Michigan Math. J. 63 (2014), 687–700

Dilatation, Pointwise Lipschitz Constants, and Condition N on Curves

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Abstract. Let $f : X \to Y$ be a homeomorphism between locally compact Ahlfors Q-regular metric spaces, $Q > 1$. We prove that finiteness of either $\operatorname{lip}_f(x)$, $h_f(x)$, or $h^*_f(x)$ for every $x \in X \setminus E$ implies that f satisfies Lusin's condition N on p-almost every curve (in the sense of curve modulus), provided that the exceptional set E has σ-finite Hausdorff $(n-p)$-measure. Here $h_f(x)$ and h^*_f are the linear dilatations of f and f^{-1} at x, and $\operatorname{lip}_f(x)$ is the pointwise Lipschitz constant (each defined with lim inf rather than lim sup).

As a corollary, we improve a theorem of Balogh, Koskela, and Rogovin on the Sobolev regularity of mappings of finite and essentially bounded dilatation.

Furthermore, we show that for nonhomeomorphic continuous mappings into arbitrary targets, finiteness of $\operatorname{lip}_f(x)$ away from E still implies condition N on p-almost every curve.

1. Introduction

Let $f : X \to Y$ be a homeomorphism, where $X = (X, d_X, \mu)$ and $Y = (Y, d_Y, \nu)$ are locally Ahlfors Q-regular metric measure spaces X and Y, with $Q > 1$. Recall that local Q-regularity merely means that the measure of a small ball of radius r is comparable to r^Q (see Section 2 for precise definitions).

We are primarily interested in the pointwise constants

$$\operatorname{lip}_f(x) := \liminf_{r\to 0} \frac{L_f(x,r)}{r} \quad \text{and} \quad h_f(x) := \liminf_{r\to 0} h_f(x,r),$$

where

$$L_f(x,r) = \sup_{x'\in B(x,r)} d(f(x'), f(x)), \qquad l_f(x,r) = \inf_{x'\in X\setminus B(x,r)} d(f(x'), f(x)),$$

$$h_f(x,r) = \frac{L_f(x,r)}{l_f(x,r)}.$$

We also let $h^*_f(x) = h_{f^{-1}}(f(x))$. One can define Lip_f, H_f, and H^*_f similarly, with lim sup replacing lim inf, but we shall mostly be interested only in the lim inf case.

Received April 10, 2013. Revision received July 4, 2014.

Partially supported by the Academy of Finland, Grant #131477. This research was conducted at the University of Jyväskylä, the Institute for Pure and Applied Mathematics at UCLA, and Kansas State University.

The main result of this paper concerns the conditions under which finiteness of the above quantities, without integrability assumptions, implies that f satisfies Lusin's condition N on almost every curve in the sense of curve modulus (and a fortiori, in the Euclidean setting, on almost every line segment parallel to each coordinate axis).

THEOREM 1.1. *Let $f : X \to Y$ be a homeomorphism between locally compact, locally Ahlfors Q-regular metric measure spaces $X = (X, d_X, \mu)$ and $Y = (Y, d_Y, \nu)$, let $1 \leq p \leq Q$ and $Q > 1$, and let $E \subset X$ have σ-finite Hausdorff $(Q - p)$-measure. Suppose that at every $x \in X \backslash E$, at least one of the quantities $\operatorname{lip}_f(x)$, $h_f(x)$, or $h_f^*(x)$ is finite. Then on p-almost every curve γ in X, f satisfies Lusin's condition N, and $f(\gamma)$ has σ-finite length.*

Here, "almost every" is defined in the sense of modulus of curve families, and Q-regularity of X and Y means that the spaces are essentially Q-dimensional. The terminology in Theorem 1.1 is made precise in Section 2.

Sobolev Regularity

Theorem 1.1 is in the spirit of earlier developments establishing Sobolev regularity, differentiability, and absolute continuity in measure for a mapping, under various finiteness and integrability assumptions on either the dilatation [BKR07; KM02; KR05; HKST01; Tys98; HK95; HK98; Han12] or the pointwise Lipschitz constant [BC06; Zü07; BRZ04; Han12; WZ13]. Indeed, while Theorem 1.1 itself does not assume any integrability for lip_f, it may be combined with integrability assumptions to establish Sobolev regularity, allowing us to recover and sometimes strengthen many of the results in the preceding works.

In [BKR07], Balogh, Koskela, and Rogovin studied homeomorphisms whose dilatation was finite off an exceptional set and also essentially bounded. Under these circumstances, they established Sobolev regularity in the following result [BKR07, Theorem 1.1, Remark 4.1].

