G \subset \mathrm{Aut}(X)$ be a finite subgroup such that the pair (X, G) is minimal. Then the following are equivalent*:

(i) $H^1(G_1, \mathrm{Pic}(X)) = 0$ *for any subgroup* $G_1 \subset G$,

(ii) *any element of G does not fix a curve of positive genus*,

Received April 7, 2014. Revision received December 8, 2014.
Partially supported by RScF grant no. 14-21-00053.

(iii) *either*

(a) $K_X^2 \geq 5$, *or*

(b)[1] *X is a quartic del Pezzo surface given by*

$$x_1^2 + \zeta_3 x_2^2 + \zeta_3^2 x_3^2 + x_4^2 = x_1^2 + \zeta_3^2 x_2^2 + \zeta_3 x_3^2 + x_5^2 = 0, \tag{1.1}$$

where $\zeta_3 = \exp(2\pi i/3)$, *and* $G \simeq (\mathbb{Z}/3\mathbb{Z}) \rtimes (\mathbb{Z}/4\mathbb{Z})$ *is generated by the following two transformations*:

$$\begin{aligned} \gamma &: (x_1, x_2, x_3, x_4, x_5) \longmapsto (x_2, x_3, x_1, \zeta_3 x_4, \zeta_3^2 x_5), \\ \beta' &: (x_1, x_2, x_3, x_4, x_5) \longmapsto (x_1, x_3, x_2, -x_5, x_4). \end{aligned} \tag{1.2}$$

The conic bundle case is considered in Section 8. The main results are Theorems 8.5 and 8.10.

Note that there are only a few subgroups $G \subset \mathrm{Cr}_2(\Bbbk)$ that are not linearizable and satisfy the equivalent conditions (i)–(iii) of the theorem (see [DI09, §8]).

The plan of the proof of Theorem 1.2 is the following. The most difficult part of the proof is the implication (ii)⇒(iii). It is proved in Sections 4–7. The implication (i)⇒(ii) is exactly the statement of Corollary 2.10, and (iii)⇒(i) is a consequence of Proposition 3.4 and Corollary 3.5.

We tried to make the paper self-contained as much as possible, so in the proofs, we do not use detailed lists from the classification of finite subgroups of $\mathrm{Cr}_2(\Bbbk)$ [DI09]. Instead, we tried to use just *general* facts and principles of this classification.

2. Preliminaries

NOTATION 2.1. • $\mathfrak{S}_n$ is the symmetric group.
- $\mathrm{sgn} : \mathfrak{S}_n \to \{\pm 1\}$ is the sign map.
- $\mathfrak{A}_n$ is the alternating group.
- $\mathfrak{D}_n$ is a dihedral group of order $2n$, $n \geq 2$ (in particular, $\mathfrak{D}_2 \simeq (\mathbb{Z}/2\mathbb{Z})^2$). We will use the following presentation:

$$\mathfrak{D}_n = \langle r, s \mid r^n = s^2 = 1, srs = r^{-1} \rangle. \tag{2.1}$$

- $\sigma : \mathfrak{D}_n \to \{\pm 1\}$ is the homomorphism defined by $\sigma(r) = 1$, $\sigma(s) = -1$.
- $\tilde{\mathfrak{D}}_n$ is the binary dihedral group (see e.g. [Spr77]). We identify $\tilde{\mathfrak{D}}_n$ with the subgroup of $SL_2(\Bbbk)$ generated by the matrices

$$\tilde{r} = \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \qquad \tilde{s} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.2}$$

Note that $\tilde{\mathfrak{D}}_n$ is a nontrivial central extension of $\mathfrak{D}_n$ by $\mathbb{Z}/2\mathbb{Z}$.
- ζ_n is a primitive nth root of unity.
- $\Phi_n(t)$ is the nth cyclotomic polynomial.

[1] This case is missing in [DI09, Th. 6.9]. This is because the arguments on p. 489 (case 3) are incorrect. However, X has an equivariant rational curve fibration (see Remark 4.8). So, the description of the group appears in [DI09, Th. 5.7]. Note that the groups $(\mathbb{Z}/2\mathbb{Z})^2{}_\bullet\mathfrak{S}_3$ and $(\mathbb{Z}/2\mathbb{Z})^3{}_\bullet\mathfrak{S}_3$ are also missing in [DI09, Thm. 6.9].

- $\mathrm{Eu}(X)$ is the topological Euler number of X.
- $\mathrm{diag}(a_1, \dots, a_n)$ is the diagonal matrix.
- X^G is the fixed point locus of an action of G on X.

2.1. G-varieties

Throughout this paper, G denotes a finite group. We use the standard language of G-varieties (see e.g. [DI09]). In particular, we systematically use the following fact: for any projective nonsingular G-surface X, there exists a birational G-equivariant morphism $X \to X_{\min}$ such that the G-surface $X_{\min}$ is *G-minimal,* that is, any birational G-equivariant morphism $f : X_{\min} \to Y$ is an isomorphism. In this situation, $X_{\min}$ is called *G-minimal model* of X. If the surface X is additionally rational, then one of the following holds [Isk80]:

- $X_{\min}$ is a del Pezzo surface whose invariant Picard number $\mathrm{Pic}(X_{\min})^G$ is of rank 1, or
- X admits a structure of G-conic bundle, that is, there exists a surjective G-equivariant morphism $f : X_{\min} \to \mathbb{P}^1$ such that $f_*\mathscr{O}_{X_{\min}} = \mathscr{O}_{\mathbb{P}^1}$, $-K_{X_{\min}}$ is f-ample, and $\mathrm{rk}\,\mathrm{Pic}(X_{\min})^G = 2$.

2.2. Stable Conjugacy

We say that G-varieties (X, G) and (Y, G) are *stably birational* if for some n and m, there exists an equivariant birational map $X \times \mathbb{P}^n \dashrightarrow Y \times \mathbb{P}^m$, where actions on $\mathbb{P}^n$ and $\mathbb{P}^m$ are trivial. This is equivalent to the conjugacy of subgroups $G \subset \Bbbk(X)(t_1, \dots, t_n)$ and $G \subset \Bbbk(Y)(t_1, \dots, t_m)$.

By the *no-name* lemma we have the following.

Remark 2.2. Let V, W be *faithful linear* representations of G. Then the G-varieties (V, G) and (W, G) are stably conjugate. Indeed, let $n := \dim V$, $m := \dim W$. Consider trivial linear representations V' and W' with $\dim V' = n$ and $\dim W' = m$. According to the *no-name lemma* (see e.g. [Sha94, App. 3]) we can choose invariant coordinates for semilinear action of G on $V \otimes \Bbbk(W)$. This means that two embeddings $G \subset \mathrm{Cr}_{n+m}(\Bbbk)$ induced by actions on $V \times W$ and $V' \times W$ are conjugate. Similarly, the embeddings $G \subset \mathrm{Cr}_{n+m}(\Bbbk)$ induced by actions on $V \times W$ and $V \times W'$ are also conjugate. Hence, (V, G) and (W, G) are stably conjugate.

Definition 2.3. We say that a G-variety (X, G) (or, by abuse of language, a group G) is *stably linearizable* if it is stably birational to (V, G), where $V = \Bbbk^m$ is some faithful linear representation.

Remark 2.4. One can define stable linearizability is several other ways:

(i) if (X, G) is stably birational to $(\mathbb{P}^N, G)$ for some N;
(ii) if (X, G) is stably birational to $(\mathbb{P}^N, G)$ for $N = \dim X$;
(iii) if there exists a G-birational map $X \times \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ for some N where the action on $\mathbb{P}^n$ is trivial.

In view of Remark 2.2, our Definition 2.3 seems to be a most natural one. Clearly, we have the following implications:

$$\text{Definition 2.3} \Longrightarrow \text{(iii)} \Longrightarrow \text{(i)}, \qquad \text{(ii)} \Longrightarrow \text{(i)}.$$

The example below shows that, in general, the implications (i), (ii), (iii) $\Longrightarrow$ Definition 2.3 do not hold.

EXAMPLE 2.5. Let $\mathbf{Q}_8$ be the quaternion group of order 8, and let V be its faithful two-dimensional irreducible representation. Then, for any r, the $(2r-1)$-dimensional projective space $\mathbb{P}(V^{\oplus r})$ is a G-variety, where $G = \mathbf{Q}_8/[\mathbf{Q}_8, \mathbf{Q}_8] \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It is easy to see that there is no fixed point on this $\mathbb{P}(V^{\oplus r})$. Applying Lemma 2.6 (below), we can see that the G-variety $(\mathbb{P}^{2r-1}, G)$ is not stably linearizable. Similar examples can be constructed for the group $G = (\mathbb{Z}/n\mathbb{Z})^2$ (e.g., instead of $\mathbf{Q}_8$, we can start with the Heisenberg group of order p^3).

LEMMA 2.6 (see [KS00]). *For any finite Abelian group G and any G-birational map $X \dashrightarrow Y$ of complete G-varieties, the set X^G is nonempty if and only if so is Y^G.*

2.3. *Stable Conjugacy and $H^1(G, \mathrm{Pic}(X))$*

DEFINITION 2.7. We say that a nonsingular G-variety (X, G) is *H^1-trivial* if $H^1(G_1, \mathrm{Pic}(X)) = 0$ for any subgroup $G_1 \subset G$.

THEOREM 2.8 [BP13]. *Let (X, G) be a smooth projective G-variety. If (X, G) is stably linearizable, then (X, G) is H^1-trivial.*

Note that the inverse implication is not true in general (see Remark 8.17). Note also that the assertion of the theorem holds for any other definition of stable linearizability Remark 2.4(i)–(iii).

Our basic tool is the following theorem proved in [BP13].

THEOREM 2.9 [BP13]. *Let (X, G) be a nonsingular projective rational G-surface, where G is a cyclic group G of prime order p. Assume that G fixes (pointwise) a curve of genus $g > 0$. Then*

$$H^1(G, \mathrm{Pic}(X)) \simeq (\mathbb{Z}/p\mathbb{Z})^{2g}.$$

If $H^1(G, \mathrm{Pic}(X)) = 0$, then (X, G) is linearizable.

COROLLARY 2.10. *Let (X, G) be a nonsingular projective rational G-surface, where G is an arbitrary finite group. If (X, G) is H^1-trivial, then any nontrivial element of G does not fix a curve of positive genus.*

3. Group Actions on del Pezzo Surfaces

NOTATION 3.1. Let X be a del Pezzo surface of degree $d \le 6$, that is, $K_X^2 = d$. It is well known that X can be realized as the blowup $X \to \mathbb{P}^2$ of $r := 9 - d$ points in general position. The group $\mathrm{Pic}(X) \simeq \mathbb{Z}^{r+1}$ has a basis $\mathbf{h}, \mathbf{e}_1, \ldots, \mathbf{e}_r \in \mathrm{Pic}(X)$, where $\mathbf{h}$ is the pull-back of the class of a line on $\mathbb{P}^2$ and the $\mathbf{e}_i$ are the classes of exceptional curves.

Put

$$\Delta_r := \{\mathbf{x} \in \mathrm{Pic}(X) \mid \mathbf{x}^2 = -2, \mathbf{x} \cdot K_X = 0\}.$$

Then Δ_r is a root system in the orthogonal complement to K_X in $\mathrm{Pic}(X) \otimes \mathbb{R}$. Depending on d, the type of Δ_r is the following [Man74]:

d	1	2	3	4	5	6
Δ_r	E_8	E_7	E_6	D_5	A_4	$\mathrm{A}_1 \times \mathrm{A}_2$

REMARK 3.2. There is a natural homomorphism

$$\rho: \mathrm{Aut}(X) \longrightarrow \mathrm{W}(\Delta_r), \tag{3.1}$$

where $\mathrm{W}(\Delta_r)$ is the Weyl group of Δ_r. This homomorphism is injective if $d \le 5$ (see e.g. [Dol12, Corollary 8.2.32]).

