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The Additive Problem with One Cube and Three Cubes of Primes

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Abstract. In this paper, we establish that all positive integers up to N but at most $O(N^{25/27+\varepsilon})$ exceptions can be represented as the sum of a cube and three cubes of primes. This improves upon the earlier result $O(N^{17/18+\varepsilon})$ obtained by Ren and Tsang [4].

1. Introduction

In 1949, Roth [5] investigated the expression of positive integers n as the sum of a cube and three cubes of primes, that is,

$$n = x^3 + p_1^3 + p_2^3 + p_3^3, \tag{1.1}$$

where x is a positive integer, and p_1, p_2, p_3 are primes. The philosophy of the Hardy–Littlewood circle method suggests that every sufficiently large integer n can be expressed in the form (1.1). Roth [5] proved that almost all positive integers n can be written as (1.1). In order to introduce Roth's theorem more precisely, we denote by $r(n)$ the number of representations of n in the form (1.1) and define

$$E(N) = |\{1 \le n \le N: \ r(n) = 0\}|. \tag{1.2}$$

Roth's theorem actually states that $E(N) \ll N \log^{-A} N$ for arbitrary large constant $A > 0$. Roth's theorem has been refined by Ren [2] to

$$E(N) \ll N^{169/170}. \tag{1.3}$$

Recently, further improvement has been obtained in a series of papers by Ren and Tsang [3; 4]. In particular, it was proved in [3] that $E(N) \ll N^{1{,}271/1{,}296+\varepsilon}$, and it was established in [4] that

$$E(N) \ll N^{17/18+\varepsilon}. \tag{1.4}$$

In this paper, we establish the following result.

Theorem 1.1. *Let $E(N)$ be defined in (1.2). Then for any $\varepsilon > 0$, we have*

$$E(N) \ll N^{25/27+\varepsilon}. \tag{1.5}$$

We establish Theorem 1.1 by the Hardy–Littlewood circle method. We employ the technique developed by Vaughan [6; 7]. This technique was recently used by Koichi Kawada to prove that all large even integers can be written as the sum of seven cubes of primes and a cube with at most two prime factors. In prior

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works [2; 3; 4], the proof is to investigate the expression $n = x^3 + p_1^3 + p_2^3 + p_3^3$, where $P < x, p_1 \le 2P$, $P^{5/6} < p_2, p_3 \le 2P^{5/6}$, and $n^{1/3} \ll P \ll n^{1/3}$. In this paper, we investigate the representation $n = x^3 + p_1^3 + p_2^3 + p_3^3$ with $P < x \le 2P$, $P^{5/6} < p_1 \le 2P^{5/6}$, and $P^{25/36} < p_2, p_3 \le 2P^{25/36}$. For the contribution from the minor arcs, we shall essentially consider the generating function

$$\sum_{h \ll \sqrt{P}} \sum_{P < x \le 2P} e(((x+h)^3 - x^3)\alpha).$$

When $|q\alpha - a| \le P^{-3/2-\varepsilon}$ for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $1 \le q \le P^{1-\varepsilon}$ and $(a, q) = 1$, we consider cancelations not only from the summation over x but also from the summation over h. For the technical simplification, we introduce a smooth weight. Then we can obtain very nice approximations to the generating functions (see Lemma 3.1 in Section 3).

As usual, we abbreviate $e^{2\pi i z}$ to $e(z)$. The letter p, with or without a subscript, always denotes a prime number. We use ε to denote a sufficiently small positive number. We denote by $\phi(n)$ the Euler function.

2. Preliminaries

Suppose that N is a sufficiently large real number. Let

$$P = (N/2)^{1/3}, \quad S_1 = P^{5/6}, \quad S_2 = P^{25/36}.$$

We define the smooth function

$$w_0(t) = \begin{cases} \exp\left(\frac{1}{(t-3/2)^2 - 1/4}\right) & \text{if } 1 < t < 2, \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$w(x) = w_0(x/P).$$

We shall investigate

$$R(n) = \sum_{\substack{P < x \le 2P \\ S_1 < p_1 \le 2S_1 \\ S_2 < p_2, p_3 \le 2S_2 \\ x^3 + p_1^3 + p_2^3 + p_3^3 = n}} w(x) \left(\prod_{j=1}^{3} \log p_j \right).$$

In order to apply the circle method, we introduce the generating functions. Let

$$f(\alpha) = \sum_{x \in \mathbb{Z}} w(x) e(x^3 \alpha). \tag{2.1}$$

For $1 \le j \le 2$, we define

$$g_j(\alpha) = \sum_{S_j < p \le 2S_j} (\log p) e(p^3 \alpha). \tag{2.2}$$

Let

$$\mathfrak{M} = \bigcup_{q \le P^{5/36}} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{P^{13/18}}{qN}, \frac{a}{q} + \frac{P^{13/18}}{qN} \right],$$

and let

$$\mathfrak{m} = \left[\frac{1}{P^{3/2}}, 1 + \frac{1}{P^{3/2}} \right] \Big\backslash \mathfrak{M}.$$

LEMMA 2.1 (Lemma 3.1 [4]). *Suppose that α is a real number and that $|\alpha - a/q| \le q^{-2}$ for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$. Let $\beta = \alpha - a/q$. Then for $1 \le j \le 2$, we have*

$$g_j(\alpha) \ll q^{\varepsilon} (\log S_j)^c \left(S_j^{1/2} \sqrt{q(1 + S_j^3 |\beta|)} + S_j^{4/5} + \frac{S_j}{\sqrt{q(1 + S_j^3|\beta|)}} \right),$$

where c is a constant.

