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The Trace Map of Frobenius and Extending Sections for Threefolds

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Abstract. In this paper, by using the trace map of Frobenius, we consider problems on extending sections for positive characteristic threefolds.

0. Introduction

In characteristic zero, by the Kodaira vanishing theorem and its generalizations, we can establish some results on adjoint divisors, such as the Kawamata–Shokurov basepoint-free theorem (see, e.g., [Kollár–Mori, Theorem 3.3]) and the Hacon–McKernan extension theorem [HM, Theorem 5.4.21]. These theorems claim, under suitable conditions, that an adjoint divisor $m(K_X + \Delta + A)$ has good properties, where $m \in \mathbb{Z}_{>0}$, (X, Δ) is a pair, and A is an ample divisor. In this paper we only consider the following very simple situation: X is a smooth projective variety, $\Delta = S$ is a smooth prime divisor, and A is an ample Cartier divisor. The following fact immediately follows from the Kodaira vanishing theorem.

Fact 0.1. *Let k be an algebraically closed field of characteristic zero. Let X be a smooth projective variety over k. Let S be a smooth prime divisor on X, and let A be an ample Cartier divisor on X such that $K_X + S + A$ is nef. Fix $m \in \mathbb{Z}_{>0}$. Then, by the Kodaira vanishing theorem, we obtain*

$$H^1(X, K_X + A + (m-1)(K_X + S + A)) = 0.$$

Thus, the natural restriction map

$$H^0(X, m(K_X + S + A)) \to H^0(S, m(K_S + A|_S))$$

is surjective.

It is natural to consider whether this fact also holds in positive characteristic. Unfortunately, however, there exists the following example.

Example 0.2 (cf. Example 4.4). Let k be an algebraically closed field of positive characteristic. Then, there exist a smooth projective surface X over k, a smooth prime divisor C on X, and an ample Cartier divisor A on X such that $K_X + C + A$ is nef and the natural restriction map

$$H^0(X, K_X + C + A) \to H^0(C, K_C + A|_C)$$

is not surjective.

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Thus, we would like to find a suitable analogue of Fact 0.1 in positive characteristic. In this paper, we prove the following two theorems.

THEOREM 0.3 (cf. Corollary 4.3). *Let k be an algebraically closed field of positive characteristic. Let X be a smooth projective surface over k. Let C be a smooth prime divisor on X, and let A be an ample Cartier divisor on X. If $H^0(C, K_C + A|_C) \neq 0$, then the natural restriction map*

$$H^0(X, K_X + C + A) \to H^0(C, K_C + A|_C)$$

is a nonzero map.

THEOREM 0.4 (cf. Theorem 7.3). *Let k be an algebraically closed field of positive characteristic. Let X be a smooth projective threefold over k. Let S be a smooth prime divisor on X, and let A be an ample Cartier divisor on X. Assume the following two conditions*:

(1) $K_X + S + A$ *is nef.*
(2) $\kappa(S, K_S + A|_S) \neq 0$.

Then, there exists $m_0 \in \mathbb{Z}_{>0}$ such that, for every integer $m \geq m_0$, the natural restriction map

$$H^0(X, m(K_X + S + A)) \to H^0(S, m(K_S + A|_S))$$

is surjective.

To show these two theorems, we use the trace map of Frobenius. This strategy is essentially the same as that used in [Schwede2, Proposition 5.3] and its proof. Let us see the idea of the proofs. Let X be a smooth projective variety. Let S be a smooth prime divisor on X, and let A be an ample Cartier divisor on X. Then, for every $e \in \mathbb{Z}_{>0}$, we obtain the following commutative diagram by using the trace map of Frobenius:

$$\begin{tikzcd} H^0(X, K_X + S + p^e A) \arrow[r] \arrow[d] & H^0(S, K_S + p^e A|_S) \arrow[r] \arrow[d, "\mathrm{Tr}^e_S(A|_S)"] & H^1(X, K_X + p^e A) \\ H^0(X, K_X + S + A) \arrow[r, "\rho"] & H^0(S, K_S + A|_S), & \end{tikzcd}$$

where the lower horizontal arrow ρ is the natural restriction map, and the upper horizontal sequence is exact. By the Serre vanishing theorem, for large $e \gg 0$, we obtain the vanishing $H^1(X, K_X + p^e A) = 0$. Thus, to prove that the restriction map ρ is surjective (resp. a nonzero map), it is sufficient to show that the trace map $\mathrm{Tr}^e_S(A|_S)$ is surjective (resp. a nonzero map). Therefore, to prove Theorems 0.3 and 0.4, we establish the following results on the trace map of Frobenius.

THEOREM 0.5 (cf. Theorem 4.1). *Let k be an algebraically closed field of positive characteristic. Let C be a smooth projective curve over k. Let A be an ample*

Cartier divisor on C. If $H^0(C, K_C + A) \neq 0$, then the trace map

$$\mathrm{Tr}^e_C(A) : H^0(C, K_C + p^e A) \to H^0(C, K_C + A)$$

is a nonzero map for every $e \in \mathbb{Z}_{>0}$.

THEOREM 0.6 (cf. Theorem 7.1). *Let k be an algebraically closed field of positive characteristic. Let S be a smooth projective surface over k. Let A be an ample Cartier divisor on S. Assume the following two conditions:*

(1) *$K_S + A$ is nef.*
(2) *$\kappa(S, K_S + A) \neq 0$.*

Then, there exists $m_1 \in \mathbb{Z}_{>0}$ such that the trace map

$$\begin{aligned}\mathrm{Tr}^e_S(A + m(K_S + A)) &: H^0(S, K_S + p^e(A + m(K_S + A))) \\ &\to H^0(S, K_S + (A + m(K_S + A)))\end{aligned}$$

is surjective for every integer $m \geq m_1$ and for every $e \in \mathbb{Z}_{>0}$.

We also consider whether Theorem 0.4 and Theorem 0.6 hold for the case where $\kappa(S, K_S + A) = 0$. Let us compare Theorem 0.4 with the following basepoint-free conjecture (cf. [Kollár–Mori, Theorem 3.3]).

CONJECTURE 0.7. *Let k be an algebraically closed field of positive characteristic. Let X be a smooth projective threefold over k. Let S be a smooth prime divisor on X, and let A be an ample Cartier divisor on X such that $K_X + S + A$ is nef. Then, $|b(K_X + S + A)|$ is basepoint-free for every $b \gg 0$.*

REMARK 0.8. After the first version of this paper was circulated, [BW, Theorem 1.2] proved that, in the above situation, $|b(K_X + S + A)|$ is basepoint-free for every sufficiently large and divisible b in characteristic $p > 5$.

If Conjecture 0.7 holds, then Theorem 0.4 also holds when $\kappa(S, K_S + A) = 0$. It is natural to ask whether Theorem 0.6 also holds when $\kappa(S, K_S + A) = 0$. Unfortunately, the answer is negative. We can construct the following example in characteristic two.

THEOREM 0.9 (cf. Theorem 8.3). *Let k be an algebraically closed field of characteristic two. Then, there exists a smooth projective surface S over k such that:*

(1) *$-K_S =: A$ is ample. In particular, $\kappa(S, K_S + A) = 0$.*
(2) *For every $e \in \mathbb{Z}_{>0}$, the trace map*

$$\mathrm{Tr}^e_S(A) : H^0(S, K_S + 2^e A) \to H^0(S, K_S + A)$$

is the zero map.

Moreover, Theorem 0.9 also shows that we cannot generalize Theorem 0.5 to dimension two.

In the Appendix, we establish the following analogue of the Hacon–McKernan extension theorem for surfaces.

THEOREM 0.10 (cf. Theorem A.1). *Let k be an algebraically closed field of positive characteristic. Let X be a smooth projective surface over k, and let C be a smooth prime divisor on X. Let $\Delta := C + B$, where B is an effective $\mathbb{Q}$-divisor on X that satisfies the following properties*:

(1) $C \not\subset \operatorname{Supp} B$, $\llcorner B \lrcorner = 0$ *and* (X, Δ) *is plt*;
(2) $B \sim_{\mathbb{Q}} A + F$, *where A is an ample $\mathbb{Q}$-divisor, and F is an effective $\mathbb{Q}$-divisor such that* $C \not\subset \operatorname{Supp} F$; *and*
(3) *no prime component of Δ is contained in the stable base locus of $K_X + \Delta$.*

Then, there exists an integer $m_0 > 0$ such that, for every integer $m > 0$, the restriction map

$$H^0(X, mm_0(K_X + \Delta)) \to H^0(C, mm_0(K_X + \Delta)|_C)$$

is surjective.

However, the proof of Theorem 0.10 does not use the trace map of Frobenius. We use instead results on the minimal model theory established in [T2] and [T3].

0.1. Overview of Contents

In Section 1, we summarize the notation. In Section 2, we give the definition and some basic properties of the trace map of Frobenius. The trace map of Frobenius is obtained by applying the functor $Hom_{\mathcal{O}_X}(-, \omega_X)$ to the Frobenius map $\mathcal{O}_X \to F_*\mathcal{O}_X$. In Section 3, we recall some known facts about the Cartier operator. We can consider the trace map of Frobenius as a Cartier operator. The Cartier operator is defined by the de Rham complex. We use the Cartier operator to consider the relation between the trace map of Frobenius and étale base change. In Section 4, we prove Theorem 0.3 and Theorem 0.5. In Section 5, we prove Theorem 0.6 when $\kappa = 1$. In Section 6, we prove Theorem 0.6 when $\kappa = 2$. In Section 7, by using Theorem 0.6 we show Theorem 0.4. In Section 8, we prove Theorem 0.9. In the Appendix, we prove Theorem 0.10.

0.2. Overview of Related Literature

We give some general references to the literature related to this paper in connection with the basepoint-free theorem, the extension theorem, the trace map of Frobenius, and the minimal model theory in positive characteristic.

Basepoint-Free Theorem and Extension Theorem. The motivation of this paper comes from the basepoint-free theorem and the extension theorem in characteristic zero. Thus, let us summarize some known results on this topic. Kawamata and Shokurov established the basepoint-free theorem for klt pairs (cf. [KMM; Kollár–Mori]). [Ambro] generalized this result (cf. [Fujino]). The extension theorem is established by Hacon and McKernan [HM, Theorem 5.4.21]. This theorem is a key to proving the existence of flips [BCHM]. For related topics, see [DHP] and [FG].

The Trace Map of Frobenius. At the heart of this paper is the trace map of Frobenius. This map plays a crucial role in the theory of F-singularities (cf. [BST; Schwede1; Schwede2]). Moreover, [CHMS] and [Mustata] establish results related to birational geometry by using the trace map of Frobenius and the theory of F-singularities. For related topics, see [BSTZ] and [TW].

Minimal Model Theory in Positive Characteristic. For results on the minimal model theory in positive characteristic, we refer to [Fujita3; KK; T2], and [T3] for the case of surfaces and to [Birkar; BW; CTX; HX; Kawamata; Keel; Kollár], and [Xu] for the case of threefolds.

1. Notation

We freely use the notation and terminology in [Kollár–Mori]. We do not distinguish in notation between invertible sheaves and divisors. For example, we often write $L+M$ for invertible sheaves L and M. For a coherent sheaf F and a Cartier divisor L, we define $F(L) := F \otimes \mathcal{O}_X(L)$.

Throughout this paper, we work over an algebraically closed field k of positive characteristic and let $\operatorname{char} k =: p > 0$. In this paper, *a variety* means an integral scheme, separated and of finite type over k. A *curve* (resp. a *surface*) means a variety of dimension one (resp. two).

2. The Trace Map of Frobenius

In this section, we define the trace map of Frobenius and we discuss some fundamental properties. We only use the smooth case. For the singular case, see [Schwede2, Section 2].

PROPOSITION 2.1. *Let X be a smooth projective variety. Let E be an effective $\mathbb{Z}$-divisor, and let D be a $\mathbb{Z}$-divisor. Then, for every positive integer e, there exists a natural $\mathcal{O}_X$-module homomorphism*

$$\mathrm{Tr}^e_{X,E}(D) : F^e_*(\omega_X(E+p^eD)) \to \omega_X(E+D).$$

We call this a trace map.

