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Upper Bounds for the Minimal Number of Singular Fibers in a Lefschetz Fibration over the Torus

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Abstract. In this paper, we give some relations in the mapping class groups of oriented closed surfaces in the form that a product of a small number of right-hand Dehn twists is equal to a single commutator. Consequently, we find upper bounds for the minimal number of singular fibers in a Lefschetz fibration over the torus.

1. Introduction

Lefschetz fibrations were originally introduced for studying topological properties of smooth complex projective varieties and afterwards generalized to differentiable category. Furthermore Donaldson and Gompf revealed the close relationship between Lefschetz fibrations and 4-dimensional symplectic topology in the late 1990s, and since then they have been extensively studied.

The information about the number of singular fibers in a Lefschetz fibration provides us important information about the topological invariants of its total space such as the Euler number, the signature, the Chern numbers, and so on. In addition, it has been known that the number of singular fibers in a Lefschetz fibration cannot be arbitrary, so it makes sense to ask what the minimal number of singular fibers in a Lefschetz fibration is. We denote by $N(g,h)$ the minimal number of singular fibers in a nontrivial relatively minimal genus g Lefschetz fibration over the oriented closed surface of genus h. This minimal number has been studied by various authors. Table 1 shows previous studies about $N(g,h)$. Korkmaz and Ozbagci [8] proved that (1) $N(g,h)=1$ if and only if $g \geq 3$ and $h \geq 2$, (2) $N(1,h)=12$ for all $h \geq 0$, and (3) $5 \leq N(2,h) \leq 8$ for all $h \geq 0$. The upper bound for $N(2,h)$ in (3) follows from the existence of a genus 2 Lefschetz fibration over the sphere with eight singular fibers, which was constructed by Matsumoto [11]. In addition, for $g=2$, Korkmaz and Stipsicz [10] showed that $N(2,h)=5$ for $h \geq 6$, and furthermore Monden [12] improved their results by showing that (1) $N(2,h)=5$ for all $h \geq 3$, (2) $N(2,2) \leq 6$, and (3) $6 \leq N(2,1) \leq 7$. Ozbagci [13] proved that the number of singular fibers in a genus 2 Lefschetz fibration over the sphere cannot be equal to 5 or 6, and Xiao [15] constructed a genus 2 Lefschetz fibration over the sphere with seven singular fibers; hence, $N(2,0)=7$. For $h=0$, some estimates for $N(g,0)$ are known. Cadavid [1] and Korkmaz [6] independently generalized Matsumoto's genus 2 Lefschetz fibration as above to genus g Lefschetz fibrations over the sphere with $2g+10$ singular fibers for g odd or with $2g+4$ singular fibers for g even. This

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Table 1 Previous results for $N = N(g, h)$

$g \backslash h$	0	1	2	3	4	5	6
7	$6 \leq N \leq 24$	$2 \leq N \leq 24$	1	1	1	1	1
6	$6 \leq N \leq 16$	$2 \leq N \leq 16$	1	1	1	1	1
5	$5 \leq N \leq 20$	$2 \leq N \leq 20$	1	1	1	1	1
4	$4 \leq N \leq 12$	$2 \leq N \leq 12$	1	1	1	1	1
3	$3 \leq N \leq 16$	$2 \leq N \leq 16$	1	1	1	1	1
2	7	6 or 7	5 or 6	5	5	5	5
1	12	12	12	12	12	12	12

fact shows $N(g, 0) \leq 2g + 10$ for g odd and $N(g, 0) \leq 2g + 4$ for g even. Stipsicz [14] proved that for any $g \geq 2$, the number of irreducible singular fibers in a genus g Lefschetz fibration over the sphere is bounded below by $(4g + 2)/5$. Therefore, we have

$$\frac{1}{5}(4g + 2) \leq N(g, 0).$$

In particular, for $N(g, 0)$, there is no universal upper bound that is independent of g.

