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On Factoriality of Threefolds with Isolated Singularities

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Abstract. We investigate the existence of complete intersection threefolds $X \subset \mathbb{P}^n$ with only isolated, ordinary multiple points and we provide some sufficient conditions for their factoriality.

0. Introduction

Grothendieck–Lefschetz's theorem ([Gro68, Exposé XII, Corollaire 3.7; Har70, Chapter IV, Corollary 3.3; BS95, Corollary 2.3.4]) says that if X is an effective, ample divisor of a smooth variety Y defined over a field of characteristic 0, then the restriction map of Picard groups

$$\operatorname{Pic} Y \longrightarrow \operatorname{Pic} X$$

is injective if $\dim X \geq 2$ and is an isomorphism if $\dim X \geq 3$. One might ask what happens, with the same hypotheses, to the restriction map $\mathrm{CH}^1(Y) \to \mathrm{CH}^1(X)$ between rational equivalence classes of codimension 1 subvarieties. Under some mild assumptions on the singularities of X (e.g., if X is normal), this is equivalent to asking whether or not the conclusions of Grothendieck–Lefschetz's theorem for Picard groups remain true for the restriction map

$$\operatorname{Cl} Y \longrightarrow \operatorname{Cl} X, \tag{1}$$

where, as usual, $\operatorname{Cl} X$ denotes the *class group* of X, namely the group of linear equivalence classes of Weil divisors. When X is smooth, there is nothing new to say since the groups $\operatorname{Pic} X$ and $\operatorname{Cl} X$ are isomorphic; however, when X is singular, the problem becomes a delicate one.

We will restrict ourselves to the case where $Y \subset \mathbb{P}^n$ ($n \geq 4$) is a smooth, complete intersection fourfold and $X \subset Y$ is a threefold with isolated singularities. Since X is projectively normal and nonsingular in codimension 1, the map (1) is an isomorphism precisely when $\operatorname{Pic} X = \operatorname{Cl} X = \mathbb{Z}$, generated by the class of $\mathcal{O}_X(1)$. This is in turn equivalent to the fact that the homogeneous coordinate ring of X is a UFD or that any hypersurface in X is the complete intersection of X with a hypersurface of $\mathbb{P}^n$. In this case we say that X is *factorial.*

In the recent years, the study of factoriality of threefolds in $\mathbb{P}^4$ having only ordinary double points ("nodes") has attracted the attention of several authors. In particular, the following result was conjectured, and proven in a weaker form, by Ciliberto and Di Gennaro ([CDG04a]). The proof of the general case is due to Cheltsov ([Che10b; Che10a]).

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Theorem. *Let $X \subset \mathbb{P}^4$ be a nodal threefold of degree d, and set $k = |\operatorname{Sing}(X)|$. If $k < (d-1)^2$, then X is factorial, whereas when $k = (d-1)^2$, then X is factorial if and only if the nodes are not contained in a plane. Moreover, if the nodes are contained in a plane π, then necessarily $\pi \subset X$, and this explains the lack of factoriality in this case.*

In the present paper we deal with the situation where the singularities involved are not necessarily nodes but, more generally, ordinary m-ple points with $m \geq 2$. Here "ordinary" means that the corresponding tangent cone is a cone over a smooth surface in $\mathbb{P}^3$. We first prove the following Lefschetz-type result; see Theorem 3.4.

Theorem A. *Let $Y \subseteq \mathbb{P}^n$ be a smooth, complete intersection fourfold, and let $X \subseteq Y$ be a reduced and irreducible threefold that is the intersection of Y with a hypersurface of $\mathbb{P}^n$. Suppose that the singular locus Σ of X consists of isolated, ordinary multiple points and denote by $\tilde{X} \subset \tilde{Y}$ the strict transform of X in the blowing-up $\tilde{Y} := \mathrm{Bl}_\Sigma Y$ of Y at Σ. If $\tilde{X}$ is ample in $\tilde{Y}$, then X is factorial.*

The first consequence is that if X has "few" singularities, which are all ordinary points, then X is factorial. More precisely, we have the following result; see Theorem 4.1.

Theorem B. *Let $Y \subset \mathbb{P}^n$ be a smooth, complete intersection fourfold, and $X \subset Y$ be a reduced, irreducible threefold that is the complete intersection of Y with a hypersurface of degree d. Assume that the singular locus of X consists of k ordinary multiple points $p_1, \dots, p_k$ of multiplicities $m_1, \dots, m_k$. If*

$$\sum_{i=1}^{k} m_i < d, \tag{2}$$

then X is factorial.

We also give a different proof of Theorem B in the case $X \subset \mathbb{P}^4$ and $k = 1$ (see the Appendix) because we find it of independent interest.

Theorem B provides the first factoriality criterion for complete intersection threefolds in $\mathbb{P}^n$ with ordinary singularities. Previously, only results for *nodal* threefolds in $\mathbb{P}^4$ and $\mathbb{P}^5$ were known ([Che10b; Kos09]).

When $X \subset \mathbb{P}^4$ and the inequality (2) is not satisfied, we can still give a factoriality criterion, provided that the singularities of X are in general position and they all have the same multiplicity. In fact, using Theorem A together with a result of Ballico (Theorem 1.5), we deduce the following result; see Theorem 4.4.

Theorem C. *Let $\Sigma := \{p_1, \dots, p_k\}$ be a set of k distinct, general points in $\mathbb{P}^4$, and let d, m be positive integers with $d \geq m$.*

(i) *If*

$$\left\lfloor \frac{d+5}{m+4} \right\rfloor^4 > k, \tag{3}$$

then there exists a hypersurface $X \subset \mathbb{P}^4$ of degree d with k ordinary m-ple points at $p_1, \dots, p_k$ and no other singularities.

(ii) *If the stronger condition*

$$\min\left\{\left\lfloor \frac{d+5}{m+4} \right\rfloor^4, \left\lfloor \frac{d}{m} \right\rfloor^4\right\} > k \tag{4}$$

holds, then any hypersurface X as in part (i) *is factorial.*

Using Theorem C, we can easily provide new examples of singular, factorial projective varieties.

In the last part of the paper we construct some nonfactorial threefolds $X \subset \mathbb{P}^4$ of degree d with only k isolated, ordinary m-ple points as singularities. In all these examples the equality $k(m-1)^2 = (d-1)^2$ is satisfied. On the other hand, in [Sab05] it is proven that if the singular locus of X consists of k_2 ordinary double points and k_3 ordinary triple points and if $k_2 + 4k_3 < (d-1)^2$, then any smooth surface contained in X is a complete intersection in X. Motivated by this fact, we make the following conjecture, which generalizes the results of Ciliberto, Di Gennaro, and Cheltsov.

Conjecture D. *Let $X \subset \mathbb{P}^4$ be a hypersurface of degree d whose singular locus consists of k ordinary multiple points $p_1, \dots, p_k$ of multiplicities $m_1, \dots, m_k$. If*

$$\sum_{i=1}^{k} (m_i - 1)^2 < (d-1)^2,$$

then X is factorial.

We hope to come back to this problem in a sequel to this paper.

Let us now explain how this work is organized. In Section 1 we fix notation and terminology, and we collect some preliminary results that are needed in the sequel of the paper. In Section 2 we discuss the notion of factoriality of projective varieties and its relations with the close concepts of local factoriality and $\mathbb{Q}$-factoriality, providing several examples and counterexamples.

