A Half-Space Theorem for Ideal Scherk Graphs in $M \times \mathbb{R}$

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ABSTRACT. We prove a half-space theorem for an ideal Scherk graph $\Sigma \subset M \times \mathbb{R}$ over a polygonal domain $D \subset M$, where M is a Hadamard surface whose curvature is bounded above by a negative constant. More precisely, we show that a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ is a translate of Σ.

1. Introduction

A well-known result in the global theory for proper minimal surfaces in the Euclidean 3-space is the so-called *half-space theorem* due to Hoffman and Meeks [11], which says that if a properly immersed minimal surface S in $\mathbb{R}^3$ lies on one side of some plane P, then S is a plane parallel to P. They also proved the *strong half-space theorem:* two properly immersed minimal surfaces in $\mathbb{R}^3$ that do not intersect must be parallel planes.

The problem of giving conditions that force two minimal surfaces of a Riemannian manifold to intersect has received considerable attention, and many people have worked on this subject.

Notice that there is no half-space theorem in Euclidean spaces of dimensions greater than 4 since there exist rotational proper minimal hypersurfaces contained in a slab.

Similarly, there exists no half-space theorem for horizontal slices in $\mathbb{H}^2 \times \mathbb{R}$ since rotational minimal surfaces (catenoids) are contained in a slab [13; 14]. However, there are half-space theorems for constant mean curvature (CMC) 1/2 surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [10; 15]. For instance, Hauswirth, Rosenberg, and Spruck [10] proved that if S is a properly immersed CMC 1/2 surface in $\mathbb{H}^2 \times \mathbb{R}$, contained on the mean convex side of a horocylinder C, then S is a horocylinder parallel to C; and if S is embedded and contains a horocylinder C on its mean convex side, then S is also a horocylinder parallel to C. Nelli and Sa Earp [15] showed that in $\mathbb{H}^2 \times \mathbb{R}$ the mean convex side of a simply connected rotational CMC 1/2 surface cannot contain a complete CMC 1/2 surface besides the rotational simply connected ones.

Other examples of homogeneous manifolds where there are half-space theorems for minimal surfaces are Nil_3 and Sol_3 [1; 4; 5]. For instance, we know that if a properly immersed minimal surface S in Nil_3 lies on one side of some entire minimal graph Σ, then S is the image of Σ by a vertical translation.

Mazet [12] proved a general half-space theorem for constant mean curvature surfaces. Under certain hypothesis, he proved that in a Riemannian 3-manifold of

bounded geometry, a constant mean curvature H surface on one side of a parabolic constant mean curvature H surface Σ is an equidistant surface to Σ.

In this paper, we consider the half-space problem for an ideal Scherk graph Σ over a polygonal domain $D \subset M$, where M denotes a Hadamard surface whose curvature is bounded above by a negative constant, that is, M is a complete simply connected Riemannian surface with curvature $K_M \leq -a^2 < 0$ for some constant $a \in \mathbb{R}$. More precisely, we prove the following result.

THEOREM 1. *Let M denote a Hadamard surface with curvature bounded above by a negative constant, and let $\Sigma = \mathrm{Graph}(u)$ be an ideal Scherk graph over an admissible polygonal domain $D \subset M$. If S is a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ, then S is a translate of Σ.*

We remark that Mazet's theorem does not apply in our case for Scherk surfaces. In fact, in the case of minimal surfaces, one of his hypotheses on the geometry of equidistant surfaces to the parabolic one is that the mean curvature points away from the original surface. However, since an end of a Scherk surface is asymptotic to some vertical plane $\gamma \times \mathbb{R}$, where γ is a geodesic, we know that an equidistant surface is asymptotic to $\gamma_s \times \mathbb{R}$, where γ_s is an equidistant curve to γ. Hence, in the case of a Scherk surface, the mean curvature vector of an equidistant surface points toward the Scherk surface.