Proof. Consider the Frobenius map

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X,$$

that is, the p^eth power map $a \mapsto a^{p^e}$. Since E is effective, we obtain by tensoring with $\mathcal{O}_X(-E)$

$$\mathcal{O}_X(-E) \to F^e_*(\mathcal{O}_X(-p^eE)) \hookrightarrow F^e_*(\mathcal{O}_X(-E)).$$

Tensoring with $\mathcal{O}_X(-D)$ we obtain

$$\begin{aligned}\mathcal{O}_X(-E-D) &\to F^e_*(\mathcal{O}_X(-E)) \otimes \mathcal{O}_X(-D)\\ &\simeq F^e_*(\mathcal{O}_X(-E-p^eD)).\end{aligned}$$

By the duality theorem for finite morphisms we obtain

$$\mathcal{H}om_{\mathcal{O}_X}(F^e_*(\mathcal{O}_X(-E-p^eD)),\omega_X)\simeq F^e_*(\omega_X(E+p^eD)).$$

Then, we apply the functor $\mathcal{H}om_{\mathcal{O}_X}(-,\omega_X)$ and obtain

$$F^e_*(\omega_X(E+p^eD))\to\omega_X(E+D).$$

This is the trace map $\mathrm{Tr}^e_{X,E}(D)$. $\square$

REMARK 2.2. By this construction, $\mathrm{Tr}^e_{X,E}(D)$ factors through $\mathrm{Tr}^e_X(E+D):=\mathrm{Tr}^e_{X,0}(E+D)$:

$$\mathrm{Tr}^e_{X,E}(D):\ F^e_*(\omega_X(E+p^eD))\hookrightarrow F^e_*(\omega_X(p^eE+p^eD))\xrightarrow{\mathrm{Tr}^e_X(E+D)}\omega_X(E+D).$$

REMARK 2.3. Let X be a smooth projective variety. Let $\operatorname{Spec}R\subset X$ be an affine open subset such that R has a p-basis $\{x_1,\dots,x_n\}$. Then, we obtain

$$\Gamma(\operatorname{Spec}R,\omega_X)=R\,dx_1\wedge\cdots\wedge dx_n$$

and

$$\Gamma(\operatorname{Spec}R,F^e_*\omega_X)=\bigoplus_{0\le i_j<p^e}R^{p^e}x_1^{i_1}\cdots x_n^{i_n}\,dx_1\wedge\cdots\wedge dx_n.$$

The trace map

$$\mathrm{Tr}^e_X:\Gamma(\operatorname{Spec}R,F^e_*\omega_X)\to\Gamma(\operatorname{Spec}R,\omega_X)$$

is described as follows:

(1) $\mathrm{Tr}^e_X(x_1^{p^e-1}\cdots x_n^{p^e-1}\,dx_1\wedge\cdots\wedge dx_n)=dx_1\wedge\cdots\wedge dx_n$.

(2) $\mathrm{Tr}^e_X(x_1^{i_1}\cdots x_n^{i_n}\,dx_1\wedge\cdots\wedge dx_n)=0$ if $0\le i_j<p^e-1$ for some $1\le j\le n$.

The following two lemmas give some fundamental properties.

LEMMA 2.4. *Let X be a smooth projective variety, and let E be an effective $\mathbb{Z}$-divisor. If D_1 and D_2 are linearly equivalent $\mathbb{Z}$-divisors, then the two trace maps $\mathrm{Tr}^e_X(D_1)$ and $\mathrm{Tr}^e_X(D_2)$ are the same for every positive integer e, that is, there exists a commutative diagram*:

$$\begin{tikzcd}
F^e_*(\omega_X(E+p^eD_2)) \arrow[r, "\mathrm{Tr}^e_X(D_2)"] \arrow[d, "\simeq"] & \omega_X(E+D_2) \arrow[d, "\simeq"] \\
F^e_*(\omega_X(E+p^eD_1)) \arrow[r, "\mathrm{Tr}^e_X(D_1)"] & \omega_X(E+D_1).
\end{tikzcd}$$

Proof. The assertion follows from $\mathrm{Tr}^e_X(D_i)=\mathrm{Tr}^e_X\otimes_{\mathcal{O}_X}\mathcal{O}_X(D_i)$. $\square$

LEMMA 2.5. *Let X be a smooth projective variety, and let E be an effective $\mathbb{Z}$-divisor. Let D be a $\mathbb{Z}$-divisor. Then, for every positive integer e,*

$$\mathrm{Tr}^{e+1}_{X,E}(D)=\mathrm{Tr}^e_{X,E}(D)\circ F^e_*(\mathrm{Tr}^1_{X,E}(p^eD)),$$

that is,

$$\mathrm{Tr}^{e+1}_{X,E}(D) : F^{e+1}_*(\omega_X(E+p^{e+1}D)) \xrightarrow{F^e_*(\mathrm{Tr}^1_{X,E}(p^eD))} F^e_*(\omega_X(E+p^eD)) \xrightarrow{\mathrm{Tr}^e_{X,E}(D)} \omega_X(E+D).$$

Proof. Consider the Frobenius maps

$$\mathcal{O}_X(-E) \to F^e_*(\mathcal{O}_X(-E)) \to F^{e+1}_*(\mathcal{O}_X(-E)).$$

Tensoring with $\mathcal{O}_X(-D)$ we obtain

$$\mathcal{O}_X(-E-D) \to F^e_*(\mathcal{O}_X(-E-p^eD)) \to F^{e+1}_*(\mathcal{O}_X(-E-p^{e+1}D)).$$

Applying the functor $\mathcal{H}om_X(-,\omega_X)$, we obtain the assertion. □

In this paper, we often use the following two commutative diagrams.

Lemma 2.6. *Let X be a smooth projective variety, and let S be a smooth prime divisor. Then, there exist the following commutative diagrams*:

(1)

$$\begin{CD}
0 @>>> F^e_*\omega_X @>>> F^e_*(\omega_X(S)) @>>> F^e_*\omega_S @>>> 0\\
@. @VV{\mathrm{Tr}^e_X}V @VV{\mathrm{Tr}^e_{X,S}}V @VV{\mathrm{Tr}^e_S}V\\
0 @>>> \omega_X @>>> \omega_X(S) @>>> \omega_S @>>> 0,
\end{CD}$$

(2)

$$\begin{CD}
0 @>>> F^e_*\omega_X @>>> F^e_*(\omega_X(p^eS)) @>>> F^e_*(\omega_{p^eS}) @>>> 0\\
@. @VV{\mathrm{Tr}^e_X}V @VV{\mathrm{Tr}^e_X(S)}V @VV{\Psi}V\\
0 @>>> \omega_X @>>> \omega_X(S) @>>> \omega_S @>>> 0.
\end{CD}$$

Moreover, Tr^e_S *factors through* Ψ:

$$\mathrm{Tr}^e_S : F^e_*\omega_S \to F^e_*(\omega_{p^eS}) \xrightarrow{\Psi} \omega_S.$$

Proof. (1) Consider the following commutative diagram:

$$\begin{CD}
0 @>>> F^e_*(\mathcal{O}_X(-S)) @>>> F^e_*\mathcal{O}_X @>>> F^e_*\mathcal{O}_S @>>> 0\\
@. @AAA @AA{F^e_X}A @AA{F^e_S}A\\
0 @>>> \mathcal{O}_X(-S) @>>> \mathcal{O}_X @>>> \mathcal{O}_S @>>> 0.
\end{CD}$$

Applying the functor $\mathcal{H}om_X(-,\omega_X)$, we obtain the assertion.

(2) Consider the following commutative diagram:

$$\begin{CD}
0 @>>> F^e_*(\mathcal{O}_X(-p^eS)) @>>> F^e_*\mathcal{O}_X @>>> F^e_*\mathcal{O}_{p^eS} @>>> 0\\
@. @AAA @AA{F^e_X}A @AAA\\
0 @>>> \mathcal{O}_X(-S) @>>> \mathcal{O}_X @>>> \mathcal{O}_S @>>> 0.
\end{CD}$$

Applying the functor $Hom_X(-,\omega_X)$, we obtain the required commutative diagram. Since

$$F_S^e : \mathcal{O}_S \to F_*^e\mathcal{O}_S$$

factors through $F_*^e\mathcal{O}_{p^eS}$, we obtain the last assertion in the lemma. $\square$

REMARK 2.7. Given a smooth projective variety X and an ample $\mathbb{Z}$-divisor A on X, it is natural to ask whether for every $e \in \mathbb{Z}_{>0}$, the trace map $\mathrm{Tr}_X^e(A)$ is surjective. However, this has a negative answer. Indeed, [Tango] constructs a smooth projective curve X and an ample $\mathbb{Z}$-divisor A on X such that the trace map $\mathrm{Tr}_X^e(A)$ is not surjective for $e = 1$.

On the other hand, we obtain an affirmative answer for the following two cases: Abelian varieties and F-split varieties.

PROPOSITION 2.8. *If X is an Abelian variety and A is an ample $\mathbb{Z}$-divisor, then the trace map*

$$\mathrm{Tr}_X^e(A) : H^0(X,\omega_X(p^eA)) \to H^0(X,\omega_X(A))$$

is surjective for every $e \in \mathbb{Z}_{>0}$.

Proof. Fix $e \in \mathbb{Z}_{>0}$. For $m \in \mathbb{Z}$, let $m_X : X \to X$ be the m-multiplication map of the Abelian variety X. If $n \in \mathbb{Z}_{>0}$ is not divisible by p, then

$$n_X : X \to X$$

is a finite morphism whose degree is not divisible by p. Thus,

$$\mathcal{O}_X \to (n_X)_*\mathcal{O}_X$$

is split as an $\mathcal{O}_X$-module homomorphism (cf. [Kollár–Mori, Proposition 5.7(2)]). We obtain the following commutative diagram:

$$\begin{tikzcd}
F_*^e(n_X)_*(\mathcal{O}_X(p^e(n_X)^*A)) & (n_X)_*(\mathcal{O}_X((n_X)^*A)) \arrow[l] \\
F_*^e(\mathcal{O}_X(p^eA)) \arrow[u] & \mathcal{O}_X(A). \arrow[l] \arrow[u, "\widetilde{n_X}"']
\end{tikzcd}$$

Applying the functor $Hom_{\mathcal{O}_X}(-,\omega_X)$ (cf. the proof of Proposition 2.1) and taking global sections gives

$$\begin{tikzcd}
H^0(X,\omega_X(p^e(n_X)^*A)) \arrow[r, "\mathrm{Tr}_X^e((n_X)^*A)"] \arrow[d] & H^0(X,\omega_X((n_X)^*A)) \arrow[d, "\widetilde{n_X}'"] \\
H^0(X,\omega_X(p^eA)) \arrow[r, "\mathrm{Tr}_X^e(A)"] & H^0(X,\omega_X(A)).
\end{tikzcd}$$

Here, $\widetilde{n_X}'$ is surjective by the splitting of $\mathcal{O}_X \to (n_X)_*\mathcal{O}_X$. Therefore, it is sufficient to show that $\mathrm{Tr}_X^e((n_X)^*A)$ is surjective. By [Mumford, Corollary 3 in Section 6] we obtain

$$n_X^* A = \frac{n^2+n}{2} A + \frac{n^2-n}{2} (-1)_X^* A.$$

Note that, since $(-1)_X$ is an automorphism, $(-1)_X^* A$ is ample. Therefore, by the Fujita vanishing theorem ([Fujita1, Theorem (1)], [Fujita2, Section 5]), we can find $n \in \mathbb{Z}_{>0}$ such that $\mathrm{Tr}_X^e((n_X)^* A)$ is surjective. $\square$

DEFINITION 2.9. Let X be a smooth projective variety. We say that X is *F-split* if the Frobenius map

$$\mathcal{O}_X \to F_* \mathcal{O}_X$$

is split as an $\mathcal{O}_X$-module homomorphism.

PROPOSITION 2.10. *Let X be an F-split smooth projective variety, and let D be a $\mathbb{Z}$-divisor. Then, the trace map*

$$\mathrm{Tr}_X^e(D) : H^0(X, \omega_X(p^e D)) \to H^0(X, \omega_X(D))$$

is surjective for every $e \in \mathbb{Z}_{>0}$.

Proof. By the definition of F-splitting we see that the Frobenius map

$$\mathcal{O}_X(-D) \to F_*^e(\mathcal{O}_X(-p^e D))$$

is split. Applying the functor $\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X)$, we obtain the assertion. $\square$

3. Facts on Cartier Operator

In this section, we collect some facts on the Cartier operator. By Remark 3.4 we consider the trace map of Frobenius as the Cartier operator.

DEFINITION 3.1. Let X be a smooth variety. Consider the de Rham complex of X

$$\Omega_X^\bullet : \mathcal{O}_X \xrightarrow{d_0} \Omega_X^1 \xrightarrow{d_1} \Omega_X^2 \xrightarrow{d_2} \cdots,$$

where $\Omega_X^i := \Omega_{X/k}^i$. Applying F_*, we obtain a complex

$$F_*\Omega_X^\bullet : F_*\mathcal{O}_X \xrightarrow{F_*d_0} F_*\Omega_X^1 \xrightarrow{F_*d_1} F_*\Omega_X^2 \xrightarrow{F_*d_2} \cdots.$$

Then, it is easy to see that F_*d_i is an $\mathcal{O}_X$-module homomorphism. We define

$$B_X^i := \mathrm{Image}(F_*d_{i-1} : F_*\Omega_X^{i-1} \to F_*\Omega_X^i),$$
$$Z_X^i := \mathrm{Ker}(F_*d_i : F_*\Omega_X^i \to F_*\Omega_X^{i+1}).$$

Note that B_X^i and Z_X^i are coherent sheaves.

