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The Simplicity of the First Spectral Radius of a Meromorphic Map

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Abstract. Let X be a compact Kähler manifold, and let $f : X \to X$ be a dominant rational map that is 1-stable (in the sense of Fornaess–Sibony). Let λ_1 and λ_2 be the first and second dynamical degrees of f. If $\lambda_1^2 > \lambda_2$, then we show that λ_1 is a simple eigenvalue of $f^* : H^{1,1}(X) \to H^{1,1}(X)$ and moreover the unique eigenvalue of modulus $> \sqrt{\lambda_2}$. A variant of the result, where we consider the first spectral radius in the case the map f may not be 1-stable, is also given. An application is stated for bimeromorphic selfmaps of 3-folds. For another application, we estimate the second dynamical degree of a class of birational maps that arises in lattice statistical mechanics and is related to matrix inverses.

1. Introduction

Let X be a compact Kähler manifold of dimension k with a Kähler form ω_X, and let $f : X \to X$ be a dominant meromorphic map. For $0 \le p \le k$, the pth dynamical degree $\lambda_p(f)$ of f is defined as follows:

$$\lambda_p(f) = \lim_{n\to\infty}\left(\int_X (f^n)^*(\omega_X^p) \wedge \omega_X^{k-p}\right)^{1/n} = \lim_{n\to\infty} r_p(f^n)^{1/n},$$

where $r_p(f^n)$ is the spectral radius of the linear map $(f^n)^* : H^{p,p}(X) \to H^{p,p}(X)$ (see Russakovskii and Shiffman [29] for the case where $X = \mathbb{P}^k$ and Dinh and Sibony [16; 15] for the general case). The dynamical degrees are log-concave; in particular, $\lambda_1(f)^2 \ge \lambda_2(f)$. In the case $f^* : H^{2,2}(X) \to H^{2,2}(X)$ preserves the cone of psef classes (i.e. those (2, 2) cohomology classes that can be represented by positive closed (2, 2) currents), we have an analog $r_1(f)^2 \ge r_2(f)$ (see Theorem 1.4).

The present paper concerns the first dynamical degree $\lambda_1(f)$ and more generally the first spectral radius $r_1(f)$. We will say that f is 1-stable if for any $n \in \mathbb{N}$, $(f^n)^* = (f^*)^n$ on $H^{1,1}(X)$ (the first use of this notion appeared in Fornaess and Sibony [18] in the case of rational selfmaps of projective spaces). When f is 1-stable, we have $\lambda_1(f) = r_1(f)$. There are many examples of 1-stable maps, for example, those that are pseudoautomorphisms.

The first main result of this paper is the following

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THEOREM 1.1. *Let X be a compact Kähler manifold of dimension k, and let $f : X \to X$ be a dominant meromorphic 1-stable map. Assume that $\lambda_1(f)^2 > \lambda_2(f)$. Then $\lambda_1(f)$ is a simple eigenvalue of $f^* : H^{1,1}(X) \to H^{1,1}(X)$. Further, $\lambda_1(f)$ is the only eigenvalue of modulus greater than $\sqrt{\lambda_2(f)}$.*

REMARK. The condition $\lambda_1(f)^2 > \lambda_2(f)$ in Theorem 1.1 is needed, as can be seen from automorphisms of complex torus. For more details, see the remark at the end of this paper.

Theorem 1.1 answers Question 3.3 in Guedj [20]. An immediate consequence of Theorem 1.1 is that if f is 1-stable and $\lambda_1(f)^2 > \lambda_2(f)$, then the "degree growth" of f satisfies $\deg(f^n) = c\lambda_1(f)^n + O(\tau^n)$ for some constants $c > 0$ and $\tau < \lambda_1(f)$. (Here, the degree of a map f with respect to a Kähler form ω is defined by $f^*(\omega).\omega^{k-1}$. In the case $X = \mathbb{P}^k$ and ω has the cohomology class of a hyperplane, this degree is the same as the usual "algebraic degree.") In the case X is a surface, the same estimate for the degree growth was obtained in Boucksom, Favre, and Jonsson [12], where the condition that f is 1-stable is not needed. The conclusion of Theorem 1.1 that $\lambda_1(f)$ is simple is very helpful in constructing Green currents and proving equidistribution properties toward it (see e.g. Guedj [20], Diller and Guedj [14], and Bayraktar [3]).