2. Preliminaries

In this section, we present some basic properties of Hadamard manifolds and state some previous results. For more details, see [9] or [6; 7; 8].

Let M be a Hadamard manifold, that is, a complete simply connected Riemannian manifold with nonpositive sectional curvature. We say that two geodesics γ_1, γ_2 of M, parameterized by arc length, are *asymptotic* if there exists a constant $c > 0$ such that the distance between them satisfies

$$d(\gamma_1(t), \gamma_2(t)) \leq c \quad \text{for all } t \geq 0.$$

Note that to be asymptotic is an equivalence relation on the oriented unit speed geodesics of M. We call each of these classes a point at infinity. We denote by $M(\infty)$ the set of points at infinity and by $\gamma(+\infty)$ the equivalence class of the geodesic γ. Throughout this section, we only consider oriented unit speed geodesics.

Let us assume that M has sectional curvature bounded from above by a negative constant. Then we have two important facts:

1. For any two asymptotic geodesics γ_1, γ_2, the distance between the two curves $\gamma_1|_{[t_0,+\infty)}$, $\gamma_2|_{[t_0,+\infty)}$ is zero for any $t_0 \in \mathbb{R}$.
2. Given $x, y \in M(\infty)$, $x \neq y$, there exists a unique geodesic γ such that $\gamma(+\infty) = x$ and $\gamma(-\infty) = y$, where $\gamma(-\infty)$ denotes the corresponding point at infinity when the orientation of γ is changed.

For any point $p \in M$, there is a bijective correspondence between the set of unit vectors in the tangent plane $T_p M$ and $M(\infty)$, where a unit vector v is mapped

to the point at infinity $\gamma_v(\infty)$, γ_v denoting the geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Analogously, given a point $p \in M$ and a point at infinity $x \in M(\infty)$, there exists a unique geodesic γ such that $\gamma(0) = p$ and $\gamma(+\infty) = x$. In particular, $M(\infty)$ is bijective to a sphere.

There exists a topology on $M^* = M \cup M(\infty)$ satisfying that the restriction to M agrees with the topology induced by the Riemannian distance. This topology is called the cone topology of M^* (see [9] for instance).

In order to define horospheres, we consider Busemann functions. Given a unit vector v, the Busemann function $B_v : M \to \mathbb{R}$ associated to v is defined as

$$B_v(p) = \lim_{t \to +\infty} (d(p, \gamma_v(t)) - t).$$

This is a C^2 convex function on M, and it satisfies the following properties.

Property 1. The gradient $\nabla B_v(p)$ is the unique unit vector w in $T_p M$ such that $\gamma_v(\infty) = \gamma_w(-\infty)$.

Property 2. If w is a unit vector such that $\gamma_v(\infty) = \gamma_w(\infty)$, then $B_v - B_w$ is a constant function on M.

DEFINITION 1. Given a point at infinity $x \in M(\infty)$ and a unit vector v such that $\gamma_v(\infty) = x$, the *horospheres at x* are defined as the level sets of the Busemann function B_v.

We have the following important facts with respect to horospheres.

- By *Property 2,* the horospheres at x do not depend on the choice of the vector v.
- The horospheres at x give a foliation of M, and since B_v is a convex function, each bounds a convex domain in M called a *horoball*.
- The intersection between a geodesic γ and a horosphere at $\gamma(\infty)$ is always orthogonal by *Property 1.*
- Take a point $p \in M$ and let H_x denote a horosphere at x. If γ is the geodesic passing through p with $\gamma(+\infty) = x$, then $H_x \cap \gamma$ is the closest point on H_x to p.
- Given $x, y \in M(\infty)$, if γ is a geodesic with these points at infinity, and H_x, H_y are disjoint horospheres, then the distance between H_x and H_y coincides with the distance between the points $H_x \cap \gamma$ and $H_y \cap \gamma$.

From now on, we restrict M to be a Hadamard surface with curvature bounded above by a negative constant, and by horocycle and horodisk we mean horosphere and horoball, respectively.