LEMMA 2.2 (Lemma 8.5 [9]). *Suppose that α is a real number and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a,q) = 1, \quad 1 \le q \le S_1^{3/2}, \quad \textit{and} \quad |q\alpha - a| \le S_1^{-3/2}.$$

Then for $1 \le j \le 2$, we have

$$g_j(\alpha) \ll S_j^{1-1/12+\varepsilon} + \frac{q^{-1/6+\varepsilon} S_j^{1+\varepsilon}}{(1 + S_j^3 |\alpha - a/q|)^{1/2}}.$$

LEMMA 2.3. *Suppose that α is a real number and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a,q) = 1, \quad 1 \le q \le S_1^{3/2}, \quad \textit{and} \quad |q\alpha - a| \le S_1^{-3/2}.$$

Then for $1 \le j \le 2$, we have

$$g_j(\alpha) \ll S_j^{1-1/12+\varepsilon} + \frac{S_j^{1+\varepsilon}}{q^{1/2}(1 + S_j^3 |\alpha - a/q|)^{1/2}}.$$

Proof. This follows from Lemma 2.1 and Lemma 2.2 by the standard argument. □

LEMMA 2.4. *For $\alpha \in \mathfrak{m}$, we have*

$$g_1(\alpha)^2 g_2(\alpha)^2 \ll S_1^2 S_2^2 P^{-2/9+\varepsilon}.$$

Proof. By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$(a,q) = 1, \quad 1 \le q \le S_1^{3/2}, \quad \text{and} \quad |q\alpha - a| \le S_1^{-3/2}.$$

If $q \le P^{5/36}$ and $|\alpha - a/q| \le S_2^{1/6}/(qS_2^3)$, then we have $1 \le a \le q$ and $|\alpha - a/q| > q^{-1}P^{13/18}N^{-1}$ due to $\alpha \in \mathfrak{m}$. By Lemma 2.1,

$$g_1(\alpha) \ll S_1^{5/6+\varepsilon} + \frac{S_1^{1+\varepsilon}}{\sqrt{q(1+S_1^3|\alpha - a/q|)}} \ll S_1 P^{-1/9+\varepsilon}.$$

Otherwise, by Lemma 2.3 we have

$$g_j(\alpha) \ll S_j^{1-1/12+\varepsilon} \quad \text{for } 1 \le j \le 2.$$

We conclude from the above that $g_1(\alpha)^2 g_2(\alpha)^2 \ll S_1^2 S_2^2 P^{-2/9+\varepsilon}$ for any $\alpha \in \mathfrak{m}$. □

We define

$$S(q,a,m) = \sum_{b=1}^{q} e\left(\frac{ab^3 - mb}{q}\right), \qquad S(q,a) = S(q,a,0),$$

$$S^*(q,a) = \sum_{\substack{b=1 \\ (b,q)=1}}^{q} e\left(\frac{ab^3}{q}\right),$$

and

$$C_h(q,a) = \sum_{x=1}^{q} e\left(\frac{a(3hx^2 + 3h^2x + h^3)}{q}\right).$$

We introduce the multiplicative function $\varpi(q)$ by taking

$$\varpi(p^{3u+v}) = \begin{cases} 3p^{-u-1/2} & \text{when } u \ge 0 \text{ and } v = 1, \\ p^{-u-1} & \text{when } u \ge 0 \text{ and } 2 \le v \le 3. \end{cases}$$

Whenever $(a,q) = 1$, we have

$$q^{-1/2} \ll |S(q,a)|/q \ll \varpi(q) \ll q^{-1/3}. \tag{2.3}$$

LEMMA 2.5. *We have*

$$\sum_{b=1}^{q} C_b(q,a) e(-bm/q) = |S(q,a,m)|^2.$$

Proof. By the definition of $C_b(q,a)$,

$$\sum_{b=1}^{q} C_b(q,a) e(-bm/q) = \sum_{x=1}^{q} \sum_{b=1}^{q} e\left(\frac{a(3bx^2 + 3b^2x + b^3) - bm}{q}\right).$$

We deduce by changing variables that

$$\begin{aligned}\sum_{b=1}^{q} C_b(q,a)e(-bm/q)\\ &= \sum_{x=1}^{q}\sum_{y=1}^{q} e\left(\frac{a(3(y-x)x^2+3(y-x)^2x+(y-x)^3)-(y-x)m}{q}\right)\\ &= \sum_{x=1}^{q}\sum_{y=1}^{q} e\left(\frac{a(y^3-x^3)-(y-x)m}{q}\right) = |S(q,a,m)|^2.\end{aligned}$$

The desired conclusion is established. □

LEMMA 2.6. *If* $(a,q)=1$, *then we have*

$$S(q,a,m) \ll q^{1/2+\varepsilon}(q,m)^{1/4}.$$

Proof. If $(q_1,q_2)=1$, then

$$S(q_1q_2,a,m) = S(q_1,aq_2^2,m)S(q_2,aq_1^2,m).$$

Therefore, it suffices to prove that

$$S(p^\alpha,a,mp^\beta) \ll p^{\alpha/2+\varepsilon}(p^\alpha,p^\beta)^{1/4} \quad \text{if } (am,p)=1. \tag{2.4}$$

In view of Lemma 4.1 in [8] and (2.3), the above estimate holds when $\alpha \le \beta$ or $\beta = 0$. Then we assume that $1 \le \beta < \alpha$. By changing variables,

$$\sum_{1\le x\le p^\alpha} e\left(\frac{ax^3-mp^\beta x}{p^\alpha}\right) = \sum_{1\le y\le p^{\alpha-1}} e\left(\frac{ay^3-mp^\beta y}{p^\alpha}\right)\sum_{1\le x\le p} e\left(\frac{3axy^2}{p}\right). \tag{2.5}$$