Denote by $Q = \mathrm{Q}(\Delta_r)$ the sublattice of $\mathrm{Pic}(X)$ generated by the roots. Clearly, $\mathrm{Q}(\Delta_r)$ coincides with the lattice of integral points in $K_X^{\perp} \subset \mathrm{Pic}(X) \otimes \mathbb{R}$.

For an element $\delta \in \mathrm{W}(\Delta_r)$ or $\mathrm{Aut}(X)$, denote by $\mathrm{tr}(\delta)$ its trace on Q. Let $G \subset \mathrm{Aut}(X)$ be a (finite) subgroup, and let n be the order of G. Computing the character of the trivial subrepresentation, we get

$$\mathrm{rk}\,\mathrm{Pic}(X)^G = 1 + \frac{1}{n}\sum_{\delta \in G} \mathrm{tr}(\delta). \tag{3.2}$$

On the other hand, since $\mathrm{Tr}_{H^2(X,\mathbb{R})}(\delta) = 1 + \mathrm{tr}(\delta)$, by the Lefschetz fixed point formula we have

$$\mathrm{Eu}(X^{\delta}) = \mathrm{tr}(\delta) + 3. \tag{3.3}$$

Now we prove the implication (iii)⇒(i) of Theorem 1.2. By [Man74, Prop. 31.3] we have the following.

COROLLARY 3.3. *Let (X, G) be a projective G-surface. Let $\{C_i\}$ be a finite G-invariant set of irreducible curves whose classes generate* $\mathrm{Pic}(X)$. *If G acts on $\{C_i\}$ transitively, then $H^1(G, \mathrm{Pic}(X)) = 0$.*

PROPOSITION 3.4. *Let (X, G) be a projective nonsingular rational surface with $K_X^2 \ge 5$. Then $H^1(G, \mathrm{Pic}(X)) = 0$.*

Proof. To show that $H^1(G, \mathrm{Pic}(X)) = 0$, we may assume that (X, G) is G-minimal (otherwise, we replace X with its minimal model). If $K_X^2 \ge 8$, then X is either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_e$, and G acts on $\mathrm{Pic}(X)$ by (possibly trivial) permutation of the extremal rays. Hence, $\mathrm{Pic}(X)$ is a permutation G-module, and $H^1(G, \mathrm{Pic}(X)) = 0$. Thus, $K_X^2 = 6$ or 5, and X is a del Pezzo surface with $\mathrm{rk}\,\mathrm{Pic}(X)^G = 1$ (see [Isk80]).

If $K_X^2 = 6$, then X contains exactly six lines $C_1, \dots, C_6 \subset X$. Since $\mathrm{Pic}(X)^G = \mathbb{Z} \cdot K_X$, these lines form one G-orbit. By Corollary 3.3 we conclude that $H^1(G, \mathrm{Pic}(X)) = 0$.

Finally, consider the case $K_X^2 = 5$. Then $\mathrm{Aut}(X) \simeq \mathrm{W}(\mathrm{A}_4) \simeq \mathfrak{S}_5$ (see e.g. [Dol12, Thm. 8.5.8]). Let $\mathscr{L} := \{L_1, \dots, L_{10}\}$ be the set of lines on X. The action of G on $\mathscr{L}$ is faithful (see Remark 3.2). Let $\mathscr{L} = O_1 \cup \dots \cup O_l$ be the decomposition in G-orbits, and let r_i be the cardinality of O_i. Then $\sum r_i = 10$. Since $\mathrm{Pic}(X)^G = \mathbb{Z} \cdot K_X$, each number r_i is divisible by 5. By Corollary 3.3 we have only one possibility, $r_1 = r_2 = 5$. In particular, the order of G is divisible by 5. Then both O_1 and O_2 form anticanonical divisors, and the corresponding dual graphs are combinatorial cycles. In this case, G contains no elements of order 3. Hence, the order of G divides 20, and G has a normal subgroup $\langle \delta \rangle$ of order 5. Since $\mathrm{tr}(\delta) = -1$, by the Lefschetz fixed point formula $\mathrm{Eu}(X^\delta) = 2$. Write $X^\delta = V_1 \cup V_0$, where $V_0 \cap V_1 = \emptyset$, $\dim V_0 = 0$, and V_1 is of pure dimension one. The action of G preserves this decomposition. If $V_1 \neq \emptyset$, then V_1 meets the cycle of lines corresponding to O_1. But then δ acts on O_1 trivially, a contradiction. Hence, $V_1 \neq \emptyset$, and so δ has exactly two isolated fixed points $P_1, P_2 \in X$. By blowing $\{P_1, P_2\}$ up we get a cubic surface $\tilde{X}$ containing a G-invariant pair of skew lines. Then a well-known classical construction gives us a birational equivariant transformation $\tilde{X} \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ (cf. [DI09, §8]). Then by the considered case $K_X^2 = 8$ we have $H^1(G, \mathrm{Pic}(X)) = 0$. □

COROLLARY 3.5. *Let (X, G) be a G-del Pezzo surface described in* (1.1) *and* (1.2). *Then (X, G) is H^1-trivial.*

Proof. If $G' \subset G$ is a proper subgroup, then (X, G') is not minimal, and $H^1(G', \mathrm{Pic}(X)) = 0$ by Proposition 3.4. It is easy to see that the set of lines on X has exactly two G-orbits consisting of 4 and 12 elements. Then $H^1(G, \mathrm{Pic}(X)) = 0$ by [Man74, Ch. 4, Sect. 31, Table 2]. □

The implication (ii)⇒(iii) of Theorem 1.2 is an immediate consequence of the following proposition which will be proved in Sections 4–7.

PROPOSITION 3.6. *Let (X, G) is a minimal G-del Pezzo surface of degree ≤ 4 such that any nonidentity element of G does not fix a curve of positive genus. Then (X, G) is isomorphic to a G-surface described in* (1.1) *and* (1.2).

4. Quartic del Pezzo Surfaces

NOTATION 4.1. Throughout this section, let X be a del Pezzo surface of degree 4. It is well known that the anticanonical linear system embeds X to $\mathbb{P}^4$ so that the image is a complete intersection of two quadrics. In a suitable coordinate system in $\mathbb{P}^4$, the equations of X can be written in the form

$$\sum_{i=0}^{4} x_i^2 = \sum_{i=0}^{4} \theta_i x_i^2 = 0, \tag{4.1}$$

where the θ_i are distinct constants (see e.g. [Dol12, Lemma 8.6.1]). We regard these constants $\theta_i \in \Bbbk$ as points of a projective line. In other words, quadrics passing through X form a pencil $\mathscr{Q}$ and the points θ_i correspond to degenerate members of $\mathscr{Q}$. Five commuting involutions $\tau_i : x_i \mapsto -x_i$ generate a normal Abelian subgroup $A \subset \mathrm{Aut}(X)$ with a unique relation $\tau_1 \cdots \tau_5 = \mathrm{id}$. Thus,

$$A = \{1, \tau_k, \tau_i\tau_j \mid 1 \le k \le 5, 1 \le i < j \le 5\}, \qquad A \simeq (\mathbb{Z}/2\mathbb{Z})^4.$$

4.1. Root System D_5

It is well known (see e.g. [Bou02]) that the root system of type D_5 can be realized as the set $\pm\mathbf{r}_i \pm \mathbf{r}_j$, where $\mathbf{r}_1, \ldots, \mathbf{r}_5$ is the standard basis of $\mathbb{R}^5$. The Weyl group $W(D_5)$ is the semidirect product $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{S}_5$, where $(\mathbb{Z}/2\mathbb{Z})^4$ acts on $\mathbb{R}^5$ by $\mathbf{r}_i \mapsto (\pm 1)_i \mathbf{r}_i$ so that $\prod_i (\pm 1)_i = 1$, and $\mathfrak{S}_5$ acts on $\mathbb{R}^5$ by permutations of the $\mathbf{r}_i$.

The image $\rho(A) \subset W(D_5)$ under the injection (3.1) coincides with $(\mathbb{Z}/2\mathbb{Z})^4 \subset (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{S}_5$. Thus, we identify $\rho(A)$ with $(\mathbb{Z}/2\mathbb{Z})^4$ and $\rho(\tau_i)$ with τ_i. Note the fixed point locus of each τ_i is an elliptic curve that cuts out on X by the hyperplane $\{x_i = 0\}$ (and so the τ_i are de Jonquières involutions of genus 1). The fixed point loci of other involutions in A consist of exactly four points. Therefore,

$$\mathrm{tr}(\tau_i) = -3 \quad \forall i, \qquad \mathrm{tr}(\tau_i\tau_j) = 1 \quad \forall i \ne j. \tag{4.2}$$

Another, intrinsic description of the τ_i is as follows. On X, there are 10 pencils of conics $\mathscr{C}_1, \ldots, \mathscr{C}_5, \mathscr{C}'_1, \ldots, \mathscr{C}'_5$ satisfying the conditions $\mathscr{C}_i \cdot \mathscr{C}'_i = 2$, $\mathscr{C}_i \cdot \mathscr{C}_j = \mathscr{C}_i \cdot \mathscr{C}'_j = 1$ for $i \ne j$ and $\mathscr{C}_i + \mathscr{C}'_i \sim -K_X$. Two "conjugate" pencils $\mathscr{C}_i$ and $\mathscr{C}'_i$ define a double cover $\psi_i : X \to \mathbb{P}^1 \times \mathbb{P}^1$. Then τ_i is the Galois involution of ψ_i. Note that ψ_i coincides with the projection of X from the vertex of a singular quadric of the pencil generated by (4.1). Thus, there are the following canonical bijections:

$$\{\tau_i\} \longleftrightarrow \{\psi_i\} \longleftrightarrow \{(\mathscr{C}_i, \mathscr{C}'_i)\} \longleftrightarrow \{\theta_i\}, \quad i = 1, \ldots, 5. \tag{4.3}$$

The group $\mathrm{Aut}(X)$ acts on the pencil of quadrics $\mathscr{Q}_\lambda$ in $\mathbb{P}^4$ generated by (4.1) so that the set of degenerate quadrics corresponding to the values $\lambda = \theta_i$, $i = 1, \ldots, 5$, is preserved. Hence, there exist homomorphisms

$$\rho_1 : \mathrm{Aut}(X) \to PGL_2(\Bbbk), \qquad \rho_2 : \mathrm{Aut}(X) \to \mathfrak{S}_5$$

with $\ker(\rho_1) = \ker(\rho_2) = A$. This immediately gives us the following possibilities for the group $\mathrm{Aut}(X)/A$ (see [DI09, Sect. 6]):

$$\{1\}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathfrak{S}_3, \mathfrak{D}_5. \tag{4.4}$$

4.2. Assumption

Now let a finite group G faithfully act on X so that (X, G) is minimal (i.e. $\mathrm{Pic}(X)^G \simeq \mathbb{Z}$) and any nonidentity element of G does not fix a curve of positive genus. Denote $A_G := G \cap A$. For short, we identify $\rho(G)$ with G.

Recall that $K_X^2 = 4$. Let $\mathscr{L} := \{L_1, \dots, L_{16}\}$ be the set of lines on X. Let $\mathscr{L} = O_1 \cup \dots \cup O_l$ be the decomposition in G-orbits, and let r_i be the cardinality of O_i. Then $\sum r_i = 16$. Since $\mathrm{Pic}(X)^G = \mathbb{Z} \cdot K_X$, each number r_i is divisible by 4.

By our assumption in 4.2 we have the following.

COROLLARY 4.2. *$G \not\ni \tau_i$ for $i = 1, \dots, 5$.*

The following lemma is an immediate consequence of the description of A.