Let Γ be an ideal polygon of M, that is, Γ is a polygon all of whose sides are geodesics and the vertices are at infinity $M(\infty)$. We assume that Γ has an even number of sides $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k$. Let D be the interior of the convex hull of the vertices of Γ, so $\partial D = \Gamma$, and D is a topological disk. We call D an ideal polygonal domain.

DEFINITION 2. An ideal Scherk graph over D is a minimal surface that is the graph of a function defined on D and taking the values $+\infty$ on each side α_i and $-\infty$ on each side β_i.

For the sake of completeness and in order to understand the hypothesis on our main result (Theorem 3), let us describe the necessary and sufficient conditions on the domain D, proved by Gálvez and Rosenberg [9], for the existence of an ideal Scherk graph over D.

At each vertex a_i of Γ, place a horocycle H_i so that $H_i \cap H_j = \emptyset$ if $i \neq j$.

Each α_i meets exactly two horodisks. Denote by $\tilde{\alpha}_i$ the compact arc of α_i outside the two horodisks and denote by $|\alpha_i|$ the length of $\tilde{\alpha}_i$, that is, the distance between these horodisks. Analogously, we can define $\tilde{\beta}_i$ and $|\beta_i|$.

Now define

$$a(\Gamma) = \sum_{i=1}^{k} |\alpha_i|$$

and

$$b(\Gamma) = \sum_{i=1}^{k} |\beta_i|.$$

Observe that $a(\Gamma) - b(\Gamma)$ does not depend on the choice of the horocycles because given two horocycles H_1, H_2 at a point $x \in M(\infty)$ and a geodesic γ with x as a point at infinity, the distance between H_1 and H_2 coincides with the distance between the points $\gamma \cap H_1$ and $\gamma \cap H_2$.

DEFINITION 3. An ideal polygon $\mathcal{P}$ is said to be inscribed in D if the vertices of $\mathcal{P}$ are among the vertices of Γ. Hence, its edges are either interior in D or equal to some α_i or β_j.

The definition of $a(\Gamma)$ and $b(\Gamma)$ extends to inscribed polygons:

$$a(\mathcal{P}) = \sum_{\alpha_i \in \mathcal{P}} |\alpha_i| \quad \text{and} \quad b(\mathcal{P}) = \sum_{\beta_i \in \mathcal{P}} |\beta_i|.$$

We denote by $|\mathcal{P}|$ the length of the boundary arcs of $\mathcal{P}$ exterior to the horodisks bounded by H_i at the vertices of $\mathcal{P}$. We call this the truncated length of $\mathcal{P}$.

DEFINITION 4. An ideal polygon Γ is said to be admissible if the two following conditions are satisfied.

1. $a(\Gamma) = b(\Gamma)$;
2. For each inscribed polygon $\mathcal{P}$ in D, $\mathcal{P} \neq \Gamma$, and for some choice of the horocycles at the vertices, we have

$$2a(\mathcal{P}) < |\mathcal{P}| \quad \text{and} \quad 2b(\mathcal{P}) < |\mathcal{P}|.$$

Moreover, an ideal polygonal domain D is said to be admissible if its boundary $\Gamma = \partial D$ is an admissible polygon.

The properties of an admissible polygon are necessary and sufficient conditions for the existence of an ideal Scherk graph over $D \subset M$ [9].

An important tool for studying minimal (and more generally, constant mean curvature) surfaces are the formulas for the flux of appropriately chosen ambient vector fields across the surface.

Let u be a function defined in D whose graph is a minimal surface, and consider $X = \frac{\nabla u}{W}$ defined on D, where $W^2 = 1 + |\nabla u|^2$. For an open domain $A \subset D$ and a boundary arc α of A, we define the flux formula across α as

$$F_u(\alpha) = \int_\alpha \langle X, \nu \rangle \, ds;$$

here α is oriented as the boundary of A, and ν is the outer conormal to A along α.