First of all, we suppose that $p \ne 3$. Then we get

$$S(p^\alpha,a,mp^\beta) = p\sum_{1\le y\le p^{\alpha-2}} e\left(\frac{apy^3-mp^{\beta-1}y}{p^{\alpha-2}}\right).$$

Clearly, (2.4) holds for $\alpha=2$, and next we consider $\alpha \ge 3$. If $\beta=1$, then by a change of variables we can obtain $S(p^\alpha,a,mp^\beta)=0$. If $\beta \ge 2$, then

$$S(p^\alpha,a,mp^\beta) = p^2\sum_{1\le y\le p^{\alpha-3}} e\left(\frac{ay^3-mp^{\beta-2}y}{p^{\alpha-3}}\right) = p^2S(p^{\alpha-3},a,mp^{\beta-2}).$$

The desired estimate follows from the iterative argument.

Now suppose that $p=3$, and we only need to consider $\beta \ge 2$. By (2.5) and a change of variable we get

$$\begin{aligned}S(p^\alpha,a,mp^\beta) &= p\sum_{1\le y\le p^{\alpha-2}} e\left(\frac{ay^3-mp^\beta y}{p^\alpha}\right)\sum_{1\le x\le p} e\left(\frac{3ap^{\alpha-2}xy^2}{p^\alpha}\right)\\ &= p^2S(p^{\alpha-3},a,mp^{\beta-2}).\end{aligned}$$

The desired estimate follows again from the iterative argument. We complete the proof. □

LEMMA 2.7. *For $y \geq 1$, we have*

$$\sum_{1\leq h<y} C_h(q,a) = \frac{y}{q}|S(q,a)|^2 + O(q^{1+\varepsilon}).$$

Proof. We introduce the congruence condition to deduce that

$$\begin{aligned}\sum_{1\leq h<y} C_h(q,a) &= \sum_{1\leq b\leq q} C_b(q,a) \sum_{\substack{1\leq h<y\\ h\equiv b (\mathrm{mod}\ q)}} 1\\ &= \frac{1}{q}\sum_{1\leq m\leq q}\sum_{1\leq b\leq q} C_b(q,a)\sum_{1\leq h<y} e\Big(\frac{m(h-b)}{q}\Big)\\ &= \frac{1}{q}\sum_{1\leq m\leq q}\sum_{1\leq b\leq q} C_b(q,a)e(-bm/q)\sum_{1\leq h<y} e(hm/q).\end{aligned}$$

By Lemma 2.5 we get

$$\sum_{1\leq h<y} C_h(q,a) = \frac{1}{q}\sum_{1\leq m\leq q} |S(q,a,m)|^2 \sum_{1\leq h<y} e(hm/q).$$

Applying the estimate $\sum_{h<y} e(hm/q) \ll 1/\|m/q\|$ for $1 \leq m \leq q-1$, we obtain

$$\sum_{1\leq h<y} C_h(q,a) = \frac{y+O(1)}{q^2}|S(q,a)|^2 + \frac{O(1)}{q}\sum_{1\leq m\leq q-1} |S(q,a,m)|^2\frac{1}{\|m/q\|}.$$

We complete the proof by applying Lemma 2.6. □

Let

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty}\frac{1}{q\phi^3(q)}\sum_{\substack{a=1\\ (a,q)=1}}^{q} S(q,a)S^*(q,a)^3 e(-an/q).$$

According to (2.5) in [3], for even numbers $n \geq 2$, we have

$$(\log\log n)^{-c} \ll \mathfrak{S}(n) \ll \log n \tag{2.6}$$

for some constant $c > 0$.

LEMMA 2.8. *We have*

$$\begin{aligned}\sum_{q\leq P^{5/36}}\sum_{\substack{a=1\\ (a,q)=1}}^{q} &\frac{S(q,a)}{q} g_1(a/q)g_2(a/q)^2 e(-an/q)\\ &= \mathfrak{S}(n)S_1S_2^2 + O(S_1S_2^2(\log N)^{-A}),\end{aligned}$$

where A is a sufficiently large constant.

Proof. By introducing the Dirichlet characters, for $1 \le j \le 2$, we have

$$g_j(a/q) = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} C_q(\chi, a) g_j(\chi),$$

where

$$C_q(\chi, a) = \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \overline{\chi}(b) e\left(\frac{ab^3}{q}\right) \quad \text{and} \quad g_j(\chi) = \sum_{S_j < p \le 2S_j} (\log p)\chi(p).$$

Therefore, we have

$$\sum_{q \le P^{5/36}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S(q,a)}{q} g_1(a/q) g_2(a/q)^2 e(-an/q)$$
$$= \sum_{q \le P^{5/36}} \frac{1}{q\phi^3(q)}$$
$$\times \sum_{\chi_1 (\mathrm{mod}\, q)} \sum_{\chi_2 (\mathrm{mod}\, q)} \sum_{\chi_3 (\mathrm{mod}\, q)} B(q, \chi_1, \chi_2, \chi_3, n) g_1(\chi_1) g_2(\chi_2) g_3(\chi_3),$$

where

$$B(q, \chi_1, \chi_2, \chi_3, n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S(q,a) C_q(\chi_1, a) C_q(\chi_2, a) C_q(\chi_3, a) e(-an/q).$$

We first consider the contribution from the principal character χ^0 modulo q. Using the bound $B(q, \chi^0, \chi^0, \chi^0, n) \ll q^{5/2+\varepsilon}(q,n)^{1/2}$ (see p. 277 in [5]), we have

$$\sum_{q \le P^{5/36}} \frac{1}{q\phi^3(q)} B(q, \chi^0, \chi^0, \chi^0, n) = \mathfrak{S}(n) + O(P^{-5/72+\varepsilon}). \tag{2.7}$$

