Smoothly Slice Boundary Links Whose Derivative Links Have Nonvanishing Milnor Invariants

HYE JIN JANG, MIN HOON KIM, & MARK POWELL

ABSTRACT. We give an example of a 3-component smoothly slice boundary link, each of whose components has a genus one Seifert surface, such that any metaboliser of the boundary link Seifert form is represented by three curves on the Seifert surfaces that form a link with nonvanishing Milnor triple linking number. We also give a generalization to m-component links and higher Milnor invariants. We prove that our examples are ribbon and that all ribbon links are boundary slice.

1. Introduction

The topological four-dimensional surgery conjecture for free groups states that the surgery sequence discussed in [FQ90, Section 11.3] is exact when the fundamental group is free. A key test case is the question of whether the Whitehead double of the Borromean rings is a topologically slice link [Fre84; CF84; Fre93; Kru08]. One strategy to slice a boundary link $L = L_1 \sqcup \cdots \sqcup L_m$ (a boundary link is a link whose components bound disjoint Seifert surfaces in S^3) is to push these Seifert surfaces $F = F_1 \sqcup \cdots \sqcup F_m$, $\partial F_i = L_i$, into the 4-ball B^4 and then to ambiently surger the Seifert surfaces to discs by finding a set of curves generating a half-rank submodule of $H_1(F_i; \mathbb{Z})$ for each $i = 1, \ldots, m$ and finding framed discs, pairwise disjoint, embedded in $B^4 \smallsetminus F$, and with boundary these curves.

In order for such framed discs to exist, such a set of curves must be a *metaboliser* for the boundary link Seifert form (see Definitions 2.4 and 2.5). Following [CHL10], we consider simple closed curves on the Seifert surfaces representing a metaboliser M as a link in S^3, also denoted by M, and call this the *derivative* of L with respect to the metaboliser.

If a derivative is itself a slice link, then the programme works, and the original link is slice. On the other hand, if we have a boundary link with Seifert surfaces and we know that all of the metabolisers are not slice links, we can wonder if this implies that the link is not slice. In the famous case of the Whitehead double of the Borromean rings, with their obvious Seifert surfaces, all derivatives are the Borromean rings, which are well known not to be slice; for example, they have a nonzero Milnor triple linking number $\overline{\mu}(123)$. Also, the Whitehead double of the

Borromean rings is known not to be smoothly slice, at least with the right choices of clasping sign [Lev12]. Indeed, with all clasping negative, according to [CT13], it is not even 0-positive in the Cochran–Harvey–Horn filtration [CHH13].

Nevertheless, in the topological category, having derivatives with nonzero Milnor invariants is known to be insufficient to deduce that a link is not slice. One can take the Whitehead double of the Whitehead link. As proven in [Fre88], this is a topologically slice link, and the metabolisers on the obvious Seifert surfaces form a Whitehead link. The Whitehead link has a nonvanishing Milnor invariant $\overline{\mu}(1122)$ and so is not slice.

The main aim of this paper is to produce a smoothly slice example and to use the Borromean rings instead of the Whitehead link.

Our example shows that it is unlikely that it is possible to use derivatives with nontrivial Milnor invariant $\overline{\mu}(123)$ to give an obstruction to a link being smoothly slice. Note that any obstruction to topological slicing is also an obstruction to smooth slicing.

THEOREM 1.1. *There exists a smoothly slice (in fact ribbon) 3-component boundary link $S = S_1 \sqcup S_2 \sqcup S_3$ with Seifert surfaces $F = F_1 \sqcup F_2 \sqcup F_3$, each of which is genus one and has an unknotted metabolising curve, such that for every metaboliser M, for the boundary link Seifert form, and for any three simple closed curves $M_1 \sqcup M_2 \sqcup M_3$ representing M, where $M_i \subset F_i$, we have $\overline{\mu}_M(123) \neq 0$.*

Moreover, for $i = 1, 2, 3$, we have that F_i is a unique minimal genus Seifert surface for S_i.

It is natural to wonder whether our links are boundary slice. In the appendix we show (Theorem A.3) the apparently well-known result that ribbon boundary links are boundary ribbon (Definition A.2) and therefore boundary slice (Definition 2.3).

We were inspired by the recent work of Cochran and Davis [CD13]. They found counterexamples to the thirty-year-old Kauffman conjecture, which was that any genus one slice knot has a genus one Seifert surface on which there is a metabolising curve J that has $\mathrm{Arf}(J) = 0$. The question has also been asked with the signature $\sigma(J)$ replacing $\mathrm{Arf}(J)$; see also [GL13].

THEOREM 1.2 (Cochran–Davis). *There exists a smoothly slice knot K with a unique genus one Seifert surface F such that for every metaboliser M for Seifert form and for any simple closed curve J representing M, where $J \subset F$, we have $\mathrm{Arf}(J) \neq 0$ and $\sigma(J) \neq 0$.*

Crucially, their examples were found in a conceptual fashion, enabling their techniques to be applied to our situation.

Until the work of Cochran and Davis appeared, there was plenty of evidence for this conjecture to be true, making their discovery all the more interesting. The following archetypal theorem of [CHL10] is one of the strongest known; it follows previous similar theorems of D. Cooper (from his thesis but unpublished), Gilmer [Gil83; Gil93], and Cochran, Orr, and Teichner [COT04].

THEOREM 1.3 (Cochran–Harvey–Leidy [CHL10]). *If K is a genus one slice knot, then on any genus one Seifert surface there exists a homologically essential simple closed curve of self-linking zero that has a vanishing zeroth-order signature and a vanishing first-order signature.*

We refer to [CHL10] for the definitions of the zeroth- and first-order signatures of a knot.

We can generalize Theorem 1.1 to replace the triple linking number with many other Milnor invariants. In the following theorem we let I be a multi-index for which there exists a link with nonvanishing Milnor invariant $\overline{\mu}(I)$. Let m be the number of distinct integers in I.

THEOREM 1.4. *Let I be a multi-index as above for which $|I|$ is odd. Then there exists a smoothly slice (in fact ribbon) m-component boundary link $S = S_1 \sqcup \cdots \sqcup S_m$ with Seifert surfaces $F = F_1 \sqcup \cdots \sqcup F_m$, each of which is genus one and has an unknotted metabolising curve, such that for every metaboliser M for the boundary link Seifert form and for any m simple closed curves $M_1 \sqcup \cdots \sqcup M_m$ representing M, where $M_i \subset F_i$, we have $\overline{\mu}_M(I) \neq 0$.*

Moreover, for $i = 1, 2, 3$, we have that F_i is a unique minimal genus Seifert surface for S_i.

REMARK 1.5. We have the following questions.

(1) Any two genus one Seifert surfaces for each component of our example in Theorem 1.1 are isotopic. Is the collection F of the three boundary link Seifert surfaces together the unique minimal genus collection?
(2) Do there exist examples for multi-indices of even length? For example, is there a smoothly slice 2-component link with genus one Seifert surfaces whose derivatives all have nonvanishing $\overline{\mu}(1122)$?
(3) Does there exist an example with unknotted components?
(4) Does there exist a slice knot, with a minimal genus Seifert surface, all of whose derivatives have nonvanishing Milnor invariants?

Organization of the Paper

Section 2 recalls some relevant definitions. Section 3 generalizes a result of [CD13] to the case of infection by string links, which shows that a certain operation on a slice link yields another slice link. Section 4 proves a formula detailing how the Milnor invariants change under infection by a string link. Section 5 gives the construction of our example and therefore the proof of Theorem 1.1, with the exception of the parenthetically claimed fact that the constructed link is ribbon. This fact is shown in Section 6. The proof of Theorem 1.4 is given in Section 7. Finally, the appendix contains a proof of the theorem that ribbon boundary links are boundary ribbon.

2. Definitions

Definition 2.1 (boundary link). An m-component oriented link $L = L_1 \sqcup \cdots \sqcup L_m \subset S^3$ is said to be a *boundary link* if there is a smooth embedding of m pairwise disjoint Seifert surfaces $F = F_1 \sqcup \cdots \sqcup F_m \subset S^3$ such that $\partial F_i = L_i$ for $i = 1, \ldots, m$.

Definition 2.2 (slice link). An m-component oriented link $L = L_1 \sqcup \cdots \sqcup L_m \subset S^3$ is *smoothly slice* if there is a smooth embedding of m pairwise disjoint discs $D_1 \sqcup \cdots \sqcup D_m \subset B^4$ such that $\partial D_i = L_i \subset S^3$ for $i = 1, \ldots, m$.

Definition 2.3 (boundary slice link). A pair (L, F) consisting of an m-component oriented smoothly slice boundary link L as in Definition 2.2, and a collection of Seifert surfaces F for L as in Definition 2.1 is a *boundary slice pair* if there is a smooth embedding of pairwise disjoint 3-manifolds $N = N_1 \sqcup \cdots \sqcup N_m \subset B^4$ such that $\partial N_i = D_i \cup_{L_i} F_i$.