In Section 3 we prove Theorem A, whereas in Section 4 we present some of its consequences, including Theorem B and Theorem C. Finally, in Section 5 we describe our nonfactorial examples and state Conjecture D.

Notation and conventions. We work over the field $\mathbb{C}$ of complex numbers.

If X is a projective variety and D_1, D_2 are divisors on X, we write $D_1 \equiv_{\mathrm{hom}} D_2$ for homological equivalence and $D_1 \equiv_{\mathrm{lin}} D_2$ for linear equivalence.

The group of linear equivalence classes of Weil divisors on X is denoted by $\mathrm{Cl}\, X$, whereas the group of linear equivalence classes of Cartier divisors is denoted by $\mathrm{Pic}\, X$.

We write $\mathrm{Sing}\, X$ for the singular locus of X and $b_k(X)$ for its kth Betti number.

1. Preliminaries

In this section we collect, for the reader's convenience, some preliminary results that are used in the sequel of the paper. We start by stating some versions of Lefschetz's theorem on hyperplane sections, both for cohomology groups and for Picard groups.

Theorem 1.1. *Let Y be a smooth, projective variety of dimension n, and $X = X_1 \cap \cdots \cap X_e \subseteq Y$ a smooth complete intersection of effective, ample divisors on Y. Then the restriction map*

$$H^i(Y, \mathbb{Z}) \longrightarrow H^i(X, \mathbb{Z})$$

is an isomorphism for $i \leq \dim X - 1 = n - e - 1$ and is injective with torsion-free cokernel for $i = \dim X = n - e$.

Proof. See [Laz04, Remark 3.1.32]. □

Theorem 1.2. *Let $X \subset \mathbb{P}^n$ be a complete intersection that has only isolated singular points. Then the restriction map*

$$H^i(\mathbb{P}^n, \mathbb{C}) \longrightarrow H^i(X, \mathbb{C})$$

is an isomorphism for $\dim X + 2 \leq i \leq 2 \dim X$.

Proof. See [Dim92, Theorem 2.11, p. 144]. □

Theorem 1.3. *Let $X \subset \mathbb{P}^n$ be a reduced complete intersection with $\dim X \geq 3$. Then the restriction map*

$$\operatorname{Pic} \mathbb{P}^n \longrightarrow \operatorname{Pic} X$$

is an isomorphism. In particular, $\operatorname{Pic} X = \mathbb{Z}$, generated by $\mathcal{O}_X(1)$.

Proof. See [Har70, Chapter IV, Corollary 3.2] or [Gro68, Exposé XII, Corollaire 3.7]. □

Theorem 1.3 admits the following generalization.

Theorem 1.4. *Let Y be a smooth, projective variety, and $X \subset Y$ a reduced, effective ample divisor. Then the restriction map*

$$\operatorname{Pic} Y \longrightarrow \operatorname{Pic} X$$

is an isomorphism if $\dim X \geq 3$ and is injective with torsion-free cokernel if $\dim X = 2$.

Proof. See [BS95, Corollary 2.3.4] and [Laz04, Example 3.1.25]. □

We will also need the following ampleness criterion for the blow-up of $\mathbb{P}^n$ at a finite number of general points.

THEOREM 1.5. *Fix integers n, k, d with $n \geq 2$, $d \geq 2$, and $k > 0$; if $n = 2$, then assume that $d \geq 3$. Let $p_1, \ldots, p_k \in \mathbb{P}^n$ be general points, denote by $\pi : \tilde{\mathbb{P}}^n \longrightarrow \mathbb{P}^n$ the blow-up of $\mathbb{P}^n$ at $p_1, \ldots, p_k$, with exceptional divisors $E_1, \ldots, E_k$, and set $H := \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$. Then the divisor*

$$L := dH - \sum_{i=1}^{k} E_i$$

is ample in $\tilde{\mathbb{P}}^n$ if and only if $L^n > 0$ or, equivalently, if and only if $d^n > k$.

Proof. See [Bal99]. □

COROLLARY 1.6. *With the notation of Theorem 1.5, if a, b are positive integers such that $\lfloor \frac{a}{b} \rfloor^n > k$, then the divisor*

$$L := aH - b \sum_{i=1}^{k} E_i$$

is ample.

Proof. Write $a = \lfloor \frac{a}{b} \rfloor \cdot b + r$, where r is an integer such that $0 \leq r < b$. Then we have

$$\begin{aligned} L = aH - b\sum_{i=1}^{k} E_i &= \left(\left\lfloor \frac{a}{b} \right\rfloor \cdot b + r \right) H - b \sum_{i=1}^{k} E_i \\ &= b \left(\left\lfloor \frac{a}{b} \right\rfloor H - \sum_{i=1}^{k} E_i \right) + rH. \end{aligned}$$

The first summand is ample by Theorem 1.5, and the second is nef, so L is ample by [Laz04, Corollary 1.4.10, p. 53]. □

2. Factoriality

DEFINITION 2.1. An integral domain A is called a *unique factorization domain* (abbreviated to UFD) if any element, which is neither 0 nor a unit, factors uniquely (up to order and units) into a product of irreducible elements.

By [Mat89, Theorem 20.1], a Noetherian integral domain is a UFD if and only if every height 1 prime ideal is principal. Moreover a Noetherian, integrally closed domain A is a UFD if and only if $\mathrm{Cl}\, A = 0$, where $\mathrm{Cl}\, A$ denotes the *divisor class group* of A, namely the group of divisorial fractional ideals modulo the subgroup of principal fractional ideals, see [Mat89, p. 165].

PROPOSITION 2.2. *If A is a Noetherian* UFD *and $S \subset A$ is a multiplicative part, then $S^{-1}A$ is a* UFD. *In particular, if $\mathfrak{p} \subset A$ is a prime ideal, then the local ring $A_{\mathfrak{p}}$ is a* UFD.

Proof. Take a height 1 prime ideal $P \subset S^{-1}A$; then there exists a prime ideal I of A such that $P = S^{-1}I$. Localization does not change height, so I has height 1 in A, and since A is a UFD, we conclude that I is principal, say $I = \langle a \rangle$. Then $P = S^{-1}\langle a \rangle = \langle \frac{a}{1} \rangle$, so P is principal, and this concludes the proof. □

Let $(X, \mathcal{O}_X) = (\operatorname{Spec} A, \tilde{A})$ be the affine scheme associated with a commutative ring A; then we write $\operatorname{Pic} A$ in place of $\operatorname{Pic} X$. Assuming that A is a Noetherian domain with only a finite number of maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ such that $A_{\mathfrak{m}_i}$ is not a UFD, there is a short exact sequence

$$0 \longrightarrow \operatorname{Pic} A \longrightarrow \operatorname{Cl} A \longrightarrow \operatorname{Cl}(S^{-1}A) \longrightarrow 0, \tag{5}$$

where $S = A - \bigcup_{i=1}^{k} \mathfrak{m}_i$. See [Fos73, Chapter V] for more details.