THEOREM 2 (Flux theorem). *Let $A \subset D$ be an open domain.*

1. *If ∂A is a compact cycle, then $F_u(\partial A) = 0$.*
2. *If α is a compact arc of A, then $F_u(\alpha) \leq |\alpha|$.*
3. *If α is a compact arc of A on which u diverges to $+\infty$, then*

$$F_u(\alpha) = |\alpha|.$$

4. *If α is a compact arc of A on which u diverges to $-\infty$, then*

$$F_u(\alpha) = -|\alpha|.$$

Another useful result related to the flux formula is the following.

PROPOSITION 1. *Let D be a domain whose boundary is an ideal polygon, and let u, v be functions defined on D whose graphs are minimal surfaces. If $u \leq v$ on D and $u = v$ on ∂D, then $F_u(\partial D) \leq F_v(\partial D)$. Moreover, equality holds if and only if $u \equiv v$ on D.*
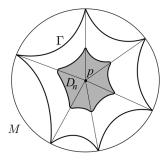
For a proof of this result, see, for example, the proof of the generalized maximum principle in [2, Theorem 2].

3. Main Result

In this section, we consider a Hadamard surface M whose curvature is bounded above by a negative constant, that is, M is a complete simply connected Riemannian surface with curvature $K_M \leq -a^2 < 0$ for some constant $a \in \mathbb{R}$. We now can establish our main result.

THEOREM 3. *Let M denote a Hadamard surface with curvature bounded above by a negative constant, and let $\Sigma = \text{Graph}(u)$ be an ideal Scherk graph over an admissible polygonal domain $D \subset M$. If S is a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ, then S is a translate of Σ.*

The idea to prove this result is based on the proof of the classical half-space theorem in the Euclidean three-space due to Hoffman and Meeks [11]. In their proof, they use as barrier a family of minimal surfaces (obtained from the catenoid by homothety) that converges to the plane minus a point, where the plane is the minimal surface for which they want to prove the half-space theorem. Hence, in

Figure 1 Convex smooth domain D_n

order to prove our result, using their ideas, we need a family of minimal surfaces that play the role of barriers and converge to our ideal Scherk graph, at least outside a compact set. To construct such a family, we follow an idea of Rosenberg, Schulze, and Spruck [16] by constructing a discrete family of minimal graphs in $D \times \mathbb{R}$.
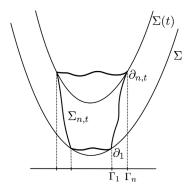
Let $\Sigma = \text{Graph}(u)$ be an ideal Scherk graph over D with $\Gamma = \partial D$. Given any point $p \in D$, consider the geodesics starting at p and going to the vertices of Γ. Take the points over each of these geodesics that are at a distance n from p. Now consider the geodesics joining two consecutive points. The angle at which two of these geodesics meet is less than π; hence, we can smooth the corners to obtain a convex domain D_n with smooth boundary $\Gamma_n = \partial D_n$ and such that $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ is an exhaustion of D. (See Figure 1.)

Denote by A_n the annular-type domain $D_n \setminus \bar{D}_1$ and by Σ_n the graph of u restrict to A_n. Hence, Σ_n is a stable minimal surface, and any sufficiently small perturbation of $\partial \Sigma_n$ gives rise to a smooth family of minimal surfaces $\Sigma_{n,t}$ with $\Sigma_{n,0} = \Sigma_n$. We use this fact to the deformation of $\partial \Sigma_n$ that is the graph over ∂A_n given by $\partial_1 \cup \partial_{n,t}$ for $t \geq 0$, where $\partial_1 = (\Gamma_1 \times \mathbb{R}) \cap \Sigma$, $\partial_{n,t} = (\Gamma_n \times \mathbb{R}) \cap T(t)(\Sigma)$, and $T(t)$ is the vertical translation by height t. Then for t sufficiently small, there exists a minimal surface $\Sigma_{n,t}$ that is the graph of a smooth function $u_{n,t}$ defined on A_n with boundary $\partial_1 \cup \partial_{n,t}$ (see Figure 2). Note that $u_{n,t}$ satisfies the minimal surface equation on A_n, and, by the maximum principle, $\Sigma_{n,t}$ stays between Σ and $\Sigma(t) = T(t)(\Sigma)$. We show that there exists a uniform interval of existence for $u_{n,t}$, that is, we prove that there exists $\delta_0 > 0$ such that for all n and $0 \leq t \leq \delta_0$, the minimal surfaces $\Sigma_{n,t} = \text{Graph}(u_{n,t})$ exist.