THEOREM (Balogh–Koskela–Rogovin). *Let $f : X \to Y$ be a homeomorphism between locally Ahlfors Q-regular metric measure spaces $X = (X, d, \mu)$ and $Y = (Y, d, \nu)$, let $1 \leq p \leq Q$, $1 < Q$, and let $E \subset X$ have σ-finite Hausdorff $(Q - p)$-measure. Suppose that at every $x \in X \backslash E$, $h_f(x) < \infty$, and that either $1 \leq p < Q$ and $\operatorname{esssup}_{x \in X} h_f(x) < \infty$, or $p = Q$ and $\sup_{x \in X} h_f(x) < \infty$. Then $f \in N^{1,p}(X, Y)$.*

If X satisfies a $(1, Q)$-Poincaré inequality, as introduced in [HK98], they further proved that one can use esssup rather than sup in the preceding result, even for the case $p = Q$ [BKR07, Theorem 5.2, Remark 5.3].

Theorem 1.1 allows for a substantial strengthening of the preceding theorem, via the second of the following two corollaries.

Corollary 1.2. *Let $f : X \to Y$, E, $1 \le p \le Q$, satisfy the conditions in Theorem 1.1. Then* lip_f *is a p-weak upper gradient of f. In particular, if* $\operatorname{lip}_f \in L^p_{\mathrm{loc}}(X)$, *then* $f \in N^{1,p}_{\mathrm{loc}}(X, Y)$.

Corollary 1.3. *Let $f : X \to Y$, E, and $1 \le p \le Q$ satisfy the conditions in Theorem 1.1. Suppose in addition that $h_f \in L^{p^*}_{\mathrm{loc}}(X)$, where $p^* = pQ/(Q-p)$ if $p < Q$ and $p^* = \infty$ if $p = Q$. Then $f \in N^{1,p}_{\mathrm{loc}}(X, Y)$.*

Corollary 1.3 can also be viewed as a generalization of similar results for the Euclidean case studied in [KR05; KM02] (cf. Remark 3.2 below).

Remark 1.4. Though [BKR07] dealt only with the case that $h_f \in L^\infty(X)$, their covering argument could be modified in a straightforward way to obtain L^p-regularity for $1 \le p < Q$, provided that $h_f \in L^{2p^*}(X)$. Thus, our methods offer an improvement here as well; the reason for the exponent $2p^*$ under the previous method is that the covering argument in [BKR07] (cited below in the proof of Proposition 3.1) introduces an extraneous factor of $1/h_f(x)$. With our methods, this is no problem since we only apply the covering argument on sets where h_f is bounded.

Remark 1.5. Notice that the preceding corollary improves the borderline case $p = Q$ in that we still need only bound the essential supremum of h_f, rather than the actual supremum (though we still require finiteness of $h_f(x)$ everywhere). We thus eliminate the aforementioned rather strong assumption of a $(1, Q)$-Poincaré inequality, which was needed to obtain this conclusion in [BKR07]. This is of particular importance in the context of the next result, which shows that under the same relaxed assumptions, we still obtain the "geometric" definition of quasiconformality.

Theorem 1.6. *Let $f : X \to Y$ be a homeomorphism between Ahlfors Q-regular metric measure spaces, $Q > 1$, such that $h_f(x) < \infty$ for each $x \in X$ and $h_f(x) \le h < \infty$ μ-almost everywhere for some constant h. Then for every curve family Γ in X,*

$$\frac{\operatorname{mod}_Q(\Gamma)}{K} \le \operatorname{mod}_Q(f(\Gamma)) \le K \operatorname{mod}_Q(\Gamma), \tag{1}$$

where $K = Ch^Q$ for some constant C depending only on the constants of Q-regularity for X and Y.

Remark 1.7. Absolute continuity in measure need not hold in our setting without a Poincaré inequality on X, and so the proof for Theorem 1.6 given below is not symmetric – the proof of second inequality requires a bit more care. It should also be noted that without a Poincaré inequality the first inequality does not imply the second, nor vice versa [Wil12, Remark 4.1].

Euclidean Domains

In the case that $X = \Omega$ and $Y = \Omega'$ are domains in $\mathbb{R}^n$, $n \geq 2$, the Besicovitch covering theorem will allow us to obtain a somewhat stronger result. To formulate it, we consider the pointwise constant $k_f(x)$ introduced in [KR05], given at each $x \in \Omega$ by $k_f(x) := \liminf_{r\to 0} k_f(x,r)$, where

$$k_f(x,r) = \left(\frac{L_f(x,r)^n}{|f(B(x,r))|}\right)^{1/(n-1)}.$$

THEOREM 1.8. *Let $\Omega \subseteq \mathbb{R}^n$, $n > 1$ be a Euclidean domain, and let $f : \Omega \to \Omega' \subseteq \mathbb{R}^n$ be a homeomorphism. Let $1 \leq p \leq n$, and let $E \subset \mathbb{R}^n$ have σ-finite Hausdorff $(n-p)$-measure. Suppose that at every $x \in \Omega \backslash E$, $\min\{\mathrm{lip}_f(x), h_f(x), h^*_f(x), k_f(x)\} < \infty$.*

Then on p-almost every curve γ in Ω, f satisfies condition N, and $f(\gamma)$ has σ-finite length. In particular, if E has σ-finite Hausdorff $(n-1)$-measure, then f satisfies condition N on almost every line parallel to the coordinate axes, and the image of almost every line has σ-finite length.