PROPOSITION 2.3. *Let $(A, \mathfrak{m})$ be a Noetherian, normal local ring with* $\dim A \geq 2$ *and set $U := \operatorname{Spec} A - \mathfrak{m}$. Then*

(i) *there is a monomorphism* $\operatorname{Pic} U \longrightarrow \operatorname{Cl} A$;
(ii) *if $A_{\mathfrak{p}}$ is a* UFD *for all $\mathfrak{p} \in U$, then* $\operatorname{Pic} U \longrightarrow \operatorname{Cl} A$ *is an isomorphism.*

Proof. See [Fos73, Proposition 18.10]. □

DEFINITION 2.4. Let $X \subset \mathbb{P}^n$ be a projective variety. We say that X is *factorial* if its homogeneous coordinate ring $S(X) = \mathbb{C}[x_0, \dots, x_n]/I_X$ is a UFD.

PROPOSITION 2.5. *If X is projectively normal and nonsingular in codimension* 1, *then X is factorial if and only if the group* $\operatorname{Cl} X$ *is isomorphic to* $\mathbb{Z}$, *generated by* $\mathcal{O}_X(1)$. *Equivalently, X is factorial if and only if the restriction map*

$$\operatorname{Cl} \mathbb{P}^n \longrightarrow \operatorname{Cl} X$$

is an isomorphism.

Proof. See [Har77, Exercise 6.3 (c), p. 147]. □

REMARK 2.6. Using Proposition 2.5 and Theorem 1.3, we see that if X is a complete intersection, nonsingular in codimension 1 and such that $\dim X \geq 3$, then X is factorial if and only if $\operatorname{Pic} X = \operatorname{Cl} X = \mathbb{Z}$, generated by $\mathcal{O}_X(1)$.

PROPOSITION 2.7. *Let $X \subset \mathbb{P}^n$ be a complete intersection such that* $\dim(\operatorname{Sing} X) < \dim X - 3$. *Then X is factorial.*

Proof. This follows from Grothendieck's proof of Samuel's conjecture, see [Gro68, Exp. XI, Corollaire 3.14] and [Mat89, p. 168]. □

Notice that Proposition 2.7 implies that any complete intersection of dimension at least 4 and with only isolated singularities is necessarily factorial. This explains why in the sequel we will restrict ourselves to the case where X is a threefold.

Let us provide now a couple of examples showing that factoriality is a subtle property, which cannot be detected by merely looking at the type of singularities of X.

EXAMPLE 2.8. Take a hypersurface $X \subset \mathbb{P}^4$ of degree d with a unique ordinary double point and no other singularities. If $d \geq 3$, then X is factorial; see [Che10b]. By contrast, if $d = 2$, then X is a cone over a smooth quadric surface in $\mathbb{P}^3$, which is not factorial because any plane contained in X is a Weil divisor that is not Cartier. Notice that, since all ordinary double points are analytically isomorphic, it is impossible to tell locally analytically the difference between the two cases $d \geq 3$ and $d = 2$, see also [Deb01, pp. 160–161].

EXAMPLE 2.9. Take a hypersurface $X \subset \mathbb{P}^4$ of degree d with exactly $(d-1)^2$ ordinary double points and no other singularities. Then X is factorial if and only if the nodes are not contained in a plane; see [Che10a]. Up to change of coordinates, the fact that the nodes are contained in a plane is equivalent to the fact that the equation of X can be written as $x_0F + x_1G = 0$, and hence the whole plane $\{x_0 = x_1 = 0\}$ is contained in X. Such a plane is a Weil divisor that is not Cartier, and this explains the lack of factoriality in this case.

DEFINITION 2.10. We say that $X \subset \mathbb{P}^n$ is *locally factorial* if the local ring $\mathcal{O}_{X,p}$ is a UFD for any $p \in X$. We say that $X \subset \mathbb{P}^n$ is *locally analytically factorial* if the complete local ring $\widehat{\mathcal{O}}_{X,p}$ is a UFD for any $p \in X$.

Since every regular local ring is a UFD ([Mat89, Theorem 20.3]) and the completion of a regular local ring is again regular ([Eis94, Exercise 19.1, p. 488]), it suffices to check the UFD property only at the points $p \in \operatorname{Sing} X$. An immediate consequence of Proposition 2.2 is that if an irreducible projective variety $X \subset \mathbb{P}^n$ is factorial, then it is locally factorial. By using Remark 2.6 and [Har77, Chapter II, Proposition 6.11] we can prove the following more precise result.

PROPOSITION 2.11. *Let $X \subset \mathbb{P}^n$ be a complete intersection, nonsingular in codimension* 1, *and such that* $\dim X \geq 3$. *Then X is factorial if and only if X is locally factorial.*

The condition $\dim X \geq 3$ in the statement of Proposition 2.11 is an essential one. In fact, take any smooth surface $V \subset \mathbb{P}^3$ of degree at least 2 and containing a line. Then V is not factorial (since the line is a divisor that is not an integer multiple of the hyperplane section), but it is locally factorial because it is nonsingular.

By Mori's theorem ([Fos73, Corollary 6.12]) there is a monomorphism $\operatorname{Cl} \mathcal{O}_{X,p} \longrightarrow \operatorname{Cl} \widehat{\mathcal{O}}_{X,p}$; this implies that if X is locally analytically factorial, then X is locally factorial. The converse is in general not true, as shown by the following examples.

EXAMPLE 2.12. Take a factorial hypersurface $X \subset \mathbb{P}^4$ with a node p. Then X is locally factorial (Proposition 2.11), so the ring $\mathcal{O}_{X,p}$ is a UFD. However, its completion $\widehat{\mathcal{O}}_{X,p}$ is not a UFD since it is isomorphic to $\mathbb{C}[[x, y, z, w]]/(xy - zw)$ and the equality $xy = zw$ is a product of irreducibles in two different ways.

EXAMPLE 2.13 ([Lip75]). Let X be a cone over a smooth, projectively normal variety $V \subset \mathbb{P}^{n-1}$. Then X is factorial if and only if $\operatorname{Pic} V = \mathbb{Z}$, generated by

$\mathcal{O}_V(1)$. Moreover, X is locally analytically factorial if and only if it is factorial and, in addition,

$$H^1(V, \mathcal{O}_V(k)) = 0 \quad \text{for all } k > 0. \tag{6}$$

Condition (6) is satisfied, for example, if V is a complete intersection and $\dim V \geq 2$.

Definition 2.14. A projective variety X is called $\mathbb{Q}$-*factorial* if every Weil divisor on X has an integer multiple that is a Cartier divisor.

Setting $G(X) := \mathrm{Cl}\, X / \mathrm{Pic}\, X$, we see that X is factorial if and only if $G(X) = 0$ and X is $\mathbb{Q}$-factorial if and only if $G(X)$ is a torsion group. In particular, if X is factorial, then it is $\mathbb{Q}$-factorial. For threefolds that are a complete intersection in a smooth ambient space, the converse also holds.