As a consequence, we obtain the following:

COROLLARY 1.2. *Let X be a compact Kähler manifold of dimension 3. Let $f : X \to X$ be a bimeromorphic map such that both f and f^{-1} are 1-stable. Assume moreover that $\lambda_1(f) > 1$. Then either f or f^{-1} satisfies the conclusions of Theorem 1.1.*

Proof. Observe that $\lambda_1(f^{-1}) = \lambda_2(f)$ and $\lambda_2(f^{-1}) = \lambda_1(f)$. Hence, when $\lambda_1(f) > 1$, at least one of the following conditions hold: $\lambda_1(f)^2 > \lambda_2(f)$ and $\lambda_1(f^{-1})^2 > \lambda_2(f^{-1})$. □

Corollary 1.2 can be applied to pseudoautomorphisms $f : X \to X$ of a 3-fold X with $\lambda_1(f) > 1$. By definition (see e.g. [17]), a bimeromorphic map $f : X \to X$ is pseudoautomorphic if there are subvarieties V, W of codimension at least 2 such that $f : X - V \to X - W$ is biholomorphic. If X has dimension 3, then any pseudoautomorphism $f : X \to X$ is both 1-stable and 2-stable (see Bedford and Kim [4]). The first examples of pseudoautomorphisms with first dynamical degree larger than 1 on blowups of $\mathbb{P}^3$ were given in [4] by studying linear fractional maps in dimension 3. There are now several other examples in any dimension (see e.g. Perroni and Zhang [27], Blanc [9], and Oguiso [26]).

Whereas the first dynamical degrees can be computed explicitly in various examples (e.g. by making a map 1-algebraic stable and computing the spectral radius of the action on $H^{1,1}$), it is much more difficult to compute the second dynamical degrees (for instance, unlike the case of 1-algebraic stability, currently there is no general criterion to check whether a map is 2-algebraic stable). In this aspect, Theorem 1.1 can be used to estimate the second dynamical degrees. We illustrate this here for a class of birational maps that arises in lattice statistical mechanics and is related to matrix inverses (see Boukraa and Maillard [11], Bellon and Viallet [8], Boukraa, Hassani, and Maillard [10], Auriac, Maillard, and Viallet [1; 2], Bedford and Kim [5; 6], Preissmann, Auriac, and Maillard [28], Bedford and Truong [7], and Truong [30]).

COROLLARY 1.3. *Let $q \geq 5$ be an integer. Let $\mathcal{M}_q$ be the space of $q \times q$ matrices with complex coefficients. Let $I : \mathcal{M}_q \to \mathcal{M}_q$ be the inverse map $I(x) = (x)^{-1}$ for $x \in \mathcal{M}_q$, and let $J : \mathcal{M}_q \to \mathcal{M}_q$ be the Hadamard inverse $J(x) = (1/x_{i,j})$ for $x = (x_{i,j}) \in \mathcal{M}_q$. The map $K = I \circ J$ defines a birational map $K : \mathbb{P}(\mathcal{M}_q) \to \mathbb{P}(\mathcal{M}_q)$, where we identify $\mathbb{P}(\mathcal{M}_q)$ with the projective space $\mathbb{P}^{q^2-1}$. Let δ_1 be the largest root of the equation $t^2 - (q^2 - 4q + 2)t + 1 = 0$, and let δ_2 be the largest root of the equation $t^2 - (q-2)t + 1 = 0$. Then*

$$\lambda_1(K) = \delta_1,$$
$$\delta_2^2 \leq \lambda_2(K) \leq \delta_1^2.$$

In particular, the growth of $\lambda_2(K)$, as a function of q, is of order at least q^2 and at most q^4.

Proof. By results in [7], there is a finite composition of blowups along smooth centers $\pi : X \to \mathbb{P}(\mathcal{M}_q)$ such that the lifting map $K_X = \pi^{-1} \circ K \circ \pi : X \to X$ is 1-algebraic stable. Moreover, the characteristic of the pullback $K_X^* : H^{1,1}(X) \to H^{1,1}(X)$ is

$$P(t)Q(t)(t-1)^{q^2-q+2}(t+1)^{q^2-3q+2},$$

where $P(t) = t^2 - (q^2 - 4q + 2)t + 1$ and $Q(t) = (t^2 - (q-2)t + 1)(t^2 + (q-2)t + 1)$. Hence, the spectral radius of $K_X^* : H^{1,1}(X) \to H^{1,1}(X)$ is δ_1, and hence $\lambda_1(K) = \lambda_1(K_X) = \delta_1$. We finish the proof by estimating $\lambda_2(K)$. First, by the log-concavity of $\lambda_1(K)$ we have $\delta_1^2 \geq \lambda_2(K)$. If $\lambda_1(K_X)^2 = \delta_1^2 > \lambda_2(K) = \lambda_2(K_X)$, then Theorem 1.1 applied to the map K_X implies that any other eigenvalue of $K_X^* : H^{1,1}(X) \to H^{1,1}(X)$ is $\leq \sqrt{\lambda_2(K_X)} = \sqrt{\lambda_2(K)}$. In particular, we have $\lambda_2(K) \geq \delta_2^2$. □

When X is a compact Kähler surface, Diller and Favre [13] proved a stronger conclusion than that of Theorem 1.1, where the condition of 1-stability is dropped. The following variant of Theorem 1.1 gives a generalization of Diller and Favre's result to higher dimensions. Recall that $r_1(f)$ is the spectral radius of $f^* : H^{1,1}(X) \to H^{1,1}(X)$ and $r_2(f)$ is the spectral radius of $f^* : H^{2,2}(X) \to H^{2,2}(X)$.