Notice that finiteness of h_f implies the same for k_f, so that the inclusion of h_f in the statement of Theorem 1.8 is redundant. We have included it all the same, to highlight that the theorem is an improvement to Theorem 1.1 for the Euclidean case. Our purpose for studying finiteness of k_f is to demonstrate a connection to the results of [KR05], which can now be deduced also as a consequence of Theorem 1.8 (cf. Remark 3.2).

REMARK 1.9. The size of the exceptional set E in Theorem 1.8 is quite sharp, as a simple example along the lines of [KR05, Remark 1.2 (b)] demonstrates. The exponent $n-1$ is the optimal dimension for removing sets of σ-finite Hausdorff measure, but in fact, the theorem is sharper than that. Recall that for any continuous function $\phi : [0,\infty) \to [0,\infty)$, the Hausdorff ϕ-measure $\mathcal{H}^\phi(A)$ of a subspace $A \subseteq X$ of a metric space X is defined by

$$\mathcal{H}^\phi(A) = \sup_{\delta>0} \mathcal{H}^\phi_\delta(A),$$

where

$$\mathcal{H}^\phi_\delta(A) = \inf\left\{\sum_{i=1}^\infty \phi(\mathrm{diam}(S_i)) : \ A \subset \bigcup_{i=1}^\infty S_i, \mathrm{diam}(S_i) \leq \delta\right\}.$$

In particular, $\phi(t) = t^s$ reduces to the usual s-dimensional Hausdorff measure $\mathcal{H}^s$.

If $\lim_{r\to 0} \psi(r) > 0$, where $\psi(r) := \phi(r)/r^{n-1}$, then a set E with σ-finite ϕ-measure has σ-finite $(n-1)$-measure as well and thus satisfies the hypotheses of Theorem 1.8 with $p = 1$. If, on the other hand, $\lim_{r\to 0} \psi(r) = 0$, then let $S \subset [0,1]$ be a Cantor set with $\mathcal{H}^\psi(S) = \mathcal{H}^1(S) = 0$. (To see that such a Cantor set exists, we construct S via the usual procedure: Let $S_0 = [0,1]$, obtain S_{k+1} by removing an open interval from the interior of each component of S_k, and let $S = \bigcap_{k=1}^{\infty} S_k$. The Cantor set will have the desired property, provided that at the kth stage, we remove intervals large enough so that $\mathcal{H}^{\psi}_{1/k}(S_k) + H^1_{1/k}(S_k) < 1/k$.) Let $\eta : [0,1] \to [0,1]$ be the Cantor staircase function associated to S, and let $f : [0,1]^n \to [0,2] \times [0,1]^{n-1}$ be given by $f(x_1, \ldots, x_n) = (x_1 + \eta(x_1), x_2, \ldots, x_n)$. Let $E = S \times [0,1]^{n-1}$. Then $\mathcal{H}^{\phi}(E) = 0$, and f is a local isometry on $[0,1]^n \backslash E$, but f fails to satisfy condition N along any of the line segments $[0,1] \times \{(x_2, \ldots, x_n)\}$.

Lipschitz Mappings

We may also consider the case where f is an arbitrary continuous mapping, rather than a homeomorphism. In this case, finiteness of the pointwise Lipschitz constant allows us to prove a version of Theorem 1.1 for arbitrary mappings $f : X \to Y$, where X is locally Q-regular, but Y may be any metric space. Finiteness of lip_f is a more straightforward condition to work with in many ways than finite dilatation, and the proof of the following theorem is very much along the lines of [BC06, Theorem 1.5], though the conclusion of the latter result is different.

THEOREM 1.10. *Let $f : X \to Y$ be a continuous mapping from a locally compact locally Ahlfors Q-regular metric measure space $X = (X, d, \mu)$ to an arbitrary metric space Y. Let $1 \le p \le Q$ and $Q > 1$, and let $E \subset X$ have σ-finite Hausdorff $(Q - p)$-measure. Suppose that at every $x \in X \backslash E$, $\mathrm{lip}_f(x) < \infty$. Then on p-almost every curve γ in X, f satisfies condition N, and $f(\gamma)$ has σ-finite length. In particular, if $X = \Omega$ is a domain in $\mathbb{R}^n$, then on almost every line segment l parallel to the coordinate axis, f satisfies condition N, and $f(l)$ has σ-finite length.*

Moreover, if $p = Q$, so that E is countable, then on every *curve γ, f satisfies condition N, and $f(\gamma)$ has σ-finite length.*

REMARK 1.11. Using similar reasoning as we use in the proof of Corollary 1.2 below, Zürcher proved that under the assumptions of Theorem 1.10, lip_f is a p-weak upper gradient (not necessarily satisfying any integrability conditions) [Zü07, Lemma 3.9], so that under the additional assumption that $\mathrm{lip}_f \in L^p_{\mathrm{loc}}(X)$, one obtains that $f \in N^{1,p}_{\mathrm{loc}}(X, Y)$.