Consider $\delta_0 > 0$ sufficiently small so that $u_{2,t}$ exists for any $t \in [0, \delta_0]$. We will show this δ_0 works for all $n \geq 2$, that is, we will prove that for $n > 2$, the set $B_n = \{\tau \in [0, \delta_0]; u_{n,t} \text{ exists for } 0 \leq t \leq \tau\}$ is in fact the interval $[0, \delta_0]$.

CLAIM. The set B_n is open and closed in the interval $[0, \delta_0]$. Hence, $B_n = [0, \delta_0]$.

Proof. Consider an (increasing) sequence $\tau_k \in B_n$ such that $\tau_k \to \tau$ as $k \to \infty$. First, observe that the family of minimal surfaces $\Sigma_{n,\tau_k} = \text{Graph}(u_{n,\tau_k})$ is contained in the region bounded by Σ and $\Sigma(\tau)$, in particular, $|u_{n,\tau_k}| \leq C_n$ for all k

Figure 2   Minimal surface $\Sigma_{n,t}$

and that the boundary component ∂_1 is contained in the boundary $\partial \Sigma_{n,\tau_k}$ for all k. Then there exists a minimal surface $\Sigma_{n,\tau}$ that is the limit of the surfaces Σ_{n,τ_k} with $\partial_1 \subset \partial \Sigma_{n,\tau}$. It remains to prove that $\Sigma_{n,\tau}$ is a graph.

Since $D_2 \subset D_n$, using the maximum principle with vertical translations of u_{2,δ_0}, we get that $u_{n,\tau_k} \leq u_{2,\delta_0}$ in a neighborhood of Γ_1. Then the gradient of u_{n,τ_k} is uniformly bounded in a neighborhood of Γ_1. We affirm that we also have a uniform bound for points in Γ_n. In fact, suppose that this is not true, so there exists a sequence $p_k \in \Gamma_n$ with $u_{n,\tau_k}(p_k) \to p \in \partial_{n,\tau}$ such that $|\nabla u_{n,\tau_k}(p_k)| \to \infty$. This implies that the minimal surface $\Sigma_{n,\tau}$ is vertical at p. Considering the horizontal geodesic γ that passes through p and is tangent to $\partial_{n,\tau}$ (recall that $\partial_{n,\tau}$ is convex), we can apply the maximum principle with boundary to $\Sigma_{n,\tau}$ and $\gamma \times (-\infty, \tau]$ to conclude that they coincide, which is impossible. Thus, there exists a constant C_n' such that $|\nabla u_{n,\tau_k}(p)| \leq C_n'$ for all k and $p \in \partial A_n = \Gamma_1 \cup \Gamma_n$.

Since u_{n,τ_k} are uniformly bounded functions and we have uniform gradient estimates for u_{n,τ_k} in ∂A_n, we get uniform gradient estimates for u_{n,τ_k} in the whole domain A_n (for instance, see [3], Lemma 2.5). Thus, the DeGiorgi–Nash–Moser and Schauder estimates imply locally uniform estimates for all higher derivatives. Then, using Arzelà–Ascoli's theorem, there is some subsequence of $\{u_{n,\tau_k}\}$ that converges to a function $u_{n,\tau}$ defined on A_n, which also satisfies the minimal surface equation; hence, its graph $\hat{\Sigma} = \text{Graph}(u_{n,\tau})$ is a minimal surface with boundary $\partial_1 \cup \partial_{n,\tau}$. By the uniqueness of the limit of Σ_{n,τ_k} we conclude that $\Sigma_{n,\tau} = \hat{\Sigma} = \text{Graph}(u_{n,\tau})$. Then $\tau \in B_n$, and B_n is closed.