THEOREM 1.4. *Let X be a compact Kähler manifold, and let $f : X \to X$ be a dominant meromorphic map. Assume that $f^* : H^{2,2}(X) \to H^{2,2}(X)$ preserves the cone of psef classes. Then*

1) *We have $r_1(f)^2 \geq r_2(f)$.*

2) *Assume moreover that $r_1(f)^2 > r_2(f)$. Then $r_1(f)$ is a simple eigenvalue of $f^* : H^{1,1}(X) \to H^{1,1}(X)$. Further, $r_1(f)$ is the only eigenvalue of modulus greater than $\sqrt{r_2(f)}$.*

The key tools in the proofs of Theorems 1.1 and 1.4 are the Hodge index theorem (Hodge–Riemann bilinear relations), Hironaka's elimination of indeterminacies for meromorphic maps, and a pull–push formula for blowups along smooth centers. Section 2 is devoted to the proofs of Theorems 1.1 and 1.4.

All of the above results have analogues in the algebraic setting, where X is a projective manifold over an algebraic closed field of characteristic zero, and $f : X \to X$ is a rational map. This will be done in a forthcoming paper. Further applications of the results in the current paper are being explored in ongoing projects. These include pseudoautomorphisms of dimension at most 4, automorphisms of complex 3-tori, and dynamics over non-Archimedean fields.

2. Proofs of Theorems 1.1 and 1.4

Let X and Y be compact Kähler manifolds, and let $h : X \to Y$ be a dominant meromorphic map. By Hironaka's elimination of indeterminacies (see e.g. Corollary 1.76 in Kollár [24] and Theorem 7.21 in Harris [21] for the case of projective X, and see Hironaka [22] and Moishezon [25] for the general case), there is a compact Kähler manifold Z, a map $\pi : Z \to X$ which is a finite sequence of blowups along smooth centers, and a surjective holomorphic map $g : Z \to Y$ such that $h = g \circ \pi^{-1}$. (Since the analytic case of Hironaka's elimination of indeterminacies is less known, we give here a sketch of its proof, cf. the paper Ishii and Milman [23] for related ideas. We thank Pierre Milman for his generous help with this. Consider Γ a resolution of singularities of the graph Γ_h, and let $p, \gamma : \Gamma \to X, Y$ be the induced holomorphic maps. In particular, $p : \Gamma \to X$ is a modification. By global Hironaka's flattening theorem, we can find a finite sequence of blowups $\pi : X' \to X$ along smooth centers, and let $\pi_\Gamma : \Gamma' \to \Gamma$ be the corresponding blowup along the ideals that are pullbacks by p of the ideals of the centers of the blowup π, so that the induced map $p' : \Gamma' \to X'$ is still holomorphic, bimeromorphic, and flat. A priori, Γ' may be singular. But a holomorphic, bimeromorphic, and flat map must actually be a biholomorphic map. Therefore, Γ' is also smooth, p' is biholomorphic, and the holomorphic maps $\pi : Z = X' \to X$ and $g = \gamma \circ \pi_\Gamma \circ p'^{-1} : Z = X' \to Y$ are what we need.)

For our purpose here, it is important to study the blowups whose center is a smooth submanifold of codimension exactly 2. We consider first the case of a single blowup. We use the conventions that if W is a subvariety, then $[W]$ denotes the current of integration along W, and if T is a closed current, then $\{T\}$ denotes its cohomology class (for the case $T = [W]$ where W is a subvariety, we write $\{W\}$ instead of $\{[W]\}$ for convenience). For two cohomology classes u and v, we denote by $u.v$ the cup product.

We have the following pull–push formulas for a single blowup (a more precise version of this for birational surface maps was given in [13]).