REMARK 1.12. The final statement in Theorem 1.10 is completely elementary (and in fact, is an immediate consequence of the elementary portion of our main technical tool, Proposition 3.1), so much so that we might avoid mentioning it, save for the following observation.

On the suggestion of the referee, we may compare our results to recent work of Wildrick and Zürcher [WZ13] regarding Lorentz–Sobolev mappings. While an in-depth treatment of these mappings is beyond the scope of this paper, we do wish to point out one simple connection with [WZ13]. Namely, [WZ13, Theorem 1.2] establishes that, given a suitable Poincaré inequality on X, the conditions that $\operatorname{lip}_f \in L^{Q,1}(X,\mu)$ and that $\operatorname{lip}_f < \infty$ away from a suitably small set $E \subseteq X$ together guarantee that f has an upper gradient in $L^{Q,1}(X,\mu)$.

If lip_f is finite everywhere away from a countable set E, then lip_f is a *genuine* upper gradient (this is essentially shown in [Zü07, Lemma 3.9] – when $p = Q$, E itself is countable, so Γ_E is empty). If lip_f is additionally assumed to be in the Lorentz space $L^{Q,1}$, then we immediately obtain the corresponding Lorentz–Sobolev regularity for f. This could be thought of as a very weak analogue to [WZ13, Theorem 1.2], illustrating that the necessity of a Poincaré inequality in that result is entirely due to the allowance of an exceptional set E on which lip_f might be infinite. One could also apply this same principle to other Lorentz–Sobolev spaces, as well as other generalizations of Newton–Sobolev spaces, for example, Newton–Orlicz–Sobolev spaces $N^{1,\Psi}$ introduced in [Tuo04], or even more generally, to the Sobolev-type spaces $N^{1,B}$ introduced by Mocanu [Moc10].

2. Definitions and Notation

Throughout the paper, $f : X \to Y$ will be a continuous mapping from a locally compact metric measure space $X = (X,\mu)$ to a metric space Y. Often Y will also be locally compact and equipped with a measure ν. The open ball of radius r centered at x is denoted $B(x,r)$.

We assume throughout that μ and ν are *locally Ahlfors Q-regular,* that is, there is a constant C such that for every $x \in X$, there is a radius $R_x > 0$ such that for every $x' \in B(x, R_x)$ and every $r < R_x$,

$$r^Q/C \le \mu(B(x,r)) \le Cr^Q. \tag{2}$$

When Y is equipped with the measure ν, the *volume derivative* of f at x is given by

$$J_f(x) = \limsup_{r\to 0} \frac{\nu(f(B(x,r)))}{\mu(B(x,r))}. \tag{3}$$

In our setting, J_f gives a precise representative of the Radon–Nikodym derivative $\frac{df^*\nu}{d\mu}$, where $f^*\nu = f_*^{-1}\nu$, the pushforward of ν along f^{-1} [Fed69, 2.9.7].

We always assume that $1 \le p \le Q$ and $Q > 1$. In the case $X = \Omega$ and $Y = \Omega'$ are domains in $\mathbb{R}^n$, we assume that $1 \le p \le n$ and that $n > 1$. In this setting, for each subset $A \subset \mathbb{R}^n$, $|A|$ denotes the n-dimensional Lebesgue outer measure of A (we also use this notation to denote the Lebesgue 1-measure on a parametrizing interval of a curve).

REMARK 2.1. The assumption that $Q > 1$ (or $n > 1$ in the Euclidean case) in our theorems is unavoidable since there are quasisymmetric self-maps of the unit interval that fail to satisfy condition N [BA56]. In our proofs, the necessity for this restriction manifests itself in the invocation of reflexivity in the Mazur's lemma construction in Proposition 3.1 below.

We take the convention that curves are compact, that is, a *curve* in X is a continuous mapping $\gamma : [a,b] \to X$. We will also denote the image of γ, $\gamma([a,b])$, simply by γ, when there is no potential for confusion. A curve γ is *rectifiable* if it has length $l(\gamma) < \infty$.

The map f is said to satisfy Lusin's condition N on a rectifiable curve γ if $f \circ \gamma_0$ satisfies condition N, where $\gamma_0 : [0, l(\gamma)] \to X$ is the arc-length parametrization. That is, f satisfies condition N on γ if for every subset $S \subseteq [0, l(\gamma)]$ with Lebesgue measure $|S| = 0$, $\mathcal{H}^1(f(\gamma_0(S))) = 0$. We define "absolute continuity" and "bounded variation" of f on γ similarly, remarking that assuming that γ is rectifiable, $f(\gamma)$ is rectifiable if and only if f has bounded variation on γ. As a result, by the continuity of $f \circ \gamma_0$, f is absolutely continuous on a rectifiable curve γ if and only if f satisfies condition N on γ and $f(\gamma)$ is rectifiable. Other standard definitions of bounded variation and absolute continuity for curves coincide with the ones we use here; see [Dud07] for a detailed exposition of these issues.