From the previous discussion we know that, as a consequence of the maximum principle, if $\tau \in B_n$, then the tangent plane to the boundary of $\Sigma_{n,\tau} = \text{Graph}(u_{n,\tau})$ is never vertical. Then $\Sigma_{n,\tau}$ is strictly stable, and, in particular, a sufficiently small perturbation of its boundary $\partial \Sigma_{n,\tau} = \partial_1 \cup \partial_{n,\tau}$ to $\partial_1 \cup \partial_{n,\tau+t}$ gives rise to a smooth family of minimal surfaces $\Sigma_{n,\tau+t}$ with boundary $\partial_1 \cup \partial_{n,\tau+t}$. Thus, B_n is open. $\qquad\square$

Therefore, we have proved that for all $n \geq 2$ and $0 \leq t \leq \delta_0$, there exists a function $u_{n,t}$ defined on A_n such that $\Sigma_{n,t} = \mathrm{Graph}(u_{n,t})$ is a minimal surface with boundary $\partial \Sigma_{n,t} = \partial_1 \cup \partial_{n,t}$.

Fix $t \in (0, \delta_0]$. For a fixed n_0, consider the sequence $\{u_{n,t}|_{A_{n_0}}\}$ for $n > n_0$. We already know that $u_{n,t} \leq u_{n_0,t}$ in a neighborhood of Γ_1; hence, we have uniform gradient estimates in such a neighborhood. Moreover, since we have uniform curvature estimates for points far from the boundary [17] and $\Gamma_n \not\subset A_{n_0}$ for all $n > n_0$, we get that the sequence of minimal surfaces $\Sigma_{n,t}$, $n > n_0$, restricted to the domain A_{n_0} has uniform curvature estimates up to the boundary. Then the DeGiorgi–Nash–Moser and Schauder estimates imply locally uniform estimates for all higher derivatives. Thus, there exists a subsequence $\{u_{n_j,t}|_{A_{n_0}}\}$ that converges to a function $\hat{u}_{n_0}$ defined over A_{n_0} whose graph $\hat{\Sigma}_{n_0}$ is a minimal surface with $\partial_1 \subset \partial \hat{\Sigma}_{n_0}$ and $u \leq \hat{u}_{n_0} \leq u + t$ over A_{n_0}.

Now consider the subsequence $\{u_{n_j,t}\}$ restricted to A_{2n_0} for $n_j > 2n_0$. Using the same argument as before, the sequence $\{u_{n_j,t}|_{A_{2n_0}}\}$ for $n_j > 2n_0$ has a subsequence $\{u_{n_{jk},t}|_{A_{2n_0}}\}$ that converges to a function $\hat{u}_{2n_0}$ defined over A_{2n_0} whose graph $\hat{\Sigma}_{2n_0}$ is a minimal surface with $\partial_1 \subset \partial \hat{\Sigma}_{2n_0}$ and $u \leq \hat{u}_{2n_0} \leq u + t$ over A_{2n_0}.

Since the sequence $\{u_{n_{jk},t}|_{A_{2n_0}}\}$ is a subsequence of $\{u_{n_j,t}|_{A_{n_0}}\}$, we conclude, by the uniqueness of the limit, that $\hat{u}_{2n_0} = \hat{u}_{n_0}$ in A_{n_0}.