A smoothly slice boundary link L is said to be *boundary slice* if there exists a collection of Seifert surfaces F such that (L, F) forms a boundary slice pair.

It is unknown whether all slice boundary links are boundary slice. The next definition follows [Ko87; Fri06].

Definition 2.4 (boundary link Seifert form). Let $g_1, \ldots, g_m$ be nonnegative integers. A (boundary link) Seifert matrix is a matrix A with entries $\{A_{ij}\}_{i,j=1,\ldots,m}$ where each entry A_{ij} is a $2g_i \times 2g_j$ matrix, $A_{ij} = A_{ji}^T$ for $i \neq j$, and $\det(A_{ii} - A_{ii}^T) = 1$.

Such a matrix arises geometrically as follows. Let $F = F_1 \sqcup \cdots \sqcup F_m$ be oriented boundary link Seifert surfaces for a boundary link L, and let g_i be the genus of F_i. Each Seifert surface F_i has a positive side inherited from its orientation. Given a curve $\gamma \subset F_i$, let γ^+ be the curve obtained by pushing γ off F_i in the positive normal direction. Choose oriented curves $\gamma_1^i, \ldots, \gamma_{2g_i}^i$ on F_i that form a basis of $H_1(F_i; \mathbb{Z})$. A boundary link Seifert matrix is then defined by

$$(A_{ij})_{k\ell} = \mathrm{lk}(\gamma_k^i, (\gamma_\ell^j)^+).$$

The boundary link Seifert form is the form

$$V : \bigoplus_{i=1}^m H_1(F_i; \mathbb{Z}) \times \bigoplus_{i=1}^m H_1(F_i; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

represented by this matrix. Note that any change of basis must respect the direct sum decomposition.

The next definition follows [Ko87; CHL10].

Definition 2.5 (metaboliser and derivative). (1) A *metaboliser* M of the boundary link Seifert form is a half-rank submodule of each $H_1(F_i; \mathbb{Z})$ with a basis represented by curves $\beta_1^1, \ldots, \beta_{g_1}^1, \beta_1^2, \ldots, \beta_{g_2}^2, \ldots, \beta_1^m, \ldots, \beta_{g_m}^m$, where
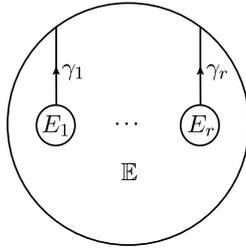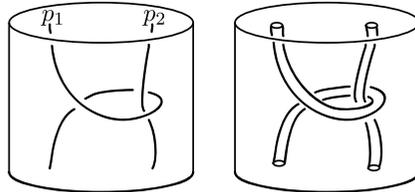
Figure 1 An r-multidisc

Figure 2 A 2-component string link and its exterior

$\beta_j^i \subset F_i$, satisfying that $V(\beta_k^i, \beta_\ell^j) = 0$ for $i, j = 1, \ldots, m$, $k = 1, \ldots, g_i$, $\ell = 1, \ldots, g_j$.

(2) If the boundary link Seifert form of a boundary link has a metaboliser, then that boundary link is said to be *algebraically slice*.

(3) Considering the β curves as a link in S^3 in their own right, we refer to this link as a *derivative* link of L, with respect to the Seifert surfaces F and the metaboliser M, and denote this link by $M_1 \sqcup \cdots \sqcup M_{\sum_{i=1}^m g_i}$.

The next definition follows [HL90].

DEFINITION 2.6 (string link). (1) An *r-multidisc* $\mathbb{E}$ is an oriented disc D^2 with an ordered collection of r embeddings of open discs $E_1, \ldots, E_r \hookrightarrow \mathbb{E}$ (see Figure 1). We abuse notation and identify E_i with its image in $\mathbb{E}$. For $i = 1, \ldots, r$, we choose a path $\gamma_i : I \to \mathbb{E} \smallsetminus (E_1 \sqcup \cdots \sqcup E_r)$ such that $\gamma_i(0) \in \partial E_i$ and $\gamma_i(1) \in \partial \mathbb{E}$ and such that the images of the γ_i are pairwise disjoint.

(2) Let p_i be a point in E_i for each $i = 1, \ldots, r$. An *r-component string link J* is a smooth, proper embedding

$$J : \bigsqcup_{i=1}^{r} p_i \times I \longrightarrow D^2 \times I$$

such that for $t = 0, 1$, we have $J(p_i \times t) = p_i \times t \in D^2 \times I$ for $i = 1, \ldots, r$ (e.g., see Figure 2). Since J is smooth, there is an embedding (called a *tubular*

neighbourhood of J)

$$\nu(J) : \bigsqcup_{i=1}^{r} E_i \times I \longrightarrow D^2 \times I$$

such that $\nu(J)$ coincides with J on $p_i \times I$, and for $t = 0, 1$, we have $\nu(J)|_{E_i \times t} = \mathrm{id}_{E_i \times t}$ for $i = 1, \ldots, r$. Abusing notation, we will often write the image of J (resp. the image of $\nu(J)$) as J (resp. $\nu(J)$).

(3) For $i = 1, \ldots, r$, choose a 0-framed parallel $\delta_i : I \to \partial E_i \times I$ of $J(p_i \times I)$ such that $\delta_i(t) = \gamma_i(0) \times \{t\} \in \mathbb{E} \times I$ for $t = 0, 1$. The *ith longitude* ℓ_i of J is defined as the following concatenation of arcs:

$$\ell_i = \delta_i \cup (\gamma_i \times \{1\}) \cup (\gamma_i(1) \times I) \cup (-\gamma_i \times \{0\}).$$

(4) Let $f : D^2 \times I \to D^2 \times I$ be the orientation-reversing diffeomorphism given by $f(x, y, t) = (x, y, 1 - t)$ for $(x, y) \in D^2$, $t \in I$. For a string link J, we define its inverse $-J$ by the composition $f \circ J$:

$$-J : \bigsqcup_{i=1}^{r} p_i \times I \xrightarrow{J} D^2 \times I \xrightarrow{f} D^2 \times I.$$

The string link $-J$ is easily seen to be the concordance inverse of J.

REMARK 2.7. If $X(J) = (D^2 \times I) \smallsetminus \nu(J)$ is the exterior of the string link J with standard orientation from $D^2 \times I$, then f restricts to an orientation-preserving diffeomorphism between $-X(J)$ and $X(-J)$.

The following definition follows [CFT09]; a similar earlier construction also appeared in [CO94].

DEFINITION 2.8 (infection by a string link). Let L be a link in S^3, and J be an r-component string link in $D^2 \times I$.

(1) An embedding $\phi : E \to S^3$ is a *proper r-multidisc* in (S^3, L) if $\phi(\mathbb{E})$ intersects L transversely and only inside $E_1, \ldots, E_r$. We also denote by ϕ the embedding of a thickening $\phi : \mathbb{E} \times I \to S^3$.

(2) Let $\mathbb{E}_\phi$ be the image of ϕ, and let $E_\phi := \phi(E_1 \sqcup \cdots \sqcup E_r)$. Note that $(\mathbb{E}_\phi \smallsetminus E_\phi) \times I$ is homeomorphic to the exterior of the trivial r-component string link.

(3) Let $S(L, J, \phi)$ be the link that is the image of L under the following homeomorphism:

$$(S^3 \smallsetminus (\mathrm{int}(\mathbb{E}_\phi \smallsetminus E_\phi) \times I)) \cup ((D^2 \times I) \smallsetminus \nu(J))$$
$$= (S^3 \smallsetminus (\mathbb{E}_\phi \times I)) \cup (((D^2 \times I) \smallsetminus \nu(J)) \cup (\overline{E_\phi} \times I))$$
$$\cong D^3 \cup D^3 \cong S^3.$$

We say that $S(L, J, \phi)$ is the *string link infection with seed L, infection link J, and axis $\phi(\mathbb{E})$*.

We use the letter S since this construction is a variant of the well-known satellite construction. The effect of infection by a string link is to tie the string link J into the strands of L that pass through E_ϕ.

3. Infections on Slice Links That Produce Slice Links

First, we have a straightforward lemma, which describes infections that, starting with a boundary link, produce a boundary link.