FACT 3.2. *Let X be a smooth variety. For every $i \in \mathbb{Z}$ such that $1 \leq i \leq \dim X$, consider the map*

$$C_X^{-1} : \Omega_X^i \to Z_X^i / B_X^i$$

locally defined by

$$C_X^{-1}|_{\operatorname{Spec} R} : \Gamma(\operatorname{Spec} R, \Omega_X^i) \to \Gamma(\operatorname{Spec} R, Z_X^i/B_X^i),$$
$$da_1 \wedge \cdots \wedge da_i \mapsto a_1^{p-1} \cdots a_i^{p-1}\, da_1 \wedge \cdots \wedge da_i,$$

where Spec R *is an open affine subset of* X, *and* $a_1, \ldots, a_i \in R$. *This map* C_X^{-1} *is a well-defined* $\mathcal{O}_X$*-module isomorphism. We call* $C_X := (C_X^{-1})^{-1}$ *the Cartier operator.*

Proof. See, for example, [EV, Theorem 9.14]. □

REMARK 3.3. Let X be an n-dimensional smooth variety. We obtain the following exact sequences:

(1) $0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B_X^1 \to 0$,
(2) $0 \to Z_X^i \to F_*\Omega_X^i \to B_X^{i+1} \to 0$ for $1 \le i \le n$, and
(3) $0 \to B_X^i \to Z_X^i \xrightarrow{C_X^i} \Omega_X^i \to 0$ for $1 \le i \le n$.

By (2) for $i = n$, we obtain $Z_X^n \simeq F_*\omega_X$. By (3) for $i = n$, we obtain

$$0 \to B_X^n \to F_*\omega_X \xrightarrow{C_X^n} \omega_X \to 0.$$

REMARK 3.4. Let X be an n-dimensional smooth projective variety. By Remark 2.3 the Cartier operator C_X^n and the trace map of Frobenius are the same: $C_X^n = \operatorname{Tr}_X^1$.

LEMMA 3.5. *Let* $\gamma : X \to Y$ *be a finite étale morphism between smooth varieties. Then, for every* i,

$$\gamma^* B_Y^i \simeq B_X^i \quad \textit{and} \quad \gamma^* Z_Y^i \simeq Z_X^i.$$

Proof. We may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Let

$$\phi : A \to B$$

be the homomorphism induced by γ. Let

$$F_A : A \to A \quad \text{and} \quad F_B : B \to B$$

be the respective pth-power maps. Since ϕ is étale, the following diagram is a tensor product:

$$\begin{tikzcd} A \arrow[r, "F_A"] \arrow[d, "\phi"] & A \arrow[d, "\phi"] \\ B \arrow[r, "F_B"] & B. \end{tikzcd}$$

Thus, we see that the natural B-module homomorphism

$$\theta^i : ((F_A)_*\Omega_A^i) \otimes_A B \to (F_B)_*(\Omega_A^i \otimes_A B),$$
$$\left(\sum_J a_J\, dx_J\right) \otimes_A b \mapsto \left(\sum_J a_J\, dx_J\right) \otimes_A b^p$$

is an isomorphism where $a_J \in A$ and $dx_J := dx_{j_1} \wedge \cdots \wedge dx_{j_i}$ for some $x_{j_l} \in A$. Since ϕ is étale, the natural B-module homomorphism

$$\rho^i : \Omega^i_A \otimes_A B \to \Omega^i_B,$$
$$\left(\sum_J a_J\, dx_J\right) \otimes_A b \mapsto \sum \phi(a_J) b\, d(\phi(x_J))$$

is an isomorphism. Since ϕ is flat, the isomorphisms in the lemma follow from the commutativity of the following diagram:

$$\begin{tikzcd}
(F_B)_*\Omega^i_B \arrow[r, "d"] & (F_B)_*\Omega^{i+1}_B \\
(F_B)_*(\Omega^i_A \otimes_A B) \arrow[r, "d"] \arrow[u, "(F_B)_*\rho^i"'] & (F_B)_*(\Omega^{i+1}_A \otimes_A B) \arrow[u, "(F_B)_*\rho^{i+1}"'] \\
((F_A)_*\Omega^i_A) \otimes_A B \arrow[r, "d"] \arrow[u, "\theta^i"'] & ((F_A)_*\Omega^{i+1}_A) \otimes_A B, \arrow[u, "\theta^{i+1}"']
\end{tikzcd}$$

which is easy to check. □

We state the following vanishing result on F-split varieties for later use.

Proposition 3.6. *If X is an n-dimensional F-split smooth projective variety and A is an ample $\mathbb{Z}$-divisor, then*

$$H^1(X, B^n_X(A)) = 0.$$

Proof. Consider the exact sequence

$$0 \to B^n_X \to F_*\omega_X \xrightarrow{C^n_X} \omega_X \to 0.$$

Then, by Proposition 2.10, the trace map

$$C^n_X = \mathrm{Tr}^1_X(A) : H^0(X, \omega_X(pA)) \to H^0(X, \omega_X(A))$$

is surjective. Therefore, we obtain the exact sequence

$$0 \to H^1(X, B^n_X(A)) \to H^1(X, \omega_X(pA)).$$

Since F-split varieties satisfy the Kodaira vanishing theorem [MR, Proposition 2], we obtain the vanishing $H^1(X, B^n_X(A)) = 0$. □

4. The Trace Map of Frobenius for Curves

In this section, we calculate the trace map

$$\mathrm{Tr}^e_X(A) : H^0(X, \omega_X(p^e A)) \to H^0(X, \omega_X(A))$$

when X is a curve. By Remark 2.7, $\mathrm{Tr}^e_X(A)$ is not surjective in general. However, we show that $\mathrm{Tr}^e_X(A)$ is almost always a nonzero map.

THEOREM 4.1. *Let X be a smooth projective curve whose genus $g(X)$ is not zero. Let A be an ample $\mathbb{Z}$-divisor. Then, for every $e \in \mathbb{Z}_{>0}$, the trace map*

$$\mathrm{Tr}_X^e(A) : H^0(X, \omega_X(p^e A)) \to H^0(X, \omega_X(A))$$

is a nonzero map.

Proof. Fix $e \in \mathbb{Z}_{>0}$. Since A is ample, we have $\deg A \geq 1$. We consider two cases, $\deg A > 1$ and $\deg A = 1$.

STEP 1. In this step, we assume that $\deg A > 1$, and we prove the assertion. The following argument follows the proof of [Schwede2, Theorem 3.3].

Fix a point $Q \in X$. By Lemma 2.6 we obtain the following commutative diagram:

$$\begin{array}{ccccc} H^0(X, K_X + p^e A) & \longrightarrow & H^0(p^e Q, K_{p^e Q} + p^e(A - Q)) & \to & H^1(X, K_X + p^e(A - Q)) \\ \big\downarrow {\scriptstyle \mathrm{Tr}_X^e(A)} & & \big\downarrow {\scriptstyle \Psi} & & \\ H^0(X, K_X + A) & \xrightarrow{\rho} & H^0(Q, K_Q + A - Q). & & \end{array}$$

By the Serre duality we obtain the vanishing

$$H^1(X, K_X + p^e(A - Q)) = 0.$$

On the other hand, Ψ is surjective because the composition

$$\begin{aligned} \mathrm{Tr}_Q^e : H^0(Q, K_Q + p^e(A - Q)) &\to H^0(p^e Q, K_{p^e Q} + p^e(A - Q)) \\ &\xrightarrow{\Psi} H^0(Q, K_Q + A - Q) \end{aligned}$$

is surjective. Therefore, the composition map $\rho \circ \mathrm{Tr}_X^e(A)$ is surjective. We conclude that $\mathrm{Tr}_X^e(A)$ is a nonzero map since $H^0(Q, K_Q + A - Q) \neq 0$.

STEP 2. In this step, we prove that if $\deg A = 1$, then there exists a point $Q \in X$ such that the natural injective map

$$H^0(X, \omega_X(p^e A - p^e Q)) \to H^0(X, \omega_X(p^e A - (p^e - 1)Q))$$

is not surjective.

Since $H^0(Q, L|_Q) \simeq k$ for every invertible sheaf L on X, we obtain the following exact sequence:

$$\begin{aligned} 0 \to H^0(X, \omega_X(p^e A - p^e Q)) &\to H^0(X, \omega_X(p^e A - (p^e - 1)Q)) \to k \\ &\to H^1(X, \omega_X(p^e A - p^e Q)). \end{aligned}$$

Therefore, it is sufficient to show that

$$h^1(X, \omega_X(p^e A - p^e Q)) = h^0(X, -(p^e A - p^e Q)) = 0$$

for some point $Q \in X$. Note that the first equality follows from the Serre duality. Assume the contrary, that is, assume that $p^e A \sim p^e Q$ for every point $Q \in X$. Since the genus $g(X)$ is not zero, there exists a nonzero l-torsion divisor D for a prime number $l \neq p$. Note that D is not a p^e-torsion. Take the prime decomposition

$$D = \sum m_i Q_i - \sum n_j R_j.$$

Since $\deg D = \sum m_i - \sum n_j = 0$, we obtain the following contradiction:

$$\begin{aligned} p^e D &= \sum m_i p^e Q_i - \sum n_j p^e R_j \\ &\sim \sum m_i p^e A - \sum n_j p^e A \\ &= \left(\sum m_i - \sum n_j\right) p^e A \\ &= 0. \end{aligned}$$

STEP 3. In this step, we assume that $\deg A = 1$ and prove the assertion in the theorem.

We fix a point $Q \in X$ as in Step 2. If $A \sim A'$, then the corresponding trace maps are the same by Lemma 2.4. Therefore, we may assume that $Q \notin \operatorname{Supp} A$. By Step 2 there exists an element

$$\eta \in H^0(X, \omega_X(p^e A - (p^e - 1)Q)) \setminus H^0(X, \omega_X(p^e A - p^e Q)).$$

Take the local ring $(R, \mathfrak{m})$ corresponding to the point Q. Note that $F^e_* R$ is a free R-module. Let $\{x\}$ be a p-basis. Then, we obtain

$$\omega_R = \bigoplus_{0 \le i < p^e} R^{p^e} x^i \, dx.$$

Thus, we can write

$$\eta|_{\operatorname{Spec} R} = \sum_{0 \le i < p^e} f_i^{p^e} x^i \, dx.$$

The fact that $\eta \notin H^0(X, \omega_X(p^e A - p^e Q))$ means that $f_i \notin \mathfrak{m}$ for some $0 \le i < p^e$. Since $\eta \in H^0(X, \omega_X(p^e A - (p^e - 1)Q))$, we have $f_i \in \mathfrak{m}$ for every $0 \le i < p^e - 1$. Therefore, we obtain $f_{p^e - 1} \notin \mathfrak{m}$. Then, we can find $c \in k^\times$ and $\mu \in \mathfrak{m}$ such that

$$f_{p^e - 1} = c + \mu.$$

By Remark 2.3 we see that $\operatorname{Tr}^e_X(A)(\eta)|_{\operatorname{Spec} R} \neq 0$. $\square$

COROLLARY 4.2. *Let X be a smooth projective curve. Let A be an ample $\mathbb{Z}$-divisor. If $H^0(X, \omega_X(A)) \neq 0$, then for every $e \in \mathbb{Z}_{>0}$, the trace map*

$$\operatorname{Tr}^e_X(A) : H^0(X, \omega_X(p^e A)) \to H^0(X, \omega_X(A))$$

is a nonzero map.

Proof. If $g(X) \ge 1$ where $g(X)$ is the genus of X, then the assertion follows from Theorem 4.1. Thus, we may assume that $X \simeq \mathbb{P}^1$. Since $\mathbb{P}^1$ is F-split, the trace map is surjective by Proposition 2.10. $\square$

In characteristic zero, the following result follows using the Kodaira vanishing theorem. In positive characteristic, we obtain the following result by the trace map of Frobenius.

COROLLARY 4.3. *Let X be a smooth projective surface, and let C be a smooth prime divisor on X. Let A be an ample $\mathbb{Z}$-divisor on X. If $H^0(C, K_C + A|_C) \neq 0$, then the natural restriction map*

$$H^0(X, K_X + C + A) \to H^0(C, K_C + A|_C)$$

is a nonzero map.