More generally, if $A \subseteq X$, then we will say that f satisfies condition N^A on γ if for every subset $S \subset \gamma^{-1}(A)$ such that $|S| = 0$, $H^1(f(\gamma(S))) = 0$.

When $S \subseteq [0, l(\gamma)]$ is measurable and $\rho : X \to \mathbb{R}$ is Borel, we will use the notation

$$\int_{\gamma(S)} d\rho := \int_0^{l(\gamma)} \chi_S(t) \rho(\gamma_0(t))\, dt.$$

The *p-modulus* of a family Γ of curves in X is given by

$$\mathrm{mod}_p(\Gamma) = \inf_\rho \int \rho^p \, d\mu,$$

where the infimum is taken over all Borel functions $\rho : X \to [0, \infty]$ that are *admissible* for Γ, that is, $\int_\gamma \rho\, ds \geq 1$ for each $\gamma \in \Gamma$. A property holds for *p-almost every curve* in X if it fails only on a curve family Γ such that $\mathrm{mod}_p(\Gamma) = 0$. A Borel function $g : X \to [0, \infty]$ is an *upper gradient* for f if

$$d(f(x_1), f(x_2)) \leq \int_\gamma g\, ds \tag{4}$$

for every curve γ in X. If inequality (4) only holds for p-almost every curve, we call g a *p-weak upper gradient* of f.

The continuous mapping f is in the local Newton–Sobolev class $N^{1,p}_{\mathrm{loc}}(X, Y)$ if it has a p-weak upper gradient $g \in L^p_{\mathrm{loc}}(\mu)$.

For background on curves, curve modulus, upper gradients, Newton–Sobolev spaces, and analysis on metric spaces in general, we refer the reader to [Dud07; HKST01; Sha00; Hei01; HK98].

3. Proofs

Our methods are inspired by an approximation argument that has become a quite popular in recent years [BKR07; Wil12; HK95; HK98; HKST01; Tys98; Cri06]. The idea is that one constructs "approximate" upper gradients, then passes to the limit to obtain a p-weak upper gradient in $L^p_{\mathrm{loc}}(X)$, ensuring local Newton–Sobolev regularity. (Some of these authors proved modulus inequalities, rather than Sobolev regularity, but the former imply the latter—this follows from a close connection between upper gradients and curve modulus [Wil12, Theorem 3.10].)

The key additional feature in this paper is that we construct what is essentially an "upper gradient relative to A" for any set A on which either lip_f, h_f, h^*_f, or k_f is bounded and apply this method to larger and larger sets. What becomes apparent from this approach is that the covering arguments are only needed for condition N, whereas the integrability conditions are ultimately only needed for proving bounded variation. The method of approaching each issue separately turns out to be substantially more powerful than tackling them both simultaneously, as can be seen with Corollary 1.3 and Theorem 1.6 (cf. Remark 1.5).

Theorems 1.1, 1.8, and 1.10 all follow quickly from the following proposition.

PROPOSITION 3.1. *Let $f : X \to Y$ be a continuous mapping from a locally Ahlfors Q-regular metric measure space $X = (X, d, \mu)$, $Q > 1$, to an arbitrary metric space Y. Let $A \subseteq X$ be a Borel set, and assume that there is a number $M < \infty$ such that one of the following conditions is satisfied*:

1. *For every $x \in A$, $\mathrm{lip}_f(x) < M$.*
2. *f is a homeomorphism, Y is locally Ahlfors Q-regular, and for every $x \in A$, $h_f(x) < M$.*
3. *f is a homeomorphism, Y is locally Ahlfors Q-regular, and for every $x \in A$, $h^*_f(x) < M$.*
4. *f is a homeomorphism, $X = \Omega$ and $Y = \Omega'$ are domains in $\mathbb{R}^n$, and for every $x \in A$, $k_f(x) < M$.*

Then on Q-almost every curve γ in X ($Q = n$ for case (4)), $f(\gamma \cap A)$ has finite length, and f satisfies condition N^A. Moreover, if condition (1) is satisfied, then the conclusion holds for every *γ in X.*

Proof. Case 1 is quite elementary. It is enough for our purposes to show that for every measurable subset $S \subseteq [0, l(\gamma)] \cap \gamma^{-1}(A)$, $H^1(f(\gamma(S))) \leq 5M|S|$, though the observant reader will note that upon proving this, and a fortiori proving condition N^A for γ, a straightforward application of the Vitali covering theorem yields the inequality $H^1(f(\gamma(S))) \leq M|S|$.