LEMMA 3.1. *Suppose that L is a boundary link with disjoint Seifert surfaces F and $\phi : \mathbb{E} \to S^3$ is a proper r-multidisc in (S^3, L). If $\phi(\mathbb{E} \setminus (E_1 \sqcup \cdots \sqcup E_r))$ is disjoint from F, then $S(L, J, \phi)$ is also a boundary link.*

Proof. The images of the Seifert surfaces F under the homeomorphism of Definition 2.8 (3) are again Seifert surfaces for $S(L, J, \phi)$. $\square$

The main goal of this section is to prove the following lemma, which is the string link infection version of [CD13, Theorem 3.1]. It describes a pair of string link infections on a slice link that produces a slice link. Suppose that we start with a slice link R and do two string link infections using multidiscs ϕ_0 and ϕ_1 such that the associated handlebodies $\phi_i((\mathbb{E} \setminus (E_1 \sqcup \cdots \sqcup E_r)) \times I)$ ($i = 0, 1$) cobound an embedding of $(\mathbb{E} \setminus (E_1 \sqcup \cdots \sqcup E_r)) \times I \times I$ in the complement of slice discs for R. Moreover, suppose that we use J and $-J$ as infection links for ϕ_0 and ϕ_1, respectively. Then the resulting double infection link is still slice.

LEMMA 3.2 (link version of Theorem 3.1 of [CD13]). *Let R be a slice link bounding slice discs D in the 4-ball B^4. Let $\phi_i : \mathbb{E} \times I \to S^3$ be (thickenings of) disjoint proper r-multidiscs in (S^3, R) for $i = 0, 1$. Define $\Delta := \mathbb{E} \setminus (E_1 \sqcup \cdots \sqcup E_r)$, an r-punctured disc, and suppose that there is a smooth embedding $\Phi : \Delta \times I \times I \to B^4 \setminus \nu D$ with*

- *$\Phi|_{\Delta \times I \times \{i\}} = \phi_i|_{\Delta \times I}$ for $i = 0, 1$ and*
- *$\Phi(x, s, t) \in \partial B^4$ if and only if $t = 0, 1$.*

Then, for any r-component string link J, the result of the double string link infection $S(S(R, J, \phi_0), -J, \phi_1)$ is smoothly slice.

The proof closely follows that of [CD13, Section 4]. For the sake of our consciences, we check the details in our string link infection case.

Proof of Lemma 3.2. Let $X(J) := (D^2 \times I) \setminus \nu(J)$ be the exterior of the string link J and consider $\overline{B^4 \setminus \operatorname{Im} \Phi}$. Decompose the boundary $\partial(\operatorname{Im} \Phi) = \partial_+ \cup \partial_-$, where

$$\partial_+ := (\phi_0(\Delta) \sqcup \phi_1(\Delta)) \times I \quad \text{and} \quad \partial_- = \Phi(\partial(\Delta \times I) \times I).$$

Note that $\partial(\overline{B^4 \setminus \operatorname{Im} \Phi}) = (\overline{S^3 \setminus \partial_+}) \cup \partial_-$ and that ∂_- is diffeomorphic to $\partial X(J) \times I$.

We are about to define a 4-manifold $\mathcal{B}$ by extending the infections to an infection on $B^4 \setminus \nu D$, using the cobordism defined by Φ. More precisely, let $\mathcal{B}$ be the

4-manifold obtained by identifying $\overline{B^4 \smallsetminus \mathrm{Im}\,\Phi}$ and $X(J) \times I$ along a diffeomorphism ψ:

$$\psi : \Phi(\partial(\Delta \times I) \times I) \longrightarrow \partial X(J) \times I,$$

which satisfies:

- $\psi(\Phi(x, s, t)) = (\nu(J)(x, s), t)$ if $(x, s) \in \partial(E_1 \sqcup \cdots \sqcup E_r) \times I, t \in I$;
- $\psi(\Phi(x, s, t)) = (x, s, t)$ otherwise.

Recall that ∂_+ is the disjoint union of two genus r handlebodies, so its boundary is two genus r surfaces. This is also the boundary of the exterior of an r-component string link. The boundary $\partial \mathcal{B}$ is obtained by gluing $\overline{S^3 \smallsetminus \partial_+}$ to $X(J)$ and $-X(J)$ along two genus r surfaces and $\partial \mathcal{B}$ is diffeomorphic to S^3 as in Definition 2.8 (3).

The image of R under the map $S^3 \smallsetminus \partial_+ \to \partial \mathcal{B} \cong S^3$ is $S(S(R, J, \phi_0), -J, \phi_1)$ by definition. (Here, we used Remark 2.7 to identify $-X(J)$ with $X(-J)$.) Since the slice discs D are disjoint from the image of Φ, their images in $\mathcal{B}$ are slice discs. The following claim therefore implies that $S(S(R, J, \phi_0), -J, \phi_1)$ is smoothly slice, and so the proof of the lemma is complete once the claim has been verified.

CLAIM. *The 4-manifold $\mathcal{B}$ is diffeomorphic to the 4-ball.*

The rest of the proof of Lemma 3.2 comprises the proof of the claim. Recall that $E_{\phi_i} := \phi_i(E_1 \sqcup \cdots \sqcup E_r)$. Let

$$V := \mathrm{Im}\,\Phi \cup \nu(E_{\phi_0}) \cup \nu(E_{\phi_1})$$

and

$$W := X(J) \times I \cup \nu(E_{\phi_0}) \cup \nu(E_{\phi_1}).$$

Here, $\nu(E_{\phi_i})$ is the tubular neighbourhood of E_{ϕ_i} in B^4 for $i = 0, 1$. Also, $\nu(E_{\phi_i})$ is glued to $X(J) \times I$ via ψ. In more detail, we have

$$V = \Phi(\Delta \times I \times I) \cup h_0^1 \cup \cdots \cup h_0^r \cup h_1^1 \cup \cdots \cup h_1^r$$

and

$$W = X(J) \times I \cup h_0^1 \cup \cdots \cup h_0^r \cup h_1^1 \cup \cdots \cup h_1^r,$$

where h_t^i is a four-dimensional 2-handle glued along the circle $\phi_t(\partial E_i \times 0)$ for $i = 1, \ldots, r$ and $t = 0, 1$ with product framing. Note that

$$X(J) \times I \cup h_0^1 \cup \cdots \cup h_0^r$$

is diffeomorphic to B^4. With this identification, the image of the attaching circles of the h_1^i forms a 0-framed r-component unlink in ∂B^4. To see this, note that the circle $\phi_1(\partial E_i \times 0)$ bounds a disc, namely the core of h_0^i together with the annulus $\partial E_i \times I$ for all i. This observation shows that W is diffeomorphic to $\natural_{i=1}^r S^2 \times D^2$, where $\natural$ denotes the boundary connected sum. The same argument shows that V is diffeomorphic to $\natural_{i=1}^r S^2 \times D^2$.

Note that $\partial V \cap \partial B^4$ is the disjoint union of two 3-balls $B_0 \sqcup B_1$, where

$$B_i = \phi_i(\Delta \times I) \cup (\nu(E_{\phi_i}) \cap \partial B^4)$$

for $i = 0, 1$. Then $\mathcal{B} = \overline{(B^4 \smallsetminus V)} \cup W$, where the gluing map is given by an embedding

$$\psi_0 : \overline{\partial V \smallsetminus (B_0 \cup B_1)} \longrightarrow \partial W.$$

Here, ψ_0 is the restriction of ψ to $\partial(\Delta \times I) \times I \subset \overline{\partial V \smallsetminus (B_0 \cup B_1)}$, together with the identity map on the intersection of the 2-handles h_t^i with $\overline{\partial V \smallsetminus (B_0 \cup B_1)}$ for $t = 0, 1$ and $i = 1, \ldots, r$. We prove that $\mathcal{B}$ is diffeomorphic to the 4-ball by showing that ψ_0 extends to a diffeomorphism between V and W.

First, we extend ψ_0 to a diffeomorphism $\psi_1 : \partial V \to \partial W$. Note that $\partial V \cong \partial W \cong \#_{i=1}^r S^2 \times S^1$.

To see that we can extend to such a diffeomorphism ψ_1, note that $\partial V \smallsetminus (B_0 \cup B_1)$ and ∂W can be understood via the handle decomposition described by the standard Heegaard diagram for $\#_{i=1}^r S^2 \times S^1$, with a genus r surface and the α_j and β_j curves parallel longitudes on the jth torus connected summand of the surface, with B_0 and B_1 the 0-handle and the 3-handle of the handle decomposition, respectively. The boundaries $\partial(\Delta \times I)$ and $\partial(X(J))$ are both diffeomorphic to a genus r surface Σ, and so $\partial(\Delta \times I) \times I$ and $\partial(X(J)) \times I$ are both diffeomorphic to $\Sigma \times I$, and these are identified by ψ_0. Each 2-handle h_t^i can be thought of as a three-dimensional 2-handle $\widetilde{h}_t^i$ product with an interval, with $\widetilde{h}_t^i \times \{0\} \subset \partial B^4$. The lower boundary $\widetilde{h}_t^i \times \{1\}$ is then the 2-handle attached to α_i for $t = 0$ and to β_i for $t = 1$.