Proposition 2.15. *Let $Y \subset \mathbb{P}^n$ be a smooth fourfold, and let $X \subset Y$ be a reduced and irreducible threefold with isolated singularities that is the intersection of Y with a hypersurface of $\mathbb{P}^n$. Then X is factorial if and only if X is $\mathbb{Q}$-factorial.*

Proof. Let us assume that $G(X)$ is a torsion group; we want to prove that $G(X) = 0$. According to [BS81, §1] and [HP13, Section 2], from (5) we obtain the so-called *Jaffe's exact sequence,* namely

$$0 \longrightarrow \mathrm{Pic}\, X \longrightarrow \mathrm{Cl}\, X \longrightarrow \bigoplus_{p \in \mathrm{Sing}\, X} \mathrm{Cl}\, \mathcal{O}_{X,p}, \tag{7}$$

so we have a monomorphism

$$G(X) \longrightarrow \bigoplus_{p \in \mathrm{Sing}\, X} \mathrm{Cl}\, \mathcal{O}_{X,p}. \tag{8}$$

Let $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{O}_{X,p}$ and set $U_p := \mathrm{Spec}\, \mathcal{O}_{X,p} - \mathfrak{m}_p$; then there is an isomorphism $\mathrm{Pic}\, U_p \longrightarrow \mathrm{Cl}\, \mathcal{O}_{X,p}$; see Proposition 2.3. On the other hand, $\mathrm{Pic}\, U_p$ is torsion-free by [Rob76]; see also [Dao12]. It follows that $\mathrm{Cl}\, \mathcal{O}_{X,p}$ is torsion-free, so (8) yields $G(X) = 0$. □

Remark 2.16. It is possible to give a different proof of Proposition 2.15 using Theorem 1.4; see [Sab09]. It is also true that any $\mathbb{Q}$-factorial Gorenstein threefold with terminal singularities is factorial ([Cut88]); however, notice that an ordinary threefold singularity of multiplicity m is terminal if and only if $m \leq 3$ ([Rei87, p. 351]). For the relevance of the concept of $\mathbb{Q}$-factoriality in the setting of birational geometry, we refer the reader to [Mel04].

Remark 2.17. If X is not a complete intersection, then Proposition 2.15 is in general not true. For instance, the cone over the Veronese surface $V \subset \mathbb{P}^5$ is $\mathbb{Q}$-factorial but not factorial; see [BS95, p. 20].

3. A Lefschetz-type Result

In this section, which is devoted to the proof of Theorem A, we use the following notation.

Let $Y \subseteq \mathbb{P}^n$ be a smooth, complete intersection fourfold, and let $X \subseteq Y$ be a reduced and irreducible threefold that is the intersection of Y with a hypersurface of $\mathbb{P}^n$. We suppose that the only singularities of X are isolated multiple points, and we denote by $\Sigma = \{p_1, \dots, p_k\}$ the singular locus of X.

We also assume that the tangent cone of X at each point p_i is a cone over a *smooth* surface of degree m_i in $\mathbb{P}^3$, and we express this condition by saying that p_i is an *ordinary multiple point* of multiplicity m_i, or *an ordinary m_i-ple point.*

Let $\tilde{Y} := \mathrm{Bl}_\Sigma(Y)$ be the blow-up of Y at Σ, let $\eta : \tilde{Y} \longrightarrow Y$ be the blowing-up map, with exceptional divisors $E_1, \dots, E_k$, and write $\tilde{X} \subset \tilde{Y}$ for the strict transform of X; notice that $\tilde{X}$ is a smooth threefold. Moreover, let H be the pullback on $\tilde{Y}$ of the hyperplane section of Y, namely $H = \eta^*\mathcal{O}_Y(1)$.

Finally, we denote by $\pi : \tilde{X} \longrightarrow X$ the restriction of η to X and by $\mathcal{E}_i \subset \tilde{X}$ the exceptional divisor of π over the point p_i, that is, $\mathcal{E}_i = \tilde{X} \cap E_i$. Since each $p_i \in X$ is an ordinary m_i-ple point, $\mathcal{E}_i$ is a smooth surface of degree m_i in $\mathbb{P}^3$, and we obtain

$$H^1(\mathcal{E}_i, \mathbb{C}) = 0, \qquad H^3(\mathcal{E}_i, \mathbb{C}) = 0. \tag{9}$$

We can summarize the situation by means of the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}_i \arrow[r, hook] \arrow[d, hook] & \tilde{X} \arrow[r, "\pi"] \arrow[d, hook] & X \arrow[d, hook] \\
E_i \arrow[r, hook] & \tilde{Y} \arrow[r, "\eta"] & Y.
\end{tikzcd}$$

Proposition 3.1. *We have*

$$b_4(\tilde{X}) = b_4(X) + k.$$

Proof. Let $\underline{\mathbb{C}}$ be the constant sheaf relative to $\mathbb{C}$ on $\tilde{X}$ and consider the corresponding Leray spectral sequence for $\pi : \tilde{X} \longrightarrow X$, namely

$$\mathrm{E}_2^{p,q} = H^p(X, R^q\pi_*\underline{\mathbb{C}}), \qquad \mathrm{E}_\infty \Rightarrow H^*(\tilde{X}, \mathbb{C}).$$

Observe that $R^0\pi_*\underline{\mathbb{C}} = \underline{\mathbb{C}}$, where by abuse of notation we continue to write $\underline{\mathbb{C}}$ for the constant sheaf relative to $\mathbb{C}$ on X. Moreover, since any semialgebraic set has locally a conic structure ([BCR98, Theorem 9.3.6, p. 225]), it follows that X is locally contractible and

$$R^q\pi_*\underline{\mathbb{C}} = \bigoplus_{i=1}^{k} H^q(\mathcal{E}_i, \mathbb{C})_{p_i}$$

for every $q > 0$, where the subscript denotes the skyscraper sheaf supported at the point p_i. Summing up, we obtain

$$\mathrm{E}_2^{p,q} = \begin{cases} H^p(X,\mathbb{C}), & p \geq 0, q = 0, \\ \bigoplus_{i=1}^k H^q(\mathcal{E}_i,\mathbb{C}), & p = 0, q > 0, \\ 0, & p > 0, q > 0. \end{cases} \tag{10}$$

The relevant part of $\mathrm{E}_2^{p,q}$ is shown in Table 1.

Table 1 The table $\mathrm{E}_2^{p,q}$

$q \backslash p$	0	1	2	3	4	5	6
4	$\bigoplus_{i=1}^k H^4(\mathcal{E}_i)$	0	$\dots$	$\dots$	$\dots$	$\dots$	0
3	$\bigoplus_{i=1}^k H^3(\mathcal{E}_i) = 0$	0	$\dots$	$\dots$	$\dots$	$\dots$	0
2	$\bigoplus_{i=1}^k H^2(\mathcal{E}_i)$	0	$\dots$	$\dots$	$\dots$	$\dots$	0
1	$\bigoplus_{i=1}^k H^1(\mathcal{E}_i) = 0$	0	$\dots$	$\dots$	$\dots$	$\dots$	0
0	$H^0(X)$	$H^1(X)$	$H^2(X)$	$H^3(X)$	$H^4(X)$	$H^5(X)$	$H^6(X)$

Computing the differentials, it is not difficult to check that $\mathrm{E}_6^{p,q} = \mathrm{E}_\infty^{p,q}$, so there is a direct sum decomposition

$$H^4(\tilde{X},\mathbb{C}) = \mathrm{E}_6^{4,0} \oplus \mathrm{E}_6^{3,1} \oplus \mathrm{E}_6^{2,2} \oplus \mathrm{E}_6^{1,3} \oplus \mathrm{E}_6^{0,4}. \tag{11}$$