The prime number theorem implies

$$g_j(\chi^0) = \sum_{S_j < p \le 2S_j} (\log p) = S_j + O(S_j (\log N)^{-A}). \tag{2.8}$$

Let

$$E = \sum_{q \le P^{5/36}} \frac{1}{q\phi^3(q)} \sideset{}{^*}\sum_{\substack{\chi_1 (\mathrm{mod}\, q) \\ \chi_2 (\mathrm{mod}\, q) \\ \chi_3 (\mathrm{mod}\, q)}} B(q, \chi_1, \chi_2, \chi_3, n) g_1(\chi_1) g_2(\chi_2) g_3(\chi_3), \tag{2.9}$$

where $\sum^*$ means that at least one of $\chi_j (1 \le j \le 3)$ is nonprincipal. We have

$$E = \sum_{\substack{r_1, r_2, r_3 \le P^{5/36} \\ r_1 + r_2 + r_3 > 3}} \sum_{\chi_1 (\mathrm{mod}\, r_1)^*} \sum_{\chi_2 (\mathrm{mod}\, r_2)^*} \sum_{\chi_3 (\mathrm{mod}\, r_3)^*} T([r_1, r_2, r_3])$$
$$\times g_1(\chi_1) g_2(\chi_2) g_3(\chi_3),$$

where $[r_1, r_2, r_3]$ is the least common multiple of r_1, r_2, r_3, $\chi \pmod r)^*$ means that the summation is taken over primitive characters modulo r, and

$$T(r) = \sum_{\substack{q \le P^{5/36} \\ r|q}} \frac{1}{q\phi^3(q)} B(q, \chi_1\chi^0, \chi_2\chi^0, \chi_3\chi^0, n).$$

Lemma 4.3 in [3] yields $T(r) \ll r^{-5/6+\varepsilon} \log P$. Then we conclude from Lemmas 4.1 and 4.2 in [1] that

$$E \ll S_1 S_2^2 (\log N)^{-A}. \tag{2.10}$$

The proof is completed by applying (2.7), (2.8), (2.9), and (2.10). □

LEMMA 2.9. *Let $g_1(\alpha)$ and $g_2(\alpha)$ be defined in (2.2). Then we have*

$$\int_0^1 |g_1(\alpha)^2 g_2(\alpha)^4| \, d\alpha \ll S_1^{1+\varepsilon} S_2^2.$$

Proof. This follows from the theorem of Vaughan [8] by considering the underlying Diophantine equation. □

3. Approximations to Generating Functions

For $h \in \mathbb{Z}$, we define

$$F_h(\alpha) = \sum_{x \in \mathbb{Z}} w(x) w(x+h) e((3hx^2 + 3h^2x + h^3)\alpha).$$

LEMMA 3.1. *Let h be a nonzero integer. Suppose that $|\alpha - a/q| \le P^{1-\varepsilon}/(|h|qP^2)$ for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $1 \le q \le P^{1-\varepsilon}$ and $(a,q) = 1$. Then we have*

$$F_h(\alpha) = \frac{C_h(q,a)}{q} \int w(x) w(x+h) e((3hx^2 + 3h^2x + h^3)\beta) \, dx + O(P^{-A}),$$

where $\beta = \alpha - a/q$, and A is a sufficiently large constant.

Proof. Set $\rho_h(x) = 3hx^2 + 3h^2x + h^3$. We have

$$\begin{aligned} F_h(a/q + \beta) &= \sum_{x \in \mathbb{Z}} w(x) w(x+h) e(\rho_h(x)(a/q + \beta)) \\ &= \sum_{b=1}^{q} e\left(\frac{a\rho_h(b)}{q}\right) \sum_{x \equiv b \pmod q} w(x) w(x+h) e(\rho_h(x)\beta) \\ &= \sum_{b=1}^{q} e\left(\frac{a\rho_h(b)}{q}\right) \sum_{m \in \mathbb{Z}} w(b+mq) w(b+mq+h) e(\rho_h(b+mq)\beta). \end{aligned}$$

We apply the Poisson formula to conclude that

$$\begin{aligned}F_h(a/q+\beta) &= \sum_{b=1}^{q} e\left(\frac{a\rho_h(b)}{q}\right)\sum_{n\in\mathbb{Z}}\int w(b+yq)w(b+yq+h) \\ &\quad \times e(\rho_h(b+yq)\beta)e(-ny)\,dy \\ &= \frac{1}{q}\sum_{n\in\mathbb{Z}}\sum_{b=1}^{q} e\left(\frac{a\rho_h(b)+nb}{q}\right)\int w(x)w(x+h)e(\rho_h(x)\beta)e(-nx/q)\,dx.\end{aligned}$$

Note that

$$\frac{d^k}{dx^k}(w(x)w(x+h)e(\rho_h(x)\beta)) \ll P^{-k}+|hP\beta|^k.$$

When $n\neq 0$, we deduce from the integration by parts k times that

$$\begin{aligned}\int w(x)w(x+h)e(\rho_h(x)\beta)e(-nx/q)\,dx &\ll (P^{-k}+|hP\beta|^k)q^k|n|^{-k}P \\ &\ll P^{-k\varepsilon+1}.\end{aligned}$$

The desired conclusion follows from the above by choosing a sufficiently large k. □