Now recall that the map ψ_0 is the identity when restricted to the intersection of the 2-handles h_t^i with $\overline{\partial V \smallsetminus (B_0 \cup B_1)}$ for $t = 0, 1$ and $i = 1, \ldots, r$. Thus, the surface $\Sigma \times I$ and the three-dimensional 1- and 2-handles of $\#_{i=1}^r S^2 \times S^1$ are all identified. Therefore, ∂B_i is sent to the boundary of the 0-handle of ∂W for $i = 0$ and to the boundary of the 3-handle for $i = 1$. We have that any diffeomorphism of the 2-sphere extends over the 3-ball, by [Mun60; Sma59]. It follows that the diffeomorphism $\psi_0(B_i)$ extends for $i = 0, 1$, so that as claimed ψ_0 extends to a diffeomorphism $\psi_1 : \partial V \to \partial W$.

We want to extend ψ_1 to a diffeomorphism between V and W. For $i = 1, \ldots, r$, each circle $\Phi(\partial(\gamma_i \times I \times \{1\}))$, which is the image under Φ of a longitude of the ith component of the trivial r-component string link, is isotopic in ∂V to the belt sphere of 2-handle h_1^i where γ_i is the chosen arc in Definition 2.6 (1) and Figure 1. (Note that each circle $\Phi(\partial(\gamma_i \times I \times \{1\}))$ links attaching circle of h_1^i once.) Recall that

$$V = \Phi(\Delta \times I) \cup h_0^1 \cup \cdots \cup h_0^r \cup h_1^1 \cup \cdots \cup h_1^r.$$

For $i = 1, \ldots, r$, let D_i be the cocore of the 2-handle h_1^i in V. Therefore,

$$V \smallsetminus (\nu(D_1) \sqcup \cdots \sqcup \nu(D_r)) \cong \Phi(\Delta \times I) \cup h_0^1 \cup \cdots \cup h_0^r \cong B^4.$$

Now we consider W. Let ℓ_i be the ith longitude of J in $X(J)$, and let m_i be the attaching circle of h_1^i. As illustrated in Figure 3, after sliding $\ell_i \times \{1\}$ over the 2-handles $h_1^1, \ldots, h_1^r$ (sufficiently many times), we can see that $\ell_i \times \{1\}$ is isotopic in ∂W to ℓ_i' that is a meridian of m_i. That is, $\ell_i \times \{1\}$ is isotopic to the belt sphere of the 2-handle h_1^i. Moreover, these isotopies can be achieved disjointly. In particular, $\ell_i \times \{1\}$ bounds a disc $\mathcal{D}_i$ that is the cocore of the 2-handle h_1^i in W.
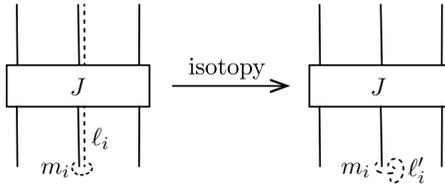
Figure 3   An isotopy of $\ell_i \times \{1\}$ in ∂W

To see the isotopies, consider that $\partial(X(J) \cup h_0^1 \cup \cdots \cup h_0^r) \cong S^3$. The longitudes $\ell_1 \cup \cdots \cup \ell_r$ define a link in this S^3. Attaching the 2-handles $h_1^1 \cup \cdots \cup h_1^r$ performs zero surgery on the meridians m_i to the ℓ_i. Standard Kirby calculus moves now convert the link to an unlink with zero surgery on the meridians. The h_1^i can be thought of as helper circles to undo the link formed by the ℓ_i.

We have just seen that, up to isotopies, ψ_1 identifies the framed boundary of D_i with the framed boundary of $\mathcal{D}_i$ for $i = 1, \ldots, r$. So, ψ_1 extends to a diffeomorphism

$$\psi_1' : \partial V \cup \nu(D_1) \cup \cdots \cup \nu(D_r) \longrightarrow \partial W \cup \nu(\mathcal{D}_1) \cup \cdots \cup \nu(\mathcal{D}_r).$$

Since $V \smallsetminus (\nu(D_1) \sqcup \cdots \sqcup \nu(D_r))$ and $W \smallsetminus (\nu(\mathcal{D}_1) \sqcup \cdots \sqcup \nu(\mathcal{D}_r))$ are 4-balls and since any diffeomorphism of ∂B^4 extends to a diffeomorphism of B^4 by [Cer68], ψ_1' extends to a diffeomorphism $V \xrightarrow{\cong} W$. This completes the proof that $\mathcal{B}$ is diffeomorphic to B^4 and therefore completes the proof of the claim and Lemma 3.2. $\hfill\square$

4. The Effect of String Link Infection on Milnor Invariants

The aim of this section is to prove the following lemma, which enables us to compute Milnor invariants of links obtained by string link infection. The Milnor invariants $\overline{\mu}_L(I)$ of a link L were introduced by Milnor [Mil57]. Later in the paper (Sections 5 and 7), we will apply the next lemma to compute the values of $\overline{\mu}(I)$ for the derivatives associated to our examples.

LEMMA 4.1. *Let I be a multi-index that contains the indices $\{1, \ldots, m\}$, and let k_i be the number of occurrences of the index i in I ($k_i \geq 1$). Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link with $\overline{\mu}_L(I') = 0$ whenever $|I'| < |I|$, and let J be an m-component string link whose closure $\widehat{J}$ has $\overline{\mu}_{\widehat{J}}(I') = 0$ whenever $|I'| < |I|$. Let $\phi : \mathbb{E} \to S^3$ be a proper m-multidisc in (S^3, L) such that for each $i = 1, \ldots, m, \phi(E_i)$ only intersects L at L_i. Denote the algebraic intersection number between $\phi(E_i)$ and L_i by n_i for $i = 1, \ldots, m$. Then $S(L, J, \phi)$ is also a link with $\overline{\mu}_{S(L,J,\phi)}(I') = 0$ whenever $|I'| < |I|$, and*

$$\overline{\mu}_{S(L,J,\phi)}(I) = \overline{\mu}_L(I) + \overline{\mu}_{\widehat{J}}(I) \cdot \prod_{i=1}^{m} n_i^{k_i}.$$
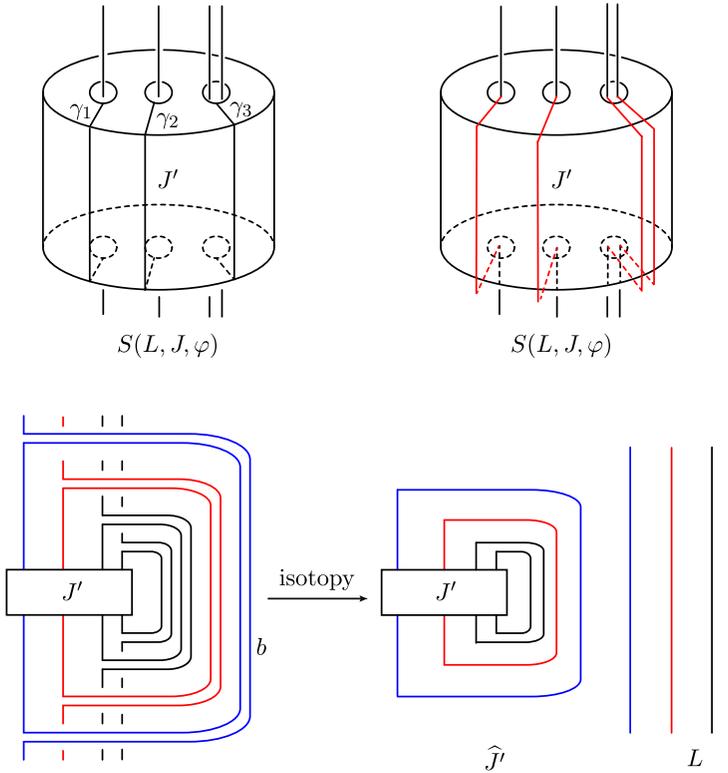
Figure 4 Internal band sums of $S(L, J, \phi)$ that produce L'

Proof. Suppose that all intersection points in $\phi(E_i) \cap L_i$ have been made transverse and that $\phi(E_i) \cap L_i$ contains α_i positive and $\beta_i = \alpha_i - n_i$ negative intersection points, $i = 1, \ldots, m$. Let J' be the oriented string link generated by taking α_i parallel copies of ith component J_i of J and β_i parallel copies of $r(J_i)$, which is J_i with opposite orientation, for $i = 1, \ldots, m$. Let L' be the split union of L and $\widehat{J'}$, the closure of J'. Then $S(L, J, \phi)$ can be considered as the outcome of performing certain internal band sums of L'. This is shown in Figure 4, which we now explain.