LEMMA 2.1. *Let X be a compact Kähler manifold of dimension k. Let $\pi : Z \to X$ be a blowup of X along a smooth submanifold $W = \pi(E)$ of codimension exactly* 2. *Let E be the exceptional divisor, and let L be a general fiber of π over W.*

(i) *There is a constant $c_E > 0$ such that*

$$(\pi)_*(\{E\}.\{E\}) = -\{W\}.$$

(ii) *If α is a closed smooth* $(1,1)$ *form with complex coefficients on Z, then*

$$\pi^*(\pi)_*(\alpha) = \alpha + (\{\alpha\}.\{L\})[E].$$

(iii) *If α is a closed smooth* $(1,1)$ *form with complex coefficients on Z, then*

$$(\pi)_*(\alpha \wedge [E]) = (\{\alpha\}.\{L\})[W].$$

(iv) *If α is a closed smooth* $(1,1)$ *form with complex coefficients on Z, then*

$$(\pi)_*((\pi)^*(\pi)_*(\alpha) \wedge \overline{\alpha}) - (\pi)_*(\alpha \wedge \overline{\alpha}) = |\{\alpha\}.\{L\}|^2[W].$$

REMARKS. Parts (i), (iii), and (iv) of Lemma 2.1 are trivially true when the center of blowup $W = \pi_1(E)$ has codimension at least 3. For example, then in (i) we have $\pi_*(\{E\}.\{E\}) = 0$. In fact, by the same argument as in the subsequent proof of (i), the cohomology class $\pi_*(\{E\}.\{E\})$ can be represented by a difference of two positive closed $(2,2)$ currents supported in $W = \pi(E)$. Since W has codimension at least 3, it follows that $\pi_*(\{E\}.\{E\}) = 0$.

Proof of Lemma 2.1. (i) Let θ be a smooth closed $(1,1)$ form on Z representing the cohomology class of E. We can write $\theta = \theta^+ - \theta^{-1}$, where $\theta^\pm$ are Kähler forms. Let $\alpha = \theta^+ \wedge [E]$ and $\beta = \theta^- \wedge [E]$. Then α and β are positive closed $(2,2)$ currents with support in E and in cohomology $\{\alpha - \beta\} = \{E\}.\{E\}$. Therefore, $\pi_*(\{E\}.\{E\})$ can be represented by the difference $\pi_*(\alpha) - \pi_*(\beta)$ of two positive closed $(2,2)$ currents $\pi_*(\alpha)$ and $\pi_*(\beta)$. Each of the latter has support in $W = \pi(E)$; hence, since W has codimension exactly 2, each of them must be a multiple of the current of integration $[W]$ by the support theorem for normal currents. We infer

$$\pi_*(\{E\}.\{E\}) = -c_E\{W\}$$

for a constant c_E. It remains to show that $c_E = 1$. To this end, we let ω_X be a Kähler form on X. Then we get

$$\{E\}.\{E\}.\{\pi^*(\omega_X^{k-2})\} = (\pi)_*(\{E\}.\{E\}).\{\omega_X^{k-2}\} = -c_E\{W\}.\{\omega_X^{k-2}\}.$$

Since $\{W\}.\{\omega_X^{k-2}\} = \{[W] \wedge \omega_X^{k-2}\}$ is a positive number (equal the mass of W), to show that $c_E = 1$, it suffices to show that $\{E\}.\{E\}.\{\pi^*(\omega_X^{k-2})\} = -\{W\}.\{\omega_X^{k-2}\}$. If we can show that $\{E\}.\{\pi^*(\omega_X^{k-2})\} = a\{L\}$ for $a = \{W\}.\{\omega_X^{k-2}\}$, then $\{E\}.\{E\}.\{\pi^*(\omega_X^{k-2})\} = a\{E\}.\{L\} = -\{W\}.\{\omega_X^{k-2}\}$, as desired. To this end, we first observe that $\{E\}.\{\pi^*(\omega_X^{k-2})\} = a\{L\}$ for some constant a because $H^{k-1,k-1}(Z)$ is generated by $\pi^*H^{k-1,k-1}(X)$ and $\{L\}$, and by the projection formula $(\pi)_*(\{E\}.\{\pi^*(\omega_X^{k-2})\}) = (\pi)_*(\{E\}).\{\omega_X^{k-2})\} = 0$. It remains to show that the constant a is exactly $\{W\}.\{\omega_X^{k-2}\}$. In fact, if $\iota_E : E \to X$ and $\iota : W \to Y$ are the inclusion maps and $\pi_E : E \to W$ is the projection, then

$$\begin{aligned}\{E\}.\{\pi^*(\omega_X^{k-2})\} &= (\iota_E)_*(\iota_E^*\pi^*\{\omega_X^{k-2}\})\\ &= (\iota_E)_*(\pi_E^*(\omega_X^{k-2}|_W)).\end{aligned}$$

The cohomology class $\{\omega_X^{k-2}\}|_W$ is a positive multiple of the class of a point, the multiple constant being $\{W\}.\{\omega_X^{k-2}\}$. In fact, if $\iota_W : W \subset X$ is the inclusion of W in X, then the multiple constant is

$$(\iota_W)_*(\{\omega_X^{k-2}\}|_W) = (\iota_W)_*(\iota_W^*(\omega_X^{k-2})) = \{\omega_X^{k-2}\}.\{W\}.$$

Now, since the map π_E is a fibration, $\pi_E^*\{point\} = \{L\}$. Therefore, $\{E\}.\{\pi^*(\omega_X^{k-2})\} = a\{L\}$, where $a = \{\omega_X^{k-2}\}.\{W\} > 0$. Thus, the proof of part (i) is finished.