LEMMA 3.2. *Suppose that* $|\alpha-a/q|\le P^{1-\varepsilon}/(qP^3)$ *for some* $a\in\mathbb{Z}$ *and* $q\in\mathbb{N}$ *with* $1\le q\le P^{1-\varepsilon}$ *and* $(a,q)=1$. *Then we have*

$$f(\alpha)=\frac{S(q,a)}{q}\int w(x)e(x^3\beta)\,dx+O(P^{-A}),$$

where $\beta=\alpha-a/q$, *and* A *is a sufficiently large constant.*

Proof. The proof is the same as that of Lemma 3.1. We omit the details. □

Let

$$\upsilon(y):=\upsilon_\beta(y)=\int w(x)w(x+y)e((3yx^2+3y^2x+y^3)\beta)\,dx.$$

LEMMA 3.3. *Suppose that* $1\le|y|\le P$. *Then we have*

$$\upsilon(y)'\ll|y|^{-1}P \tag{3.1}$$

and for any $k\in\mathbb{N}$ *that*

$$\upsilon(y)\ll P(|y|P^2|\beta|)^{-k}. \tag{3.2}$$

Proof. We have

$$\begin{aligned}\upsilon(y)' &= \int w(x)w(x+y)'e((3yx^2+3y^2x+y^3)\beta)\,dx \\ &\quad +\beta\int w(x)w(x+y)(2\pi i)(3x^2+6yx+3y^2) \\ &\quad \times e((3yx^2+3y^2x+y^3)\beta)\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int w(x)w(x+y)' e((3yx^2+3y^2x+y^3)\beta)\,dx \\
&\quad + \int w(x)w(x+y)\frac{3x^2+6yx+3y^2}{6yx+3y^2} \\
&\quad \times \left(\frac{d}{dx}e((3yx^2+3y^2x+y^3)\beta)\right)dx \\
&= \int w(x)w(x+y)' e((3yx^2+3y^2x+y^3)\beta)\,dx \\
&\quad - \int \left(\frac{d}{dx}w(x)w(x+y)\frac{3x^2+6yx+3y^2}{6yx+3y^2}\right) \\
&\quad \times e((3yx^2+3y^2x+y^3)\beta)\,dx.
\end{aligned}
$$

Then (3.1) follows easily. Estimate (3.2) follows from the integration by parts k times. □

Let $H = 6P^{1/2}$. We define

$$\mathcal{F}^+(\alpha) = \sum_{1\le h\le H} F_h(\alpha), \qquad \mathcal{F}^-(\alpha) = \sum_{1\le h\le H} F_{-h}(\alpha),$$

and

$$\mathcal{F}(\alpha) = \sum_{H<|h|<P} F_h(\alpha).$$

LEMMA 3.4. *Suppose that $|\alpha - a/q| \le P^{1-\varepsilon}/(qP^{5/2})$ for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $1 \le q \le P^{1-\varepsilon}$ and $(a,q) = 1$. Then we have*

$$\mathcal{F}^+(\alpha) = \frac{|S(q,a)|^2}{q^2}\int_1^H \upsilon_{\alpha-a/q}(y)\,dy + O(P^{1+\varepsilon})$$

and

$$\mathcal{F}^-(\alpha) = \frac{|S(q,a)|^2}{q^2}\int_{-H}^{-1} \upsilon_{\alpha-a/q}(y)\,dy + O(P^{1+\varepsilon}).$$

Similarly, if $|\alpha - a/q| \le P^{1-\varepsilon}/(qP^3)$ for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $1 \le q \le P^{1-\varepsilon}$ and $(a,q) = 1$, then

$$\mathcal{F}(\alpha) = \frac{|S(q,a)|^2}{q^2}\int_{H<|y|<P} \upsilon_{\alpha-a/q}(y)\,dy + O(P^{1+\varepsilon}).$$

Proof. Write $\beta = \alpha - a/q$. In view of Lemma 3.1,

$$\mathcal{F}^+(\alpha) = \frac{1}{q}\sum_{1\le h\le H} C_h(q,a)\upsilon_\beta(h) + O(P^{-A}).$$

Then by partial summation,

$$\mathcal{F}^{+}(\alpha) = \frac{1}{q}\sum_{1\le h\le H} C_h(q,a)\upsilon_\beta(H) - \frac{1}{q}\int_1^H\Big(\sum_{1\le h<y} C_h(q,a)\Big)\upsilon_\beta(y)'\,dy + O(P^{-A}).$$

Applying Lemma 2.7 and (3.1), we obtain

$$\begin{aligned}\mathcal{F}^{+}(\alpha) &= \frac{H}{q^2}|S(q,a)|^2\upsilon_\beta(H) - \int_1^H \frac{y}{q^2}|S(q,a)|^2\upsilon_\beta(y)'\,dy + O(P^{1+\varepsilon})\\ &= \frac{1}{q^2}|S(q,a)|^2\int_1^H \upsilon_\beta(y)\,dy + O(P^{1+\varepsilon}).\end{aligned}$$

The desired conclusions for $\mathcal{F}^{-}(\alpha)$ and $\mathcal{F}(\alpha)$ can be established in the same way. □

4. The Minor Arcs Estimates

Define

$$\mathcal{M} = \bigcup_{q\le P^{1-\varepsilon}}\ \bigcup_{\substack{a=1\\(a,q)=1}}^{q}\left[\frac{a}{q} - \frac{P^{1-\varepsilon}}{qP^{5/2}}, \frac{a}{q} + \frac{P^{1-\varepsilon}}{qP^{5/2}}\right]$$

and

$$\mathcal{R} = \bigcup_{q\le P^{3/4}}\ \bigcup_{\substack{a=1\\(a,q)=1}}^{q}\left[\frac{a}{q} - \frac{P^{3/4}}{qP^{5/2}}, \frac{a}{q} + \frac{P^{3/4}}{qP^{5/2}}\right].$$