Our argument is in the same spirit as the one for the analogous statement using metric differentials [Dud07, Lemma 2.3] and similar statements elsewhere. Let $\varepsilon > 0$. By the Vitali covering lemma we may cover S with a sequence of intervals $I_j = [t_j - s_j, t_j + s_j]$, centered at points in S and with $s_j < \varepsilon/2$, such that $|\bigcup_{i=1}^{\infty} I_j| < |S| + \varepsilon$, such that the intervals $I_j/5 = [t_j - s_j/5, t_j + s_j/5]$ are pairwise disjoint, and such that $\mathrm{diam}(f(\gamma(I_j))) \leq 2L_f(\gamma(t_j), s_j) < 2Ms_j = M|I_j|$.

We then have

$$H^1_\varepsilon(f(\gamma(S))) \le \sum_{j=1}^\infty \operatorname{diam}(f(\gamma(I_j))) < M\sum_{j=1}^\infty |I_j|$$
$$\le 5M\sum_{j=1}^\infty |I_j/5| \le 5M\left|\bigcup_{i=1}^\infty I_j\right| < 5M(|S|+\varepsilon).$$

Passing to the limit as ε approaches 0 gives the desired inequality $H^1(f(\gamma(S))) \le 5M|S|$.

We now turn to the homeomorphic cases. Throughout, we will use C to denote various constants that depend only on the constants of local Q-regularity for X and Y. In different expressions, C may denote different constants, even within a single string of equations or inequalities. Whenever we consider balls in either X or Y, we tacitly assume that radii have been chosen small enough so that the local Q-regularity condition (2) is satisfied. Also, since our considerations are entirely local, we will assume throughout that $\mu(X) < \infty$ and $\nu(Y) < \infty$.

Moving to Case 2, we suppose that $h_f(x) < M$ on A and fix $\varepsilon > 0$. By assumption, there is at each $x \in A$ a radius $r_x < \varepsilon$ such that $h_f(x, r_x) < M$. Let $B_x = B(x, r_x)$ and $L_x = L_f(x, r_x)$. By the covering results [BKR07, Lemmas 2.2, 2.3] there is a sequence of points $x_i \in A$ such that the balls $B_i = B_{x_i}$ satisfy the following properties (to ease notation, let $r_i = r_{x_i}$ and $L_i = L_{x_i}$):

(i) For each $i \ne j$, $B_i/3 \cap B_j/3 = \emptyset$, and

$$B\left(f(x_i), \frac{L_i}{10M^2}\right) \cap B\left(f(x_j), \frac{L_j}{10M^2}\right) = \emptyset.$$

(ii) $\bigcup_{i=1}^\infty B_i \supseteq A$.

We now define $\rho_\varepsilon : X \to \mathbb{R}$ by

$$\rho_\varepsilon = 2\sum_{i=1}^\infty \left(\frac{L_i}{r_i}\right)\chi_{2B_i}.$$

Then for every rectifiable curve γ in X such that $\operatorname{diam}(\gamma) \ge 4\varepsilon$, we may conclude that for each i such that $B_i \cap \gamma \ne \emptyset$, γ joins B_i with $X\backslash 2B_i$, so that $\int_\gamma \chi_{2B_i}\, ds \ge r_i$. We thus have the inequality

$$\int_\gamma \rho_\varepsilon\, ds \ge \sum_{\gamma\cap B_i \ne \emptyset} 2L_i \ge H^1_\varepsilon(f(\gamma\cap A)). \tag{5}$$

Moreover, we have

$$\int_X \rho_\varepsilon^Q\, d\mu \le C\int_X \left(\sum_{i=1}^\infty \left(\frac{L_i}{r_i}\right)\chi_{B_i}\right)^Q d\mu \le C\int_X \left(\sum_{i=1}^\infty \frac{L_i}{r_i}\chi_{B_i/3}\right)^Q d\mu$$
$$= C\sum_{i=1}^\infty \left(\frac{L_i}{r_i}\right)^Q \mu(B_i/3) \le C\sum_{i=1}^\infty L_i^Q$$

$$\leq CM^{2Q} \sum_{i=1}^{\infty} \nu\left(B\left(f(x_i), \frac{L_i}{10M^2} \right) \right)$$
$$\leq CM^{2Q} \nu(f(X)).$$

Here the second inequality is a well-known consequence of the boundedness of the Hardy–Littlewood maximal operator; see, for example, [Boj88, Lemma 4.2]. Because the preceding estimate is independent of ε, a standard application of reflexivity, Mazur's lemma, and a theorem of Fuglede (see, e.g., the proof of [BKR07, Theorem 1.1]) gives a sequence of convex combinations of the functions ρ_ε, converging in $L^Q(X)$ to some function ρ, such that on Q-almost every curve γ, $\int_\gamma \rho\, ds < \infty$, and

$$\int_\gamma \rho\, ds \geq H^1(f(\gamma \cap A)), \tag{6}$$

so that $f(\gamma \cap A)$ has finite length. Moreover, it follows immediately from the basic properties of modulus [Fug57, p. 177, (c)] that on Q-almost every curve γ, inequality (6) holds for every subcurve of γ, whereby for every $S \subseteq \gamma^{-1}(A)$,

$$\int_{\gamma(S)} \rho\, ds \geq H^1(f(\gamma(S) \cap A)),$$

from which condition N^A immediately follows, and so the proof of the second case is complete.