We start by looking at $S(L, J, \phi)$. Recall, from Definition 2.6 (1), the paths γ_i ($i = 1, \ldots, m$) in the m-multidisc ϕ that connect $\phi(\partial E_i)$ and the boundary of the m-multidisc. These paths γ_i determine $\sum_{i=1}^{m} (\alpha_i + \beta_i)$ arcs as shown on the top right of Figure 4. These arcs induce bands, which we denote b, connecting components of $S(L, J, \phi)$ to themselves, as shown on the bottom left of Figure 4. The result of band sums along these bands is the split union of $\widehat{J'}$ and L, i.e., L', the salient part of which can be seen on the bottom right of Figure 4.

Note that a band sum can be annulled by *dual band sum,* that is, another band sum along the cocore of original band (see Figure 5). Hence, reversing the above
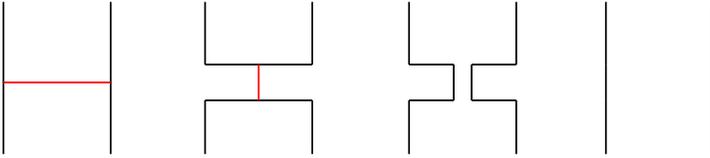
Figure 5   A dual band sum annuls the original band sum

operation with the dual band sum, $S(L, J, \phi)$ can be seen to be the result of internal band sums of L' along the dual bands of b.

Using this interpretation of the operation of string link infection as performing band sums (cf. [Coc04, Section 10]), we can apply [Coc90, Theorem 8.13], which states that the first nonvanishing Milnor invariants are additive under exterior band sums, namely band sums that join a split link. In fact, the band sums we use are slightly more general than those of [Coc90, Definition 8.7] since one component of L is connected to several components of $\widehat{J'}$. However, the proof of [Coc90, Theorem 8.13] easily generalizes, once the contribution to the Milnor invariant from $\widehat{J'}$ is interpreted in the appropriate sense, which we describe now.

In $\widehat{J'}$, each parallel copy of $\widehat{J_i}$ is labeled with an index j for $j \in \{1, \dots, \sum_i \alpha_i + \sum_i \beta_i\}$. Define a function $h : \{1, \dots, \sum_i \alpha_i + \sum_i \beta_i\} \to \{1, \dots, m\}$ that sends the index of a parallel copy of $\widehat{J_i}$ to i. The integer $h(j)$ records which component of $S(L, J, \phi)$ the parallel copy labeled j will become part of after the band sums since all parallel copies of $\widehat{J_i}$ will belong to the ith component of $S(L, J, \phi)$. We need to take the sum

$$\sum_{\{I' | h(I') = I\}} \overline{\mu}_{\widehat{J'}}(I'),$$

where we can apply the function h to a multi-index in the obvious way.

First, we note that reversing the orientation on a single component L_i of a link changes the sign of a Milnor invariant $\overline{\mu}_L(I)$ by $(-1)^{k_i}$, where k_i is the number of times that i appears in I [Mil57, page 296].

Choose a parallel copy of $\widehat{J_i}$ for each occurrence of i in I and do this for all $i = 1, \dots, m$. Let I' be the multi-index that arises by replacing each occurrence of i in I by the index of the parallel copy of $\widehat{J_i}$ chosen. Note that $h(I') = I$.

By [Mil57, Theorem 7], the Milnor invariant of the collection of parallel copies is

$$\overline{\mu}_{\widehat{J}}(I) \cdot \prod_{j \in I'} r_j^{\lambda_j},$$

where $r_j \in \{\pm 1\}$ is -1 if the parallel copy of a component of $\widehat{J}$ with index j chosen uses the reverse orientation and is $+1$ otherwise, and λ_j is defined to be the number of times that j appears in I'. Note that if λ_j is even, then r_j makes no difference to the Milnor invariant. We need to sum over all possible choices of parallel copies, to obtain the contribution to $\overline{\mu}_{S(L, J, \phi)}(I)$. For each i in I, we have to make k_i choices, and there are $\alpha_i + \beta_i$ choices each time.

Thus, there are $\prod_{i=1}^{m}(\alpha_i + \beta_i)^{k_i}$ possible choices, whose contribution with sign is $\prod_{i=1}^{m}(\alpha_i - \beta_i)^{k_i} = \prod_{i=1}^{m} n_i^{k_i}$.

Therefore, the total contribution of $\widehat{J'}$ to the Milnor invariant is

$$\overline{\mu}_{\widehat{J}}(I) \cdot \prod_{i=1}^{m} n_i^{k_i}.$$

Combining this with the additivity of the first nonvanishing Milnor invariants under exterior band sums, it follows that

$$\overline{\mu}_{S(L,J,\phi)}(I) = \overline{\mu}_L(I) + \overline{\mu}_{\widehat{J}}(I) \cdot \prod_{i=1}^{m} n_i^{k_i},$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

5. Construction of Our Example

In this section, we construct the slice boundary link promised in Theorem 1.1. Let $R = R_1 \sqcup R_2 \sqcup R_3$ be a 3-component split link each of whose components are the seed ribbon knot used by Cochran and Davis [CD13, Figures 6 and 10] (see Figure 7). The Cochran–Davis seed knot was shown to have a unique genus one Seifert surface, up to isotopy, in [Hor10, page 2213]. We denote the unique Seifert surface for R_i by F_i', $i = 1, 2, 3$.

Our example is obtained by a double string link infection on R that satisfies the conditions in Lemmas 3.1 and 3.2, using a string link J for our infection link whose closure has unknotted components, pairwise linking numbers vanishing, and $\overline{\mu}_{\widehat{J}}(123) = u \neq 0$. We take J to be a string link representative of the Borromean rings, to be definite. We choose an orientation of J so that $\overline{\mu}_{\widehat{J}}(123) = 1$.

Using Lemmas 3.1 and 3.2, we will prove that the resulting link is a smoothly slice boundary link. Then, using Lemma 4.1, we will calculate that the Milnor triple linking numbers of the derivatives with respect to the genus one Seifert surfaces for the infected link induced from the F_i' are nonzero.

Let $X_R := S^3 \smallsetminus \nu R$ be the exterior of R. For each $i = 1, 2, 3$, choose two disjoint, oriented curves x_i, y_i and an arc z_i between x_i and y_i in X_R as in Figure 6. These are (three copies of) the same curves used by Cochran and Davis in [CD13, Section 5]. Our Figure 6 is copied from [CD13, Figure 6]; we include it here for the convenience of the reader. The isotopy between the two diagrams in Figure 6 is explained in [CD13, Figure 7]. Note that x_i, y_i correspond to the curves η_1, η_2 in [CD13, Figure 6] and z_i corresponds to the core of the band between η_1 and η_2 in [CD13, Figure 8].

From the proof of [CD13, Proposition 5.1], there are an annulus A_i ($i = 1, 2, 3$) and slice discs D of R such that $x_i \sqcup -y_i$ forms the oriented boundary of A_i and A_i lies in $B^4 \smallsetminus \nu D$. Note that it can be arranged that each A_i contains the arc z_i.

Let $Y = \bigvee_{i=1}^{3} I$, where the wedge uses basepoints $\{0\} \in I$ for each I. Let Y_0 be the three end points of Y coming from $\{1\} \in I$. We fix the genus one Seifert surface $F' = F_1' \sqcup F_2' \sqcup F_3'$ of R as in Figure 6. For each $i = 1, 2, 3$, choose an arc
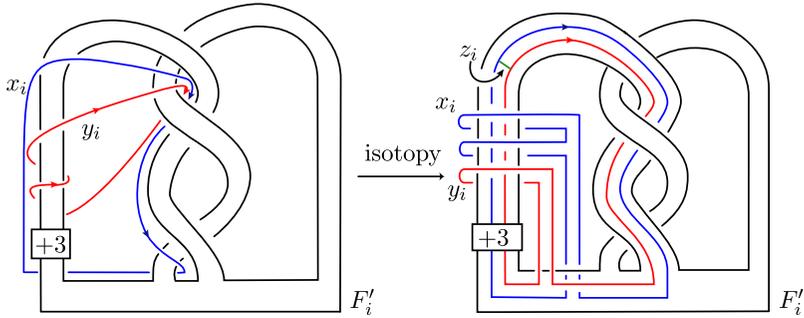
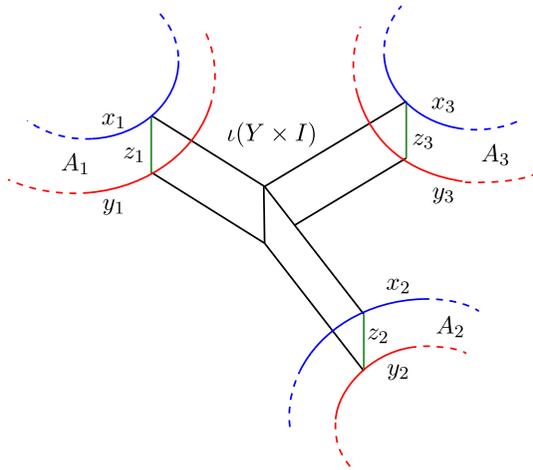**Figure 6**   An isotopy of the curves x_i and y_i

Figure 7   The Seifert surface F' of R and $\iota(Y \times I)$

connecting the base point of S^3 to an interior point of z_i that intersects neither R, the x_j, the y_j, D, nor F'. This extends to an embedding

$$\iota : Y \times I \longrightarrow X_R$$

such that

- $\iota(Y_0 \times I)$ is $z_1 \sqcup z_2 \sqcup z_3$;
- $\iota((Y \smallsetminus Y_0) \times I)$ is disjoint from x_i, y_i, R, and F'.