Let

$$\mathfrak{n}_1 = \mathfrak{m}\setminus\mathcal{M},\quad \mathfrak{n}_2 = \mathcal{M}\setminus\mathcal{R},\quad \text{and}\quad \mathfrak{n}_3 = \mathfrak{m}\cap\mathcal{R}.$$

LEMMA 4.1. *For $\alpha\in\mathfrak{n}_1$, we have*

$$|\mathcal{F}^{+}(\alpha)| + |\mathcal{F}^{-}(\alpha)| \ll P^{1+\varepsilon}. \tag{4.1}$$

Proof. By Dirichlet's approximation theorem, there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ such that $1\le q\le P^{3/2+\varepsilon}$, $(a,q)=1$, and $|q\alpha - a|\le P^{-3/2-\varepsilon}$. In view of the proof of the lemma in [6], we have

$$|\mathcal{F}^{+}(\alpha)| + |\mathcal{F}^{-}(\alpha)| \ll (P^{3/2}q^{-1/2} + P + P^{1/4}q^{1/2})P^{\varepsilon}.$$

Since $\alpha\in\mathfrak{n}_1$, we have $1\le a\le q$ and $q > P^{1-\varepsilon}$. Estimate (4.1) easily follows from the above. □

LEMMA 4.2. *For $\alpha\in\mathfrak{n}_2$, we have*

$$|\mathcal{F}^{+}(\alpha)| + |\mathcal{F}^{-}(\alpha)| \ll P^{1+\varepsilon}. \tag{4.2}$$

Proof. For $\alpha \in \mathcal{M}$, there exist a and q such that $1 \le a \le q \le P^{1-\varepsilon}$, $(a,q)=1$, and $|\alpha - a/q| \le P^{1-\varepsilon}/(qP^{5/2})$. By (2.3), (3.2), and Lemma 3.4,

$$|\mathcal{F}^+(\alpha)| + |\mathcal{F}^-(\alpha)| \ll HPq^{-2/3}(1 + HP^2|\alpha - a/q|)^{-1} + P^{1+\varepsilon}.$$

We conclude from $\alpha \notin \mathcal{R}$ that either $q > P^{3/4}$ or $|\alpha - a/q| > P^{3/4}/(qP^{5/2})$. Estimate (4.2) follows from the above immediately. □

LEMMA 4.3. *For* $\mathfrak{n} \in \{\mathfrak{n}_1, \mathfrak{n}_2\}$, *we have*

$$\int_{\mathfrak{n}} (|\mathcal{F}^+(\alpha)| + |\mathcal{F}^-(\alpha)|)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha \ll P^{1+\varepsilon} S_1 S_2^2.$$

Proof. This follows from Lemma 2.9 and Lemmas 4.1 and 4.2. □

LEMMA 4.4. *We have*

$$\int_{\mathfrak{n}_3} (|\mathcal{F}^+(\alpha)| + |\mathcal{F}^-(\alpha)|)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha \ll P^{1+\varepsilon} S_1 S_2^2.$$

Proof. We define $\Phi(\alpha)$ on $\mathcal{R}$ by taking $\Phi(\alpha) = HP(1+HP^2|\alpha - a/q|)^{-1}\varpi(q)^2$ if $|q\alpha - a| \le P^{3/4-5/2}$ for some a and q with $(a,q)=1$ and $1 \le a \le q \le P^{3/4}$. Lemma 2.4, (3.2), and Lemma 3.4 together imply

$$\begin{aligned}\int_{\mathfrak{n}_3} &(|\mathcal{F}^+(\alpha)| + |\mathcal{F}^-(\alpha)|)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha \\ &\ll S_1^2 S_2^2 P^{-2/9+\varepsilon} \int_{\mathcal{R}} \Phi(\alpha)|g_2(\alpha)^2|\,d\alpha \\ &\quad + P^{1+\varepsilon}\int_0^1 |g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha.\end{aligned}$$

Applying Lemma 2.2 in [9], we have $\int_{\mathcal{R}} \Phi(\alpha)|g_2(\alpha)^2|\,d\alpha \ll S_2^{2+\varepsilon}P^{-1}$. Then by Lemma 2.9 we obtain

$$\int_{\mathfrak{n}_3} (|\mathcal{F}^+(\alpha)| + |\mathcal{F}^-(\alpha)|)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha \ll S_1^2 S_2^4 P^{-2/9-1+\varepsilon} + P^{1+\varepsilon} S_1 S_2^2.$$

The proof is completed. □

LEMMA 4.5. *We have*

$$\int_{\mathfrak{m}} (|\mathcal{F}^+(\alpha)| + |\mathcal{F}^-(\alpha)|)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha \ll P^{1+\varepsilon} S_1 S_2^2.$$

Proof. This follows from Lemmas 4.3 and 4.4 by observing that $\mathfrak{m} = \mathfrak{n}_1 \cup \mathfrak{n}_2 \cup \mathfrak{n}_3$. □

LEMMA 4.6. *We have*

$$\int_{\mathfrak{m}} \mathcal{F}(\alpha)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha \ll P^{1+\varepsilon} S_1 S_2^2.$$

Proof. Note that

$$\int_{\mathfrak{m}} \mathcal{F}(\alpha)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha$$
$$= \int_0^1 \mathcal{F}(\alpha)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha - \int_{\mathfrak{M}} \mathcal{F}(\alpha)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha.$$