Proof. By Lemma 2.6, we obtain the following commutative diagram:

$$\begin{array}{ccccc} H^0(X, K_X + C + p^e A) & \longrightarrow & H^0(C, K_C + p^e A|_C) & \longrightarrow & H^1(X, K_X + p^e A) \\ \Big\downarrow{\scriptstyle \mathrm{Tr}^e_{X,C}(A)} & & \Big\downarrow{\scriptstyle \mathrm{Tr}^e_C(A|_C)} & & \\ H^0(X, K_X + C + A) & \longrightarrow & H^0(C, K_C + A|_C). & & \end{array}$$

We see $H^1(X, K_X + p^e A) = 0$ for $e \gg 0$ by the Serre vanishing theorem. Thus, the assertion follows from Corollary 4.2. □

In characteristic zero, in the above situation, the restriction map is surjective by the Kodaira vanishing theorem. However, in positive characteristic, the restriction map is not surjective in general.

EXAMPLE 4.4. There exists a smooth projective surface X, a smooth prime divisor H on X, and an ample $\mathbb{Z}$-divisor A such that:

(1) $|K_X + H + A|$ is basepoint free and

(2) the natural restriction map

$$H^0(X, K_X + H + A) \to H^0(H, K_H + A|_H)$$

is not surjective.

Construction. Let X be a smooth projective surface, and let A be an ample $\mathbb{Z}$-divisor on X such that

$$H^1(X, K_X + A) \neq 0.$$

We can find such a surface by [Raynaud]. Take a smooth hyperplane section H of X such that $|K_X + H + A|$ is basepoint-free and

$$H^1(X, K_X + H + A) = 0.$$

Consider the exact sequence

$$0 \to \mathcal{O}_X(K_X + A) \to \mathcal{O}_X(K_X + H + A) \to \mathcal{O}_H(K_H + A|_H) \to 0.$$

Then, we obtain the following exact sequence:

$$H^0(X, K_X + H + A) \to H^0(H, K_H + A|_H) \to H^1(X, K_X + A) \to 0.$$

Since $H^1(X, K_X + A) \neq 0$, the restriction map is not surjective. □

We use the following corollary in Section 8.

Corollary 4.5. *Let X be a smooth projective surface. Let L be a $\mathbb{Z}$-divisor on X such that*

$$L = C + M,$$

where C is a smooth prime divisor, and M is a nef and big $\mathbb{Z}$-divisor such that $M|_C$ is ample. If $H^0(C, K_C + M|_C) \neq 0$, then the trace map

$$\mathrm{Tr}^e_X(L) : H^0(X, \omega_X(p^e L)) \to H^0(X, \omega_X(L))$$

is a nonzero map.

Proof. By Lemma 2.6 we obtain the following commutative diagram:

$$\begin{tikzcd}
H^0(X, K_X + p^e L) \arrow[r] \arrow[d, "\mathrm{Tr}^e_X(L)"] & H^0(p^e C, K_{p^e C} + p^e M|_C) \arrow[r] \arrow[d, "\Psi"] & H^1(X, K_X + p^e M) \\
H^0(X, K_X + L) \arrow[r] & H^0(C, K_C + M|_C). &
\end{tikzcd}$$

By [T1, Theorem 2.6] we have $H^1(X, K_X + p^e M) = 0$ for $e \gg 0$. By Corollary 4.2, Ψ is a nonzero map because the composition

$$\begin{aligned}\mathrm{Tr}^e_C(M|_C) : H^0(C, K_C + p^e M|_C) &\to H^0(p^e C, K_{p^e C} + p^e M|_{p^e C}) \\ &\xrightarrow{\Psi} H^0(C, K_C + M|_C)\end{aligned}$$

is nonzero. Therefore, the trace map $\mathrm{Tr}^e_X(L)$ also is a nonzero map. □

5. Surjectivity of the Trace Maps for Surfaces ($\kappa = 1$)

In this section, we show the surjectivity of the trace map

$$H^0(X, \omega_X(p^e(A + m(K_X + A)))) \to H^0(X, \omega_X(A + m(K_X + A)))$$

when X is a surface and $\kappa(X, K_X + A) = 1$. For this, we establish the following vanishing result.

Proposition 5.1. *Let C be a smooth curve. Let $Y := \mathbb{P}^1 \times C$, and let $\pi : Y \to C$ be the projection. Let $f : X \to Y$ be the blowup at one point, and let*

$$\theta : X \xrightarrow{f} Y \xrightarrow{\pi} C.$$

If A_X is a θ-ample $\mathbb{Z}$-divisor on X, then

$$R^1\theta_*(B^2_X(A_X)) = 0.$$

Proof.

Step 1. In this step, we assume that C is rational and prove the assertion.

Since the assertion is local on C, we may assume that $C \simeq \mathbb{P}^1$. For an arbitrary ample $\mathbb{Z}$-divisor A_C on C, by the Leray spectral sequence we obtain the following exact sequence:

$$\begin{aligned}0 &\to H^1(C, \theta_*(B^2_X(A_X)) \otimes \mathcal{O}_C(A_C)) \\ &\to H^1(X, B^2_X(A_X + \theta^* A_C)) \\ &\to H^0(C, R^1\theta_*(B^2_X(A_X)) \otimes \mathcal{O}_C(A_C)) \to 0.\end{aligned}$$

Let A_C be an ample $\mathbb{Z}$-divisor on C such that

(1) $A_X + \theta^* A_C$ is ample and
(2) $R^1\theta_*(B_X^2(A_X)) \otimes \mathcal{O}_C(A_C)$ is generated by global sections.

Then, it is sufficient to show

$$H^1(X, B_X^2(A_X + \theta^* A_C)) = 0.$$

Since X is a toric variety hence F-split, this follows from Proposition 3.6.

STEP 2. In this step, we prove that the following assertions are equivalent:

(1) $R^1\theta_*(B_X^2(A_X)) = 0$;
(2) $H^1(X_c, B_X^2(A_X)|_{X_c}) = 0$ for every closed point $c \in C$, where $X_c = \theta^{-1}(c)$.

By [Hartshorne, Theorem 12.11] there exists an isomorphism

$$R^1\theta_*(B_X^2(A_X)) \otimes k(c) \simeq H^1(X_c, B_X^2(A_X)|_{X_c}).$$

By Nakayama's lemma, if

$$R^1\theta_*(B_X^2(A_X)) \otimes k(c) = 0,$$

then $R^1\theta_*(B_X^2(A_X))|_U = 0$ for some open neighborhood $c \in U \subset C$.

STEP 3. In this step, we show that $H^1(X_c, B_X^2(A_X)|_{X_c}) = 0$ for the closed points $c \in C$ other than the one corresponding to the singular fiber. Fix such a closed point $c \in C$. We can shrink C around $c \in C$. Thus, we may assume that $X = Y = \mathbb{P}^1 \times C \to C$. We can find a Cartesian diagram

$$\begin{array}{ccc} X' := \mathbb{P}^1 \times C' & \xleftarrow{\gamma_X} & \mathbb{P}^1 \times C = X \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} \\ C' & \xleftarrow{\gamma_C} & C \end{array}$$

such that $C' \simeq \mathbb{A}^1$ and γ_C and γ_X are finite étale morphisms. Note that $\gamma_X^* B_{X'}^2 \simeq B_X^2$ by Lemma 3.5. We can find a θ'-ample $\mathbb{Z}$-divisor $A_{X'}$ on X' such that $(\gamma_X^*(\mathcal{O}_{X'}(A_{X'})))|_{X_c} \simeq \mathcal{O}_X(A_X)|_{X_c}$. Therefore, we obtain

$$\begin{aligned} H^1(X_c, B_X^2(A_X)|_{X_c}) &= H^1(X_c, \gamma_X^* B_{X'}^2(\gamma_X^* A_{X'})) \\ &= H^1(X'_{c'}, B_{X'}^2(A_{X'})) \\ &= 0. \end{aligned}$$

The last equality follows from Step 1 and Step 2.

STEP 4. In this step, we show that $H^1(X_c, B_X^2(A_X)|_{X_c}) = 0$ for the closed point $c \in C$ corresponding to the singular fiber.

We can replace C by a neighborhood of c. We find a commutative diagram

$$\begin{array}{ccc} X' & \xleftarrow{\gamma_X} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xleftarrow{\gamma_Y} & Y \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ C' & \xleftarrow{\gamma_C} & C \end{array}$$

where $C' \simeq \mathbb{A}^1$, each square is a fiber product, and γ_C, γ_Y, and γ_X are finite étale morphisms. Let $\theta' := \pi' \circ f'$ and $c' := \gamma_C(c)$. Note that $\gamma_X^* B_{X'}^2 \simeq B_X^2$ by Lemma 3.5. Then, $\gamma_X|_{X_c} : X_c \to X'_{c'}$ is an isomorphism.

We show that there exists a θ'-ample $\mathbb{Z}$-divisor $A_{X'}$ on X' such that $(\gamma_X^*(\mathcal{O}_{X'}(A_{X'})))|_{X_c} \simeq \mathcal{O}_X(A_X)|_{X_c}$. The fiber $X_c \simeq X'_{c'} =: D$ is a reduced simple normal crossing divisor $D_1 \cup D_2$ with $D_i \simeq \mathbb{P}^1$. We can assume that D_1 is the f'-exceptional curve and D_2 is the proper transform of a fiber of $Y' \to C'$. We consider the following exact sequence:

$$1 \to \mathcal{O}_D^\times \to \mathcal{O}_{D_1}^\times \times \mathcal{O}_{D_2}^\times \to \mathcal{O}_{D_1 \cap D_2}^\times \to 1.$$

Since the intersection $D_1 \cap D_2$ is one point,

$$H^0(X, \mathcal{O}_{D_1}^\times \times \mathcal{O}_{D_2}^\times) \to H^0(X, \mathcal{O}_{D_1 \cap D_2}^\times)$$

is surjective. Thus, we obtain the following group isomorphism:

$$\operatorname{Pic} D \xrightarrow{\simeq} \operatorname{Pic} D_1 \times \operatorname{Pic} D_2, \qquad L \mapsto (L|_{Y_1}, L|_{Y_2}).$$

Therefore, it suffices to show that, for given positive integers $a_1, a_2 \in \mathbb{Z}_{>0}$, there exists an invertible sheaf $A_{X'}$ on X' such that $A_{X'}|_{D_i} \simeq \mathcal{O}_{D_i}(a_i)$ for each $i = 1, 2$. Consider a section S' of $Y' \to C'$ passing through the blown-up point and its proper transform $T' \subset X'$. Then, $\mathcal{O}_{X'}(T')|_{Y_1} \simeq \mathcal{O}_{D_1}(1)$ and $\mathcal{O}_{X'}(T')|_{Y_2} \simeq \mathcal{O}_{D_2}$. Since

$$\mathcal{O}_{X'}(nT' + mD_1)|_{D_1} \simeq \mathcal{O}_{D_1}(n - m), \qquad \mathcal{O}_{X'}(nT' + mD_1)|_{D_2} \simeq \mathcal{O}_{D_1}(m),$$

we are done.

Thus, we obtain

$$\begin{aligned} H^1(X_c, B_X^2(A_X)) &= H^1(X_c, \gamma_X^* B_{X'}^2(\gamma_X^* A_{X'})) \\ &= H^1(X'_{c'}, B_{X'}^2(A_{X'})) \\ &= 0. \end{aligned}$$

The last equality follows from Step 1 and Step 2.

The assertion in the proposition follows from Step 2, Step 3, and Step 4. □

Let us prove the main theorem in this section.

THEOREM 5.2. *Let X be a smooth projective surface, and let A be an ample $\mathbb{Z}$-divisor on X such that $\kappa(X, K_X + A) = 1$ and $K_X + A$ is nef. Then there exists $m_1 \in \mathbb{Z}_{>0}$ such that the trace map*

$$\begin{aligned}\mathrm{Tr}_X^e(A + m(K_X + A)) : H^0(X, K_X + p^e(A + m(K_X + A)))\\ \to H^0(X, K_X + (A + m(K_X + A)))\end{aligned}$$

is surjective for every integer $m \geq m_1$ and for every $e \in \mathbb{Z}_{>0}$.

Proof.

STEP 1. We see that $K_X + A$ is semiample by [Fujita3, The second theorem in Introduction].

STEP 2. In this step, we prove that, for some $n_0 \in \mathbb{Z}_{>0}$, the complete linear system

$$\Phi_{|n_0(K_X+A)|} = \theta : X \to C$$

gives a ruled surface structure, that is, θ is a projective morphism to a smooth projective curve such that $\theta_*\mathcal{O}_X = \mathcal{O}_C$ and a general fiber is $\mathbb{P}^1$.