Case 3 is almost entirely the same as Case 2, except that since our assumption is equivalent to a bound on $h_{f^{-1}}(y)$ on $f(A)$, we apply [BKR07, Lemmas 2.2, 2.3] to f^{-1} to get "roundish" sets in the domain and balls in the target, rather than vice versa as before. More precisely, we follow the preceding argument with the following changes: We choose at each $y \in f(A)$ a radius $r_y < \varepsilon$ so that $h_{f^{-1}}(y, r_y) < M$ and let $B_y = B(y, r_y)$ and $L_y = L_{f^{-1}}(y, r_y)$. Then [BKR07, Lemmas 2.2, 2.3] gives us a sequence $y_i \in f(A)$ such that the family of balls $B_i = B_{y_i}$ satisfies conditions (i) and (ii), but with x, f, and A replaced by y, f^{-1}, and $f(A)$, respectively. We define $\rho_\varepsilon : X \to \mathbb{R}$ by

$$\rho_\varepsilon = 2 \sum_{i=1}^{\infty} \left(\frac{r_i}{L_i} \right) \chi_{B(f^{-1}(y_i), 2L_i)}$$

with $r_i = r_{y_i}$ and $L_i = L_{y_i}$. The remainder of the argument proceeds entirely as in the proof for Case 2.

Finally, the proof of Case 4 is also like that of Case 2, except that we invoke the Besicovitch covering theorem instead of [BKR07, Lemmas 2.2, 2.3]. More precisely, we assume without loss of generality that Ω and Ω' are bounded and choose the radii $r_x < \varepsilon$ such that $k_f(x, r_x) < M$ and $B(x, 2r_x) \subseteq \Omega$.

By the Besicovitch covering theorem there are a constant C_n depending only on n and a sequence of points $x_i \in A$ such that the balls $B_i = B(x_i, r_{x_i})$ satisfy the following properties:

(i) $\sum_{i=1}^{\infty} \chi_{B_i} \le C_n$.
(ii) $\bigcup_{i=1}^{\infty} B_i \supseteq A$.

We define ρ_ε exactly as in the proof of Case 2 and obtain the estimate

$$\begin{aligned}\int_\Omega \rho_\varepsilon^n\,dx &\le C\int_\Omega\left(\sum_{i=1}^{\infty}\left(\frac{L_i}{r_i}\right)\chi_{B_i}\right)^n dx \le C\int_\Omega C_n^{n-1}\sum_{i=1}^{\infty}\left(\frac{L_i}{r_i}\right)^n \chi_{B_i}\,dx\\ &= C\sum_{i=1}^{\infty}\left(\frac{L_i}{r_i}\right)^n |B_i| = C\sum_{i=1}^{\infty} L_i^n \le C\sum_{i=1}^{\infty} M^{n-1}|f(B_i)|\\ &\le CM^{n-1}|\Omega'|.\end{aligned}$$

The remainder of the proof is the same as before. □

Proof of Theorems 1.8, 1.1, and 1.10. It follows from [Zü07, Proposition 3.2] that for p-almost every curve γ in X, $\mathcal{H}^1(f(\gamma)\cap f(E)) = 0$, whereby f satisfies condition N^E on p-almost every curve in X.

It is elementary that $L_f(x,r)$, $\nu(f(B(x,r)))$, and $\mu(B(x,r))$ are continuous from the left in r and lower semicontinuous in x. That the same is true (for small r) of $l_f(x,r)$ follows easily from local compactness. Therefore, $L_f(x,r)/r$, $h_f(x,r)$, $h^*(x,r)$, and $k_f(x,r)$ are Borel measurable in x and continuous from the left in r, whereby

$$\operatorname{lip}_f(x) = \liminf_{\mathbb{Q}\ni r\to 0} L_f(x,r)/r$$

is a Borel function, and likewise for h_f, h_f^*, and k_f.

We now apply Proposition 3.1 repeatedly with $A = \operatorname{lip}_f^{-1}([0,M])$, $A = h_f^{-1}([0,M])$, $A = (h_f^*)^{-1}([0,M])$, and $A = k_f^{-1}([0,M])$ for each $M \in \mathbb{N}$, so from the countable subadditivity of modulus and measure we have that f satisfies condition $N^{X\setminus E}$ on Q-almost every γ. Since, by Hölder's inequality, a curve family Γ with $\operatorname{mod}_Q(\Gamma) = 0$ also satisfies $\operatorname{mod}_p(\Gamma) = 0$ for every $p \le Q$, it follows that f satisfies condition $N^{X\setminus E}$ on p-almost every curve as well. Having already established condition N^E on p-almost every curve, all three theorems now follow immediately from the subadditivity of modulus and measure. □