Considering the underlying Diophantine equation, we have

$$\int_0^1 \mathcal{F}(\alpha)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha$$
$$= \sum_{H<|h|<P} \sum_{x} \sum_{\substack{p_1,p_2,p_3,p_4,p_5,p_6 \\ S_1<p_1,p_2\le 2S_1,\, S_2<p_3,p_4,p_5,p_6\le 2S_2 \\ 3hx^2+3h^2x+h^3=p_1^3-p_2^3+p_3^3-p_4^3+p_5^3-p_6^3}} w(x)w(x+h)\prod_{j=1}^{6}\log p_j.$$

If $w(x)w(x+h)\neq 0$, then $x, x+h \ge P$ and

$$|3hx^2+3h^2x+h^3| = |h|(x^2+x(x+h)+(x+h)^2) \ge 18P^{5/2}.$$

However, we have $|p_1^3-p_2^3+p_3^3-p_4^3+p_5^3-p_6^3| \le 8S_1^3 = 8P^{5/2}$. Therefore,

$$\int_0^1 \mathcal{F}(\alpha)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha = 0,$$

and it suffices to prove that

$$\int_{\mathfrak{M}} \mathcal{F}(\alpha)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha \ll P^{1+\varepsilon} S_1 S_2^2. \tag{4.3}$$

We deduce from Lemma 3.4 that

$$\int_{\mathfrak{M}} \mathcal{F}(\alpha)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha$$
$$= \sum_{q\le P^{5/36}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{|S(q,a)|^2}{q^2}$$
$$\times \int_{|\beta|\le P^{13/18}/(qN)} \int_{H<|y|<P} \upsilon_\beta(y)\,dy \left| g_1\left(\frac{a}{q}+\beta\right)^2 g_2\left(\frac{a}{q}+\beta\right)^4\right| d\beta$$
$$+ O(P^{1+\varepsilon}) \int_{\mathfrak{M}} |g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha.$$

Then by Lemma 2.9 we arrive at

$$\int_{\mathfrak{M}} \mathcal{F}(\alpha)|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha$$
$$= \sum_{\substack{p_1,p_2,p_3,p_4,p_5,p_6 \\ S_1<p_1,p_2\le 2S_1 \\ S_2<p_3,p_4,p_5,p_6\le 2S_2}} \left(\prod_{j=1}^{6}\log p_j\right) \sum_{q}\sum_{a} e\left(\frac{a}{q}\Delta(\mathbf{p})\right)$$

$$\times \frac{|S(q,a)|^2}{q^2}\mathcal{I}(\mathbf{p}) + O(P^{1+\varepsilon}S_1S_2^2),$$

where

$$\Delta(\mathbf{p}) = p_1^3 - p_2^3 + p_3^3 - p_4^3 + p_5^3 - p_6^3$$

and

$$\mathcal{I}(\mathbf{p}) = \int_{|\beta|\le P^{13/18}/(qN)} \int_{H<|y|<P} \upsilon_\beta(y)\,dy e(\Delta(\mathbf{p})\beta)\,d\beta.$$

In view of (3.2), we have

$$\begin{aligned}&\int_{|\beta|>P^{13/18}/(qN)} \int_{H<|y|<P} \upsilon_\beta(y)\,dy e(\Delta(\mathbf{p})\beta)\,d\beta \\ &\qquad \ll \int_{|\beta|>P^{13/18}/(qN)} \int_{H<|y|<P} (|y|P^2|\beta|)^{-k}\,dy\,d\beta \\ &\qquad \ll P^{-A}.\end{aligned}$$

Then we obtain

$$\mathcal{I}(\mathbf{p}) = \mathcal{J}(\mathbf{p}) + O(P^{-A}),$$

where

$$\begin{aligned}\mathcal{J}(\mathbf{p}) = \int_{-\infty}^{+\infty} \int_{H<|y|<P} \int w(x)w(x+y) \\ \times e((3yx^2+3y^2x+y^3+\Delta(\mathbf{p}))\beta)\,dx\,dy\,d\beta.\end{aligned}$$

Note that $\mathcal{J}(\mathbf{p})$ is essentially the measure of the surface defined by the equation $(3yx^2+3y^2x+y^3)+\Delta(\mathbf{p})=0$ with $H<|y|<P$ and $P\le|x|\le 2P$. Recalling the conditions $S_1 < p_1, p_2 \le 2S_1$, and $S_2 < p_3, p_4, p_5, p_6 \le 2S_2$, we obtain $\mathcal{J}(\mathbf{p}) = 0$. Thus, (4.3) is established, and the proof is completed. □

LEMMA 4.7. *We have*

$$\int_{\mathfrak{m}} |f(\alpha)g_1(\alpha)g_2(\alpha)^2|^2\,d\alpha \ll P^{1+\varepsilon}S_1S_2^2.$$

Proof. By the definition of $f(\alpha)$ we have

$$\begin{aligned}|f(\alpha)^2| &= \sum_x\sum_y w(x)w(y)e((y^3-x^3)\alpha) \\ &= \sum_x\sum_h w(x)w(x+h)e(((x+h)^3-x^3)\alpha) \\ &= \sum_h\sum_x w(x)w(x+h)e(((x+h)^3-x^3)\alpha) = \sum_h F_h(\alpha).\end{aligned}$$

If $w(x)w(x+h)\ne 0$, then $|h|<P$. Therefore, we have

$$\begin{aligned}|f(\alpha)^2| &= \mathcal{F}^+(\alpha) + \mathcal{F}^-(\alpha) + \mathcal{F}(\alpha) + \sum_x w(x)^2 \\ &= \mathcal{F}^+(\alpha) + \mathcal{F}^-(\alpha) + \mathcal{F}(\alpha) + O(P),\end{aligned}$$

and consequently,

$$\int_{\mathfrak{m}} |f(\alpha)g_1(\alpha)g_2(\alpha)^2|^2\,d\alpha$$
$$= \int_{\mathfrak{m}} (\mathcal{F}^+(\alpha) + \mathcal{F}^-(\alpha) + \mathcal{F}(\alpha) + O(P))|g_1(\alpha)^2 g_2(\alpha)^4|\,d\alpha.$$