We can find $n_0 \in \mathbb{Z}_{>0}$ such that $\Phi_{|n_0(K_X+A)|} = \theta : X \to C$ is a projective morphism to a smooth projective curve such that $\theta_*\mathcal{O}_X = \mathcal{O}_C$. By [Bădescu, Corollary 7.3] general fibers are integral. Since a general fiber F satisfies

$$0 = (K_X + A) \cdot F > (K_X + F) \cdot F,$$

we see $F \simeq \mathbb{P}^1$. Thus, θ gives a ruled surface structure.

STEP 3. In this step, we prove that it is sufficient to show that

$$R^1\theta_*(B_X^2(A')) = 0$$

for every ample $\mathbb{Z}$-divisor A'.

By Remark 3.3 and Remark 3.4 we obtain the following exact sequence:

$$0 \to B_X^2 \to F_*\omega_X \xrightarrow{\mathrm{Tr}_X^1} \omega_X \to 0.$$

By Lemma 2.5 it suffices to find $m_0 \in \mathbb{Z}_{>0}$ such that

$$H^1(X, B_X^2(p^d(A + m(K_X + A)))) = 0$$

for every $d \geq 0$ and $m \geq m_0$. By the Frujita vanishing theorem we can find $d_0 \in \mathbb{Z}_{>0}$ such that if $d > d_0$, then

$$H^1(X, B_X^2(p^d(A + m(K_X + A)))) = 0$$

for every $m \geq 0$. Therefore, we fix an integer $0 \leq d \leq d_0$, and it is enough to find $n_d \in \mathbb{Z}_{>0}$, depending on d, such that

$$H^1(X, B_X^2(p^d(A + m(K_X + A)))) = 0$$

for every $m \geq n_d$. We can write $K_X + A = \theta^* H$ where H is an ample $\mathbb{Z}$-divisor on C. By the Leray spectral sequence we obtain

$$\begin{aligned}
0 &\to H^1(C, \theta_*(B_X^2(p^d A)) \otimes_{\mathcal{O}_C} \mathcal{O}_C(p^d m H)) = 0 \\
&\to H^1(X, B_X^2(p^d(A + m(K_X + A)))) \\
&\to H^0(C, R^1\theta_*(B_X^2(p^d A)) \otimes_{\mathcal{O}_C} \mathcal{O}_C(p^d m H)) \to 0.
\end{aligned}$$

If $m \gg 0$, then the first term vanishes by the Serre vanishing theorem. Thus, it is sufficient to show that $R^1\theta_*(B_X^2(A')) = 0$ for every ample $\mathbb{Z}$-divisor A'.

STEP 4. In this step, we prove the assertion in the theorem. By Step 3 it is sufficient to show that

$$R^1\theta_*(B_X^2(A')) = 0$$

for every ample $\mathbb{Z}$-divisor A'.

Since $X \to C$ is a ruled surface structure, after contracting (-1)-curves in fibers, we obtain morphisms

$$\theta : X \xrightarrow{f} Y \xrightarrow{\pi} C,$$

where f is a birational morphism to a smooth projective surface Y. Note that every fiber of π is irreducible; otherwise, we can find a (-1)-curve in a reducible fiber. By the adjunction formula every fiber of π is $\mathbb{P}^1$. Thus, π is a $\mathbb{P}^1$-bundle structure (cf. [Bădescu, Corollary 11.11]). Since the problem is local on C, we may assume that θ has only one singular fiber and that $Y = \mathbb{P}^1 \times C$. Let F_s be the singular fiber. We see that

$$0 = (K_X + A) \cdot F_s = -2 + A \cdot F_s.$$

Thus, F_s has at most two irreducible components. This implies that f is the blowup at one point or an isomorphism. Then, the equation $R^1\theta_*(B_X^2(A')) = 0$ follows from Proposition 5.1. □

6. Surjectivity of the Trace Maps for Surfaces ($\kappa = 2$)

In this section, we show the surjectivity of the trace map

$$H^0(X, \omega_X(p^e(A + m(K_X + A)))) \to H^0(X, \omega_X(A + m(K_X + A)))$$

when X is a surface and $\kappa(X, K_X + A) = 2$. Let us recall a lemma on global generation.

LEMMA 6.1. *Let X be a smooth projective variety. Let A be an ample $\mathbb{Z}$-divisor, and let G be a coherent sheaf. Then, there exists $n_0 \in \mathbb{Z}_{>0}$, depending only on A and G, such that*

$$G(n_0 A + N)$$

is generated by global sections for every nef $\mathbb{Z}$-divisor N.

Proof. The assertion immediately follows from Castelnuovo–Mumford regularity [Lazarsfeld, Theorem 1.8.5] and the Fujita vanishing theorem ([Fujita1, Theorem (1)], [Fujita2, Section 5]). □

To prove the surjectivity, we establish the following vanishing result.

PROPOSITION 6.2. *Let $h : X \to Z$ be a birational morphism between smooth projective surfaces. Let A_X be an ample $\mathbb{Z}$-divisor on X, and let A_Z be an ample $\mathbb{Z}$-divisor on Z. Then there exists $m_0 \in \mathbb{Z}_{>0}$ such that*

$$H^1(X, B_X^2(A_X + h^*(m_0 A_Z + N_Z))) = 0$$

for every nef $\mathbb{Z}$-divisor N_Z on Z.

Proof. The birational morphism h is a composition of n point blowups. We prove the assertion by induction on n.

STEP 1. If $n = 0$, then the assertion follows from the Fujita vanishing theorem ([Fujita1, Theorem (1)], [Fujita2, Section 5]). Thus, we may assume that $n > 0$ and the assertion holds for $n - 1$.

STEP 2. In this step, we prove that we may assume that $h(\mathrm{Ex}(h))$ is one point.

Assume that the assertion holds when $h(\mathrm{Ex}(h))$ is one point. The Leray spectral sequence induces a short exact sequence

$$\begin{aligned} 0 &\to H^1(Z, h_*(B_X^2(A_X + h^*(m_0 A_Z + N_Z)))) = 0 \\ &\to H^1(X, B_X^2(A_X + h^*(m_0 A_Z + N_Z))) \\ &\to H^0(Z, R^1 h_*(B_X^2(A_X + h^*(m_0 A_Z + N_Z)))) \to 0, \end{aligned}$$

where the equation $H^1(Z, h_*(B_X^2(A_X + h^*(m_0 A_Z + N_Z)))) = 0$ follows from the Fujita vanishing theorem ([Fujita1, Theorem (1)], [Fujita2, Section 5]). By Lemma 6.1 the following assertions are equivalent:

- $H^1(X, B_X^2(A_X + h^*(m_0 A_Z + N_Z))) = 0$.
- $R^1 h_*(B_X^2(A_X)) = 0$.

Set $h(\mathrm{Ex}(h)) = \{z_0, z_1, \ldots, z_m\}$ and $Z_0 := Z \setminus \{z_1, \ldots, z_m\}$. We only show that $R^1 h_*(B_X^2(A_X))|_{Z_0} = 0$. Let X' be the smooth projective surface obtained by patching $Z \setminus \{z_0\}$ and $X \setminus h^{-1}(\{z_1, \ldots, z_m\})$. We obtain a birational morphism $h' : X' \to Z$ of smooth projective surfaces such that $h'(\mathrm{Ex}(h'))$ is equal to $\{z_0\}$ and $h'|_{h'^{-1}(Z_0)} = h|_{h^{-1}(Z_0)}$. Let $A_{X'}^{(0)}$ be the closure of $A_X|_{h^{-1}(Z_0)}$. Since $A_{X'}^{(0)}$ and $A_X|_{h^{-1}(Z)}$ are the same around $\mathrm{Ex}(h')$, we see that $A_{X'}^{(0)}$ is h'-ample. We fix $n_0 \gg 0$ such that $A_{X'} := A_{X'}^{(0)} + n_0 h'^* A_Z$ is ample. By our assumption, we obtain

$$H^1(X', B_{X'}^2(A_{X'} + h'^*(m_0 A_Z + N_Z))) = 0.$$

By the previous argument using the Leray spectral sequence, this is equivalent to

$$R^1 h'_*(B_X^2(A_{X'})) = 0.$$

This implies

$$0 = R^1 h'_*(B_X^2(A_{X'}^{(0)}))|_{Z_0} = R^1 h_*(B_X^2(A_X))|_{Z_0}.$$

We are done.

From now on, we assume that $h(\mathrm{Ex}(h))$ is one point.

STEP 3. We consider the factorization

$$h : X \xrightarrow{f} Y \xrightarrow{g} Z,$$

where g is the blowup of Z at a point. Let E_Y be the g-exceptional curve. Note that $E_Y^2 = -1$. We see

$$g^* A_Z - \frac{1}{l} E_Y$$

is an ample $\mathbb{Q}$-divisor for every large integer $l \gg 0$. Thus, by replacing A_Z with a multiple we may assume that

$$A_Y := g^* A_Z - E_Y$$

is an ample $\mathbb{Z}$-divisor. In particular, we obtain

$$h^* A_Z = f^* A_Y + f^* E_Y.$$

By the induction hypothesis there exists $m_1 \in \mathbb{Z}_{>0}$ such that

$$H^1(X, B_X^2(A_X + f^*(m_1 A_Y + N_Y))) = 0$$

for every nef $\mathbb{Z}$-divisor N_Y on Y. We have

$$m_1 h^* A_Z = m_1 f^* A_Y + m_1 f^* E_Y.$$

STEP 4. Let $E_1, \ldots, E_n \subset X$ be the h-exceptional curves. where E_1 is the proper transform of E_Y. In this step, we construct a sequence of $\mathbb{Z}$-divisors

$$0 =: E(0) \leq E(1) \leq E(2) \leq \cdots \leq E(R-1) \leq E(R) := f^* E_Y$$

such that:

(a) For every $0 \leq r \leq R-1$, $E(r+1) - E(r) = E_j$ for some $1 \leq j \leq n$;
(b) $E(r) \cdot E_i \geq -1$ for every $0 \leq r \leq R$ and for every $1 \leq i \leq n$.

We consider a decomposition into one-point blowups:

$$f : X =: X^{(n)} \xrightarrow{f^{(n)}} \cdots \xrightarrow{f^{(3)}} X^{(2)} \xrightarrow{f^{(2)}} X^{(1)} := Y.$$

We may assume that, for every $2 \leq j \leq n$, $E_j \subset X$ is the proper transform of the $f^{(j)}$-exceptional curve. For $1 \leq j \leq i \leq n$, let $E_j^{(i)} \subset X^{(i)}$ be the image of E_j (e.g., the $f^{(i)}$-exceptional curve is $E_i^{(i)} \subset X^{(i)}$). Let $f^{(i+1)}(\mathrm{Ex}(f^{(i+1)})) =: P^{(i)} \in X^{(i)}$ and denote by $g^{(i)}$ the induced map

$$g^{(i)} : X^{(i)} \to Z.$$

Note that $P^{(i)} \in \mathrm{Ex}(g^{(i)})$. Since $\mathrm{Supp}(\mathrm{Ex}(g^{(i)}))$ is a simple normal crossing, there are two cases:

(1) $P^{(i)} \in E_j^{(i)}$ for some j and $P^{(i)} \notin E_{j'}^{(i)}$ for every $j' \neq j$;

(2) $P^{(i)} \in E_j^{(i)} \cap E_{j'}^{(i)}$ for some $j \neq j'$ and $P^{(i)} \notin E_{j''}^{(i)}$ for every $j'' \neq j, j'$.