(ii) This is a standard result using $\{E\}.\{L\} = -1$ (see also the proof of (iii)).

(iii) Since $(\pi)_*(\alpha \wedge [E])$ is a normal $(2,2)$ current with support in $W = \pi(E)$, which is a subvariety of codimension 2 in X, by support theorem it follows that there is a constant c such that $(\pi)_*(\alpha \wedge [E]) = c[W]$. It is clear that c depends only on the cohomology class of $(\pi)_*(\alpha \wedge [E])$. Since $H^{1,1}(Z)$ is generated by $\pi^*(H^{1,1}(X))$ and $\{E\}$, we can write $\{\alpha\} = a\pi^*(\beta) + b\{E\}$, where $\beta \in H^{1,1}(X)$. Then using (i) and the projection formula, we obtain

$$\begin{aligned}(\pi)_*\{\alpha \wedge [E]\} &= (\pi)_*(\{\alpha\}.\{E\}) = b(\pi)_*(\{E\}.\{E\})\\ &= -bc_E\{\pi(E)\}.\end{aligned}$$

Therefore, $c = -bc_E$. The constant $-b$ can be computed as follows:

$$\{\alpha\}.\{L\} = (a\pi^*(\beta) + b\{E\}).\{L\} = b\{E\}.\{L\} = -b.$$

Hence, $c = (\{\alpha\}.\{L\})c_E$, as claimed.

(iv) We have

$$\begin{aligned}(\pi)_*(\pi^*(\pi)_*(\alpha) \wedge \overline{\alpha}) &= (\pi)_*((\alpha + (\{\alpha\}.\{L\})[E]) \wedge \overline{\alpha})\\ &= (\pi)_*(\alpha \wedge \overline{\alpha}) + (\{\alpha\}.\{L\})(\pi)_*([E] \wedge \overline{\alpha})\\ &= (\pi)_*(\alpha \wedge \overline{\alpha}) + c_E|\{\alpha\}.\{L\}|^2[\pi(E)].\end{aligned}$$

Thus, (iv) is proved. □

In particular, Lemma 2.1 shows that for a single blowup $\pi : Z \to X$, if α is a closed smooth $(1,1)$ form with complex coefficients, then $(\pi)_*((\pi)^*(\pi)_*(\alpha) \wedge \overline{\alpha}) - (\pi)_*(\alpha \wedge \overline{\alpha})$ is a positive closed $(2,2)$ current. (If the center of blowup W has codimension exactly 2, then this follows from Lemma 2.1 (iv), whereas if W has codimension at least 3, then $(\pi)_*((\pi)^*(\pi)_*(\alpha) \wedge \overline{\alpha}) - (\pi)_*(\alpha \wedge \overline{\alpha}) = 0$, as observed in the remarks after the statement of Lemma 2.1.) It follows that if $u \in H^{1,1}(Z)$ is a cohomology class with complex coefficients, then $\pi_*(u).\pi_*(\overline{u}) - \pi_*(u.\overline{u})$ is a psef class, that is, can be represented by a positive closed $(2,2)$ current. In fact, let α be a closed smooth $(1,1)$ form representing u. Then, $(\pi)_*(u.\overline{u})$ is represented by $(\pi)_*(\alpha \wedge \overline{\alpha})$, and by the projection formula,

$(\pi)_*(u).(\pi)_*(\overline{u})$ is represented by $(\pi)_*(\pi^*(\pi)_*(\alpha)\wedge\overline{\alpha})$. Hence, from (iv) we infer that $\pi_*(u).\pi_*(\overline{u})-\pi_*(u.\overline{u})$ is psef, as claimed. We now give a generalization of this to the case of a finite blowup and to meromorphic maps.

PROPOSITION 2.2. 1) *Let X be a compact Kähler manifold, and let $\pi: Z\to X$ be a finite composition of blowups along smooth centers. Further, let $u\in H^{1,1}(Z)$ be a $(1,1)$ cohomology class with complex coefficients. Then $(\pi)_*(u).(\pi)_*(\overline{u})-(\pi)_*(u.\overline{u})$ is a psef class.*

2) *Let X and Y be compact Kähler manifolds, and let $h: X\to Y$ be a dominant meromorphic map. Further, let $v\in H^{1,1}(Y)$ be a cohomology class with complex coefficients on Y. Then $h^*(v).h^*(\overline{v})-h^*(v.\overline{v})$ is a psef class in $H^{2,2}(X)$.*