The proof is completed by combining Lemma 2.9 and Lemmas 4.5 and 4.6. □

5. Proof of Theorem 1.1

Proof of Theorem 1.1. Bessel's inequality yields

$$\sum_{N<n\le 2N} \left| \int_{\mathfrak{m}} f(\alpha)g_1(\alpha)g_2(\alpha)^2 e(-n\alpha)\,d\alpha \right|^2 \le \int_{\mathfrak{m}} |f(\alpha)g_1(\alpha)g_2(\alpha)^2|^2\,d\alpha.$$

Then we conclude from Lemma 4.7 that

$$\sum_{N<n\le 2N} \left| \int_{\mathfrak{m}} f(\alpha)g_1(\alpha)g_2(\alpha)^2 e(-n\alpha)\,d\alpha \right|^2 \ll P^{1+\varepsilon} S_1 S_2^2.$$

Thus, for all integers $n \in (N, 2N]$ with at most $O(N^{25/27+3\varepsilon})$ exceptions, we have

$$\left| \int_{\mathfrak{m}} f(\alpha)g_1(\alpha)g_2(\alpha)^2 e(-n\alpha)\,d\alpha \right| \ll P^{-2-\varepsilon} S_1 S_2^2. \tag{5.1}$$

Next, we consider the contribution from the major arcs. By Lemma 3.2,

$$\int_{\mathfrak{M}} f(\alpha)g_1(\alpha)g_2(\alpha)^2 e(-n\alpha)\,d\alpha$$
$$= \sum_{q\le P^{5/36}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{S(q,a)}{q} e\left(\frac{-an}{q}\right)$$
$$\times \int_{|\beta|\le P^{13/18}/(qN)} u(\beta) g_1\left(\frac{a}{q}+\beta\right) g_2\left(\frac{a}{q}+\beta\right)^2 e(-n\beta)\,d\beta$$
$$+ O(P^{-A}),$$

where

$$u(\beta) = \int w(x)e(x^3\beta)\,dx.$$

We deduce from integration by parts that $u(\beta) \ll P(P^3|\beta|)^{-k}$ for any $k \in \mathbb{N}$. Then we obtain

$$\int_{\mathfrak{M}} f(\alpha)g_1(\alpha)g_2(\alpha)^2 e(-n\alpha)\,d\alpha$$
$$= \sum_{q\le P^{5/36}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{S(q,a)}{q} e\left(\frac{-an}{q}\right)$$

$$\times \int_{|\beta|\le N^{\varepsilon}/N} u(\beta) g_1\left(\frac{a}{q}+\beta\right) g_2\left(\frac{a}{q}+\beta\right)^2 e(-n\beta)\,d\beta$$
$$+ O(P^{-A}).$$

When $|\beta| \le N^{\varepsilon}/N$, we have $g_1(\frac{a}{q}+\beta) - g_1(\frac{a}{q}) \ll S_1 P^{-1/2+\varepsilon}$ and $g_2(\frac{a}{q}+\beta) - g_2(\frac{a}{q}) \ll S_2 P^{-11/12+\varepsilon}$. Let

$$\mathcal{S}(n) = \sum_{q\le P^{5/36}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{S(q,a)}{q} e\left(\frac{-an}{q}\right) g_1\left(\frac{a}{q}\right) g_2\left(\frac{a}{q}\right)^2.$$

Then we conclude from the above that

$$\int_{\mathfrak{M}} f(\alpha) g_1(\alpha) g_2(\alpha)^2 e(-n\alpha)\,d\alpha$$
$$= \mathcal{S}(n) \int_{|\beta|\le N^{\varepsilon}/N} u(\beta) e(-n\beta)\,d\beta + O(P^{-19/9} S_1 S_2^2).$$

Applying $u(\beta) \ll P(P^3|\beta|)^{-k}$ again, we obtain

$$\int_{\mathfrak{M}} f(\alpha) g_1(\alpha) g_2(\alpha)^2 e(-n\alpha)\,d\alpha$$
$$= \mathcal{S}(n) \int_{-\infty}^{\infty} u(\beta) e(-n\beta)\,d\beta + O(P^{-19/9} S_1 S_2^2). \tag{5.2}$$

Note that

$$\int_{-\infty}^{\infty} u(\beta) e(-n\beta)\,d\beta = \lim_{\lambda\to 0^+} \frac{1}{\lambda} \int_{|x^3-n|\le\lambda} w(x)\,dx.$$

Thus, for $N < n \le 2N$, we have

$$P^{-2} \ll \int_{-\infty}^{\infty} u(\beta) e(-n\beta)\,d\beta \ll P^{-2}. \tag{5.3}$$

By Lemma 2.8,

$$\mathcal{S}(n) = \mathfrak{S}(n) S_1 S_2^2 + O(S_1 S_2^2 (\log N)^{-A}). \tag{5.4}$$

We deduce from (2.6), (5.2), (5.3), and (5.4) that

$$\int_{\mathfrak{M}} f(\alpha) g_1(\alpha) g_2(\alpha)^2 e(-n\alpha)\,d\alpha \gg P^{-2} S_1 S_2^2 (\log N)^{-1}. \tag{5.5}$$

In view of (5.5) and the argument around (5.1), we have $r(n) > 0$ for all integers $n \in (N, 2N]$ with at most $O(N^{25/27+\varepsilon})$ exceptions. The proof of Theorem 1.1 is completed by the dyadic argument. □

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