For $1 \leq i \leq n$, we construct a finite sequence $(\mathrm{Seq})_i$ of prime divisors on X inductively as follows. Every member of $(\mathrm{Seq})_i$ is equal to E_j for some j. Let

$$(\mathrm{Seq})_1 := (E_1).$$

Assume that we obtain $(\mathrm{Seq})_i$. We construct $(\mathrm{Seq})_{i+1}$ as follows. There are two cases (1) and (2) as before. Assume (1), that is, $P_i \in E_j^{(i)}$ and $P_i \notin E_{j'}^{(i)}$ for every $j' \neq j$. If

$$(\mathrm{Seq})_i = (\dots, E_j, \dots, E_{j'}, \dots),$$

then we define $(\mathrm{Seq})_{i+1}$ by

$$(\mathrm{Seq})_{i+1} := (\dots, E_{i+1}, E_j, \dots, E_{j'}, \dots).$$

In other words, we add E_{i+1} only in front of E_j (for each appearance of E_j). Assume (2), that is, $P^{(i)} \in E_j^{(i)} \cap E_{j'}^{(i)}$ for some $j \neq j'$ and $P_i \notin E_{j''}^{(i)}$ for every $j'' \neq j, j'$. If

$$(\mathrm{Seq})_i = (\dots, E_j, \dots, E_{j'}, \dots, E_{j''}, \dots),$$

then we define $(\mathrm{Seq})_{i+1}$ by

$$(\mathrm{Seq})_{i+1} := (\dots, E_{i+1}, E_j, \dots, E_{i+1}, E_{j'}, \dots, E_{j''}, \dots).$$

In other words, we add E_{i+1} only in front of E_j and $E_{j'}$ (for each appearance of E_j or $E_{j'}$). We obtain a finite sequence $(\mathrm{Seq})_i$ for $1 \leq i \leq n$. Let

$$(\mathrm{Seq})_n = (E_{a(1)}, E_{a(2)}, E_{a(3)}, \dots, E_{a(R)}),$$

where $a(l) \in \{1, \dots, n\}$. We define the finite sequence (SEQ) by

$$\begin{aligned}(\mathrm{SEQ}) &= (E_{a(1)}, E_{a(1)} + E_{a(2)}, E_{a(1)} + E_{a(2)} + E_{a(3)}, \dots) \\ &=: (E(1), E(2), E(3), \dots, E(R)).\end{aligned}$$

It suffices to show that $E(r) \cdot E_j \geq -1$ for every j and $E(R) = f^* E_Y$. By our construction we can check that $E(R) = f^* E_Y$. We prove that $E(r) \cdot E_j \geq -1$ for every j and r by induction on n.

We show that $E(r) \cdot E_n \geq -1$ for every $1 \leq r \leq R$. We consider the behavior of the sequence

$$E(1) \cdot E_n, E(2) \cdot E_n, \dots.$$

If $E(r) \cdot E_n > E(r+1) \cdot E_n$, then $E_{a(r+1)} = E_n$. Thus, we consider the subset $K := \{k_1, \dots, k_\nu\} \subset \{1, \dots, R\}$ with $k_1 < k_2 < \dots < k_\nu$ such that $E_{a(k_1)} = \dots = E_{a(k_\nu)} = E_n$ and that $E_{a(r')} \neq E_n$ for every $r' \in \{1, \dots, R\} \setminus K$:

$$\begin{aligned}(\mathrm{Seq})_n = (\dots, E_{a(k_1)} = E_n, E_{a(k_1+1)}, \dots, E_{a(k_2)} = E_n, \\ E_{a(k_2+1)}, \dots, E_{a(k_\nu)} = E_n, E_{a(k_\nu+1)}, \dots).\end{aligned}$$

By the construction we see $E_n \cap E_{a(k+1)} \neq \emptyset$ for every $k \in K$. Therefore, we obtain

$$\begin{aligned}E(k_1) \cdot E_n &\geq -1, \\ E(k_1+1) \cdot E_n = (E(k_1) + E_{a(k_1+1)}) \cdot E_n &\geq -1 + E_{a(k_1+1)} \cdot E_n \geq 0,\end{aligned}$$

$$E(k_2) \cdot E_n \geq -1,$$
$$E(k_2+1) \cdot E_n = (E(k_2) + E_{a(k_2+1)}) \cdot E_n \geq -1 + E_{a(k_2+1)} \cdot E_n \geq 0,$$
$$\cdots .$$

Thus, we see that $E(r) \cdot E_n \geq -1$ for every $1 \leq r \leq R$.

We consider the corresponding sequences $(\mathrm{Seq})_{n-1}^{(n-1)}$ and $(\mathrm{SEQ})^{(n-1)}$ on $X^{(n-1)}$, that is,

$$(\mathrm{Seq})_n^{(n-1)} := (E_{a(1)}^{(n-1)}, E_{a(2)}^{(n-1)}, E_{a(3)}^{(n-1)}, \ldots, E_{a(R)}^{(n-1)}),$$
$$(\mathrm{SEQ})^{(n-1)} := (E_{a(1)}^{(n-1)}, E_{a(1)}^{(n-1)} + E_{a(2)}^{(n-1)}, E_{a(1)}^{(n-1)} + E_{a(2)}^{(n-1)} + E_{a(3)}^{(n-1)}, \ldots)$$
$$=: (E^{(n-1)}(1), E^{(n-1)}(2), E^{(n-1)}(3), \ldots, E^{(n-1)}(R)),$$

where we set $E_n^{(n-1)} := 0$. By the induction hypothesis we obtain

$$E^{(n-1)}(r) \cdot E_i^{(n-1)} \geq -1$$

for every $1 \leq r \leq R$ and $1 \leq i \leq n$. By our construction we see that

$$(f^{(n)})^*(E^{(n-1)}(r)) = E(r)$$

for every $r \in \{1, \ldots, R\} \setminus K$. Therefore, for $r \in \{1, \ldots, R\} \setminus K$ and $1 \leq i \leq n$, we obtain

$$E(r) \cdot E_i = (f^{(n)})^*(E^{(n-1)}(r)) \cdot E_i = E^{(n-1)}(r) \cdot E_i^{(n-1)} \geq -1.$$

Thus, it suffices to show that

$$E(r) \cdot E_i \geq -1$$

for $r \in K$ and $1 \leq i \leq n-1$. This follows from

$$E(r) \cdot E_i = (E(r-1) + E_{a(r)}) \cdot E_i = (E(r-1) + E_n) \cdot E_i \geq E(r-1) \cdot E_i \geq -1.$$

We are done.

STEP 5. In this step, we construct a sequence of $\mathbb{Z}$-divisors

$$0 =: D(0) \leq D(1) \leq D(2) \leq \cdots \leq D(S-1) \leq D(S) := m_1 f^* E_Y$$

such that:

(a) For every $0 \leq s \leq S-1$, $D(s+1) - D(s) = E_j$ for some j;
(b) $D(s) + A_X + m_1 f^* A_Y$ is nef for every $0 \leq s \leq S$.

We define the sequence $\{D(s)\}_{s=0}^{S}$ by

$$E(0),$$
$$E(1), E(2), \ldots, E(R),$$
$$E(R) + E(1), E(R) + E(2), \ldots, 2E(R),$$
$$2E(R) + E(1), 2E(R) + E(2), \ldots, 3E(R),$$
$$\cdots$$
$$(m_1 - 1)E(R) + E(1), (m_1 l - 1)E(R) + E(2), \ldots, m_1 E(R).$$

Then, the sequence $\{D(s)\}_{s=0}^{S}$ satisfies (a). We now show (b). For every $0 \le s \le S$, we can write

$$D(s) + A_X + m_1 f^* A_Y = E(r) + t f^* E_Y + A_X + m_1 f^* A_Y$$

for some $0 \le r \le R$ and some $0 \le t \le m_1 - 1$. To show that this divisor is nef, it is sufficient to show that

$$(E(r) + t f^* E_Y + A_X + m_1 f^* A_Y) \cdot E_j \ge 0$$

for every $1 \le j \le n$. By Step 4, for every $2 \le j \le n$, we obtain

$$(E(r) + t f^* E_Y + A_X + m_1 f^* A_Y) \cdot E_j = (E(r) + A_X) \cdot E_j \ge 0.$$

On the other hand, when $j = 1$, we have

$$\begin{aligned}(E(r) + t f^* E_Y + A_X + m_1 f^* A_Y) \cdot E_1 &\ge (t f^* E_Y + m_1 f^* A_Y) \cdot E_1\\ &= (t E_Y + m_1 A_Y) \cdot E_Y\\ &\ge (m_1 E_Y + m_1 A_Y) \cdot E_Y\\ &= 0.\end{aligned}$$

STEP 6. For a $\mathbb{Z}$-divisor D on X and for a curve $E \simeq \mathbb{P}^1$ in X, by Lemma 2.6 we obtain the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X, \omega_X(pD)) & \longrightarrow & H^0(X, \omega_X(E + pD)) & \longrightarrow & H^0(E, \omega_E(pD)) & \longrightarrow & H^1(X, \omega_X(pD)) \\ & & \downarrow{\scriptstyle \alpha := \mathrm{Tr}_X(D)} & & \downarrow{\scriptstyle \beta := \mathrm{Tr}_{X,E}(D)} & & \downarrow{\scriptstyle \gamma := \mathrm{Tr}_E(D)} & & \\ 0 & \longrightarrow & H^0(X, \omega_X(D)) & \longrightarrow & H^0(X, \omega_X(E + D)) & \longrightarrow & H^0(E, \omega_E(D)), & & \end{array}$$

where the horizontal sequences are exact, and the vertical arrows are the trace maps. Then the following assertions hold:

(1) γ is surjective.
(2) If $H^1(X, \omega_X(pD)) = 0$ and α is surjective, then β also is surjective.
(3) If β is surjective, then the trace map

$$\mathrm{Tr}_X(E + D) : H^0(X, \omega_X(p(E + D))) \to H^0(X, \omega_X(E + D))$$

also is surjective.

The assertion in (1) holds because $E \simeq \mathbb{P}^1$ is F-split (Proposition 2.10). We deduce (2) from the snake lemma. The assertion in (3) follows from Remark 2.2.

STEP 7. Let $m_2 \in \mathbb{Z}_{>0}$ such that

$$H^1(X, \omega_X(m_2 h^* A_Z + N_X)) = 0$$

for every nef $\mathbb{Z}$-divisor N_X on X. Note that, since $h^* A_Z$ is nef and big, we can find such an integer m_2 by [T1, Theorem 2.6]. Let $m_0 := m_1 + m_2$ and fix a nef $\mathbb{Z}$-divisor N_Z on Z.

We would like to apply the diagram in Step 6 for

$$\begin{aligned} D &= D(s) + A_X + m_1 f^* A_Y + m_2 h^* A_Z + h^* N_Z,\\ E &= D(s+1) - D(s),\end{aligned}$$

where $0 \leq s \leq S-1$. Note that by Step 5 this divisor D is nef. By Step 3, $\alpha = \mathrm{Tr}_X(D)$ in Step 6 is surjective for

$$D = D(0) + A_X + m_1 f^* A_Y + m_2 h^* A_Z + h^* N_Z.$$

We have

$$H^1(X, \omega_X(p(D(s) + A_X + m_1 f^* A_Y + m_2 h^* A_Z + h^* N_Z))) = 0$$

by the choice of m_2. Therefore, by Step 5 and Step 6 we obtain the surjection

$$\mathrm{Tr}_X(D) : H^0(X, \omega_X(pD)) \to H^0(X, \omega_X(D))$$

for

$$\begin{aligned} D &= D(S) + A_X + m_1 f^* A_Y + m_2 h^* A_Z + h^* N_Z \\ &= m_1 f^* E_Y + A_X + m_1 f^* A_Y + m_2 h^* A_Z + h^* N_Z \\ &= A_X + m_1 h^* A_Z + m_2 h^* A_Z + h^* N_Z \\ &= A_X + (m_1 + m_2) h^* A_Z + h^* N_Z \\ &= A_X + h^*(m_0 A_Z + N_Z). \end{aligned}$$

Thus, the assertion in the proposition follows from

$$H^1(X, \omega_X(p(A_X + h^*(m_0 A_Z + N_Z)))) = 0. \qquad \square$$

PROPOSITION 6.3. *Let $h : X \to Z$ be a birational morphism between smooth projective surfaces. Let A_X be an ample $\mathbb{Z}$-divisor on X, and let A_Z be an ample $\mathbb{Z}$-divisor on Z. Then, there exists $m_1 \in \mathbb{Z}_{>0}$ such that the trace map* $\mathrm{Tr}_X^e(A_X + h^*(m_1 A_Z + N_Z))$

$$H^0(X, \omega_X(p^e(A_X + h^*(m_1 A_Z + N_Z)))) \to H^0(X, \omega_X(A_X + h^*(m_1 A_Z + N_Z)))$$

is surjective for every $e \in \mathbb{Z}_{>0}$ and for every nef $\mathbb{Z}$-divisor N_Z on Z.