LEMMA 4.3. *There are two kinds of nontrivial subgroups $A' \subset A$ satisfying the property $A' \not\ni \tau_i$ for $i = 1, \dots, 5$:*

- *$A_{i,j} = \{1, \tau_i\tau_j \mid i \neq j\}$, and*
- *$A_{k,l,m} = \{1, \tau_k\tau_l, \tau_l\tau_m, \tau_k\tau_m \mid k \neq l \neq m \neq k\}$.*

REMARK 4.4. Note that if $A_G = A_{i,j}$, then A_G is contained in the center of G. Using (4.2), we immediately conclude that

$$\sum_{\upsilon \in A_G} \mathrm{tr}(\upsilon) = \begin{cases} 6 & \text{if } A_G = A_{i,j}, \\ 8 & \text{if } A_G = A_{k,l,m}. \end{cases} \tag{4.5}$$

For G/A_G, we have the same possibilities (4.4) as for $\mathrm{Aut}(X)/A$. Consider these possibilities case by case. By (4.5) and (3.2), $G \neq A_G$.

4.3. Cases $G/A_G \simeq \mathbb{Z}/5\mathbb{Z}$ and $\mathfrak{D}_5$

The order of G divides 40. By Sylow's theorem the Sylow 5-subgroup $G_5 \subset G$ is normal. By Assumption 4.2 we see that $r_i \not\equiv 0 \mod 5$ for all i. Hence, G_5 is contained in the stabilizer of any line $L \in \mathscr{L}$. But then the action of G on $\mathscr{L}$ and on $\mathrm{Pic}(X)$ is not faithful, a contradiction.

4.4. Case $G/A_G \simeq \mathbb{Z}/3\mathbb{Z}$

For convenience of the reader, we reproduce here the following fact from [DI09, Sect. 6]:

LEMMA 4.5 [DI09, Sect. 6]. *Let X be a quartic del Pezzo surface, and let $\gamma \in$ $\mathrm{Aut}(X)$ be an element of order 3. Then X is isomorphic to the surface given by (1.1). Moreover,* $\mathrm{Aut}(X) \simeq A \rtimes \mathfrak{S}_3$. *The center of* $\mathrm{Aut}(X)$ *is of order 2 and generated by an element of the form $\tau_i\tau_j$, $i \neq j$.*

Proof. Since X contains exactly 16 lines, there exists at least one γ-invariant line $L \subset X$. Let $L_1, \dots, L_5 \subset X$ be (skew) lines meeting L, and let $f : X \to \mathbb{P}^2$ be the contraction of $L_1, \dots, L_5$. Let $C := f(L)$ and $P_i = f(L_i)$. Then the action of γ on X is induced by one on $C \subset \mathbb{P}^2$. Up to permutation of $L_1, \dots, L_5$, we may assume that γ fixes P_1 and P_2 and permutes P_3, P_4, P_5. Then the set $\{P_1, \dots, P_5\}$ is unique up to projective equivalence. Hence, X is unique up to isomorphism. On the other hand, it is easy to see that the surface (1.1) admits an isomorphism γ of order 3 given by (1.2). Moreover, $\mathrm{Aut}(X)$ contains the group $A \rtimes \mathfrak{S}_3$ generated by A, γ, and

$$\beta : (x_1, x_2, x_3, x_4, x_5) \longmapsto (x_1, x_3, x_2, x_5, x_4).$$

By (4.4) we see that $\mathrm{Aut}(X) = A \rtimes \mathfrak{S}_3$. □

COROLLARY 4.6. *Let $\gamma \in \mathrm{Aut}(X)$ be an element of order 3. Then X^γ consists of exactly five points.*

By Corollary 4.6 the exists a G-fixed point $P \in X$. Since in a neighborhood of P the action of $(\mathbb{Z}/2\mathbb{Z})^2$ cannot be free in codimension one, we have $A_G = A_{i,j}$ for some $i \neq j$. Hence, G is cyclic of order 6. Since the cardinality of any orbit $O_i \subset \mathscr{L}$ must be divisible by 4, we get a contradiction.

4.5. *Case $G/A_G \simeq \mathfrak{S}_3$*

We show that only the possibility (iii)(b) of Theorem 1.2 occurs here. Let G_3 (resp. G_2) be a Sylow 3-subgroup (resp. 2-subgroup) of G. Clearly, $G_2 \supset A_G$ and $G_2/A_G \simeq \mathbb{Z}/2\mathbb{Z}$. By Lemma 4.5, X is isomorphic to the surface given by (1.1), $\mathrm{Aut}(X) \simeq A \rtimes \mathfrak{S}_3$, and the center of $\mathrm{Aut}(X)$ is generated by an element $\tau_i\tau_j$, $i \neq j$.

LEMMA 4.7. *In the above settings, the image of the natural representation $\rho : \mathrm{Aut}(X) \hookrightarrow \mathrm{W}(\mathrm{D}_5) \subset GL(Q)$ is contained in $SL(Q)$.*

Proof. By the description of D_5 in 4.1 we can write the elements of A in a diagonal form so that $A \subset SL(Q)$ and the determinant of any element of $\mathrm{W}(\mathrm{D}_5)$ equals ± 1. The fixed point locus of β consists of a smooth rational curve and a pair of isolated points. Hence, $\mathrm{tr}(\beta) = 1$, and so $\det(\beta) = 1$. This implies that the image of the whole group $\mathrm{Aut}(X)$ is contained in $SL(Q)$. □

4.5.1 Assume that $A_G = A_{i,j,k}$. Since elements of A_G and G_3 do not commute, G_3 is not normal in G. By Sylow's theorem the number of Sylow 3-subgroups equals to 4. The action on the set of these subgroups induces an isomorphism $G \simeq \mathfrak{S}_4$. By Corollary 4.6 for the elements $\gamma \in G$ of order 3, we have $\mathrm{tr}(\gamma) = 2$. Hence, by (4.5) and (3.2)

$$\sum_{\upsilon \in \mathfrak{A}_4} \mathrm{tr}(\upsilon) = 24, \qquad \sum_{\upsilon \in \mathfrak{S}_4 \setminus \mathfrak{A}_4} \mathrm{tr}(\upsilon) = -24.$$

Since $\mathrm{Eu}(X^\upsilon) > 0$ for all $\upsilon \in G$, we have $\mathrm{tr}(\upsilon) = -2$ for all $\upsilon \in \mathfrak{S}_4 \setminus \mathfrak{A}_4$. In our case, $\dim Q = 5$. Hence, $\mathrm{tr}(\upsilon)$ must be odd for an element of order 2, a contradiction.

4.5.2 Thus, $A_G = A_{i,j}$. Then G_3 is normal in G, and so G is a semi-direct product $G = G_3 \rtimes G_2$ that is not a direct product because G is not abelian. For short, we identify G with its image in $\mathrm{W}(\mathrm{D}_5) \subset GL(Q)$. We claim that G_2 is cyclic. Indeed, otherwise $G \simeq \mathfrak{S}_3 \times (\mathbb{Z}/2\mathbb{Z})$. It is easy to check that in this case, Q must contain a trivial G-representation (because $G \subset SL(Q)$ by

Lemma 4.7). Since $\operatorname{Pic}(X)^G \simeq \mathbb{Z}$, this is impossible. Therefore, $G_2 \simeq \mathbb{Z}/4\mathbb{Z}$ and $G \simeq (\mathbb{Z}/3\mathbb{Z}) \rtimes (\mathbb{Z}/4\mathbb{Z})$. Up to permutations of coordinates, we may assume that the center of $\operatorname{Aut}(X)$ is generated by

$$\delta = \tau_4\tau_5 : (x_1, x_2, x_3, x_4, x_5) \longmapsto (x_1, x_2, x_3, -x_4, -x_5).$$

Clearly, the center of G commutes with all elements of $\operatorname{Aut}(X)$. Thus, $\delta \in G$. Now let $\beta^\bullet$ (resp. $\gamma^\bullet$) be an element of G of order 4 (resp. 3) whose image in $\mathfrak{S}_3$ coincides with β (resp. γ). Thus, $\beta^\bullet(x_i) = \pm\beta(x_i)$ and $\gamma^\bullet(x_i) = \pm\gamma(x_i)$ for all i. Since $\gamma^{\bullet 3} = \mathrm{id}$, replacing x_i with $\pm x_i$, we may assume that $\gamma^\bullet = \gamma$. Since $(\beta^\bullet)^2 = \delta$ and $\beta^\bullet\gamma\beta^{\bullet -1} = \gamma^{-1}$, as before, we get $\beta^\bullet = \beta'$. Thus, our group G coincides with that constructed in (1.1) and (1.2). It remains to show that this group is minimal. Let $\nu \in G$ be an element of even order $2k$. Then $\nu^k = \delta$, and so $X^\nu = (X^\delta)^\nu$. Recall that X^δ is a set of four points. Then one can easily see that $\operatorname{Eu}(X^\nu) = 1$ (resp. 2) if $k = 3$ (resp. 2). Thus, we have

$$\sum_{\upsilon \in G} \operatorname{tr}(\upsilon) = 5 + 1 + 2 \cdot 2 - 2 \cdot 2 - 6 \cdot 1 = 0.$$

By (3.2) we have $\operatorname{rk}\operatorname{Pic}(X)^G = 1$, that is, G is minimal.

REMARK 4.8. Note that our group G acts on X^{G_3} and by Corollary 4.6 there is a G-fixed point $P \in X^{G_3}$ such that P does not lie on any line. Let $\tilde{X} \to X$ be the blowup of P. Then $\tilde{X}$ is a cubic surface admitting an action of G such that $\operatorname{rk}\operatorname{Pic}(\tilde{X})^G = 2$. The exceptional divisor is an invariant line $L \subset \tilde{X}$, and the projection from L gives a structure of G-equivariant conic bundle $\tilde{X} \to \mathbb{P}^1$. Thus, we are in the situation described further in Theorem 8.5 and Construction 8.7 (with $n = 3$).

4.6. *Case $G/A_G \simeq \mathbb{Z}/2\mathbb{Z}$*

Since $\operatorname{Pic}(X)^G \simeq \mathbb{Z}$, $A_G \neq \{1\}$. Assume that $A_G = A_{i,j}$ for some i, j. Then by (4.5) we have $\sum_{\delta \in G \setminus A_G} \operatorname{tr}(\delta) = -6$. Hence, there exists $\delta \in G \setminus A_G$ such that $\operatorname{Eu}(X^\delta) \le 0$. Since $X^\delta \neq \emptyset$, the element δ fixes pointwise a curve of positive genus. This contradicts Assumption 4.2. Therefore, $A_G = A_{i,j,k}$ for some i, j, k. In particular, G is a (noncyclic) group of order 8. Again by (4.5) we have $\sum_{\delta \in G \setminus A_G} \operatorname{tr}(\delta) = -8$ and $\operatorname{Eu}(X^\delta) > 0$ for all $\delta \in G \setminus A_G$. Hence, $\operatorname{Eu}(X^\delta) = 1$ for all $\delta \in G \setminus A_G$. This means that any element $\delta \in G \setminus A_G$ has a unique fixed point and the action of G on X is free in codimension 1. Applying the holomorphic Lefschetz fixed point formula, we obtain that any $\delta \in G \setminus A_G$ has at least two fixed points, a contradiction.

4.7. *Case $G/A_G \simeq \mathbb{Z}/4\mathbb{Z}$*

Note that the stabilizer of $A_{i,j}$ (and $A_{k,l,m}$) in $\mathfrak{S}_5 = \mathrm{W}(\mathrm{D}_5)/A$ is the group $\mathfrak{S}_2 \times \mathfrak{S}_3$. Hence, neither $A_{i,j}$ nor $A_{k,l,m}$ can be a normal subgroup of G. Thus, $A_G = \{1\}$. Again we have $0 = 5 + \operatorname{tr}(\delta^2) + 2\operatorname{tr}(\delta)$, where $\operatorname{tr}(\delta)$, $\operatorname{tr}(\delta^2) \ge -2$ by (3.3) because G does not fix a curve of positive genus. We get only one possibility:

$\operatorname{tr}(\delta^2) = -1$, $\operatorname{tr}(\delta) = -2$. Hence, X^G is a point, say P, and X^{δ^2} is either a smooth rational curve or a pair of points. On the other hand, $X^{\delta^2} \ni P$ and G acts on X^{δ^2} fixing P, a contradiction.