In the case of $h = 1$, it has been known that $2 \leq N(g, 1) \leq N(g, 0)$. The former inequality follows by [8]. The latter inequality comes from the following observation. If a genus g Lefschetz fibration over the sphere is given, then by taking the fiber sum of it and the trivial Σ_g-bundle $\Sigma_g \times \Sigma_1$ over the torus, we can construct a genus g Lefschetz fibration over the torus without changing the number of the singular fibers. However, nontrivial upper bounds for $N(g, 1)$ have not been given explicitly as far as the author knows. The present paper aims at giving new upper bounds for $N(g, 1)$, i.e., in the case of Lefschetz fibrations over the torus. The main theorem of this paper is the following.

THEOREM 1.1. *For the minimal number of singular fibers in a Lefschetz fibration over the torus, the following holds*:

(1) $N(g, 1) \leq 6$ *for all* $g \geq 3$.
(2) $N(g, 1) \leq 4$ *for all* $g \geq 5$.

In particular, there is a universal upper bound for $N(g, 1)$ *that does not depend on* g.

We will prove Theorem 1.1 by concretely constructing a genus g Lefschetz fibration over the torus with six or four singular fibers for arbitrary $g \geq 3$ or $g \geq 5$, respectively. This will be done by providing new relations in the mapping class group of the surface of genus g, which are in the form that a product of six or four right-hand Dehn twists is equal to a single commutator.

The contents of this paper are as follows. Section 2 consists of the fundamental concepts on Lefschetz fibrations. In particular, we illustrate the fact that Lefschetz fibrations can be obtained from a certain type of relations in the mapping class groups of fiber surfaces. In Section 3 we introduce Matsumoto's relation and the alternative 8-holed torus relation in the mapping class groups of holed surfaces, which will be used in the proof of Theorem 1.1. Finally, in Section 4 we prove Theorem 1.1. We regard Matsumoto's relation and the alternative 8-holed torus relation as relations in the mapping class groups of other closed surfaces by embedding the original holed surfaces into the closed surfaces. Then we transform the relations to new ones, which will prove Theorem 1.1.

2. Lefschetz Fibrations

We recall some basic definitions and facts about Lefschetz fibrations (for details, see [4]). Let X^4 be a connected, oriented, and closed smooth 4-dimensional manifold, and Σ_g be the connected, oriented, and closed smooth 2-dimensional manifold with genus g. A smooth map $f : X^4 \to \Sigma_h$ is called a *Lefschetz fibration over* Σ_h if f has only finitely many critical points $p_1, p_2, \dots, p_n$, around each of which f is expressed as $(z_1, z_2) \mapsto z_1^2 + z_2^2$ by local complex coordinates compatible with the orientations of the manifolds. We assume that the critical values $b_i = f(p_i)$ are distinct. The inverse image of a regular value (or a critical value) is called a *regular fiber* (resp. a *singular fiber*). We will also assume that the regular fibers are connected. Since f is a submersion on the complement of singular fibers, the restriction $f : X^4 \setminus (f^{-1}(b_1) \cup f^{-1}(b_2) \cup \dots \cup f^{-1}(b_n)) \to \Sigma_h \setminus \{b_1, b_2, \dots, b_n\}$ is a Σ_g-bundle over $\Sigma_h \setminus \{b_1, b_2, \dots, b_n\}$ for some Σ_g. The *genus* of the Lefschetz fibration is defined to be the genus of a regular fiber. Furthermore, in this paper, we assume the relative minimality and the nontriviality. A Lefschetz fibration is said to be *relatively minimal* if there is no fiber that contains a (-1)-sphere (embedded sphere with self-intersection -1), and *nontrivial* if it has at least one singular fiber. The singular fiber $f^{-1}(b_i)$ is obtained by "crushing" a simple closed curve c_i, called the *vanishing cycle,* on a nearby regular fiber to a point. If the vanishing cycle is separating (or nonseparating), then the corresponding singular fiber is said to be *reducible* (resp. *irreducible*). Two Lefschetz fibrations $f : X \to \Sigma_h$ and $f' : X' \to \Sigma_h$ are said to be *isomorphic* if there are orientation-preserving diffeomorphisms $\Psi : X \to X'$ and $\psi : \Sigma_h \to \Sigma_h$ such that $f' \circ \Psi = \psi \circ f$.