Proof. 1) We prove by induction on the number of single blowups performed. If π is a single blowup, then this follows from the above observation. Now assume that 1) is true when the number of single blowups performed is $\leq n$. We prove that 1) is true also when the number of single blowups performed is $\leq n+1$. We can decompose $\pi=\pi_1\circ\pi_2: Z\to Y\to X$, where $\pi_2: Z\to Y$ is a single blowup, and $\pi_1: Y\to X$ is a composition of n single blowups. Applying the inductional assumption to π_1 and the cohomology class $(\pi_2)_*(u)$, we get

$$\pi_*(u).\pi_*(\overline{u})=(\pi_1)_*((\pi_2)_*(u)).(\pi_1)_*((\pi_2)_*(\overline{u}))\geq(\pi_1)_*((\pi_2)_*(u).(\pi_2)_*(\overline{u})).$$

Here $\geq$ means that the difference of the two currents is psef. Now using the result for the single blowup π_2 and the fact that push-forward by the holomorphic map π_1 preserves psef classes, we have

$$(\pi_1)_*((\pi_2)_*(u).(\pi_2)_*(\overline{u}))\geq(\pi_1)_*(\pi_2)_*(u.\overline{u})=\pi_*(u.\overline{u}).$$

Hence, $\pi_*(u).\pi_*(\overline{u})\geq\pi_*(u.\overline{u})$, as desired.

2) By Hironaka's elimination of indeterminacies (see Hironaka [22], Moishezon [25], and also the beginning of this section), we can find a compact Kähler manifold Z, a finite blowup along smooth centers $\pi: Z\to X$, and a surjective holomorphic map $g: Z\to Y$ such that $h=g\circ\pi^{-1}$. Here Z is a desingularization of the graph of h and hence has the same dimension as that of X. We will apply part 1) for the map $\pi: Z\to X$.

By definition, $h^*(v)=\pi_*g^*(v)$ and $h^*(v.\overline{v})=\pi_*(g^*(v.\overline{v}))=\pi_*(g^*(v).g^*(\overline{v}))$ (to see these equalities, we choose a smooth closed $(1,1)$ form α representing v and see immediately the equalities on the level of currents). Therefore, applying 1) to the blowup $\pi: Z\to X$ and to the $(1,1)$ cohomology class $u=g^*(v)$ on Z, we obtain

$$h^*(v).h^*(\overline{v})-h^*(v.\overline{v})=\pi_*(g^*(v)).\pi_*(\overline{g^*(v)})-\pi_*(g^*(v).\overline{g^*(v)})\geq 0.\qquad\square$$

For the proofs of Theorems 1.1 and 1.4, we need to use the famous Hodge index theorem (Hodge–Riemann bilinear relations; see e.g. the last part of Chapter 0 in Griffiths and Harris [19]). Let X be a compact Kähler manifold of dimension k. Let $w\in H^{1,1}(X)$ be the cohomology class of a Kähler form on X. We define the

Hermitian quadratic form that for cohomology classes with complex coefficients $u, v \in H^{1,1}(X)$ takes the value

$$\mathcal{H}(u, v) = u.\overline{v}.w^{k-2}.$$

The Hodge index theorem says that the signature of $\mathcal{H}$ is $(1, h^{1,1} - 1)$, where $h^{1,1}$ is the dimension of $H^{1,1}(X)$.

We are now ready for the proofs of Theorems 1.1 and 1.4.

Proof of Theorem 1.1. First, we show that there cannot be two noncollinear vectors $u_1, u_2 \in H^{1,1}(X)$ for which $f^*u_1 = \tau_1 u_1$ and $f^*u_2 = \tau_2 u_2$, where $\tau = \min\{|\tau_1|, |\tau_2|\} > \sqrt{\lambda_2(f)}$. Assuming otherwise, we will show that for any u in the complex vector space of dimension 2 generated by u_1 and u_2, $\mathcal{H}(u, u) \geq 0$, and this gives a contradiction to the Hodge index theorem. To this end, it suffices to show that $u.\overline{u}$ is psef. Let $u = a_1 u_1 + a_2 u_2$. For $n \in \mathbb{N}$, we define

$$v_n = \frac{a_1}{\tau_1^n} u_1 + \frac{a_2}{\tau_2^n} u_2.$$

Then it is easy to check that $(f^*)^n(v_n) = u$. Because f is 1-stable, we have from Proposition 2.2 that

$$u.\overline{u} = (f^*)^n(v_n).(f^*)^n(\overline{v_n}) = (f^n)^*(v_n).(f^n)^*(\overline{v_n}) \geq (f^n)^*(v_n.\overline{v_n})$$

for any $n \in \mathbb{N}$. (Here the inequality $\geq$ means that the difference of the two cohomology classes is psef.) We fix an arbitrary norm $\|\cdot\|$ on the vector space $H^{1,1}(X)$. Then $\|v_n\|$ is bounded by $1/\tau^n$; hence, the assumption that $\tau > \sqrt{\lambda_2(f)}$ implies that $(f^n)^*(v_n.\overline{v_n})$ converges to 0. Therefore, $u.\overline{u} \geq 0$, as desired.