Thus, Proposition 3.6 is proved in the case $K_X^2 = 4$.

5. Cubic Surfaces

NOTATION 5.1. Throughout this section, X denotes a cubic surface $X \subset \mathbb{P}^3$. Let $G \subset \operatorname{Aut}(X)$ be a subgroup such that (X, G) is minimal and any nonidentity element of G does not fix a curve of positive genus. Since the embedding $X \subset \mathbb{P}^3$ is anticanonical, it is G-equivariant. By our assumption, for any element $1 \neq \delta \in G$, the set $(\mathbb{P}^3)^\delta$ does not contain any hyperplane. Let $\psi(x_1, x_2, x_3, x_4) = 0$ be the equation of X. We choose homogeneous coordinates in $\mathbb{P}^3$ so that δ has a diagonal form.

CLAIM 5.2. *Let $\tau \in G$ be an element of order 2. Then in suitable coordinates, its action on $\mathbb{P}^3$ has the form $\tau = \operatorname{diag}(1, 1, -1, -1)$, and*

$$\psi = \psi_3(x_1, x_2) + x_1\psi_2(x_3, x_4) + x_2\psi_2'(x_3, x_4),$$

where $\deg \psi_3 = 3$, $\deg \psi_2 = \deg \psi_2' = 2$, and ψ_3 has no multiple factors. Furthermore, $X^\tau = L(\tau) \cup \{P_1, P_2, P_3\}$, where $L(\tau) := \{x_1 = x_2 = 0\}$ and $\{P_1, P_2, P_3\} = X \cap \{x_3 = x_4 = 0\}$. In particular, $\operatorname{Eu}(X^\tau) = 5$.

Proof. Since $(\mathbb{P}^3)^\tau$ does not contain any hyperplane, we can write $\tau = \operatorname{diag}(1, 1, -1, -1)$. Replacing τ with $-\tau$, we may assume that ψ is invariant. The rest is obvious. □

CLAIM 5.3. *Let $\tau \in G$ be an element of order 3. Then the fixed point locus X^τ is zero-dimensional, and $\operatorname{Eu}(X^\tau) \geq 3$.*

Proof. Up to permutations of coordinates, we may assume that δ has the form $\operatorname{diag}(1, 1, \zeta_3, \zeta_3)$ or $\operatorname{diag}(1, 1, \zeta_3, \zeta_3^{-1})$. Assume that $\dim X^\tau = 1$. By the preceding there exists a line $L \subset X^\tau$. It is well known that a given line L on a cubic surface meets exactly 10 other lines $L_1, \dots, L_{10}$ and up to reenumeration one can assume that the lines $\{L_1, \dots, L_5\}$ (resp. $\{L_6, \dots, L_{10}\}$) are mutually disjoint. Then each line L_i must be δ-invariant (because $L_i \cap L$ is a fixed point). In this case, the classes of $L_1, \dots, L_5$ are contained in $\operatorname{Pic}(X)^\delta$ and linearly independent there. Since the canonical class K_X is also δ-invariant, we see that the action of δ on $\operatorname{Pic}(X)$ must be trivial. This contradicts the injectivity of $\rho : \operatorname{Aut}(X) \longrightarrow \mathrm{W}(\mathrm{E}_6)$ (see Remark 3.2).

Thus, $\dim X^\tau = 0$. On the other hand, $X^\tau \neq \emptyset$ and $\operatorname{tr}(\tau) = 3$, 0, or -3. Hence, $\operatorname{Eu}(X^\delta) = 6$ or 3. □

LEMMA 5.4. *For any element $\delta \in G$, we have $\operatorname{tr}(\delta) \geq 0$ except for the following case:*

(*) $\operatorname{ord}(\delta)=6$, $\operatorname{tr}(\delta)=-1$, *$X^\delta$ consists of two points* $X^\delta = L(\delta^3)^\delta = \{R_1, R_2\}$, *where $L(\delta^3)$ is the line introduced in Claim 5.2. Moreover, in the local coordinates near R_i, the action of δ^2 is given by a scalar matrix.*

Proof. By [CCN+85] the orders of elements of $\mathrm{W(E_6)}$ are as follows: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12. Consider the possibilities for $\delta \in G$. Let $\chi(t)$ be the characteristic polynomial of δ on Q. Clearly, $\deg\chi = 6$, and χ is a product of cyclotomic polynomials Φ_d, where d divides $\operatorname{ord}(\delta)$.

If $\operatorname{ord}(\delta)\le 3$, then $\operatorname{tr}(\delta)\ge 0$ by Claims 5.2 and 5.3. Thus, we may assume that $\operatorname{ord}(\delta)\ge 4$. If $\operatorname{ord}(\delta)=5$, then the only possibility is $\chi=\Phi_5\Phi_1^2 = t^6-t^5-t+1$ and $\operatorname{tr}(\delta)=1$. If $\operatorname{ord}(\delta)=9$, then again we have $\chi=\Phi_9=t^6+t^3+1$ and $\operatorname{tr}(\delta)=0$.

It remains to consider the case where the order of δ is even, so $\operatorname{ord}(\delta)=2m$, $m=2, 3, 4, 5$, or 6. Then δ^m is described in Claim 5.2, and so

$$X^\delta = L^\delta \cup \{P_1, P_2, P_3\}^\delta,$$

where $L := L(\delta^m)$, and the points P_1, P_2, P_3 lie on one line in $\mathbb{P}^3$. Here L^δ either is a couple of points or coincides with L. Hence, $\operatorname{Eu}(L^\delta)=2$ and $\{P_1, P_2, P_3\}^\delta = \emptyset$ if and only if δ permutes all the P_i. Thus, $\operatorname{Eu}(X^\delta)\le 2$ only if $m=3$, $\operatorname{tr}(\delta)=-1$, and $X^\delta = L^\delta$. Consider the blow-down $X\to X'$ of L to a point, say R. Since δ^2 acts on X freely in codimension one (see Claim 5.3), in the local coordinates near R, the action of δ^2 can be written as $\operatorname{diag}(\zeta_3, \zeta_3^{-1})$. Then it is easy to see that in the local coordinates near R_i, the action can be written as $\operatorname{diag}(\zeta_3^k, \zeta_3^k)$, $k=1$ or 2. □

Proof of Proposition 3.6 in the case $K_X^2=3$. Since (X, G) is minimal, we have $\sum_{\delta\in G}\operatorname{tr}(\delta)=0$ by (3.2). Hence, $\operatorname{tr}(\delta)<0$ for some $\delta\in G$. By Lemma 5.4 we have $\operatorname{ord}(\delta)=6$ and $\operatorname{tr}(\delta)=-1$. Let $G_1,\dots,G_r\subset G$ be all cyclic subgroups generated by such elements δ_i of order 6. We claim that $\delta_i^3\ne\delta_j^3$ for $i\ne j$. Assume the converse: $\delta_i^3=\delta_j^3 := \tau$. The element τ is described in Claim 5.2. Put $L := L(\tau)$. The projection from L defines a $\langle\delta_i,\delta_j\rangle$-equivariant conic bundle structure $f: X\to\mathbb{P}^1$ so that the restriction $f|_L: L\to\mathbb{P}^1$ is a double cover. It has two ramification points $R_1, R_2\in L$. Since each δ_i has exactly two fixed points, we have $X^{G_i}=X^{G_j}=\{R_1, R_2\}$.

Replacing δ_j with $\delta_j^{\pm 1}$, we may assume that the action of δ_i^2 and δ_j^2 on $T_{R_1,X}$ has the form $\operatorname{diag}(\zeta_3,\zeta_3)$. Hence, $\delta_i^2=\delta_j^2$, and so $\delta_i=\delta_j$, which proves our claim. In particular, we see that for $i\ne j$, the intersection $G_i\cap G_j$ does not contain any elements of order 2. Then by (3.2)

$$0=\sum_{\delta\in G}\operatorname{tr}(\delta) > \sum_{i=1}^{r}(\operatorname{tr}(\delta_i)+\operatorname{tr}(\delta_i^{-1})+\operatorname{tr}(\delta^3))=0.$$

The contradiction proves Proposition 3.6 in the case $K_X^2=3$. □

6. Del Pezzo Surfaces of Degree 2

NOTATION 6.1. Throughout this section, X denotes a del Pezzo surface of degree 2. Recall that the anticanonical map is a double cover $X \to \mathbb{P}^2$ branched over a smooth quartic $R \subset \mathbb{P}^2$. Let $\psi(x_0, x_1, x_2) = 0$ be the equation of R. Then X can be given by the equation $y^2 = \psi(x_0, x_1, x_2)$ in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$. The Galois involution $\gamma : X \to X$ of the double cover $X \to \mathbb{P}^2$ is called the *Geiser involution.* It is contained in the center of $\mathrm{Aut}(X)$, and X^γ is a curve of genus 3. For any $\mathbf{x} \in \mathrm{Pic}(X)$, the element $\mathbf{x} + \gamma^*\mathbf{x}$ is the pull-back of some element of $\mathrm{Pic}(\mathbb{P}^2)$.

By (3.2) (cf. proof of Proposition 3.6 in the case $K_X^2 = 3$) to establish Proposition 3.6 in the case $K_X^2 = 2$, it is sufficient to prove the following.

LEMMA 6.2. *Let $G \subset \mathrm{Aut}(X)$ be a finite subgroup such that any nonidentity element of G does not fix a curve of positive genus. Then $\mathrm{tr}(\delta) \geq 0$ for any $\delta \in G$.*

Proof. It is known that the center of $\mathrm{W}(\mathrm{E}_7)$ is a cyclic group of order 2 generated by the element γ induced by the Geiser involution of X and acting as minus identity on $\mathrm{Q}(\mathrm{E}_7)$. The quotient $\mathrm{W}(\mathrm{E}_7)/\langle\gamma\rangle$ is the (unique) simple group of order 1,451,520 isomorphic to $PSp_6(\mathbb{F}_2)$. Let $\bar{G}$ be the image of G in $\mathrm{W}(\mathrm{E}_7)/\langle\gamma\rangle$. By our assumption the group G does not contain γ. Hence, $G \simeq \bar{G}$. Using the description of conjugacy classes in $PSp_6(\mathbb{F}_2)$ (see [CCN+85]), we obtain that the order of any element of G is one of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15. Consider these possibilities case by case. Let $\chi_\delta(t)$ denote the characteristic polynomial of the action of $\delta \in G$ on $Q \otimes \mathbb{Q}$.

6.1. Case: G Has an Element of Order 2

Let $\tau \in G$ be an element of order 2. For the action on $\mathbb{P}^2$, we have only one possibility $\tau : (x_0 : x_1 : x_2) \longmapsto (-x_0 : x_1 : x_2)$, and then ψ has the form $x_0^4 + x_0^2\psi_2(x_1, x_2) + \psi_4(x_1, x_2) = 0$, where ψ_4 has no multiple factors (because B is smooth). For the action on X, we have two possibilities:

$$\tau : (x_0 : x_1 : x_2 : y) \longmapsto (-x_0 : x_1 : x_2 : y), \tag{6.1}$$

$$\tau : (x_0 : x_1 : x_2 : y) \longmapsto (-x_0 : x_1 : x_2 : -y). \tag{6.2}$$

Since X^τ is an elliptic curve in the case (6.1), this case does not occur. Thus, we are in the situation of (6.2). Then X^τ consists of four points. By (3.3) we have $\mathrm{tr}(\tau) = 1$. Moreover, $\chi_\tau = \Phi_1^4\Phi_2^3$.