Using (9) and (10), we obtain

$$\begin{aligned} \mathrm{E}_6^{4,0} &= H^4(X,\mathbb{C}), \\ \mathrm{E}_6^{3,1} &= \mathrm{E}_6^{2,2} = \mathrm{E}_6^{1,3} = 0, \\ \mathrm{E}_6^{0,4} &= \ker\left\{ d_5 \colon \bigoplus_{i=1}^k H^4(\mathcal{E}_i,\mathbb{C}) \longrightarrow H^5(X,\mathbb{C}) \right\}. \end{aligned}$$

Moreover, X is a complete intersection threefold with only isolated singularities; hence, Theorem 1.2 yields $H^5(X,\mathbb{C}) \cong H^5(\mathbb{P}^n,\mathbb{C}) = 0$. Then (11) becomes

$$H^4(\tilde{X},\mathbb{C}) = H^4(X,\mathbb{C}) \oplus \bigoplus_{i=1}^k H^4(\mathcal{E}_i,\mathbb{C}).$$

Since each $\mathcal{E}_i$ is a smooth surface, by the Poincaré duality we deduce $H^4(\mathcal{E}_i,\mathbb{C}) \cong \mathbb{C}$, and this completes the proof. $\square$

Proposition 3.2. *If $b_4(X) = 1$, then X is factorial.*

Proof. Assume that $b_4(X) = 1$ and let $S \subset X$ be any reduced, irreducible surface; we must show that S is a complete intersection in X. Let S' and X' be general hyperplane sections of S and X, respectively, and let $H_X \in |\mathcal{O}_X(1)|$, $H_{X'} \in |\mathcal{O}_{X'}(1)|$. By assumption it follows that there exist integers p, q such that

$$pS \equiv_{\mathrm{hom}} qH_X$$

on X, and hence

$$pS' \equiv_{\mathrm{hom}} qH_{X'}$$

on X'. Since X' is a smooth complete intersection surface, by Theorem 1.1 the map

$$H^2(\mathbb{P}^{n-1}, \mathbb{Z}) \longrightarrow H^2(X', \mathbb{Z})$$

is injective with torsion-free cokernel, and hence there exists an integer r such that

$$S' \equiv_{\mathrm{hom}} rH_{X'} \tag{12}$$

on X'. Again by Theorem 1.1 we deduce $H^1(X', \mathbb{Z}) = 0$, and hence $H^1(X', \mathcal{O}_{X'}) = 0$. Therefore, by looking at the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{O}^*_{X'} \longrightarrow 0$$

we see that there is an injective map

$$\operatorname{Pic} X' = H^1(X', \mathcal{O}^*_{X'}) \hookrightarrow H^2(X', \mathbb{Z}),$$

so (12) implies

$$S' \equiv_{\mathrm{lin}} rH_{X'} \tag{13}$$

on X'. Since any smooth complete intersection is projectively normal ([Har77, ex. 8.4 (b), p. 188]), it follows that S' is the complete intersection of X' with a hypersurface of $\mathbb{P}^{n-1}$ of degree r, say F'. Then the Koszul resolution of $\mathcal{I}_{S'/\mathbb{P}^{n-1}}$ shows that

$$H^1(\mathbb{P}^{n-1}, \mathcal{I}_{S'/\mathbb{P}^{n-1}}(i)) = 0 \quad \text{for all } i \in \mathbb{Z}. \tag{14}$$

Applying the snake lemma ([GM99, Chapter 2]) to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^n}(i-1) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^n}(i) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^{n-1}}(i) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_S(i-1) \arrow[r] & \mathcal{O}_S(i) \arrow[r] & \mathcal{O}_{S'}(i) \arrow[r] & 0
\end{tikzcd}$$

we obtain the short exact sequence

$$0 \longrightarrow \mathcal{I}_{S/\mathbb{P}^n}(i-1) \longrightarrow \mathcal{I}_{S/\mathbb{P}^n}(i) \longrightarrow \mathcal{I}_{S'/\mathbb{P}^{n-1}}(i) \longrightarrow 0,$$

which in turn gives, passing to cohomology,

$$H^1(\mathbb{P}^n, \mathcal{I}_{S/\mathbb{P}^n}(i-1)) \longrightarrow H^1(\mathbb{P}^n, \mathcal{I}_{S/\mathbb{P}^n}(i)) \longrightarrow H^1(\mathbb{P}^{n-1}, \mathcal{I}_{S'/\mathbb{P}^{n-1}}(i)). \tag{15}$$

Since $H^1(\mathbb{P}^n, \mathcal{I}_{S/\mathbb{P}^n}) = 0$, by using (14) and (15) we find by induction that $H^1(\mathbb{P}^n, \mathcal{I}_{S/\mathbb{P}^n}(i)) = 0$ for any $i \geq 0$. In particular, the map

$$H^0(\mathbb{P}^n, \mathcal{I}_{S/\mathbb{P}^n}(r)) \longrightarrow H^0(\mathbb{P}^{n-1}, \mathcal{I}_{S'/\mathbb{P}^{n-1}}(r))$$

is surjective, so we can lift the hypersurface $F' \in H^0(\mathbb{P}^{n-1}, \mathcal{I}_{S'/\mathbb{P}^{n-1}}(r))$ to a hypersurface $F \in H^0(\mathbb{P}^n, \mathcal{I}_{S/\mathbb{P}^n}(r))$. Moreover, such a F does not contain X (since F' does not contain X'). Hence, it follows by degree reasons that S is the complete intersection of X with F. □

REMARK 3.3. If $X \subset \mathbb{P}^4$ is a hypersurface of degree d whose only singularities are ordinary double points, then the converse of Proposition 3.2 also holds, namely X is factorial if and only if $b_4(X) = 1$. In fact, in the nodal case we have $b_4(X) = 1 + \delta$, where δ is the *defect* of X, namely the number of dependent conditions imposed by the reduced subscheme Σ to the homogeneous forms of degree $2d-5$. In other words,

$$\delta = k + h^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2d-5)) - h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d-5)) = h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2d-5)),$$

and we have $\delta = 0$ precisely when X is factorial, see [Dim92, Chapter 6; Cyn01; Che10b].

If the singular points of X have higher multiplicity, then the converse of Proposition 3.2 is in general no longer true. For instance, let $X \subset \mathbb{P}^4$ be a cone over a surface $V \subset \mathbb{P}^3$ of degree $d \geq 4$ with $\operatorname{Pic} V = \mathbb{Z}$. Then X is factorial (Example 2.13), but [Dim92, formula (4.18), p. 169] shows that

$$b_4(X) = b_2(V) = d^3 - 4d^2 + 6d - 2 > 1.$$

We are now ready to prove our Lefschetz-type result, namely Theorem A of the Introduction.