We continue this argument to A_{kn_0} for all $k > 2$, obtaining a function $\hat{u}_{kn_0}$ defined on A_{kn_0} whose graph $\hat{\Sigma}_{kn_0}$ is a minimal surface with $\partial_1 \subset \partial \hat{\Sigma}_{kn_0}$, $u \leq \hat{u}_{kn_0} \leq u + t$ over A_{kn_0}, and $\hat{u}_{kn_0} = \hat{u}_{ln_0}$ in A_{ln_0} for each $1 \leq l \leq k$. Hence, using a diagonal process, we obtain a subsequence of $\{u_{n,t}\}$ that converges to a function $\hat{u}_\infty$ defined over $\Omega = D \setminus \bar{D}_1$ (the limit of the domains A_n) whose graph $\hat{\Sigma}_\infty$ is a minimal surface with $\partial \hat{\Sigma}_\infty = \partial_1$, $u \leq \hat{u}_\infty < u + t$ over Ω, and $\hat{u}_\infty = \hat{u}_{kn_0}$ in A_{kn_0} for all k.

For simplicity, let us write $\hat{u}$ and $\hat{\Sigma}$ to denote $\hat{u}_\infty$ and $\hat{\Sigma}_\infty$.

Note that, since $u \leq \hat{u} \leq u + t$ over Ω, the minimal surface $\hat{\Sigma} = \mathrm{Graph}(\hat{u})$ assumes the same infinite boundary values at Γ as the ideal Scherk graph $\Sigma = \mathrm{Graph}(u)$. Consider the restriction of u to Ω and continue denoting by Σ the graph of u restricted to Ω. We will show that Σ and $\hat{\Sigma}$ coincide by analyzing the flux of the functions $u, \hat{u}$ across the boundary of Ω, which is $\Gamma_1 \cup \Gamma$, and using Proposition 1.

Let $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k$ be the geodesic sides of the admissible ideal polygon Γ with $u(\alpha_i) = +\infty = \hat{u}(\alpha_i)$ and $u(\beta_i) = -\infty = \hat{u}(\beta_i)$. For each n, consider pairwise disjoint horocycles $H_i(n)$ at each vertex a_i of Γ such that the convex horodisk bounded by $H_i(n+1)$ is contained in the convex horodisk bounded by $H_i(n)$. For each side α_i, let us denote by α_i^n the compact arc of α_i that is the part of α_i outside the two horodisks and by $|\alpha_i^n|$ the length of α_i^n, that is, the distance between the two horodisks. Analogously, we define β_i^n for each side β_i. Denote by c_i^n the compact arc of $H_i(n)$ contained in the domain D and let $\mathcal{P}^n$ be the polygon formed by α_i^n, β_i^n, and c_i^n.

Since the function u is defined in the interior region bounded by $\mathcal{P}^n$ and $\mathcal{P}^n$ is a compact cycle, by the flux theorem it follows that $F_u(\mathcal{P}^n) = 0$. On the other hand, since $u \leq \hat{u}$, we have $F_u(\mathcal{P}^n) \leq F_{\hat{u}}(\mathcal{P}^n)$ and then $F_{\hat{u}}(\mathcal{P}^n) \geq 0$. Moreover, the flux of $\hat{u}$ across $\mathcal{P}^n$ satisfies

$$F_{\hat{u}}(\mathcal{P}^n) = \sum_i F_{\hat{u}}(\alpha_i^n) + \sum_i F_{\hat{u}}(\beta_i^n) + \sum_i F_{\hat{u}}(c_i^n)$$

$$\leq \sum_i (|\alpha_i^n| - |\beta_i^n|) + \sum_i |c_i^n|.$$

Notice that $|c_i^n| \to 0$ as $n \to \infty$, and, since Γ is an admissible polygon, we have $\sum_i |\alpha_i^n| = \sum_i |\beta_i^n|$ for any n. Hence, we conclude

$$F_{\hat{u}}(\mathcal{P}^n) \to 0 \quad \text{as } n \to \infty.$$

Then $F_u(\Gamma) = \lim_{n\to\infty} F_u(\mathcal{P}^n) = 0 = \lim_{n\to\infty} F_{\hat{u}}(\mathcal{P}^n) = F_{\hat{u}}(\Gamma)$.