6.2. Case: G Has an Element of Order 4

Assume that G contains an element δ of order 4. Then $\delta^2 = \tau$, where τ is described in 6.1. On the other hand, $\chi_\delta = \Phi_4^k\Phi_2^l\Phi_1^m$, where $k > 0$. Then $\chi_\tau = \Phi_2^{2k}\Phi_1^{7-2k}$. This contradicts 6.1. Thus, G does not contain any elements of order divisible by 4.

6.3. Case: G Has an Element of Order 3

Let $\theta \in G$ be an element of order 3. We have two possibilities for the action on X:

$$\begin{aligned} \theta : (x_0 : x_1 : x_2 : y) &\longmapsto (\zeta_3 x_0 : x_1 : x_2 : y), \\ \psi &= x_0^3 \psi_1(x_1, x_2) + \psi_4(x_1, x_2), \end{aligned} \tag{6.3}$$

$$\begin{aligned} \theta : (x_0 : x_1 : x_2 : y) &\longmapsto (x_0 : \zeta_3 x_1 : \zeta_3^2 x_2 : y), \\ \psi &= x_0^4 + a_2 x_0^2 x_1 x_2 + x_0 x_1^3 + x_0 x_2^3 + a_0 x_1^2 x_2^2. \end{aligned} \tag{6.4}$$

In the case (6.3), the intersection $X \cap \{x_0 = 0\}$ is an elliptic curve of fixed points. This contradicts our assumption.

Thus, we have case (6.4). Then X^θ consists of four points, and so $\mathrm{tr}(\theta) = 1$. Hence, $\chi_\theta = \Phi_1^3 \Phi_3^2$.

6.4. Case: G Has an Element of Order 6

Let $\delta \in G$ be an element of order 6. Then $\delta = \tau\theta$, where τ (resp. θ) is described in the case 6.1 (resp. 6.3). Hence, $\mathrm{tr}(\delta) = -5$ or 1. But in the first case, $\mathrm{Eu}(X^\delta) = -2$, and so $\dim X^\delta = 1$. On the other hand, $X^\delta \subset X^\tau$, where $\dim X^\tau = 0$. The contradiction shows that $\mathrm{tr}(\delta) = 1$.

6.5. Case: G Has an Element of Order 9

Let $\delta \in G$ be an element of order 9. Since χ_δ is divisible by the cyclotomic polynomial Φ_9, we have $\chi_\delta = \Phi_9 \Phi_1$, and so $\mathrm{tr}(\delta) = 1$. The same arguments show that $\mathrm{tr}(\delta) \geq 0$ if δ is an element of order 5 or 7.

6.6. Case: G Has an Element of Order 15

Let $\delta \in G$ be an element of order 15. As in case 6.5, we see that $\chi_\delta = \Phi_5 \Phi_3 \Phi_1$. Hence, $\chi_{\delta^5} = \Phi_3 \Phi_1^5$. This contradicts the result of 6.3.

This finishes the proof of Lemma 6.2. □

7. Del Pezzo Surfaces of Degree 1

NOTATION 7.1. Throughout this section, let X be a del Pezzo surface of degree 1. Recall that in this case, the linear system $|-2K_X|$ determines a double cover $X \to Y \subset \mathbb{P}^3$, where Y is a quadratic cone. The corresponding Galois involution $\beta : X \to X$ is called the *Bertini involution.* Its fixed point locus X^β is the union of a curve of genus 4 and a single point P. As in the case $K_X^2 = 2$, β is contained in the center of $\mathrm{Aut}(X)$, and $-\beta$ acts on $\mathrm{Pic}(X)$ as the reflection with respect to $Q = K_X^\perp$.

The linear system $|-K_X|$ is an elliptic pencil whose base locus coincides with P (a single point). The natural representation $\mathrm{Aut}(X) \to GL(T_{P,X})$ is faithful. Let $\pi : X \dashrightarrow B = \mathbb{P}^1$ be the map given by $|-K_X|$. Here B can be naturally identified with $\mathbb{P}(T_{P,X})$. Every singular member F of $|-K_X|$ is an irreducible curve of arithmetic genus 1. Hence, F is a rational curve with a unique singularity R, which is either a node or a simple cusp. Computing the topological Euler number, we obtain the following.

LEMMA 7.2. *Let* $\#_{\text{node}}$ (*resp.* $\#_{\text{cusp}}$) *be the number of nodal* (*resp. cuspidal rational curves*) *in the pencil* $|-K_X|$. *Then*

$$\#_{\text{node}} + 2\#_{\text{cusp}} = 12.$$

LEMMA 7.3. *Any element* $\iota \in \mathrm{Aut}(X)$ *of order* 2 *fixes a curve of positive genus.*

Proof. There are two choices for the action of ι on $T_{P,X}$: $\mathrm{diag}(-1,-1)$ and $\mathrm{diag}(-1,1)$. In the first case, the action coincides with the action on $T_{P,X}$ of the Bertini involution β. Hence, $\iota \circ \beta^{-1}$ acts trivially on $T_{P,X}$, and so $\iota \circ \beta^{-1}$ is the identity map. In this case, X^{ι} contains a curve of genus 4. Assume that ι acts on $T_{P,X}$ as $\mathrm{diag}(-1,1)$. Then the fixed point locus of ι contains a smooth curve C passing through P, and the action on $B \simeq \mathbb{P}(T_{P,X})$ is not trivial. Then the restriction $\pi|_C : C \to B$ cannot be dominant. Hence, C is a fiber of π, and so C is an elliptic curve. □

LEMMA 7.4. *Let* $G = \langle \delta \rangle \subset \mathrm{Aut}(X)$ *be a group of order* 3. *Assume that the representation of* G *in* $GL(T_{P,X})$ *is given by a scalar matrix. Then the pair* (X, G) *is minimal, and* X^G *contains a curve of genus* 2.

Proof. Clearly, the action of δ on $B \simeq \mathbb{P}(T_{P,X})$ is trivial. We claim that X^{δ} is the union of a smooth irreducible curve C and P. Indeed, if X^{δ} contains an isolated point $R \neq P$, then π is well defined at R, and the action of δ on $T_{R,X}$ in suitable coordinates has the diagonal form $\mathrm{diag}(\zeta_3, \zeta_3^{\pm 1})$. Let $F = \pi^{-1}(\pi(R))$ be the fiber of π passing through R. Since the action on B is trivial, the differential $d\pi : T_{R,X} \to T_{\pi(R),B}$ is not surjective. Hence, $R \in F$ is a singular point. Let $\nu : F' \to F$ be the normalization. If $R \in F$ is a node, then the cyclic group G has three fixed points $\nu^{-1}(R)$ and P on $F' \simeq \mathbb{P}^1$, a contradiction. Hence, $R \in F$ is a cusp. Then locally near R the map ν is given by $t \mapsto (t^2, t^3)$. So the action near R is not free in codimension one. Again we get a contradiction.

Thus, X^{δ} consists of P and a smooth curve C. Since $P \not\ni C$, C contains no fibers of π. Let F_1 be a degenerate fiber of π. The action of G on F_1 has exactly two fixed points: P and $R := \mathrm{Sing}(F_1)$. Hence, $C \cap F_1 = R$, and so C is connected. Since C is smooth, it must be irreducible.

Denote $r := \mathrm{rk}\,\mathrm{Pic}(X)^G$. By (3.2) and (3.3)

$$\mathrm{Eu}(X^{\delta}) = 1 + 2 - 2g(C) = 3 + \mathrm{tr}(\delta) = 2 + r - \frac{1}{2}(9 - r).$$

The only solution is $r = 1$, $g(C) = 2$. Then (X, G) is minimal. □

LEMMA 7.5. *Let* $\{1\} \neq G \subset \mathrm{Aut}(X)$ *be a group such that the induced action on the pencil* B *is trivial. Then some nonidentity element of* G *fixes a curve of positive genus.*

Proof. The group G is contained in the kernel of the composition

$$G \to GL(T_{P,X}) \to PGL(T_{P,X}).$$

Hence, the image of G in $GL(T_{P,X})$ consists of scalar matrices, and so G is a cyclic group. Let $\delta \in G$ be a generator, and let $m > 1$ be its order.

The group G acts faithfully on the general member of $|-K_X|$, which is an elliptic curve, and P is a fixed point. Then G must contain an element δ of order $m = 2$ or 3. Since the representation $G \to GL(T_{P,X})$ is faithful, δ must be either the Bertini involution β or an element of order 3 described in Lemma 7.4. The assertion follows. □

COROLLARY 7.6. *Let $G \subset \mathrm{Aut}(X)$ be a subgroup such that the natural homomorphism $G \to \mathrm{Aut}(B)$ is not injective. Then some nonidentity element of G fixes a curve of positive genus.*

Proof. Apply Lemma 7.5 to the kernel of $G \to \mathrm{Aut}(B)$. □

Now we are ready to finish the proof of Proposition 3.6 in the case $K_X^2 = 1$. Assume that any nonidentity element of G does not fix a curve of positive genus. By Corollary 7.6 the group G acts faithfully on B. By Lemma 7.3 the order of G is odd. Hence, by the classification of finite subgroups of $PGL_2(\Bbbk)$ (see e.g. [Kle56; Spr77]) G is a cyclic group. Let $\delta \in G$ be its generator. Then the pencil $|-K_X|$ has exactly two invariant members, say C_1 and C_2. We claim that G faithfully acts on C_1 and C_2. Indeed, otherwise some nonidentity element $\delta \in G$ fixes C_i (pointwise). By our assumption C_i has a (unique) singular point, say P_i. Then $T_{P_i,C_i} = T_{P_i,X}$, and so the action of G on C_i must be faithful, a contradiction. Therefore, G faithfully acts on C_1 and C_2.

First, we assume that both C_1 and C_2 are smooth elliptic curves. Then $G \simeq \mathbb{Z}/3\mathbb{Z}$, and by Lemma 7.4 the element δ acts on $T_{P,X}$ as $\mathrm{diag}(\zeta_3, \zeta_3^{-1})$. The fixed point locus X^G consists of five points P, $P_1, P_2 \in C_1 \setminus C_2$ and $P_3, P_4 \in C_2 \setminus C_1$. Then by (3.3) we have $\mathrm{tr}(\delta) = \mathrm{tr}(\delta^2) = 2$, and so (X, G) is not minimal by (3.2).

Now we assume that C_1 has a singular point, say P_1. Since G is cyclic, P_1 cannot be an ordinary double point. Hence, $P_1 \in C_1$ is a cusp. Locally near P_1 the normalization is given by $t \mapsto (t^2, t^3)$. Since the action of G on X is free in codimension one near P_1, the order of G is coprime to 3. Then C_2 cannot be an elliptic curve, so C_2 is also a cuspidal rational curve. Then G permutes singular members of $|-K_X|$ other than C_1 and C_2. By Lemma 7.2 the order of G divides $12 - 4 = 8$, a contradiction.

8. Conic Bundles

In this section, we consider G-surfaces admitting a conic bundle structure. The convenience of the reader, we recall definitions and basic facts (see [DI09]).

8.1. Setup

Let X be a projective nonsingular surface, and let $f : X \to B$ be a dominant morphism, where B is a nonsingular curve. We say that the pair f is a *conic bundle* if $f_*\mathscr{O}_X = \mathscr{O}_B$ (i.e., f has connected fibers) and $-K_X$ is f-ample. Then any fiber X_b, $b \in B$, is isomorphic to a reduced conic in $\mathbb{P}^2$. Let G be a finite group acting on X and B. We say that f is a *G-conic bundle* if f is G-equivariant. We say that a G-conic bundle $f : X \to B$ is *relatively G-minimal* if $\mathrm{rk}\,\mathrm{Pic}(X/B)^G = 1$. In this section, we assume that $B \simeq \mathbb{P}^1$ (because X is a rational surface). By Noether's formula the number of degenerate fibers equals $8 - K_X^2$. In particular, $K_X^2 \leq 8$.