THEOREM 3.4. *Let $Y \subset \mathbb{P}^n$ be a smooth, complete intersection fourfold, and let $X \subset Y$ be a reduced and irreducible threefold that is the intersection of Y with a hypersurface of $\mathbb{P}^n$. Suppose that the singular locus $\Sigma = \{p_1, \dots, p_k\}$ of X consists only of ordinary multiple points and denote by $\tilde{X} \subset \tilde{Y}$ the strict transform of X in the blowing-up $\tilde{Y} := \mathrm{Bl}_\Sigma Y$ of Y at Σ. If $\tilde{X}$ is ample in $\tilde{Y}$, then X is factorial.*

Proof. By Theorem 1.1 we have $h^2(Y, \mathbb{C}) = h^2(\mathbb{P}^n, \mathbb{C}) = 1$, so after blowing up the k points in Σ we obtain

$$h^2(\tilde{Y}, \mathbb{C}) = h^2(Y, \mathbb{C}) + k = 1 + k.$$

By assumption $\tilde{X}$ is ample in $\tilde{Y}$, so again by Theorem 1.1 we can write

$$h^2(\tilde{X}, \mathbb{C}) = h^2(\tilde{Y}, \mathbb{C}) = 1 + k. \tag{16}$$

Using Proposition 3.1, the Poincaré duality, and (16), we get

$$b_4(X) + k = b_4(\tilde{X}) = h^2(\tilde{X}, \mathbb{C}) = 1 + k;$$

hence, $b_4(X) = 1$, and X is factorial by Proposition 3.2. □

COROLLARY 3.5. *With the same notation as before, if $\tilde{X}$ is ample in $\tilde{Y}$, then the restriction maps $\mathrm{Cl}\,\mathbb{P}^n \longrightarrow \mathrm{Cl}\,X$ and $\mathrm{Cl}\,Y \longrightarrow \mathrm{Cl}\,X$ are both isomorphisms.*

Proof. The map $\mathrm{Cl}\,\mathbb{P}^n \longrightarrow \mathrm{Cl}\,X$ is an isomorphism by Theorem 3.4, whereas the map $\mathrm{Cl}\,\mathbb{P}^n \longrightarrow \mathrm{Cl}\,Y$ is an isomorphism by Theorem 1.3. Hence, $\mathrm{Cl}\,Y \longrightarrow \mathrm{Cl}\,X$ is an isomorphism as well. □

Theorem 1.4 and Corollary 3.5 imply that if $\tilde{X}$ is ample in $\tilde{Y}$, then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}\,Y & \longrightarrow & \mathrm{Cl}\,X \\ \Big\downarrow{\scriptstyle \eta^*} & & \Big\downarrow{\scriptstyle \pi^*} \\ \mathrm{Cl}\,\tilde{Y} & \longrightarrow & \mathrm{Cl}\,\tilde{X} \end{array} \tag{17}$$

whose horizontal arrows are both isomorphisms. Moreover, the pull-back map $\pi^* : \mathrm{Cl}\,X \longrightarrow \mathrm{Cl}\,\tilde{X}$ (which can be defined because $\mathrm{Cl}\,X = \mathrm{Pic}\,X$) is injective.

REMARK 3.6. The converse of Theorem 3.4 is in general not true. In fact, let $V \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$ such that $\mathrm{Pic}\,V = \mathbb{Z}$, and let $X \subset \mathbb{P}^4$ be the cone over V. Then the divisor $\tilde{X} \subset \tilde{\mathbb{P}}^4$ belongs to the linear system $|d(H-E)|$; hence, $\tilde{X}^4 = 0$, and by the Nakai–Moishezon criterion $\tilde{X}$ is not ample. However, X is factorial (see Example 2.13).

4. Applications

Let us give now some applications of the previous results. We start by showing that if a threefold hypersurface in a good ambient space has "few" singularities, which are all ordinary points, then X is factorial.

THEOREM 4.1. *Let $Y \subset \mathbb{P}^n$ be a smooth, complete intersection fourfold, and $X \subset Y$ be a reduced, irreducible threefold that is the complete intersection of Y with a hypersurface of degree d. Assume that the singular locus of X consists precisely of k ordinary multiple points $p_1, \dots, p_k$ of multiplicities $m_1, \dots, m_k$. If*

$$\sum_{i=1}^{k} m_i < d, \tag{18}$$

then X is factorial.

Proof. We use the same notation as in Section 3. By Theorem 3.4 it is sufficient to show that $\tilde{X}$ is an ample divisor in $\tilde{Y}$. Since each $p_i \in X$ is an ordinary singular point, $\tilde{X}$ is smooth, and we have

$$\tilde{X} = dH - \sum_{i=1}^{k} m_i E_i = \left(d - \sum_{i=1}^{k} m_i\right) H + \sum_{i=1}^{k} m_i (H - E_i). \tag{19}$$

By (18) the linear system $|(d - \sum_{i=1}^{k} m_i)H|$ is base-point-free; since the same is clearly true for $|m_i(H - E_i)|$, equation (19) shows that $\mathcal{O}_{\tilde{Y}}(\tilde{X})$ is a globally generated line bundle on $\tilde{Y}$. By [Laz04, Corollary 1.2.15, p. 28] it remains only to prove that $\tilde{X}$ has a positive intersection with every effective, irreducible curve

$\tilde{C} \subset \tilde{Y}$. If $\tilde{C} \subset E_i$ for some i, then this is clear. Otherwise, $C := \eta_* \tilde{C}$ is an effective and irreducible curve on Y, and by the projection formula we have

$$H \cdot \tilde{C} = \eta^* \mathcal{O}_Y(1) \cdot \tilde{C} = \mathcal{O}_Y(1) \cdot \eta_* \tilde{C} = \mathcal{O}_Y(1) \cdot C > 0.$$

On the other hand, $(H - E_i) \cdot \tilde{C} \geq 0$ because $|H - E_i|$ is base-point-free. Then (19) implies $\tilde{X} \cdot \tilde{C} > 0$, and we are done. □

As far as we know, Theorem 4.1 provides the first factoriality criterion for complete intersection threefolds in $\mathbb{P}^n$ with ordinary singularities. For *nodal* threefolds in $\mathbb{P}^4$ and $\mathbb{P}^5$, some sharper results were previously obtained in [Che10b] and [Kos09], respectively. See also [Sab05] and [CDG04b]. When $X \subset \mathbb{P}^4$ and $k = 1$, namely when we have exactly one (ordinary) singular point, Theorem 4.1 can be deduced from [Dim92, Theorem 4.17, p. 214]. Since we find this other proof of independent interest, for the sake of completeness, we include it in the Appendix.

EXAMPLE 4.2. Let $m, d \in \mathbb{N}$ with $m < d$ and take a homogeneous polynomial $f_m(x_0, x_1, x_2, x_3)$ of degree m such that $V := V(f_m) \subset \mathbb{P}^3$ is a smooth surface. Given general forms $f_{m+1}, f_{m+2}, \dots, f_d \in \mathbb{C}[x_0, x_1, x_2, x_3]$, of respective degrees $m+1, m+2, \dots, d$, the polynomial

$$f := x_4^{d-m} f_m + x_4^{d-m-1} f_{m+1} + \cdots + f_d$$

defines a hypersurface $X \subset \mathbb{P}^4$ of degree d with a unique singular point, namely $p = [0 : 0 : 0 : 0 : 1]$, which is ordinary of multiplicity m. Then X is factorial by Theorem 4.1.