Hence $\lambda_1(f)$ is the unique eigenvalue of modulus $> \sqrt{\lambda_2(f)}$ of $f^* : H^{1,1}(X) \to H^{1,1}(X)$. It remains to show that $\lambda_1(f)$ is a simple root of the characteristic polynomial of $f^* : H^{1,1}(X) \to H^{1,1}(X)$. Assuming otherwise, by using the Jordan normal form of a matrix we get that there will be two noncollinear vectors $u_1, u_2 \in H^{1,1}(X)$ for which $f^*(u_1) = \lambda_1(f) u_1$ and $f^*(u_2) = \lambda_1(f) u_2 + u_1$. Let $u = a_1 u_1 + a_2 u_2$. For any $n \in \mathbb{N}$, we define

$$v_n = \frac{a_1}{\lambda_1(f)^n} u_1 - \frac{n a_2}{\lambda_1(f)^{n+1}} u_1 + \frac{a_2}{\lambda_1(f)^n} u_2.$$

Then it is easy to check that $(f^*)^n(v_n) = u$, and we can proceed as in the first part of the proof. $\square$

Proof of Theorem 1.4. 1) First, we observe that for any $v \in H^{1,1}(X)$ with complex coefficients, $(f^*)^n(v).(f^*)^n(\overline{v}) \geq (f^*)^n(v.\overline{v})$ for all $n \in \mathbb{N}$. For example, we show how to do this for $n = 2$. Applying Proposition 2.2, we have

$$(f^*)^2(v).(f^*)^2(\overline{v}) = f^*(f^*(v)).f^*(\overline{f^*(v)}) \geq f^*(f^*(v).f^*(\overline{v})).$$

By Proposition 2.2 again and the assumption that $f^* : H^{2,2}(X) \to H^{2,2}(X)$ preserves psef classes, we obtain

$$f^*(f^*(v).f^*(\overline{v})) \geq (f^*)^2(v.\overline{v}),$$

and hence $(f^*)^2(v).(f^*)^2(\overline{v}) \geq (f^*)^2(v.\overline{v})$, as desired.

We now finish the proof of 1). Let ω_X be a Kähler form on X. Then from the first part of the proof we get

$$(f^*)^n(\omega_X).(f^*)^n(\omega_X) \geq (f^*)^n(\omega_X^2)$$

for all $n \in \mathbb{N}$. For convenience, we let $\|\cdot\|$ denote an arbitrary norm on either $H^{1,1}(X)$ or $H^{2,2}(X)$. There is a constant $C > 0$, independent of n, such that for all $n \in \mathbb{N}$, we have

$$\|(f^*)^n(\omega_X).(f^*)^n(\omega_X)\| \leq C\|(f^*)^n(\omega_X)\|^2 \leq C\|(f^*)^n|_{H^{1,1}(X)}\|^2$$

and

$$C(f^*)^n(\omega_X^2) \geq \|(f^*)^n|_{H^{2,2}(X)}\|.$$

(In the second inequality we used the assumption that $f^* : H^{2,2}(X) \to H^{2,2}(X)$ preserves the cone of psef classes.)

Therefore,

$$C^2\|(f^*)^n|_{H^{1,1}(X)}\|^2 \geq \|(f^*)^n|_{H^{2,2}(X)}\|$$

for any $n \in \mathbb{N}$. Taking the nth root and letting $n \to \infty$, we obtain $r_1(f)^2 \geq r_2(f)$.

2) Using the ideas from the proofs of Theorem 1.1 and 1), we obtain 2) immediately. □

Remark. Here we give the final remark. The condition $\lambda_1(f)^2 > \lambda_2(f)$ is necessary in Theorem 1.1. In fact, let $\zeta > 1$ be a quadratic algebraic number that is a root of $t^2 + at + 1 = 0$, where $a \in \mathbb{Z}$. Let A be a matrix in $\mathrm{GL}(2;\mathbb{Z})$ whose characteristic polynomial is $t^2 + at + 1$. Then A gives rise to an automorphism of a complex 2-torus T^2, and the product map $f = (A, A)$ gives rise to an automorphism of the 4-torus $T^2 \times T^2$. Then $\lambda_1(f) = \zeta^2$ is not a simple root of the characteristic polynomial of f. In this case, $\lambda_2(f) = \zeta^4 = \lambda_1(f)^2$.

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References

[1] J. C. Angles d'Auriac, J. M. Maillard, and C. M. Viallet, *A classification of four-state spin edge Potts models,* J. Phys. A 35 (2002), 9251–9272.

[2] ———, *On the complexity of some birational transformations,* J. Phys. A, Math. Gen. 39 (2006), 3641–3654.