8.1.1 Moreover, if a G-conic bundle $f : X \to \mathbb{P}^1$ is relatively G-minimal, then $K_X^2 \neq 7$. From now on $f : X \to B$ denotes a relatively G-minimal conic bundle with $B \simeq \mathbb{P}^1$. If $K_X^2 = 8$, then f is a $\mathbb{P}^1$-bundle, that is, X is a Hirzebruch surface $\mathbb{F}_n$. In this case, the action of G on $\mathrm{Pic}(X)$ is trivial, and so $H^1(G, \mathrm{Pic}(X)) = 0$. For $K_X^2 = 3$, 5, and 6 the pair (X, G) is not minimal: there exists an equivariant birational morphism to a G-del Pezzo surface X' with $\mathrm{Pic}(X')^G \simeq \mathbb{Z}$ and $K_{X'}^2 > K_X^2$ [Isk80]. This case was investigated in the previous sections.

Thus we have the following:

PROPOSITION 8.1. *Let $f : X \to \mathbb{P}^1$ be a G-conic bundle with $K_X^2 \geq 5$. Assume that the surface X is G-minimal. Then $K_X^2 = 8$ and $X \simeq \mathbb{F}_n$, where $n \neq 1$. Moreover, X is H^1-trivial.*

REMARK 8.2. Assume that in the notation of 8.1, the group G is abelian. Then it is linearizable if and only if it is stably linearizable and if and only if G has a fixed point (see [DI09, Sect. 8] and Lemma 2.6)

From now on we assume that $K_X^2 \leq 4$.

8.1.2 Let G_F be the largest group that acts trivially on B. We have an exact sequence

$$1 \longrightarrow G_F \longrightarrow G \xrightarrow{\pi} G_B \longrightarrow 1,$$

where G_B acts faithfully on B, and G_F acts faithfully on the generic fiber X_η. We also have a natural homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(\mathrm{Pic}(X)).$$

Since $B \simeq \mathbb{P}^1$ and $K_X^2 \leq 5$, the group $\ker(\rho)$ fixes pointwise any section with negative self-intersection. In particular, this implies that $\ker(\rho) \subset G_F$ and $\ker(\rho)$ is a cyclic group.

NOTATION 8.3. Let $f : X \to B \simeq \mathbb{P}^1$ be a relatively G-minimal G-conic bundle, and let F be a typical fiber. Let $F_1, \ldots, F_m$ be all the degenerate fibers, let R_i be the singular point of F_i, and let $P_i := f(F_i)$. Thus, $F_i = f^{-1}(P_i) = F_i' + F_i''$ and $F_i' \cap F_i'' = \{R_i\}$. Let $\Delta := \{P_1, \ldots, P_m\}$ be the discriminant locus.

LEMMA 8.4 (cf. [Bla11, Lemmas 3.9 and 3.10]). *In the notation of 8.3, assume that any nonidentity element of G does not fix a curve of positive genus. Let $\delta \in G$ be an element of order $n > 1$. Then one of the following holds*:

(i) *δ does not switch components of any degenerate fiber,*
(ii) *there are exactly two degenerate fibers whose components are switched by δ, or*
(iii) *δ switches components of exactly one degenerate fiber, say F_1. In this case, δ^2 acts on B trivially, and δ acts on B nontrivially. Moreover, δ^2 switches components of exactly two degenerate fibers* (*other than F_1*).

Proof. Let $F_1, \dots, F_r$ be all the degenerate fibers whose components are switched by δ. We assume that $r > 0$ (otherwise, we are in the situation of (i)).

First, we consider the case where the action of δ on B is trivial. Then δ has exactly two fixed points on any smooth fiber. Hence, X^δ contains a (smooth) curve C. For $i \in \{1, \dots, r\}$, each intersection point $C \cap F_i$ is a single point, which must coincide with $R_i = \mathrm{Sing}(F_i)$. So, C is connected, and the ramification locus of the double cover $f_C : C \to B$ coincides with $\{P_1, \dots, P_r\}$. In particular, r is even. If $r > 2$, then C is a curve of genus $(r-2)/2 > 0$, a contradiction. Hence, $r = 2$.

Now consider the case where the action of δ on B is nontrivial. Since δ has exactly two fixed points on B, we have $r \le 2$. Assume that $r = 1$. If any element of the group $\langle\delta\rangle$ does not switch components of any fiber except for F_1, then we can run a relative $\langle\delta\rangle$-minimal model program on X so that the resulting surface has a relatively $\langle\delta\rangle$-minimal conic bundle structure over B with exactly one degenerate fiber. It is easy to see (see e.g. [DI09, Lemma 5.1]) that this is impossible. Hence, some element δ^k, where $k > 1$, switches components of a fiber $F_2 \ne F_1$. Take k to be minimal possible. The points $f(F_2)$ and $f(F_1)$ are fixed by δ^k. By our assumption $r = 1$, the point $f(F_2)$ is not fixed by δ. This is possible only if δ^k acts trivially on B. According to the previously considered case, δ^k switches components of exactly two fibers, so the $\langle\delta\rangle$-orbit of F_2 consists of two elements. Therefore, $k = 2$. □

Now we are going to classify H^1-trivial G-conic bundles with $K_X^2 \le 4$. There are two essentially different cases: $\ker(\rho) = \{1\}$ and $\ne \{1\}$.

Case $\ker(\rho) = \{1\}$.

THEOREM 8.5. *Let $f : X \to B = \mathbb{P}^1$ be a relatively G-minimal G-conic bundle with $K_X^2 \le 4$. Assume that (X, G) is H^1-trivial and* $\ker(\rho) = \{1\}$. *Then $G \simeq \tilde{\mathfrak{D}}_n$, where $n = 6 - K_X^2$ is odd, $G_F \simeq \mathbb{Z}/2\mathbb{Z}$ is the center of G, $G/G_F \simeq \mathfrak{D}_n$, and the action is given further by Construction 8.7.*[2]

[2]For $n = 5$, see also [Tsy11, Thm. 6.5].

REMARK 8.6. In the case $n = 3$, the surface X is not G-minimal: contracting an invariant horizontal (-1)-curve, we get a quartic del Pezzo surface (see (1.1), (1.2) and Remark 4.8).

CONSTRUCTION 8.7 (cf. [DI09, 5.12], [Tsy11, 3.2]). Let $n \geq 3$ be an odd integer. The representation (2.2) induces a faithful action $\sigma_1 : \mathfrak{D}_n \longrightarrow \mathrm{Aut}(\mathbb{P}^1)$. Consider another faithful action $\sigma_2 : \mathfrak{D}_n \longrightarrow \mathrm{Aut}(\mathbb{P}^1)$:

$$\tilde{r} \mapsto \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \qquad \tilde{s} \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Clearly, we have $\lambda \circ \sigma_1 = \sigma_2 \circ \lambda$, where the map $\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ is given by $\lambda : x \mapsto x^2$. Consider also the action

$$\sigma = \sigma_1 \times \sigma_2 : \mathfrak{D}_n \longrightarrow \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1).$$

The curves

$$\Gamma := \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid x^2 = y\},$$
$$L := \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y^n = 1\}$$

are $\mathfrak{D}_n$-invariant. Let $L_k := \{(x, y) \mid y = \zeta_n^k\}$ be a component of L. It is easy to see that L_k meets Γ transversally at two points. Now we explicitly construct a double cover $\pi : Y \to \mathbb{P}^1 \times \mathbb{P}^1$ branched over $\Gamma + L$. In homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$, the curve $\Gamma + L$ is given by

$$\phi := (x_1^2 y_0 - x_0^2 y_1)(y_1^n - y_0^n) = 0.$$

For short, we put $q := (n + 1)/2$. Let $\nu : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{n+2}$ be the Segre embedding

$$\nu : ((x_0 : x_1), (y_0, y_1)) \longmapsto (t_{0,0}, \dots, t_{0,q}, t_{1,0}, \dots, t_{1,q}),$$
$$\text{where } t_{a,b} = x_0^{1-a} x_1^a y_0^{q-b} y_1^b, 0 \leq a \leq 1, 0 \leq b \leq q.$$

Clearly, ϕ can be written as a homogeneous polynomial of degree 2 in the $t_{a,b}$. Thus, we can exhibit $Y \subset \mathbb{P}^{n+3}$ as the intersection of the hypersurface

$$z^2 = \phi(t_{0,0}, \dots, t_{1,q})$$

with the projective cone that is the preimage of $\nu(\mathbb{P}^1 \times \mathbb{P}^1)$ under the projection

$$\mathbb{P}^{n+3} \dashrightarrow \mathbb{P}^{n+2} \supset \nu(\mathbb{P}^1 \times \mathbb{P}^1), \qquad (z, t_{0,0}, t_{0,1}, \dots) \longmapsto (t_{0,0}, t_{0,1}, \dots).$$

Let $\sigma : \mathfrak{D}_n \to \{\pm 1\}$ be as in 2.1. Consider the group

$$\{(\delta, \alpha) \in \mathfrak{D}_n \times \langle \zeta_4 \rangle \mid \sigma(\delta) = \alpha^2\}.$$

This group is a nontrivial central extension of $\mathfrak{D}_n$ by $\mathbb{Z}/2\mathbb{Z}$, and it is isomorphic to $\tilde{\mathfrak{D}}_n$. By the previous construction we see that $\tilde{\mathfrak{D}}_n$ acts on Y so that π is equivariant. The projection of $\mathbb{P}^1 \times \mathbb{P}^1$ to the second factor induces a rational curve fibration $Y \to \mathbb{P}^1$ whose fibers are irreducible except for those corresponding to two ramification points of the double cover $\Gamma \to \mathbb{P}^1$. Let $\bar{L}_k := \pi^{-1}(L_k)$. There are exactly $2n$ nodes $Q_1', Q_1'', \dots, Q_n', Q_n'' \in Y$, where $\{Q_k', Q_k''\} = \pi^{-1}(\Gamma \cap L_k)$. Let $\tilde{Y} \to Y$ be the minimal resolution, and let $\tilde{Y} \to X$ the contraction of all $\tilde{L}_k$, the proper transforms of the $\bar{L}_k$. Then $f : X \to \mathbb{P}^1$ is a $\tilde{\mathfrak{D}}_n$-conic bundle with $n+2$ degenerate fibers fitting to the following commutative diagram:

$$\begin{tikzcd} \tilde{Y} \arrow[rr] \arrow[d] & & Y \arrow[d, "\pi"] \\ X & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[ll] \end{tikzcd} \tag{8.1}$$

Proof of Theorem 8.5. Assume that ρ is injective. Then so is $\rho_F : G_F \to \mathrm{Aut}(\mathrm{Pic}(X))$.

LEMMA 8.8. $G_F \neq \{1\}$.

Proof. Indeed, otherwise G faithfully acts on $B = \mathbb{P}^1$. For any degenerate fiber F_i, there exists an element $\delta \in G$ switching the components of F_i. In particular, $\mathrm{ord}(\delta) = 2k$ for some k. Clearly, we may assume that $k = 2^l$. By Lemma 8.4 there exists exactly one more degenerate fiber $F_j \neq F_i$ whose components are switched by δ. Thus, $X^\delta = \{R_i, R_j\}$. If $k = 1$, then the holomorphic Lefschetz fixed point formula implies that the cardinality of X^δ equals 4, a contradiction. Hence, $k > 1$. Put $\gamma := \delta^k$. It is easy to see that $X^\gamma = F_i^\gamma \cup F_j^\gamma$. Since X^γ is δ-invariant and smooth, we can see that it is zero-dimensional and consists of exactly six points. Again, we get a contradiction by the holomorphic Lefschetz fixed point formula. This proves our lemma. □

The group G_F interchanges pairwise components of (some) degenerate fibers. So, there exists an embedding

$$G_F \hookrightarrow \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_2.$$

On the other hand, G_F acts faithfully on a typical fiber, so there exists an embedding $G_F \hookrightarrow PGL_2(\Bbbk)$. This immediately gives us either $G_F \simeq \mathbb{Z}/2\mathbb{Z}$ or $G_F \simeq (\mathbb{Z}/2\mathbb{Z})^2$ (see [DI09, Thm. 5.7]).