Proof of Corollary 1.2. By Theorem 1.1, f satisfies condition N on p-almost every rectifiable curve. On such a curve γ, either $\int_\gamma \operatorname{lip}_f\,ds = \infty$, in which case inequality (4) is trivially satisfied, or we may apply [Zü07, Lemma 3.6] (cf. the claim in the proof of [Zü07, Lemma 3.9]) to deduce that, as in the conclusion of [Zü07, Lemma 3.9], lip_f satisfies inequality (4) on γ. Thus, lip_f is a p-weak upper gradient for f. The final statement on Newton–Sobolev regularity follows immediately from the definition of $N^{1,p}_{\mathrm{loc}}(X,Y)$. □

Proof of Corollary 1.3. Since $B(x, l_f(x,r)) \subseteq f(B(x,r))$, we obtain from the definitions that

$$\operatorname{lip}_f \le C h_f J_f^{1/Q}. \tag{7}$$

It now follows immediately from Hölder's inequality that $\mathrm{lip}_f \in L^p_{\mathrm{loc}}(X)$, so that by Corollary 1.2, $f \in N^{1,p}_{\mathrm{loc}}(X, Y)$. □

REMARK 3.2. In the Euclidean case, we may apply Hölder's inequality in a manner analogous to the proof of Corollary 1.3. Suppose, for instance, that $k_f \in L^1(\Omega)$. It follows immediately from the definitions that $\mathrm{lip}_f(x) \leq Ck_f(x)^{(n-1)/n} J_f(x)^{1/n}$, and so Hölder's inequality implies $\mathrm{lip}_f \in L^1(\Omega)$. By Corollary 1.2 (substituting k_f for h_f) we recover a theorem of Koskela and Rogovin; namely, that $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \Omega')$ [KR05, Theorem 1.1]. (Recall that the class $N^{1,p}(\Omega, \Omega')$ is contained in $W^{1,p}(\Omega, \Omega')$ and that every member of the latter class has a representative in the former [Sha00; HKST01].)

Proof of Theorem 1.6. To prove the first inequality in (1), we note that by inequality (7), along with Corollary 1.2, f has a Q-weak upper gradient $g \in L^Q(X, \mu)$ (namely $g = \mathrm{lip}_f$) satisfying (for $K = Ch^Q$) the inequality

$$g^Q \leq K J_f \quad \text{for } \mu\text{-almost every } x \in X.$$

The first inequality in (1) then holds by [Wil12, Theorem 1.1].

To prove the second inequality, we may just as well prove the first inequality for the inverse map f^{-1}. To this end, we first fix $x \in X$ and $r > 0$, and let $y = f(x)$, $L = L_f(x, r)$, $l = l_f(x, r)$, and $H = h_f(x, r) = \frac{L}{l}$. Via the inclusions $B(y, l) \subseteq f(B(x, r)) \subseteq B(y, L)$, along with the Q-regularity of X and Y, we have

$$\left(\frac{L_{f^{-1}}(y, l)}{l}\right)^Q \leq \left(\frac{r}{l}\right)^Q = \left(\frac{Hr}{L}\right)^Q \leq CH^Q \frac{\mu(B(x, r))}{\nu(B(y, L))} \leq CH^Q \frac{\mu(f^{-1}(B(y, L)))}{\nu(B(y, L))}.$$

Choosing r_i so that $\lim_{r_i \to 0} h_f(x, r_i) = h_f(x) = h^*_{f^{-1}}(y) < \infty$, we obtain at *every* $y \in Y$ the inequality

$$\mathrm{lip}_{f^{-1}}(y)^Q \leq Ch^*_{f^{-1}}(y)^Q J_{f^{-1}}(y). \tag{8}$$

Now, to complete the proof of the second inequality in (1), it suffices (as before, via Corollary 1.2 and [Wil12, Theorem 1.1]) to show that for ν-almost every $y \in Y$,

$$\mathrm{lip}_{f^{-1}}(y)^Q \leq Ch^Q J_{f^{-1}}(y). \tag{9}$$

To this end, we let $F \subseteq Y$ be the set on which inequality (9) fails.

We first note that from inequality (8), together with the assumption that $h^*_{f^{-1}}(f(x)) = h_f(x) < h$ μ-almost everywhere, it follows that inequality (9) already holds for $f_*\mu$-almost every $y \in Y$. Thus, $(f^{-1})^*\mu(F) = f_*\mu(F) = 0$, so $J_{f^{-1}}(y) = 0$ for ν-almost every $y \in F$.

On the other hand, the failure of inequality (9) on F implies that $\mathrm{lip}_{f^{-1}}(y) > 0$ for every $y \in F$, from which the finiteness of $h_{f^{-1}}(y)$ and inequality (8) imply that $J_{f^{-1}}(y) > 0$, and so we conclude that $\nu(F) = 0$. □

ACKNOWLEDGMENTS. The author would like to thank Pekka Koskela for advice regarding the organization of this paper and Pietro Poggi-Corradini and Thomas Zürcher for helpful comments. I also thank the referee for suggesting a connection to results on Lorentz–Sobolev regularity.

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