[3] T. Bayraktar, *Green currents for meromorphic maps of compact Kähler manifolds,* J. Geom. Anal. 23 (2013), 970–998.
[4] E. Bedford and K.-H. Kim, *Pseudo-automorphisms of 3-space: periodicities and positive entropy in linear fractional recurrences,* arXiv:1101.1614.
[5] ———, *On the degree growth of birational mappings in higher dimension,* J. Geom. Anal. 14 (2004), 567–596.
[6] ———, *Degree growth of matrix inversion: birational maps of symmetric, cyclic matrices,* Discrete Contin. Dyn. Syst. 21 (2008), no. 4, 977–1013.
[7] E. Bedford and T. T. Truong, *Degree complexity of birational maps related to matrix inversion,* Commun. Math. Phys. 298 (2010), no. 2, 357–368.
[8] M. Bellon and C. M. Viallet, *Algebraic entropy,* Commun. Math. Phys. 204 (1999), 425–437.
[9] J. Blanc, *Dynamical degrees of (pseudo)-automorphisms fixing cubic hypersurfaces,* Indiana Univ. J. Math. (to appear), arXiv:1204.4256.
[10] S. Boukraa, S. Hassani, and J. M. Maillard, *Noetherian mappings,* Physica D 185 (2003), no. 1, 3–44.
[11] S. Boukraa and J. M. Maillard, *Factorization properties of birational mappings,* Physica A 220 (1995), 403–470.
[12] S. Boucksom, C. Favre, and M. Jonsson, *Degree growth of meromorphic surface maps,* Duke Math. J. 141 (2008), no. 3, 519–538.
[13] J. Diller and C. Favre, *Dynamics of bimeromorphic maps of surfaces,* Am. J. Math. 123 (2001), no. 6, 1135–1169.
[14] J. Diller and V. Guedj, *Regularity of dynamical Green's functions,* Trans. Am. Math. Soc. 361 (2009), no. 9, 4783–4805.
[15] T.-C. Dinh and N. Sibony, *Regularization of currents and entropy,* Ann. Sci. Éc. Norm. Super. 37 (2004), no. 6, 959–971.
[16] ———, *Une borne supérieure de l'entropie topologique d'une application rationnelle,* Ann. Math. 161 (2005), 1637–1644.
[17] I. Dolgachev and D. Ortland, *Point sets in projective spaces and theta functions,* Astérisque 165 (1988).
[18] J. E. Fornaess and N. Sibony, *Complex dynamics in higher dimensions,* Complex potential theory (Montreal, PQ, 1993), Notes partially written by Estela A. Gavosto, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 459, pp. 131–186, Kluwer Acad. Publ., Dordrecht, 1994.
[19] P. Griffiths and J. Harris, *Principles of algebraic geometry,* John Wiley and Sons, Inc., New York, 1978.
[20] V. Guedj, *Decay of volumes under iteration of meromorphic mappings,* Ann. Inst. Fourier (Grenoble) 54 (2004), no. 7, 2369–2386.
[21] J. Harris, *Algebraic geometry: a first course,* Springer-Verlag, New York, 1992.
[22] H. Hironaka, *Flattening of analytic maps,* Manifolds–Tokyo 1973 (Proc. international conf., Tokyo 1973), pp. 313–321, Univ. Tokyo Press, Tokyo, 1975.
[23] S. Ishii and P. Milman, *The geometric minimal models of analytic spaces,* Math. Ann. 323 (2002), no. 3, 437–451.
[24] J. Kollár, *Lectures on resolutions of singularities,* Annals of Mathematics Studies, Princeton University Press, Princeton, 2007.
[25] B. Moishezon, *Modifications of complex varieties and the Chow lemma,* Classification of algebraic varieties and compact complex manifolds, Lecture Notes in Mathematics, 412, pp. 133–139, Springer-Verlag, Heidelberg, 1974.
[26] K. Oguiso, *Automorphism groups of Calabi–Yau manifolds of Picard number two,* arXiv:1206.1649.

[27] F. Perroni and D.-Q. Zhang, *Pseudo-automorphisms of positive entropy on the blowups of products of projective spaces,* arXiv:1111.3546.
[28] E. Preissmann, J. C. Angles d'Auriac, and J. M. Maillard, *Birational mappings and matrix sub-algebra from the Chiral–Potts model,* J. Math. Phys. 50 (2009), no. 1, 013302, 26 pages
[29] A. Russakovskii and B. Shiffman, *Value distributions for sequences of rational mappings and complex dynamics,* Indiana Univ. Math. J. 46 (1997), 897–932.
[30] T. T. Truong, *Degree complexities of birational maps related to matrix inversions: symmetric case,* Math. Z. 270 (2012), no. 3-4, 725–738.

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