Proof. For $m \in \mathbb{Z}_{>0}$ and a nef $\mathbb{Z}$-divisor N_Z on Z, set

$$D(m, N_Z) := A_X + h^*(m A_Z + N_Z).$$

By Lemma 2.5 we obtain

$$\mathrm{Tr}_X^{d+1}(D(m, N_Z)) = \mathrm{Tr}_X^d(D(m, N_Z)) \circ F_*^d(\mathrm{Tr}_X(p^d D(m, N_Z))).$$

Thus, it suffices to find $m_1 \in \mathbb{Z}_{>0}$ such that $\mathrm{Tr}_X(p^d D(m_1, N_Z))$ is surjective for every $d \in \mathbb{Z}_{\geq 0}$ and nef $\mathbb{Z}$-divisor N_Z on Z. By the Fujita vanishing theorem we can find $d_0 \in \mathbb{Z}_{>0}$ such that $\mathrm{Tr}_X(p^d D(m_1, N_Z))$ is surjective for every $d > d_0$, $m_1 \in \mathbb{Z}_{>0}$ and nef $\mathbb{Z}$-divisor N_Z on Z. By Proposition 6.2, for every $0 \leq i \leq d_0$, we can find $n_i \in \mathbb{Z}_{>0}$ such that $\mathrm{Tr}_X(p^i D(m, N_Z))$ is surjective for every $m \geq n_i$ and nef $\mathbb{Z}$-divisor N_Z on Z. Therefore, for $m_1 := \max_{1 \leq i \leq d_0}\{n_i\}$, the trace map $\mathrm{Tr}_X(p^d D(m_1, N_Z))$ is surjective for every $d \in \mathbb{Z}_{\geq 0}$ and nef $\mathbb{Z}$-divisor N_Z on Z. $\square$

Let us prove the main theorem in this section.

THEOREM 6.4. *Let X be a smooth projective surface. Let A be an ample $\mathbb{Z}$-divisor on X such that $K_X + A$ is nef and big. Then there exists $m_1 \in \mathbb{Z}_{>0}$ such that the trace map* $\mathrm{Tr}_X^e(A + m(K_X + A))$

$$H^0(X, \omega_X(p^e(A + m(K_X + A)))) \to H^0(X, \omega_X(A + m(K_X + A)))$$

is surjective for every $m \geq m_1$ and for every $e \in \mathbb{Z}_{>0}$.

Proof. By Proposition 6.3 it is sufficient to prove that there exists a birational morphism

$$h : X \to Z$$

to a smooth projective surface Z such that $K_X + A$ is the pull-back of an ample $\mathbb{Z}$-divisor on Z. If $K_X + A$ is ample, then there is nothing to show. We may assume that $K_X + A$ is not ample. Then, by the Nakai–Moishezon criterion we can find a curve E such that $(K_X + A) \cdot E = 0$. This implies $K_X \cdot E < 0$. Moreover, since $K_X + A$ is big, the equation $(K_X + A) \cdot E = 0$ implies $E^2 < 0$. Therefore, E is a (-1)-curve. Let $g : X \to Y$ be the contraction of E. Set $A_Y := g_*A$. Then, we see that A_Y is ample and $K_X + A = g^*(K_Y + A_Y)$. We can apply the same argument to Y and A_Y, that is, $K_Y + A_Y$ is ample, or we can find a (-1)-curve E_Y on Y with $(K_Y + A_Y) \cdot E_Y = 0$. Since $\rho(Y) = \rho(X) - 1$, this procedure terminates. Thus, we obtain a birational morphism $h : X \to Z$ to a smooth projective surface such that $K_X + A = h^*(K_Z + h_*A)$, where $K_Z + h_*A$ is ample. □

7. Main Theorem for Threefolds

In this section, we prove the main theorem for threefolds. Let us summarize the results on the trace map obtained in the previous sections.

THEOREM 7.1. *Let X be a smooth projective surface. Let A be an ample $\mathbb{Z}$-divisor on X such that $K_X + A$ is nef and $\kappa(X, K_X + A) \neq 0$. Then there exists $m_1 \in \mathbb{Z}_{>0}$ such that the trace map*

$$\mathrm{Tr}_X^e(A + m(K_X + A)) : H^0(X, K_X + p^e(A + m(K_X + A))) \\ \to H^0(X, K_X + (A + m(K_X + A)))$$

is surjective for every $m \geq m_1$ and for every $e \in \mathbb{Z}_{>0}$.

Proof. If $\kappa(X, K_X + A) = -\infty$, then there is nothing to show. Thus, we may assume that $\kappa(X, K_X + A) \geq 1$. Then, the assertion follows from Theorem 5.2 and Theorem 6.4 □

REMARK 7.2. In the previous situation, one can show $\kappa(X, K_X + A) \neq -\infty$ using the abundance theorem obtained in [Fujita3, Theorem 1.4]. Indeed, by Bertini's theorem we can find an effective $\mathbb{Q}$-divisor D such that $\llcorner D \lrcorner = 0$ and $A \sim_{\mathbb{Q}} D$.

Let us prove the main theorem.

THEOREM 7.3. *Let X be a smooth projective threefold. Let S be a smooth prime divisor on X, and let A be an ample $\mathbb{Z}$-divisor on X such that*

(1) $K_X + S + A$ *is nef and*
(2) $\kappa(S, K_S + A|_S) \neq 0$.

Then there exists $m_0 \in \mathbb{Z}_{>0}$ *such that, for every integer* $m \geq m_0$, *the natural restriction map*

$$H^0(X, m(K_X + S + A)) \to H^0(S, m(K_S + A|_S))$$

is surjective.

Proof. Let $L := K_X + S + A$. By Lemma 2.6 we obtain the following commutative diagram:

$$\begin{array}{ccccc} H^0(X, \omega_X(S + p^e A + p^e mL)) & \longrightarrow & H^0(S, \omega_S(p^e A|_S + mp^e L|_S)) & \longrightarrow & H^1(X, \omega_X(p^e A + p^e mL)) \\ \Big\downarrow \mathrm{Tr}^e_{X,S}(A+mL) & & \Big\downarrow \mathrm{Tr}^e_S(A|_S+mL|_S) & & \\ H^0(X, \omega_X(S + A + mL)) & \longrightarrow & H^0(S, \omega_S(A|_S + mL|_S)). & & \end{array}$$

By (2) and Theorem 7.1 we can find $m_0 \in \mathbb{Z}_{>0}$ such that the trace map $\mathrm{Tr}^e_S(A|_S + mL|_S)$

$$H^0(S, K_S + p^e A + p^e mL) \to H^0(S, K_S + A|_S + mL|_S)$$

is surjective for every $m \geq m_0$ and $e \in \mathbb{Z}_{>0}$. Fix an integer $m \geq m_0$. By the Serre vanishing theorem we have

$$H^1(X, \omega_X(p^e A + p^e mL)) = 0$$

for $e \gg 0$. Therefore, the natural restriction map

$$\begin{aligned} &H^0(X, (m+1)(K_X + S + A)) \\ &\quad = H^0(X, \omega_X(S + A + mL)) \\ &\quad \to H^0(S, \omega_S(A|_S + mL|_S)) \\ &\quad = H^0(S, (m+1)(K_S + A|_S)) \end{aligned}$$

is surjective. □

8. Calculation for the Case $\kappa = 0$

In this section, we consider whether Theorem 7.1 holds for the case $\kappa(X, K_X + A) = 0$. Let X be a smooth projective surface, and let A be an ample $\mathbb{Z}$-divisor on X. Assume that $K_X + A$ is nef and $\kappa(X, K_X + A) = 0$. By the abundance theorem [Fujita3, The second theorem in Introduction] we see that $K_X + A \sim_{\mathbb{Q}} 0$. Then, $-K_X$ is ample. In particular, X is a rational surface. In this case, $\mathrm{Pic}(X)$ has no torsion; hence, $K_X + A \sim 0$. We consider the following question.

QUESTION 8.1. Let X be a smooth projective surface such that $-K_X$ is ample. Is the trace map

$$\mathrm{Tr}^e_X(-K_X) : H^0(X, \omega_X(-p^e K_X)) \to H^0(X, \omega_X(-K_X))$$

surjective?

If $K_X^2 \geq 4$, then we obtain an affirmative answer.

PROPOSITION 8.2. *Let X be a smooth projective surface such that $-K_X$ is ample. If $K_X^2 \geq 4$, then the trace map*

$$\mathrm{Tr}_X^e(-K_X) : H^0(X, \omega_X(-p^e K_X)) \to H^0(X, \omega_X(-K_X))$$

is surjective for every $e \in \mathbb{Z}_{>0}$.

Proof. Since $h^0(X, \omega_X(-K_X)) = 1$, it is sufficient to show that the trace map

$$\mathrm{Tr}_X^e(-K_X) : H^0(X, \omega_X(-p^e K_X)) \to H^0(X, \omega_X(-K_X))$$

is a nonzero map. Since $K_X^2 \geq 4$, X is obtained by blowing up $\mathbb{P}^2$ at ≤ 5 points. Therefore, we can find a smooth conic C_0 passing through these points. Let L_0 be a line that does not pass through these points. Let C and L be the proper transforms of C_0 and L_0, respectively. We see that $L|_C$ is ample, $H^0(C, \omega_C(L|_C)) \neq 0$, and L is nef and big. Since $C + L \in |-K_X|$, we can apply Corollary 4.5. □

If X is F-split, then the trace map $\mathrm{Tr}_X^e(-K_X)$ in Question 8.1 is surjective. Note that, by [Hara, Example 5.5] and [Smith, Proposition 4.10], if $K_X^2 \geq 4$, then X is F-split. However, since [Hara] contains no explicit proof, we decided to include the previous proof. Moreover, [Hara, Example 5.5] and [Smith, Proposition 4.10] show that if $K_X^2 = 3$ and X is not F-split, then X is a Fermat-type cubic surface in characteristic two. Indeed, this example gives a negative answer to Question 8.1 as follows.

THEOREM 8.3. *Let* $\mathrm{char}\, k = p = 2$. *Consider $\mathbb{P}^3$ and let $[x : y : z : w]$ be the homogeneous coordinates. Let*

$$X := \{[x : y : z : w] \in \mathbb{P}^3 | x^3 + y^3 + z^3 + w^3 = 0\}.$$

Then the trace map

$$\mathrm{Tr}_X^e(-K_X) : H^0(X, \omega_X(-2^e K_X)) \to H^0(X, \omega_X(-K_X))$$

is the zero map for every $e \in \mathbb{Z}_{>0}$.

Proof. By Lemma 2.5, we may assume that $e = 1$. By Lemma 2.6 we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_*\omega_{\mathbb{P}^3} & \longrightarrow & F_*(\omega_{\mathbb{P}^3}(X)) & \longrightarrow & F_*\omega_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \omega_{\mathbb{P}^3} & \longrightarrow & \omega_{\mathbb{P}^3}(X) & \longrightarrow & \omega_X & \longrightarrow & 0. \end{array}$$

Tensoring $\mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3} - X)$ and taking H^0, we obtain

$$\begin{array}{ccc} H^0(\mathbb{P}^3, \omega_{\mathbb{P}^3}(X - 2K_{\mathbb{P}^3} - 2X)) & \xrightarrow{\beta} & H^0(X, \omega_X(-2K_X)) \\ \Big\downarrow {\scriptstyle \mathrm{Tr}_{\mathbb{P}^3,X}(-K_{\mathbb{P}^3}-X)} & & \Big\downarrow {\scriptstyle \mathrm{Tr}_X(-K_X)} \\ H^0(\mathbb{P}^3, \omega_{\mathbb{P}^3}(X - K_{\mathbb{P}^3} - X)) & \xrightarrow{\alpha} & H^0(X, \omega_X(-K_X)). \end{array}$$

Since $H^1(\mathbb{P}^3, L) = 0$ for an arbitrary invertible sheaf L, β is surjective. Therefore, it is sufficient to prove that the trace map $\mathrm{Tr}_{\mathbb{P}^3,X}(-K_{\mathbb{P}^3} - X)$ is the zero map. By

Lemma 2.4 we obtain

$$\mathrm{Tr}_{\mathbb{P}^3,X}(-K_{\mathbb{P}^3}-X)=\mathrm{Tr}_{\mathbb{P}^3,X}(H),$$

where H is defined by

$$H:=\{[x:y:z:w]\in\mathbb{P}^3|w=0\}.$$

Thus, we show that

$$\mathrm{Tr}:=\mathrm{Tr}_{\mathbb{P}^3,X}(H):H^0(\mathbb{P}^3,\omega_{\mathbb{P}^3}(X+2H))\to H^0(\mathbb{P}^3,\omega_{\mathbb{P}^3}(X+H))$$

is the zero map. Let us take a k-linear basis of $H^0(\mathbb{P}^3,\omega_{\mathbb{P}^3}(X+2H))$. Note that

$$h^0(\mathbb{P}^3,\omega_{\mathbb{P}^3}(X+2H))=4.$$

Let $\operatorname{Spec} k[X,Y,Z]\subset\mathbb{P}^3$ be the affine open subset defined by $w\neq 0$. Consider the following four 3-forms:

$$\begin{aligned}\eta_1&:=\frac{1}{X^3+Y^3+Z^3+1}\,dX\wedge dY\wedge dZ,\\ \eta_X&:=\frac{X}{X^3+Y^3+Z^3+1}\,dX\wedge dY\wedge dZ,\\ \eta_Y&:=\frac{Y}{X^3+Y^3+Z^3+1}\,dX\wedge dY\wedge dZ,\\ \eta_Z&:=\frac{Z}{X^3+Y^3+Z^3+1}\,dX\wedge dY\wedge dZ.\end{aligned}$$