Consider the case $G_F \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Then $G_F = \{1, \tau_1, \tau_2, \tau_3\}$, where the τ_j are distinct elements of order 2. Fix $i \in \{1, \dots, m\}$. The point R_i is fixed by G_F. The actions of all the τ_j on $T_{R_i,X}$ cannot have the (same) form $\mathrm{diag}(-1,-1)$. Hence, at least one of them, say τ_1, is of type $\mathrm{diag}(1,-1)$ (in suitable coordinates). Then τ_1 must switch the components of F_i. Indeed, otherwise τ_1 fixes pointwise a component of F_i. But this is impossible because τ_1 acts trivially on B. Moreover, for each singular fiber F_i, exactly two elements of G_F switch the components of F_i. Taking Lemma 8.4 into account, we see that Δ consists of three elements. This contradicts our assumption $K_X^2 \leq 4$.

Therefore, $G_F \simeq \mathbb{Z}/2\mathbb{Z}$. Let $\tau \in G_F$ be the element of order 2. Since $\rho(\tau) \neq \mathrm{id}$, by Lemma 8.4 the element τ switches components of exactly two degenerate fibers, say F_{r-1} and F_r. By our assumption $K_X^2 \leq 4$, we have $r > 2$. Then the set $\{P_{r-1}, P_r\}$ is G_B-invariant. This is possible only if G_B is either cyclic or dihedral. Let C be the one-dimensional part of X^τ. As in the proof of Lemma 8.4, we see that C is a smooth rational curve and $f_C : C \to B$ is a double cover ramified over $\{P_{r-1}, P_r\}$. The group $G_B = G/G_F$ faithfully acts on C so that f_C is G_B-equivariant.

Let $\delta \in G$ be an element that switches the components of F_1. If δ does not permute F_{r-1} and F_r, then δ fixes three points $P_{r-1}, P_r, P_1 \in B = \mathbb{P}^1$. So, it trivially acts on B, that is, $\delta \in G_F$, a contradiction. Thus, δ permutes F_{r-1} and F_r. Let $\upsilon \in \mathrm{Aut}(C)$ be the Galois involution of f_C, and let $G_C \subset \mathrm{Aut}(C)$ be the (isomorphic) image of G_B. Since G_B faithfully acts on B, $\upsilon \notin G_C$. On the other hand, υ commutes with any element of G_C. Hence, G_C and υ generate a subgroup $G'_C = G_C \times \langle \upsilon \rangle \subset \mathrm{Aut}(C)$, so that the set $\{R_{r-1}, R_r\} \subset C$ is G'_C-invariant. By the classification of finite subgroups of $\mathrm{Aut}(\mathbb{P}^1)$ we see that $G'_C \simeq \mathfrak{D}_{2n}$, where n must be odd (because $\upsilon \notin \mathfrak{D}_n \subset \mathfrak{D}_{2n}$). In particular, $G_B \simeq \mathfrak{D}_n$. For $i = 1, \dots, r-2$, we have $C \cap F'_i = \{R'_i\}$ and $C \cap F''_i = \{R''_i\}$, where the points R'_i and R''_i are permuted by υ and have nontrivial stabilizers in G_C. There are only three nontrivial orbits of $\mathfrak{D}_{2n}$ on $C \simeq \mathbb{P}^1$: O_{2n}, O'_{2n}, and O_2 [Kle56; Spr77]. They have $2n$, $2n$, and 2 elements, respectively. Since υ cannot fix any element of O_{2n} and O'_{2n}, we may assume that O'_{2n} form one $\mathfrak{D}_n$-orbit and O_{2n} splits in the union of two $\mathfrak{D}_n$-orbits. Then O_{2n} coincides with $C \cap (\bigcup_{i=1}^{r-2} F_i)$, and so $n = r-2$. Recall that n is odd and G is a central extension of $G_B \simeq \mathfrak{D}_n$ by $G_F \simeq \mathbb{Z}/2\mathbb{Z}$. We claim that $G \simeq \tilde{\mathfrak{D}}_n$. Indeed, otherwise $G = G_B \times G_F \simeq \mathfrak{D}_n \times \mathbb{Z}/2\mathbb{Z}$. Take δ as before. Then δ fixes P_1. Since $G \simeq \mathfrak{D}_n \times \mathbb{Z}/2\mathbb{Z}$, we have $\mathrm{ord}(\delta) = 2$. The action of δ on $T_{R_1,X}$ has the form $\mathrm{diag}(1, -1)$. Hence, δ fixes pointwise a (smooth) curve D passing through R_1. Since δ switches the components of F_1, D is not a component of F_1. Hence, D dominates B and $\delta \in G_F$, a contradiction. Thus, $G \to G_B$ is not split, and so $G \simeq \tilde{\mathfrak{D}}_n$.

Now we construct the following G-equivariant commutative diagram:

$$
\begin{tikzcd}
\tilde{Y} \arrow[rrr, bend left] \arrow[d] & & Z \arrow[dl, "\mu"'] \arrow[dr, "\upsilon"] & Y \arrow[d, "\pi"] \\
X \arrow[r] & X/\langle \tau \rangle & & \mathbb{F}_e
\end{tikzcd}
\tag{8.2}
$$

Here $X/\langle \tau \rangle$ has $n = r-2$ nodes, which are images of $R_1, \dots, R_n$, μ is the minimal resolution, and υ is the contraction of the proper transforms of $R'_1, R''_1, \dots, R'_n, R''_n$. It is easy to see that the image of υ must be a smooth geometrically ruled surface. On the other hand, to arrive at $\mathbb{F}_e$ from X, we can first blowup the points $R_1, \dots, R_n$. We get $\tilde{Y}$. The action of G lifts to $\tilde{Y}$, and $\tilde{Y} \to Y \to \mathbb{F}_e$ is the Stein factorization. Let $E_1, \dots, E_n$ be μ-exceptional divisors, and let $L_k := \upsilon(E_k)$. Let $C_\bullet \subset \mathbb{F}_e$ be the proper transform of $C/\langle \tau \rangle \subset X/\langle \tau \rangle$. Clearly, π is a double cover branched over $C_\bullet + L_1 + \cdots + L_n$. Comparing (8.2) and (8.1), we see that it remains to show that $e = 0$, that is, $\mathbb{F}_e \simeq \mathbb{P}^1 \times \mathbb{P}^1$. We can write $C_\bullet \sim 2s + aF_\bullet$, where s is the minimal section, and $F_\bullet$ is a fiber of $\mathbb{F}_e$. Since $C_\bullet$ is an irreducible smooth rational curve, we get two possibilities: $(e, a) = (0, 1)$ and $(1, 2)$. Since the branch divisor $C_\bullet + L_1 + \cdots + L_n$ is divisible by 2 and n is odd, we see that the second case is impossible. This proves Theorem 8.5. $\square$

Case $\ker(\rho) \neq \{1\}$

DEFINITION 8.9 [DI09]. A conic bundle $f : X \to \mathbb{P}^1$ is said to be *exceptional* if for some positive integer g, the number of degenerate fibers equals $2g+2$ and there are two disjoint sections C_1 and C_2 with $C_1^2 = C_2^2 = -(g+1)$.

THEOREM 8.10. *Let $f : X \to \mathbb{P}^1$ be a relatively G-minimal G-conic bundle with $K_X^2 = 6 - 2g \leq 4$. Assume that (X, G) is H^1-trivial and* $\ker(\rho) \neq \{1\}$. *Then we have*:

(i) *f is exceptional*, *in particular, K_X^2 is even*;
(ii) *$G_F = \ker(\rho)$, and it is a nontrivial cyclic group*;
(iii) *either $G_B \simeq \mathfrak{D}_n$ or $G_B \simeq \mathfrak{S}_4$*;
(iv) *the action of G on X is given by Construction 8.11*.

The following is a particular case of the general construction [DI09, Sect. 5.2].

CONSTRUCTION 8.11 [DI09, Sects. 5.2 and 5.3]. First, we fix some data. Let $\tilde{G}_B \subset SL_2(\Bbbk)$ be a finite noncyclic subgroup, and let $G_B = \tilde{G}_B/\{\pm \mathrm{id}\}$ be its image in $PSL_2(\Bbbk)$. Fix two homomorphisms $\sigma, \chi_B : G_B \to \{\pm 1\}$, where χ_B is surjective (we assume that such a homomorphism χ_B exists). We also regard σ and χ_B as characters defined on $\tilde{G}_B$. Let $g \geq 1$, and let Y be the hypersurface in $\mathbb{P}(g+1, g+1, 1, 1)$ given by $x_1x_2 = \psi(y_1, y_2)$, where $\psi(y_1, y_2)$ is a homogeneous $\tilde{G}_B$-semiinvariant of degree $2g+2$ and weight σ. Thus, $\delta(\psi) = \sigma(\delta)\psi$ for all $\delta \in \tilde{G}_B$. We assume also that ψ has no multiple factors. Put

$$\Gamma := \{(h, \delta) \in GL_2(\Bbbk) \times \tilde{G}_B \mid h(x_1x_2) = \sigma(\delta)x_1x_2\}.$$

It is easy to see that Γ naturally acts on Y and the kernel of the action coincides with

$$K := \langle((-1)^{g+1}\,\mathrm{id}, -\mathrm{id})\rangle.$$

Thus, $\mathrm{Aut}(Y) \supset \Gamma/K$. Denote by $p : \mathrm{Aut}(Y) \to G_B$ the homomorphism induced by the projection to the second factor. The surface Y has two singular points, which are of type $\frac{1}{g+1}(1, 1)$. Let $X \to Y$ be the minimal resolution. The projection $(x_1 : x_2 : y_1 : y_2) \dashrightarrow (y_1 : y_2)$ induces a conic bundle structure $f : X \to \mathbb{P}^1 = B$ whose degenerate fibers correspond to the zeros of ψ. In particular, $K_X^2 = 6 - 2g$.

The action on the set $\mathrm{Sing}(Y) = \{(1:0:0:0), (0:1:0:0)\}$ defines a homomorphism $\chi : \mathrm{Aut}(Y) \to \{\pm 1\}$. Now, take a subgroup $G \subset \Gamma/K$ such that the restriction $\chi_G : G \to \{\pm 1\}$ and the projection $p_G : G \to G_B$ are surjective, and $\ker(p) \cap G \subset \ker(\chi)$. Thus, χ descends to a character $\chi_B : G_B \to \{\pm 1\}$.

There are the following possibilities:

No.	g	G_B	ψ	σ	χ_B
1^o	2	$\mathfrak{S}_4$	ψ_6	sgn	sgn
2^o	5	$\mathfrak{S}_4$	ψ_{12}	1	sgn
3^o	8	$\mathfrak{S}_4$	$\psi_6\psi_{12}$	sgn	sgn
4^o	≥ 1	$\mathfrak{D}_{2g+2}$	$y_1^{2g+2} - y_2^{2g+2}$	$\xi \cdot \sigma^{g-1}$	σ or ξ
5^o	≥ 1	$\mathfrak{D}_{g+1}$	$y_1^{2g+2} - y_2^{2g+2}$	σ^g	σ
6^o	≥ 1	$\mathfrak{D}_{2g}$	$y_1 y_2(y_1^{2g} - y_2^{2g})$	$\xi \cdot \sigma^{g-1}$	ξ

Here $\psi_6 = y_1 y_2(y_1^4 - y_2^4)$ and $\psi_{12} = y_1^{12} - 33y_1^8 y_2^4 - 33y_1^4 y_2^8 + y_2^{12}$, and, for even n, the homomorphism $\xi : \mathfrak{D}_n \to \{\pm 1\}$ is defined by $\xi(r) = -1$, $\xi(s) = -1$.