The image of the map ι is illustrated in Figure 7. Push $\iota((0, 1) \times Y)$ very slightly into B^4. A thickening of $\mathrm{Im}(\iota) \cup (A_1 \sqcup A_2 \sqcup A_3)$ in B^4 gives an embedding

$$\Phi : \Delta \times I \times I \longrightarrow B^4 \smallsetminus \nu D,$$

where $\Delta = \mathbb{E} \smallsetminus (E_1 \sqcup E_2 \sqcup E_3)$. Denote $\phi_t = \Phi|_{\Delta \times I \times t} : \Delta \times I \to X_R$ for $t = 0, 1$. Then $\phi_0(\partial E_i \times I)$, $\phi_1(\partial E_i \times I)$ are annulus neighborhoods of x_i, y_i in S^3,
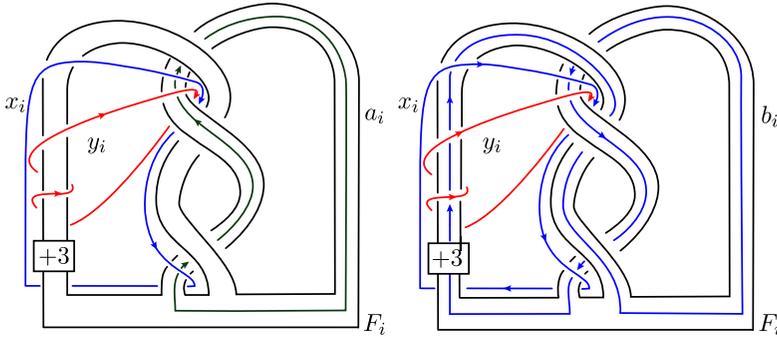
Figure 8   The Seifert surface F_i and its derivatives a_i, b_i

respectively, for $i = 1, 2, 3$. Since x_i and y_i bound discs in S^3 that intersect R transversely, we can extend ϕ_0, ϕ_1 to thickenings of two 3-multidiscs in S^3.

With this notation, by Lemma 3.2, $S(S(R, J, \phi_0), -J, \phi_1)$ is smoothly slice in B^4 for any 3-component string link J. Moreover, $S(S(R, J, \phi_0), -J, \phi_1)$ is again a boundary link by Lemma 3.1 since the image of ι does not meet F'. Thus, we have that $S(S(R, J, \phi_0), -J, \phi_1)$ is a smoothly slice boundary link for any string link J.

From now on, let $S = S_1 \sqcup S_2 \sqcup S_3$ denote the link $S(S(R, J, \phi_0), -J, \phi_1)$; recall that J is a string link representative for the Borromean rings.

Let $F = F_1 \sqcup F_2 \sqcup F_3$ be the Seifert surface of S arising as the image of F' under the homeomorphism in Definition 2.8 (3). Note that each S_i is ambient isotopic to the original Cochran–Davis seed knot, so is a slice knot, and as remarked above, the Cochran–Davis seed knot was shown to have a unique genus one Seifert surface in [Hor10, p. 2213]. There are exactly two derivative knots on each F_i, which are shown in Figure 8. The curves a_i in Figure 8 are unknots, so the components individually satisfy the strong Kauffman conjecture that there is a slice metaboliser. Once again, Figure 8 comes from [CD13], this time, their Figure 10, and we include it here for the benefit of the reader.

THEOREM 5.1. *Each derivative M of S with respect to F has a nonzero Milnor triple linking number $\overline{\mu}_M(123) \neq 0$.*

Proof. Note that there are exactly two derivatives for Cochran–Davis' seed knot with respect to its unique genus one Seifert surface. Let $\{a_i, b_i\}$ be the set of the derivatives for F_i, $i = 1, 2, 3$. Here, a_i is the unknotted curve on the Seifert surface F_i shown on the left of Figure 8, while b_i is the trefoil which passes around both bands of the copy of F_i drawn on the right of Figure 8. Hence, there are exactly eight derivatives of F, each of which is a 3-component link of the form $c_1 \sqcup c_2 \sqcup c_3$, where $c_i \in \{a_i, b_i\}$ for $i = 1, 2, 3$. The Seifert matrix for F

Table 1

$\#\{i \in \{1, 2, 3\} \mid c_i = b_i\}$	0	1	2	3
$\overline{\mu}_{c_1 \sqcup c_2 \sqcup c_3}(123)$		7	−5	1	−2

(with respect to the most natural basis) is the block sum

$$\bigoplus_{i=1}^{3} \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}.$$

Note that the links $c_1 \sqcup c_2 \sqcup c_3$ are obtained from a 3-component split link (whose components are either the unknot or the trefoil) by double string link infection along ϕ_0 and ϕ_1, by J and $-J$, respectively. So, using Lemma 4.1, we can calculate $\overline{\mu}_{c_1 \sqcup c_2 \sqcup c_3}(123)$.

From Figure 8,

$$\mathrm{lk}(a_i, x_i) = 2, \qquad \mathrm{lk}(b_i, x_i) = -1,$$
$$\mathrm{lk}(a_i, y_i) = 1, \qquad \mathrm{lk}(b_i, y_i) = 1.$$

By applying Lemma 4.1 twice (with multi-index 123) we have

$$\overline{\mu}_{c_1 \sqcup c_2 \sqcup c_3}(123) = \overline{\mu}_U(123) + \overline{\mu}_{\hat{J}}(123) \cdot \prod_{i=1}^{3} \mathrm{lk}(c_i, x_i) + \overline{\mu}_{\overline{\cap}_J}(123) \cdot \prod_{i=1}^{3} \mathrm{lk}(c_i, y_i),$$

where U is the unlink. For indices i, j, k with $\{i, j, k\} = \{1, 2, 3\}$,

$$\overline{\mu}_{a_i \sqcup a_j \sqcup a_k}(123) = 0 + 1 \cdot 2 \cdot 2 \cdot 2 - 1^4 = 7,$$
$$\overline{\mu}_{a_i \sqcup a_j \sqcup b_k}(123) = 0 + 1 \cdot 2 \cdot 2 \cdot (-1) - 1^4 = -5,$$
$$\overline{\mu}_{a_i \sqcup b_j \sqcup b_k}(123) = 0 + 1 \cdot 2 \cdot (-1)^2 - 1^4 = 1,$$
$$\overline{\mu}_{b_i \sqcup b_j \sqcup b_k}(123) = 0 + 1 \cdot (-1)^3 - 1^4 = -2.$$

Here, $\overline{\mu}_{\hat{J}}(123) = 1$ and $\overline{\mu}_{\overline{\cap}_J}(123) = -1$.

For a short version, see Table 1. Since all the numbers in the bottom row are nonzero, this completes the proof of Theorem 5.1 and therefore of Theorem 1.1.
□

REMARK 5.2. If instead of a string link representative of the Borromean rings, we had used a string link $\mathcal{J}$ as our infection link, whose closure has $\overline{\mu}_{\hat{J}}(123) = u \neq 0, 1$, then the only change would be that the numbers in the bottom row of Table 1 are multiplied by u. This is easy to see from the applications of Lemma 4.1 used to compute these numbers.