These are elements of $\omega_{k(X,Y,Z)}=(\omega_{\mathbb{P}^3})_\xi$ where ξ is the generic point of $\mathbb{P}^3$. By a direct calculation, these four elements are linearly independent, and $\eta_1,\eta_X,\eta_Y,\eta_Z\in H^0(\mathbb{P}^3,\omega_{\mathbb{P}^3}(X+2H))$. In particular, these four elements form a k-linear basis of $H^0(\mathbb{P}^3,\omega_{\mathbb{P}^3}(X+2H))$. The trace map is a p^{-1}-linear map, that is, for $a,b,c,d\in k$,

$$\begin{aligned}&\mathrm{Tr}(a\eta_1+b\eta_X+c\eta_Y+d\eta_Z)\\ &\quad=a^{1/p}\,\mathrm{Tr}(\eta_1)+b^{1/p}\,\mathrm{Tr}(\eta_X)+c^{1/p}\,\mathrm{Tr}(\eta_Y)+d^{1/p}\,\mathrm{Tr}(\eta_Z).\end{aligned}$$

Thus, it is sufficient to show

$$\mathrm{Tr}(\eta_1)=\mathrm{Tr}(\eta_X)=\mathrm{Tr}(\eta_Y)=\mathrm{Tr}(\eta_Z)=0.$$

Let us only prove $\mathrm{Tr}(\eta_X)=0$. This follows from

$$\begin{aligned}&(\mathrm{Tr}(\eta_X))|_{\operatorname{Spec} k[X,Y,Z]}\\ &=\mathrm{Tr}\left(\frac{X}{X^3+Y^3+Z^3+1}\,dX\wedge dY\wedge dZ\right)\\ &=\mathrm{Tr}\left(\frac{X(X^3+Y^3+Z^3+1)}{(X^3+Y^3+Z^3+1)^2}\,dX\wedge dY\wedge dZ\right)\\ &=\frac{1}{X^3+Y^3+Z^3+1}\,\mathrm{Tr}((X^4+XY^3+XZ^3+X)\,dX\wedge dY\wedge dZ)\\ &=0.\end{aligned}$$

The last equality follows from Remark 2.3. □

We do not know whether the conclusion of Theorem 7.3 holds when $\kappa(S, K_S + A|_S) = 0$. However, the following example shows that when this is the case, we cannot argue as in the proof of the theorem.

EXAMPLE 8.4. Let $\operatorname{char} k = p = 2$. Then, there exist smooth projective threefold X over k, a smooth prime divisor S_0 on X, and an ample $\mathbb{Z}$-divisor A on X that satisfy the following properties.

(1) $|K_X + S_0 + A|$ is basepoint free.

(2) The natural restriction map

$$H^0(X, m(K_X + S_0 + A)) \to H^0(S_0, m(K_{S_0} + A|_{S_0}))$$

is surjective for every $m \in \mathbb{Z}_{>0}$.

(3) The trace map $\mathrm{Tr}^e_{S_0}(A|_{S_0} + m(K_{S_0} + A|_{S_0}))$

$$H^0(S_0, \omega_{S_0}(2^e(A|_{S_0} + m(K_{S_0} + A|_{S_0})))) \to H^0(S_0, \omega_{S_0}(A|_{S_0} + m(K_{S_0} + A|_{S_0})))$$

is the zero map for every $m \in \mathbb{Z}_{>0}$ and for every $e \in \mathbb{Z}_{>0}$.

Proof. Let S be the surface in Theorem 8.3, and let $A_S := -K_S$. Take an arbitrary smooth projective curve C and fix an arbitrary ample $\mathbb{Z}$-divisor A_C on C. Let $X := S \times C$, and let π_S and π_C be the projections onto the first and second components, respectively. Fix a point $c_0 \in C$ and let $S_0 := S \times \{c_0\}$. Let

$$A := \pi_S^* A_S + \pi_C^* A_C.$$

Note that $A|_{S_0} = -K_{S_0}$. Thus, (3) follows from Theorem 8.3. The assertion in (1) follows from

$$\begin{aligned} K_X + S_0 + A &= \pi_S^*(K_S + A_S) + \pi_C^*(K_C + c_0 + A_C) \\ &= \pi_C^*(K_C + c_0 + A_C). \end{aligned}$$

Therefore, in order to conclude the proof, it is sufficient to show (2). This follows from

$$\begin{aligned} &H^1(X, K_X + A + (m-1)(K_X + S_0 + A)) \\ &\quad = H^1(X, \pi_C^*(K_C + A_C + (m-1)(K_C + c_0 + A_C))) \\ &\quad \simeq H^1(S, \mathcal{O}_S) \otimes_k H^0(C, K_C + A_C + (m-1)(K_C + c_0 + A_C)) \\ &\qquad \oplus H^0(S, \mathcal{O}_S) \otimes_k H^1(C, K_C + A_C + (m-1)(K_C + c_0 + A_C)) \\ &\quad = 0. \end{aligned}$$

The last equality holds since $H^1(S, \mathcal{O}_S) = 0$ and

$$\deg_C(A_C + (m-1)(K_C + c_0 + A_C)) > 0. \qquad \square$$

Appendix: Extension Theorem for Surfaces

For the surface case, we can freely use the minimal model theory (cf. [Fujita3; KK; T2]). By using results obtained in [Fujita3; T2], and [T3] we can establish an analogue of [HM, Theorem 5.4.21] as follows.

THEOREM A.1 (extension theorem). *Let X be a smooth projective surface, and let C be a smooth prime divisor on X. Let $\Delta := C + B$, where B is an effective $\mathbb{Q}$-divisor that satisfies the following properties*:

(1) $C \not\subset \operatorname{Supp} B$, $\llcorner B \lrcorner = 0$, *and* (X, Δ) *is plt*,
(2) B *is a big* $\mathbb{Q}$*-divisor, and*
(3) *no prime component of* Δ *is contained in the stable base locus of* $K_X + \Delta$.

Then, there exists an integer $m_0 > 0$ *such that, for every integer* $m > 0$, *the restriction map*

$$H^0(X, mm_0(K_X + \Delta)) \to H^0(C, mm_0(K_X + \Delta)|_C)$$

is surjective.

Proof.

STEP 1. In this step, we prove that if E is a curve in X such that $E^2 < 0$ and $(K_X + \Delta) \cdot E < 0$, then the following three assertions hold:

(a) $K_X \cdot E = E^2 = -1$,
(b) E is not a prime component of Δ, and
(c) $E \cdot C = 0$.

Since $E^2 < 0$, we obtain $(K_X + E) \cdot E \le (K_X + \Delta) \cdot E < 0$. Then, there exists a birational morphism $f : X \to Y$ to a normal $\mathbb{Q}$-factorial surface Y such that $\operatorname{Ex}(f) = E$ (cf. [T2, Theorem 6.2]). Let $\Delta_Y := f_*\Delta$ and define $d \in \mathbb{Q}$ by

$$K_X + \Delta = f^*(K_Y + \Delta_Y) + dE.$$

The inequalities $(K_X + \Delta) \cdot E < 0$ and $E^2 < 0$ imply $d > 0$. We can find an integer $l > 0$ such that $l(K_Y + \Delta_Y)$ is Cartier. Then, E is a fixed component of

$$l(K_X + \Delta) = f^*(l(K_Y + \Delta_Y)) + dlE.$$

We deduce that assumption (3) implies (b). This gives $E \cdot \Delta \ge 0$. Thus, assertion (a) follows from

$$K_X \cdot E \le (K_X + \Delta) \cdot E < 0.$$

Let us show (c). If $E \cdot C > 0$, then $E \cdot C \ge 1$. This implies the following contradiction:

$$0 > (K_X + \Delta) \cdot E = K_X \cdot E + C \cdot E + B \cdot E \ge -1 + 1 + 0 = 0.$$

STEP 2. In this step, we prove that we may assume that $K_X + \Delta$ is nef.

Assume that $K_X + \Delta$ is not nef. Then, there exists a curve E such that $(K_X + \Delta) \cdot E < 0$. By (3) there exists an integer $l > 0$ such that $|l(K_X + \Delta)| \neq \emptyset$. This implies $E^2 < 0$. We see that E is a (-1)-curve by Step 1. Let $f : X \to Y$ be the contraction of E. Let

$$\Delta_Y := f_*\Delta, \qquad C_Y := f_*C, \qquad B_Y := f_*B.$$

We can check that Y and these divisors also satisfy conditions (1), (2), and (3). Let $m_1 > 0$ be an integer such that $m_1\Delta$ is a $\mathbb{Z}$-divisor. Then we have

$$m_1(K_X + \Delta) = f^*(m_1(K_Y + \Delta_Y)) + eE$$

for some $e \in \mathbb{Z}_{>0}$. Let n be an arbitrary positive integer. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$, we have

$$f_*(\mathcal{O}_X(nm_1(K_X + \Delta))) \simeq f_*(\mathcal{O}_X(nm_1 f^*(K_Y + \Delta_Y))) \simeq \mathcal{O}_Y(nm_1(K_Y + \Delta_Y)).$$

Since $C \cap E = \emptyset$ by Step 1, we have

$$f_*(\mathcal{O}_C(nm_1(K_X+\Delta))) \simeq f_*(\mathcal{O}_C(nm_1 f^*(K_Y+\Delta_Y))) \simeq \mathcal{O}_{C_Y}(nm_1(K_Y+\Delta_Y)).$$

We conclude that we have the following commutative diagram:

$$\begin{array}{ccc}
H^0(X, nm_1(K_X + \Delta)) & \longrightarrow & H^0(C, nm_1(K_X + \Delta)|_C) \\
\| & & \| \\
H^0(Y, nm_1(K_Y + \Delta_Y)) & \longrightarrow & H^0(C_Y, nm_1(K_Y + \Delta_Y)|_{C_Y}),
\end{array}$$

where the horizontal arrows are the natural restriction maps. Thus, we can reduce the problem on X to the problem on Y. After repeating this argument finitely many times, we reduce to the case where $K_X + \Delta$ is nef.

STEP 3. By the abundance theorem [Fujita3, The second theorem in Introduction], $K_X + \Delta$ is semiample. Let

$$f := \phi_{|m_2(K_X+\Delta)|} : X \to R$$

for some $m_2 \in \mathbb{Z}_{>0}$ such that $f_*\mathcal{O}_X = \mathcal{O}_R$. In this step, we prove that $f_*\mathcal{O}_C = \mathcal{O}_{f(C)}$.

Assume that $f_*\mathcal{O}_C \neq \mathcal{O}_{f(C)}$. We run the $(K_X + \{\Delta\})$-MMP, where $\{\Delta\}$ denotes the fractional part of Δ. By [T3, Proposition 2.8] there exist morphisms

$$X \xrightarrow{g} V \to R,$$

where V is a smooth projective curve such that a general fiber G of g satisfies $G \simeq \mathbb{P}^1$ and $\llcorner\Delta\lrcorner \cdot G = 2$. Note that $B = \{\Delta\}$ and $C = \llcorner\Delta\lrcorner$. Since G is a fiber and B is big, we deduce $G \cdot B > 0$. Thus, we obtain the following contradiction:

$$0 = (K_X + \Delta) \cdot G = (K_X + B) \cdot G + 2 > (K_X + G) \cdot G + 2 = 0.$$

STEP 4. In this step, we prove the assertion in the theorem. Let

$$f := \phi_{|m_2(K_X+\Delta)|} : X \to R$$

be such that $f_*\mathcal{O}_X = \mathcal{O}_R$, and let $f(C) =: D$. By Step 3 we have $f_*\mathcal{O}_C = \mathcal{O}_D$. Let H be an ample Cartier divisor on R such that $m_2(K_X + \Delta) = f^*H$. By the Serre vanishing theorem we can find $m_3 \in \mathbb{Z}_{>0}$ such that

$$H^0(R, mm_3H) \to H^0(D, mm_3H)$$

is surjective for every $m \in \mathbb{Z}_{>0}$. Since $f_*\mathcal{O}_X = \mathcal{O}_R$ and $f_*\mathcal{O}_C = \mathcal{O}_D$, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(X, mm_2m_3(K_X+\Delta)) & \longrightarrow & H^0(C, mm_2m_3(K_X+\Delta)|_C) \\ \| & & \| \\ H^0(R, mm_3H) & \longrightarrow & H^0(D, mm_3H|_D). \end{array}$$

This implies the assertion in the theorem. □

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