Proof of Theorem 8.10(i). Since $\ker(\rho) \neq \{0\}$, the conic bundle f is exceptional by [DI09, Prop. 5.5]. In particular, we can write $m = 2g + 2$, where $g \in \mathbb{Z}_{>0}$. $\square$

Let C_1 and C_2 be disjoint $-(g+1)$-sections (see Definition 8.9).

Proof of Theorem 8.10(ii). Recall that $\ker(\rho) \subset G_F$ (see 8.1.2). If there exists an element $\delta \in G_F$ that switches C_1 and C_2, then δ switches components of all degenerate fibers. Since the number of degenerate fibers equals $2g + 2 \geq 4$, this contradicts Lemma 8.4. Hence, both C_1 and C_2 are G_F-invariant, and then any component of a degenerate fiber also must be G_F-invariant. Since K_X and the components of the fibers generate a subgroup of index 2 in $\mathrm{Pic}(X)$, we have $G_F = \ker(\rho)$. Finally, the action of G_F on a typical fiber F has two fixed points $C_1 \cap F$ and $C_2 \cap F$. Then G_F must be cyclic. $\square$

Corollary 8.12. *Let $\chi : G \to \{\pm 1\}$ be the (surjective) homomorphism induced by the action on $\{C_1, C_2\}$. Then $G_F \subset \ker(\chi)$. Thus, χ passes through a surjective homomorphism $\chi_B : G_B \to \{\pm 1\}$.*

Proof of Theorem 8.10(iii). Suppose that G_B is cyclic. By (ii) of our theorem $G_B \neq \{1\}$. Thus, G_B has exactly two fixed points $P', P'' \in B$ and acts freely on $B \setminus \{P', P''\}$. For any degenerate fiber F_i, there exists an element $\delta \in G$ that switches components of F_i. Then $P_i = f(F_i)$ must coincide with P' or P''. Hence, f has at most two degenerate fibers, a contradiction. Thus, G_B is not cyclic.

Recall that $G_B \subset PGL_2(\Bbbk)$. By the classification of finite subgroups in $PGL_2(\Bbbk)$ (see e.g. [Kle56; Spr77]) we have $G_B \simeq \mathfrak{D}_n$, $\mathfrak{A}_4$, $\mathfrak{S}_4$, or $\mathfrak{A}_5$. By Corollary 8.12 we have $G_B \not\simeq \mathfrak{A}_4, \mathfrak{A}_5$. $\square$

Lemma 8.13. *For $P_i = f(F_i)$, let $G_i \subset G_B$ be its stabilizer. Then G_i is a cyclic group generated by an element τ_i such that $\chi_B(\tau_i) = -1$.*

Proof. Since the representation of G_i on $T_{P_i, B}$ is faithful, G_i is cyclic. The components of F_i are switched by some element $\delta_i \in G$. Then $\chi(\delta_i) = -1$, and the image of δ_i is contained in G_i. $\square$

Proof of Theorem 8.10(iv). Basically, this is the third construction of exceptional conic bundles in [DI09, 5.2]. We have to prove only 1^o–6^o.

Define a homogeneous semiinvariant $\psi(y_1, y_2)$ that vanishes at $P_1, \dots, P_{2g+2} \in \mathbb{P}^1_{y_1,y_2}$ with multiplicity one and does not vanish everywhere else.

LEMMA 8.14. *Let $G_i \subset G_B$ be the stabilizer of $P_i = f(F_i)$, and let τ_i be its generator. Then the set $\Delta := \{P_1, \dots, P_{2g+2}\}$ satisfies the following property*:

- *the fixed point locus B^{τ_i} is contained in Δ.*

In particular, Δ is the union of some nontrivial G_B-orbits.

Proof. Let $\hat{\tau}_i \in G$ be a preimage of τ_i. By construction, $\hat{\tau}_i$ switches components of F_i. If F_i is the only fiber whose components are switched by $\hat{\tau}_i$, then $\hat{\tau}_i$ is as in Lemma 8.4(iii). But then $\hat{\tau}_i^2 \in G_F = \ker(\sigma)$, and so $\hat{\tau}_i^2$ does not switch components of any fiber. This contradicts Lemma 8.4(iii). Hence, $\hat{\tau}_i$ switches the components of two fibers: F_i and $F_j \neq F_i$. Therefore, $B^{\tau_i} = \{f(F_i), f(F_j)\} \subset \Delta$. □

Consider the case $G_B \simeq \mathfrak{S}_4$. Then χ coincides with the sign map $\mathrm{sgn} : \mathfrak{S}_4 \to \{\pm 1\}$. There are only three nontrivial orbits of $\mathfrak{S}_4$ on $\mathbb{P}^1$: O_{12}, O_8, and O_6 (see e.g. [Kle56; Spr77]). They have 12, 8, and 6 elements, respectively. The corresponding semiinvariants have the forms $\psi_{12} = y_1^{12} - 33y_1^8y_2^4 - 33y_1^4y_2^8 + y_2^{12}$, $\psi_8 = y_1^8 + 14y_1^4y_2^4 + y_2^8$, and $\psi_6 = y_1y_2(y_1^4 - y_2^4)$. By Lemma 8.13, for any point $P_i \in \Delta$, its stabilizer $G_i \subset G_B$ is generated by an odd permutation. So, the order of G_i equals 2 or 4, and $O_8 \not\subset \Delta$. Hence, there are the following possibilities: $\Delta = O_{12}$, $\Delta = O_6$, and $\Delta = O_6 \cup O_{12}$.

Now consider the case $G_B \simeq \mathfrak{D}_n$. We use the presentation (2.1). There are only three nontrivial orbits of $\mathfrak{D}_n$ on $\mathbb{P}^1$: O_n, O_n', and O_2 [Kle56; Spr77]. They have n, n, and 2 elements, respectively. The corresponding semiinvariants of $\mathfrak{D}_n$ have the form $\psi_n = y_1^n - y_2^n$, $\psi_n' = y_1^n + y_2^n$, $\psi_2 = y_1y_2$. Since Δ contains at least four points, $\Delta \neq O_2$. Thus, we may assume that $O_n \subset \Delta$. Assume that $\Delta \supset O_n \cup O_n'$. Then any element $\tau \in \mathfrak{D}_n \setminus \langle r \rangle$ generates the stabilizer of some point $P_i \in \Delta$. By Lemma 8.13 the character χ takes value -1 on $\mathfrak{D}_n \setminus \langle r \rangle$. Hence, $\chi(r) = 1$, r cannot generate the stabilizer of a point of Δ, and so $O_2 \not\subset \Delta$. Thus, for Δ, we have the following possibilities: $\Delta = O_n$, $O_n \cup O_n'$, and $O_n \cup O_2$, corresponding to 4^o, 5^o, and 6^o, respectively. Finally, χ_B can be computed by using Lemma 8.13. This proves Theorem 8.10. □

COROLLARY 8.15. *Let $f : X \to B = \mathbb{P}^1$ be a relatively G-minimal G-conic bundle, where G is an Abelian group. Assume that f has at least one degenerate fiber and that (X, G) is G-minimal and H^1-trivial. Then the following assertions hold*:

- *$K_X^2 = 4$, $G \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, f has exactly four degenerate fibers.*
- *The image of G in* $\mathrm{Aut}(B)$ *is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and f is an exceptional conic bundle with $g = 1$.*
- *There are two disjointed sections C_1 and C_2 that are* (-2)*-curves. Moreover, X is a weak del Pezzo surface, that is, $-K_X$ is nef and big.*

- *The anticanonical model $\bar{X} \subset \mathbb{P}^4$ is an intersection of two quadrics whose singular locus consists of two ordinary double points and the line joining them does not lie on $\bar{X}$.*

REMARK 8.16. The surface X and group G described before are extremal in many senses. According to [Bla09, Sect. 7], G is the only finite Abelian subgroup of $\mathrm{Cr}_2(\Bbbk)$ that is not conjugate to a group of automorphisms of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ but whose nontrivial elements do not fix any curve of positive genus. The intersection of two quadrics $\bar{X} \subset \mathbb{P}^4$ as before is called the *Iskovskikh surface* [CT88]. This is the only intersection of two quadrics in $\mathbb{P}^4$ for which the clean Hasse principle can fail [Isk71; CT88].

REMARK 8.17. In the notation of Corollary 8.15, it is easy to see that the group $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has no any fixed points on X. Hence, (X, G) is not stably linearizable (see Lemma 2.6). Moreover, (X, G) is not stably conjugate to $(\mathbb{P}^2, G)$ for any action of G on $\mathbb{P}^2$.

ACKNOWLEDGMENTS. I would like to thank Igor Dolgachev for useful conversations through emails. I am also grateful to the referee for constructive criticism and for pointing out a gap in the earlier version of Lemma 8.4.

References

[BCSD85] A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer, *Variétés stablement rationnelles non rationnelles,* Ann. of Math. (2) 121 (1985), no. 2, 283–318.

[Bla09] J. Blanc, *Linearisation of finite Abelian subgroups of the Cremona group of the plane,* Groups Geom. Dyn. 3 (2009), no. 2, 215–266.

[Bla11] ______, *Elements and cyclic subgroups of finite order of the Cremona group,* Comment. Math. Helv. 86 (2011), no. 2, 469–497.

[Bou02] N. Bourbaki, *Lie groups and Lie algebras,* Chapters 4–6, Elements of mathematics, Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley.

[BP13] F. Bogomolov and Y. Prokhorov, *On stable conjugacy of finite subgroups of the plane Cremona group, I,* Cent. Eur. J. Math. 11 (2013), no. 12, 2099–2105.

[CCN+85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray,* Clarendon Press, Oxford, 1985.

[CT88] D. F. Coray and M. A. Tsfasman, *Arithmetic on singular del Pezzo surfaces,* Proc. Lond. Math. Soc. (3) 57 (1988), no. 1, 25–87.

[DI09] I. V. Dolgachev and V. A. Iskovskikh, *Finite subgroups of the plane Cremona group,* Algebra, arithmetic, and geometry: In honor of Yu. I. Manin. Vol. I, Progr. Math., 269, pp. 443–548, Birkhäuser Boston Inc., Boston, MA, 2009.

[Dol12] I. V. Dolgachev, *Classical algebraic geometry,* Cambridge University Press, Cambridge, 2012.

[Isk71] V. A. Iskovskikh, *A counterexample to the Hasse principle for a system of two quadratic forms in five variables,* Math. Notes 10 (1971), no. 3, 575–577.

[Isk80] ______, *Minimal models of rational surfaces over arbitrary fields,* Math. USSR, Izv. 14 (1980), no. 1, 17–39.

[Kle56] F. Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree,* revised edition, Dover Publications Inc., New York, NY, 1956, Translated into English by George Gavin Morrice.

[KS00] J. Kollár and E. Szabó, *Fixed points of group actions and rational maps,* Canad. J. Math. 52 (2000), no. 5, 1054–1056, Appendix to "Essential dimensions of algebraic groups and a resolution theorem for G-varieties" by Z. Reichstein and B. Youssin.

[Man74] Y. I. Manin, *Cubic forms: Algebra, geometry, arithmetic,* North-Holland Mathematical Library, 4, North-Holland Publishing Co., Amsterdam, 1974, Translated from the Russian by M. Hazewinkel, North-Holland Mathematical Library, Vol. 4.

[Sha94] I. R. Shafarevich, *Basic algebraic geometry. 1,* 2nd edition, Springer-Verlag, Berlin, 1994, Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid.

[Spr77] T. A. Springer, *Invariant theory,* Lecture Notes in Math., 585, Springer-Verlag, Berlin, 1977.

[Tsy11] V. I. Tsygankov, *Equations of G-minimal conic bundles,* Sb. Math. 202 (2011), no. 11–12, 1667–1721.

Steklov Mathematical Institute
8 Gubkina street
Moscow 119991
Russia

Department of Algebra
Moscow State University
Moscow 117234
Russia

Laboratory of Algebraic Geometry
SU-HSE
7 Vavilova str.
Moscow 117312
Russia

prokhoro@gmail.com