REMARK 4.3. In the statement of Theorem 4.1, the condition that all the singularities are ordinary is an essential one, as shown by the following example of Kollár, see [Mel04, p. 108]. Consider a general quartic hypersurface $X \subset \mathbb{P}^4$ whose defining polynomial is in the span of the monomials $\{x_0^4, x_1^4, (x_4^2 x_3 + x_2^3)x_0, x_3^3 x_1, x_4^2 x_1^2\}$. Then X has the unique singular point $p = [0 : 0 : 0 : 0 : 1]$, but it is not factorial because it contains the plane $\{x_0 = x_1 = 0\}$. Notice that $p \in X$ is a nonordinary double point because the corresponding tangent cone is a cone over a singular quadric surface in $\mathbb{P}^3$ (in fact, p is a so-called cA_1-singularity, see [Rei87]).

If the bound (18) is not satisfied, we can still give a factoriality criterion for a hypersurface $X \subset \mathbb{P}^4$, provided that its singularities are in general position and they all have the same multiplicity.

THEOREM 4.4. *Let $\Sigma := \{p_1, \dots, p_k\}$ be a set of k distinct, general points in $\mathbb{P}^4$, and let d, m be positive integers with $d \geq m$.*

(i) *If*

$$\left\lfloor \frac{d+5}{m+4} \right\rfloor^4 > k, \tag{20}$$

then there exists a hypersurface $X \subset \mathbb{P}^4$ of degree d, with k ordinary m-ple points at $p_1, \dots, p_k$ and no other singularities.

(ii) *If the stronger condition*

$$\min\left\{\left\lfloor \frac{d+5}{m+4} \right\rfloor^4, \left\lfloor \frac{d}{m} \right\rfloor^4\right\} > k \tag{21}$$

holds, then any hypersurface X as in part (i) *is factorial.*

Proof. We set $Y = \mathbb{P}^4$ and we use the same notation as in Section 3. Since $d \geq m$, we have

$$\left\lfloor \frac{d+4}{m+3} \right\rfloor^4 > \left\lfloor \frac{d+5}{m+4} \right\rfloor^4 > k,$$

so by Corollary 1.6 the two divisors

$$(d+4)H - (m+3)\sum_{i=1}^{k} E_i \quad \text{and} \quad (d+5)H - (m+4)\sum_{i=1}^{k} E_i$$

are ample on $\tilde{Y}$, and by the Kodaira vanishing theorem we deduce

$$\begin{aligned} &H^1\left(\tilde{Y}, (d-1)H - m\sum_{i=1}^{k} E_i\right) = 0 \quad \text{and} \\ &H^1\left(\tilde{Y}, dH - (m+1)\sum_{i=1}^{k} E_i\right) = 0. \end{aligned} \tag{22}$$

By using the two exact sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\tilde{Y}}\left((d-1)H - m\sum_{i=1}^{k} E_i\right) &\longrightarrow \mathcal{O}_{\tilde{Y}}\left(dH - m\sum_{i=1}^{k} E_i\right) \\ \longrightarrow \mathcal{O}_H\left(dH - m\sum_{i=1}^{k} E_i\right) &\longrightarrow 0, \\ 0 \longrightarrow \mathcal{O}_{\tilde{Y}}\left(dH - (m+1)\sum_{i=1}^{k} E_i\right) &\longrightarrow \mathcal{O}_{\tilde{Y}}\left(dH - m\sum_{i=1}^{k} E_i\right) \\ \longrightarrow \mathcal{O}_{\sum_{i=1}^{k} E_i}\left(dH - m\sum_{i=1}^{k} E_i\right) &\longrightarrow 0 \end{aligned}$$

and (22), we now see that the restriction maps

$$H^0\left(\tilde{Y}, dH - m\sum_{i=1}^{k} E_i\right) \longrightarrow H^0\left(H, dH - m\sum_{i=1}^{k} E_i\right), \tag{23}$$

$$H^0\left(\tilde{Y}, dH - m\sum_{i=1}^{k} E_i\right) \longrightarrow H^0\left(E_j, dH - m\sum_{i=1}^{k} E_i\right), \quad j = 1, \dots, k, \tag{24}$$

are all surjective. This means that the linear system $|dH - m\sum_{i=1}^{k} E_i|$ restricts to a complete linear system on the general element of $|H|$ and on each E_j; therefore, $|dH - m\sum_{i=1}^{k} E_i|$ is base-point-free, and by Bertini's theorem its general element $\tilde{X}$ is smooth and irreducible. Then $X := \eta_*(\tilde{X})$ is the desired hypersurface, and this proves (i).

Finally, if (21) holds, then $\tilde{X}$ is ample in $\tilde{Y}$ by Corollary 1.6; hence, X is factorial by Theorem 3.4. This proves (ii). □

COROLLARY 4.5. *If $d \geq \frac{5}{4}m$ and (20) holds, then* any *hypersurface as in Theorem 4.4(i) is factorial.*

Proof. The assumptions imply

$$\left\lfloor \frac{d}{m} \right\rfloor^4 \geq \left\lfloor \frac{d+5}{m+4} \right\rfloor^4 > k,$$

so the claim follows by Theorem 4.4, part (ii). □

The following examples show that the numerical inequalities in Theorem 4.4 are not sharp.

- Let $X \subset \mathbb{P}^4$ be a hypersurface of degree 3 with two ordinary double points and no other singularities. Then (21) is not satisfied, but X is factorial ([Che10b]).
- Let $V \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$ such that $\operatorname{Pic} V = \mathbb{Z}$, and let $X \subset \mathbb{P}^4$ be the cone over V. Then (21) is not satisfied, but X is factorial (Example 2.13).
- Let $X \subset \mathbb{P}^4$ be a hypersurface of degree d with a unique singularity that is ordinary of multiplicity m. If $m < d < 2m + 3$, then (21) is not satisfied, but X is factorial (Theorem 4.1).

5. Nonfactorial Examples

This section is devoted to the construction of some examples of nonfactorial hypersurfaces in $\mathbb{P}^4$ with only ordinary multiple points as singularities. They generalize the examples of nonfactorial, nodal hypersurfaces described in Example 2.9. More precisely, we prove the following result.

PROPOSITION 5.1. *For any pair (t, δ) of positive integers, there exists a nonfactorial hypersurface $X \subset \mathbb{P}^4$ of degree d with k ordinary m-ple points as only singularities, where*

$$d = \delta t + 1, \quad k = \delta^2, \quad m = t + 1.$$

Proof. Consider a general pencil of curves of degree δ in the projective plane $\mathbb{P}^2$ with homogeneous coordinates $[x_2 : x_3 : x_4]$. Let $F_1, F_2, \ldots, F_t$ and $G_1, G_2, \ldots, G_t$ be general elements in the pencil; then the product of the F_i defines a plane curve of degree δt with δ^2 points of multiplicity t at the base points