6. Stabilizing to Obtain a Ribbon Derivative Link

In this section, as advertised, we show how to stabilize the Seifert surfaces F_i of S with an S-equivalence so that there is a ribbon derivative link. This will in turn imply that S is a ribbon link. Thus, the nonvanishing Milnor invariants of the
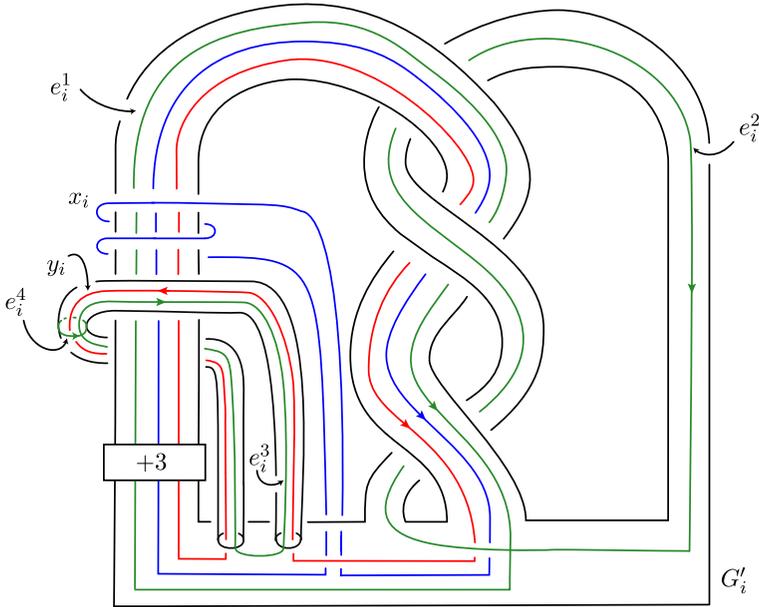
Figure 9 The Seifert surface after surgery to make it genus two, together with a basis for first homology and the infection curves x_i and y_i

derivative links are not particularly robust, as far as stabilization is concerned. We remark that ribbon links are boundary slice, as shown in Theorem A.3, so we also prove that S is boundary slice.

We note that Proposition 6.1 provides an alternative to Lemma 3.2 for showing that S is smoothly slice.

In [CD13, Section 7], it is claimed that the Seifert surface for their knot stabilizes to a surface with a slice derivative link, although details are not given in their preprint. We thank Chris Davis for helpful discussions concerning this. For the convenience of the reader, we now show how to achieve this stabilization, with the necessary adaptation to our link case.

PROPOSITION 6.1. *The link S constructed in Section 5 is a ribbon link.*

Proof. Figure 9 shows the result of an ambient surgery on $S^0 \times D^2 \subset F'_i$, together with the infection curve x_i and y_i. This is a modification of the right-hand side of Figure 6. The new cylinder $S^1 \times D^1$ has been attached so as to tube along part of the infection curve y_i. We call these new genus two Seifert surfaces G'_i for $i = 1, 2, 3$ and let $G' := G'_1 \sqcup G'_2 \sqcup G'_3$.

The curves x_i and y_i should be regarded as lying underneath the Seifert surface G'_i, except for where x_i loops around the left-hand band, and where the y_i curve passes through the tube.

A basis $\{e_i^1, e_i^2, e_i^3, e_i^4\}$ for $H_1(G_i'; \mathbb{Z})$ is also shown. The curves e_i^1 and e_i^2 were generators of $H_1(F_i; \mathbb{Z})$, whereas the curves e_i^3, e_i^4 are a meridian and longitude of the new tube.

With respect to this basis, the Seifert matrix is given by

$$V := \begin{pmatrix} 3 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

A metaboliser for the Seifert form is generated by $\{e_i^2, e_i^3\}$. Both curves are unknotted. Now note that

$$\text{lk}(e_i^2, x_i) = \text{lk}(e_i^2, y_i) = 2$$

and that

$$\text{lk}(e_i^3, x_i) = \text{lk}(e_i^3, y_i) = 1.$$

Let

$$\mathcal{L}' := e_1^2 \sqcup e_1^3 \sqcup e_2^2 \sqcup e_2^3 \sqcup e_3^2 \sqcup e_3^3$$

be the derivative link representing the metaboliser obtained by taking the direct sum of the submodules of $H_1(G'; \mathbb{Z})$ generated by $\{e_i^2, e_i^3\}$ for $i = 1, 2, 3$.

We perform the double infection by using a string link representative J of the Borromean rings and 3-multidiscs determined by the curves x_i and y_i, as in Section 5. Let G denote the image of G' under the homeomorphism of Definition 2.8 (3). The image of the link $\mathcal{L}'$ is then a derivative for the link S with respect to the Seifert surfaces G, which we denote $\mathcal{L}$:

$$\mathcal{L} := S(S(\mathcal{L}', J, \phi_0), -J, \phi_1).$$

Here, as in Section 5, the 3-multidiscs ϕ_0 and ϕ_1 are determined by the curves x_i and y_i. However, as is consistent with the linking number observations made above, for each $i = 1, 2, 3$, the curves x_i and y_i are isotopic in $S^3 \smallsetminus \nu(e_i^2 \sqcup e_i^3)$. This isotopy is not obvious, but follows along the lines of the isotopy depicted in [CD13, Figure 7], except restricted by the curves e_i^2 and e_i^3. In fact, the whole collection of curves $e_i^2 \sqcup e_i^3 \sqcup x_i \sqcup y_i$ is isotopic to the link shown in Figure 10.

Thus, $\mathcal{L}$ is isotopic to the link $S(\mathcal{L}', J \# -J, \phi_0)$, which is the result of a single infection by the string link $J \# -J$. Since this latter is a ribbon string link, the link $\mathcal{L}$ is ribbon. Using parallel copies of the ribbon discs for $\mathcal{L}$, we can surger G into a disjoint collection of ribbon discs for S. This completes the proof that S is a ribbon link. $\qquad\square$

REMARK 6.2. Recall that for each component S_i of S, one of the metabolising curves b_i on F_i is a left-handed trefoil. As in the statement of Theorem 1.1, each surface F_i has a slice metabolising curve, namely the unknotted curve a_i. Therefore, each component S_i, considered as a knot, satisfies the Kauffman conjecture in the strongest sense: for any genus one Seifert surface, there is a slice derivative knot. Recall that the genus one Seifert surface for each component is unique up to isotopy.
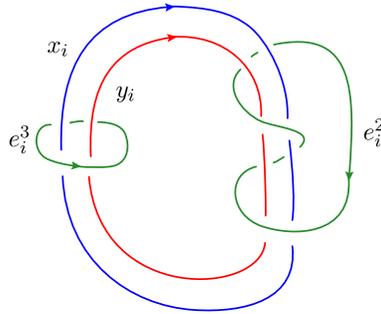
Figure 10   The curves $e_i^2 \sqcup e_i^3 \sqcup x_i \sqcup y_i$ from Figure 9 after an isotopy

If one desires, one may modify our example by tying a right-handed trefoil into the left-hand band of F_i with a satellite construction in such a way that no twists are introduced. In the language of Definition 2.8, this is achieved with infection by a 1-component string link representative of the right-handed trefoil using a 1-multidisc that intersects the left-hand band of F_i in an arc that cuts the band. When performing this infection, we must be sure to avoid affecting the curves x_i or y_i; that is, they should be disjoint from the 1-multidisc.

Let $S^\dagger$ be the new link, and let $F^\dagger$ be the induced Seifert surfaces. Now, *all* the derivative links of $F^\dagger$ comprise slice knots, but still are not slice by virtue of their Milnor invariants. The link $S^\dagger$ is still ribbon since the proof of this section, in particular, the link type of $e_i^2 \sqcup e_i^3 \sqcup x_i \sqcup y_i$, is unchanged. The only caveat is that after this alteration without further proof we no longer know that the minimal genus Seifert surface $F_i^\dagger$ for each component is unique.

7. Generalization to Other Milnor Invariants

In this section we give a generalization of Theorem 1.1 to many other Milnor invariants promised in Theorem 1.4. Cochran [Coc90, Theorem 7.2] associated an integer $\delta(I)$ to a multi-index I giving the minimal nonnegative integer that can be realized as a Milnor invariant with the given multi-index $\overline{\mu}_L(I)$ of any link L. If no link can have nonzero Milnor invariant with the given index, then $\delta(I) = 0$.

For a multi-index I, recall from Section 4 that we denote the number of occurrences of the index i by k_i. In the following theorem, we suppose that I is a multi-index with $\delta(I) \neq 0$, and we let m be the number of distinct integers in I.

THEOREM 7.1. *Let I be a multi-index as above for which $|I|$ is odd. Then there exists a smoothly slice (in fact ribbon) m-component boundary link $S = S_1 \sqcup \cdots \sqcup S_m$ with Seifert surfaces $F = F_1 \sqcup \cdots \sqcup F_m$, each of which is genus one and has an unknotted metabolising curve, such that for every metaboliser M, for the boundary link Seifert form, and for any m simple closed curves $M_1 \sqcup \cdots \sqcup M_m$ representing M, where $M_i \subset F_i$, we have $\overline{\mu}_M(I) \neq 0$.*

Moreover, for $i = 1, 2, 3$, we have that F_i is a unique minimal genus Seifert surface for S_i.