On the other hand, since $\mathcal{P}^n$ is homotopic to Γ_1, it follows that $F_{\hat{u}}(\Gamma_1) = F_{\hat{u}}(\mathcal{P}^n)$ for any n, and we conclude that $F_{\hat{u}}(\Gamma_1) = 0$. Analogously (or using the flux theorem as we did for $\mathcal{P}^n$), $F_u(\Gamma_1) = 0$. Therefore, we have proved that the functions u and $\hat{u}$ have the same flux across the boundary $\partial \Omega = \Gamma_1 \cup \Gamma$.

Since $\Sigma = \text{Graph}(u)$ and $\hat{\Sigma} = \text{Graph}(\hat{u})$ are two minimal graphs over $\Omega = D \setminus \bar{D}_1$ such that $u \leq \hat{u}$ on Ω, $u = \hat{u}$ on $\partial \Omega$, and $F_u(\partial \Omega) = F_{\hat{u}}(\partial \Omega)$, we conclude, using Proposition 1, that $u \equiv \hat{u}$ over Ω, that is, $\hat{\Sigma}$ is the Scherk graph over Ω with $\partial \hat{\Sigma} = \partial_1$.

REMARK 1. We have proved that for any $t \in (0, \delta_0]$, we can get a subsequence of the minimal surfaces $\Sigma_{n,t}$ that converges to a minimal surface $\hat{\Sigma}$, which is the Scherk graph over $D \setminus \bar{D}_1$ with $\partial \hat{\Sigma} = \partial_1$.

Now we are able to prove the theorem.

Proof of Theorem 3. Since $\Sigma \cap S = \emptyset$, we can suppose that S is entirely under Σ. Pushing down Σ by vertical translations, we have two possibilities: either a translate of Σ touches S for the first time in an interior point, and then, by the maximum principle, they coincide; or S is asymptotic at infinity to a translate of Σ. Let us analyze this last case.

Without loss of generality, assume that S is asymptotic at infinity to Σ. We want to prove that, in fact, they coincide. Suppose that this is not true. Then since S is proper, there are a point $p_0 \in \Sigma$ and a cylinder $C = B_\Sigma(p_0, r_0) \times (-r_0, r_0)$ for some $r_0 > 0$ such that $S \cap C = \emptyset$, where $B_\Sigma(p_0, r_0)$ is the intrinsic ball centered at p_0 with radius r_0. We can assume that r_0 is less than the injectivity radius of Σ at p_0. In our construction of the surfaces $\Sigma_{n,t}$, we can choose the first domain of the exhaustion D_1 sufficiently small so that $\partial_1 \subset B_\Sigma(p_0, r_0/2)$ and take $t = \min\{r_0/2, \delta_0\}$.

Observe that when we translate $\Sigma_{n,t}$ vertically downward by an amount t, the boundaries of the translates of $\Sigma_{n,t}$ stay strictly above S. Thus, by the maximum principle, all the translates remain disjoint from S. We call $\Sigma'_{n,t}$ this final translate with boundary $\partial \Sigma'_{n,t} = \partial'_1 \cup \partial'_n$, where $T(t)(\partial'_1) = \partial_1 \subset \Sigma$ and $\partial'_n \subset \Sigma$. Hence, all

the surfaces $\Sigma'_{n,t}$ lie above S, and, as we proved before, there exists a subsequence of $\Sigma'_{n,t}$ that converges to the ideal Scherk graph Σ' defined over $D \setminus \bar{D}_1$ with $T(t)(\Sigma') = \Sigma$. In particular, we conclude that S lies below Σ', which yields a contradiction since we are assuming that S is asymptotic at infinity to Σ. $\qquad\square$

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