$[a_k : b_k : c_k]$ of the pencil, and similarly for the product of the G_i. Next, let us define

$$F = \prod_{i=1}^{t} F_i(x_2, x_3, x_4) + \sum_{j=t}^{\delta t}\left(\sum_{\alpha+\beta+\gamma=\delta t-j} \Phi^j_{\alpha\beta\gamma}(x_0, x_1) x_2^{\alpha} x_3^{\beta} x_4^{\gamma}\right),$$
$$G = \prod_{i=1}^{t} G_i(x_2, x_3, x_4) + \sum_{j=t}^{\delta t}\left(\sum_{\alpha+\beta+\gamma=\delta t-j} \Psi^j_{\alpha\beta\gamma}(x_0, x_1) x_2^{\alpha} x_3^{\beta} x_4^{\gamma}\right), \tag{25}$$

where $\Phi^j_{\alpha\beta\gamma}(x_0, x_1)$, $\Psi^j_{\alpha\beta\gamma}(x_0, x_1)$ are general homogeneous forms of degree j in the variables x_0, x_1. Now, set

$$f = x_0 F + x_1 G. \tag{26}$$

We claim that the hypersurface $X = V(f) \subset \mathbb{P}^4$ (whose degree is $d = \delta t + 1$) has the desired properties. Indeed, varying F_i, G_i, $\Phi^j_{\alpha\beta\gamma}$, $\Psi^j_{\alpha\beta\gamma}$, the polynomials f define a linear system contained in $|\mathcal{O}_{\mathbb{P}^4}(d)|$, whose base locus is the plane π of equations $x_0 = x_1 = 0$. Thus, using Bertini's theorem, we easily deduce that the only singular points of the general hypersurface X constructed in this way are the δ^2 points $[0 : 0 : a_k : b_k : c_k] \in \pi$. Moreover, X is obviously not factorial since $\pi \subset X$.

It remains only to show that each $p_i \in X$ is an ordinary $(t+1)$-ple point. Up to a linear change of coordinates involving only x_2, x_3, and x_4, we may assume that $p_i = [0 : 0 : 1 : 0 : 0]$. Write

$$\prod_{i=1}^{t} F_i(1, x_3, x_4) = \prod_{i=1}^{t}(u_{3,i}x_3 + u_{4,i}x_4) + \text{higher order terms},$$
$$\prod_{i=1}^{t} G_i(1, x_3, x_4) = \prod_{i=1}^{t}(v_{3,i}x_3 + v_{4,i}x_4) + \text{higher order terms},$$

where $u_{3,i}, u_{4,i}, v_{3,i}, v_{4,i} \in \mathbb{C}$. Since the variable x_2 appears in the expression of the second summands of F and G with exponent at most $t(\delta - 1)$, the equation of the tangent cone of X at p_i is given by

$$x_0\left(\prod_{i=1}^{t}(u_{3,i}x_3 + u_{4,i}x_4) + \Phi^t_{t(\delta-1)00}(x_0, x_1)\right)$$
$$+ x_1\left(\prod_{i=1}^{t}(v_{3,i}x_3 + v_{4,i}x_4) + \Psi^t_{t(\delta-1)00}(x_0, x_1)\right) = 0,$$

and for a general choice of the parameters, this defines a cone over a smooth surface of degree $t+1$ in $\mathbb{P}^3$. Then the proof is complete. $\square$

Observe that all the examples in Proposition 5.1 satisfy

$$k(m-1)^2 = (d-1)^2.$$

On the other hand, in [Sab05] it is proven that if the singular locus of X consists of k_2 ordinary double points and k_3 ordinary triple points and if $k_2 + 4k_3 < (d-1)^2$, then any smooth surface contained in X is a complete intersection in X. Motivated by this result, we make the following conjecture, which generalizes the theorem of Ciliberto, Di Gennaro, and Cheltsov stated in the Introduction.

Conjecture 5.2. *Let $X \subset \mathbb{P}^4$ be a hypersurface of degree d, whose singular locus consists of k ordinary multiple points $p_1, \dots, p_k$ of multiplicities $m_1, \dots, m_k$. If*

$$\sum_{i=1}^{k} (m_i - 1)^2 < (d-1)^2, \tag{27}$$

then X is factorial.

Theorem 4.1 shows that Conjecture 5.2 is true for $k = 1$.

Appendix

In the present Appendix we give a different proof of Theorem 4.1 in the case $k = 1$ (i.e., for hypersurfaces with only one singularity) by applying some results from [Dim92]. More precisely, we make use of the following lemma.

Lemma A.1. *Let $(X, 0)$ be a germ of an affine hypersurface in $\mathbb{C}^4$, defined by the polynomial f, such that X has an isolated singularity of multiplicity m at 0. If the singularity is ordinary, then*

$$\mu\text{-det}(X, 0) = m,$$

where μ-det$(X, 0)$ denotes the smallest positive integer s such that the family $f_t = f + th$, $t \in [0, \varepsilon)$, is μ-constant for any germ h vanishing of order s at 0 and $\varepsilon > 0$ small enough (here μ is the Milnor number, and ε may depend on h).

Proof. If $\deg(f) = d$, then we can write

$$f = f_m + f_{m+1} + \cdots + f_d,$$

where f_i is the homogeneous piece of degree i. Since $0 \in X$ is an ordinary singularity, the tangent cone $V(f_m) \subset \mathbb{C}^4$ has an isolated singularity at the origin; then f defines a so-called *semiquasihomogeneous* (SQH) *hypersurface singularity* with principal part f_m, see [GLS07, p. 123]. So by [Dim92, p. 74] or [GLS07, Corollary 2.18] we obtain

$$\mu(f) = \mu(f_m) = (m-1)^4.$$

Let h be a general homogeneous polynomial of degree s with $s < m$. For $t \neq 0$, the polynomial $f_t = f + th$ defines an SQH hypersurface singularity with principal part th; hence,

$$\mu(f + th) = \mu(th) = (h-1)^4 < (m-1)^4 = \mu(f),$$

and the family f_t is *not* μ-constant. Therefore, we have μ-det$(X, 0) \geq m$.

It remains to show that μ-det$(X,0) \leq m$. Let h be a germ vanishing of order m at the origin; then we want to prove that $\mu(f+th) = \mu(f)$ if t is small enough. We have

$$f + th = (f_m + th_m) + g,$$

where all the monomials appearing in g have degree at least $m+1$. Moreover, if t is small enough, $V(f_m + th_m)$ has an isolated singularity at the origin. Hence, $f+th$ defines an SQH hypersurface singularity with principal part $f_m + th_m$, so

$$\mu(f+th) = \mu(f_m + th_m) = (m-1)^4 = \mu(f).$$

This completes the proof. □

Now let $Y \subset \mathbb{P}^n$ be a smooth, complete intersection fourfold, and $X \subset Y$ be a reduced, irreducible threefold that is the complete intersection of Y with a hypersurface of degree d. Assume that the singular locus of X consists precisely of one ordinary multiple point p of multiplicity $m < d$. We want to show that X is factorial. By Lemma A.1 the assumption $m < d$ becomes

$$m = \mu\text{-det}(X, p) < d,$$

and then by [Dim92, Theorem 4.17, p. 214] we obtain $\Delta_X = 1$, where Δ_X is the Alexander polynomial of X. By [Dim92, p. 146 and p. 206] this in turn implies $H^4_0(X) = 0$, where H_0 stands for the primitive cohomology, namely

$$H^4_0(X) = \operatorname{coker}\{H^4(\mathbb{P}^4, \mathbb{C}) \longrightarrow H^4(X, \mathbb{C})\}.$$

In other words, the restriction map $H^4(\mathbb{P}^4, \mathbb{C}) \longrightarrow H^4(X, \mathbb{C})$ is surjective, and hence $b_4(X) \leq 1$. Since the class of the hyperplane section of X is certainly nonzero in $H^4(X, \mathbb{C})$, it follows $b_4(X) = 1$, and thus X is factorial by Proposition 3.2.

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