Proof. The proof is analogous to the construction in Section 5. Take m copies of the Cochran–Davis seed knot as $R = R_1 \sqcup \cdots \sqcup R_m$. Let J be an m-component string link whose closure $\widehat{J}$ has the Milnor invariant $\overline{\mu}_{\widehat{J}}(I) = \delta(I)$ and $\overline{\mu}_{\widehat{J}}(I') = 0$ whenever $|I'| < |I|$. Such a string link can be produced by Cochran's realization theorem [Coc90, Theorem 7.2]. Let x_i and y_i, $i = 1, \ldots, m$, be copies of the curves shown in Figures 6 and 8. The collections $x_1 \sqcup \cdots \sqcup x_m$ and $y_1 \sqcup \cdots \sqcup y_m$ determine m-multidiscs $\phi_0(\mathbb{E})$ and $\phi_1(\mathbb{E})$, respectively. Define $\Delta := \mathbb{E} \smallsetminus (E_1 \sqcup \cdots \sqcup E_m)$. Just as in Section 5, $\phi_0(\Delta \times I)$ and $\phi_1(\Delta \times I)$ cobound an embedding $\Delta \times I \times I$ in the complement of a collection of slice discs for R. By Lemma 3.2, the double infection $S := S(S(R, J, \phi_0), -J, \phi_1)$ is a slice link.

The individual knots are still copies of the Cochran–Davis seed knot, so have unique genus one Seifert surfaces with an unknotted metaboliser, the curve a_i from Figure 8. The proof that S is a ribbon link follows analogously to the proof given in Section 6; neither the number of components nor the infection string link were relevant to that proof.

A metaboliser consists of curves $c_1 \sqcup \cdots \sqcup c_m$ where c_i is either of the type a_i or of the type b_i for each $i = 1, \ldots, m$, where the curves a_i and b_i are depicted in Figure 8. Computing as in the proof of Theorem 5.1 and using the general version of Lemma 4.1, we see that the Milnor invariants of the metabolisers are given by the formula

$$\overline{\mu}_{c_1 \sqcup \cdots \sqcup c_m}(I) = \delta(I) \cdot \left(\left(\prod_{\{i \mid c_i = a_i\}} 2^{k_i} \right) \cdot \left(\prod_{\{i \mid c_i = b_i\}} (-1)^{k_i} \right) - 1 \right).$$

Since $k_i \geq 1$ for all i, this could only be zero when $c_i = b_i$ for all i. But then

$$\prod_{\{i \mid c_i = b_i\}} (-1)^{k_i} = \prod_{i=1}^{m} (-1)^{k_i} = (-1)^{\sum_{i=1}^{m} k_i} = -1$$

since by hypothesis $\sum_{i=1}^{m} k_i = |I|$ is odd. It follows that $\overline{\mu}_{b_1 \sqcup \cdots \sqcup b_m}(I) = -2\delta(I) \neq 0$. $\square$

Appendix. Ribbon Boundary Links Are Boundary Ribbon

DEFINITION A.1. An oriented link $L = L_1 \sqcup \cdots \sqcup L_m$ is ribbon if it is the boundary of m pairwise disjoint discs $D = D_1 \sqcup \cdots \sqcup D_m$ embedded in B^4 in such a way that the radial Morse function on B^4 restricts to a Morse function on each D_i which has no critical points of index 2.

A link is a ribbon boundary link if it is both a ribbon link and a boundary link. Let $\mathcal{F}$ be the free group on m generators. Recall that we denote the exterior of a link L by $X_L := S^3 \smallsetminus \nu L$ and that a link is a boundary link if and only if its exterior admits a map $\pi_1(X_L) \to \mathcal{F}$ with the images of the meridians generating $\mathcal{F}$ [Smy66; Gut72].

Any collection of Seifert surfaces $\{F_i\}$ induces such a map: let $F_i \times [-1, 1]$ be a regular neighbourhood of F_i and map $F_i \times [-1, 1] \to [-1, 1] \to S_i^1$, where the codomain is the ith S^1 in a wedge of m circles $\bigvee_m S^1$ each of which has $-1 \in S_i^1$ as its basepoints. The first map is projection, and the last map is given by $t \mapsto e^{\pi i t}$. Map the exterior of the regular neighbourhood of the Seifert surfaces to the basepoint of $\bigvee_m S^1$. This defines a map $X_L \to \bigvee_m S^1 = B\mathcal{F}$. The induced map on fundamental groups gives a map $\pi_1(X_L) \twoheadrightarrow \mathcal{F}$ as required.

DEFINITION A.2 (boundary ribbon). A ribbon boundary link is said to be a *boundary ribbon* if there is a homomorphism $\pi_1(B^4 \smallsetminus \nu D) \to \mathcal{F}$ that extends the homomorphism $\pi_1(X_L) \to \mathcal{F}$. Thus, there is a map $B^4 \smallsetminus \nu D \to B\mathcal{F} = \bigvee_m S^1$ extending the map to $\bigvee_m S^1$ defined by the Seifert surfaces. Then by transversality there are 3-manifolds $R_1 \sqcup \cdots \sqcup R_m$ embedded in B^4, with $\partial R_i = F_i \cup_{L_i} D_i$ for $i = 1, \ldots, m$.

As far as the authors are aware, the next theorem has not appeared in the literature before, although it is well known to the experts.

THEOREM A.3. *Ribbon boundary links are boundary ribbon.*

The proof below was told to the authors by Kent Orr, whom we thank for allowing us to include this in our paper.

Proof of Theorem A.3. Let G be a group, let G_k be the kth lower central subgroup, and let $\theta_k : G/G_k \to (G/G_k)/(G_{k-1}/G_k) \overset{\cong}{\to} G/G_{k-1}$ be the canonical map. The nilpotent completion $\widehat{G}$ of a group G can be constructed as the subgroup of the infinite product $\prod_{k=1}^{\infty} G/G_k$, given by elements $(g_2, g_3, g_4, \ldots)$, $g_k \in G/G_k$, for which $\theta_k(g_k) = g_{k-1}$, $k \geq 3$. There is a homomorphism $G \to \widehat{G}$, and taking nilpotent completion is a functor; see, for example, [BS77].

Let $Y_D := B^4 \smallsetminus \nu D$. Note that the maps $X_L \to Y_D$ and $X_L \to B\mathcal{F}$, given respectively by inclusion and the fact that L is a boundary link, induce isomorphisms $H_1(X_L; \mathbb{Z}) \overset{\cong}{\to} H_1(Y_D; \mathbb{Z})$, $H_1(X_L; \mathbb{Z}) \overset{\cong}{\to} H_1(\mathcal{F}; \mathbb{Z})$ and surjections $H_2(X_L; \mathbb{Z}) \twoheadrightarrow H_2(Y_D; \mathbb{Z})$, $H_2(X_L; \mathbb{Z}) \twoheadrightarrow H_2(\mathcal{F}; \mathbb{Z}) \cong 0$. Stallings' theorem [Sta65] thus implies that all three groups $\pi_1(X_L)$, $\pi_1(Y_D)$, and $\mathcal{F}$ have isomorphic nilpotent completions with isomorphisms induced functorially. The proof is now contained in Figure 11, as explained below.

Since the upper left vertical map $\pi_1(X_L) \twoheadrightarrow \pi_1(Y_D)$ is surjective and since the diagram commutes the image in $\widehat{\mathcal{F}}$ of all three groups on the left-hand side is the same. But $\mathcal{F}$ injects into its nilpotent completion since $\mathcal{F}$ is residually nilpotent; that is, the bottom horizontal map is an injection. Thus, the image of $\mathcal{F}$ in $\widehat{\mathcal{F}}$ is $\mathcal{F}$, and so the image of $\pi_1(Y_D)$ in $\widehat{\mathcal{F}}$ is also $\mathcal{F}$. We have a map $\pi_1(Y_D) \to \mathcal{F}$ as required. $\square$

$$\begin{array}{ccc}
\pi_1(Y_D) & \longrightarrow & \widehat{\pi_1(Y_D)} \\
\uparrow & & \uparrow \cong \\
\pi_1(X_L) & \longrightarrow & \widehat{\pi_1(X_L)} \\
\downarrow & & \downarrow \cong \\
\mathcal{F} \rightarrowtail & \longrightarrow & \widehat{\mathcal{F}}.
\end{array}$$

Figure 11

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H. J. Jang
Department of Mathematics
Pohang University of Science and
  Technology
Pohang, Gyungbuk 790-784
Republic of Korea

hye.jin.jang.1986@gmail.com

M. H. Kim
Department of Mathematics
Pohang University of Science and
  Technology
Pohang, Gyungbuk 790-784
Republic of Korea

kminhoon@gmail.com

M. Powell
Department of Mathematics
Indiana University
Bloomington, IN 47405
USA

macp@indiana.edu