There is a deep connection between Lefschetz fibrations and the surface mapping class groups (for details, see [11]). We fix a regular value $b_0 \in \Sigma_h$ and an identification ι between the regular fiber $f^{-1}(b_0)$ and the model surface Σ_g. Let γ be a smooth loop in Σ_h based at b_0. Then the pull-back bundle $\gamma^*(f)$ is described as $\Sigma_g \times [0, 1]$ with $\Sigma_g \times 0$ and $\Sigma_g \times 1$ being identified via an orientation-preserving diffeomorphism ϕ from Σ_g to itself: $f^{-1}(\gamma) \cong \Sigma_g \times [0, 1]/(x, 0) \sim (\phi(x), 1)$. Let $\mathcal{M}_g$ be the mapping class group of genus g that consists of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g. Then the map $\Phi : \pi_1(\Sigma_g \setminus \{b_1, b_2, \dots, b_n\}, b_0) \to \mathcal{M}_g$ that maps γ to ϕ is well defined, and

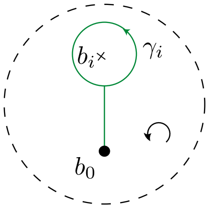

Figure 1 The oriented loop γ_i on Σ_h

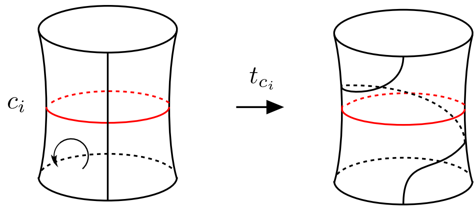

Figure 2 Right-hand Dehn twist t_{c_i} along the simple closed curve c_i

becomes an antihomomorphism. Here we do not distinguish a curve or a diffeomorphism from its homotopy or isotopy class, respectively, and we use this convention throughout this paper. We call the map Φ the *monodromy representation* of the Lefschetz fibration f. Let γ_i be a loop on $\Sigma_g \setminus \{b_1, b_2, \ldots, b_n\}$ based at b_0 that surrounds exclusively b_i with the orientation as depicted in Figure 1. The monodromy representation Φ maps γ_i to the right-hand Dehn twist t_{c_i} along the corresponding vanishing cycle c_i: $\Phi(\gamma_i) = t_{c_i}$ (see Figure 2). If we change the identification $\iota : f^{-1}(b_0) \to \Sigma_g$ to another one, then the monodromy representation Φ changes to $\rho\Phi\rho^{-1}$ for some $\rho \in \mathcal{M}_g$. A Lefschetz fibration determines the monodromy representation up to such a conjugation. Conversely, if an antihomomorphism $\Phi : \pi_1(\Sigma_h \setminus \{b_1, b_2, \ldots, b_n\}, b_0) \to \mathcal{M}_g$ that maps each γ_i to a right-hand Dehn twist is given, then we can construct a relatively minimal genus g Lefschetz fibration over Σ_h with its monodromy representation Φ. Moreover, if $g \geq 2$, then such a Lefschetz fibration is determined uniquely up to an isomorphism.

Furthermore, Lefschetz fibrations correspond to a certain type of relations in the mapping class groups. The fundamental group $\pi_1(\Sigma_h \setminus \{b_1, b_2, \ldots, b_n\}, b_0)$ has the finite presentation

$$\pi_1(\Sigma_h \setminus \{b_1, b_2, \ldots, b_n\}, b_0) = \left\langle \alpha_j, \beta_j, \gamma_i \;\middle|\; \prod_{j=1}^{h} [\alpha_j, \beta_j] = \prod_{i=1}^{n} \gamma_i \right\rangle,$$

where α_j, β_j $(j = 1, 2, \ldots, h)$ and γ_i $(i = 1, 2, \ldots, n)$ are the loops as indicated in Figure 3, and $[\alpha_j, \beta_j] = \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1}$ represents the commutator of α_j and β_j. Thus, a monodromy representation Φ satisfies $\prod_j [\Phi(\alpha_j), \Phi(\beta_j)] = \prod_i t_{c_i}$,

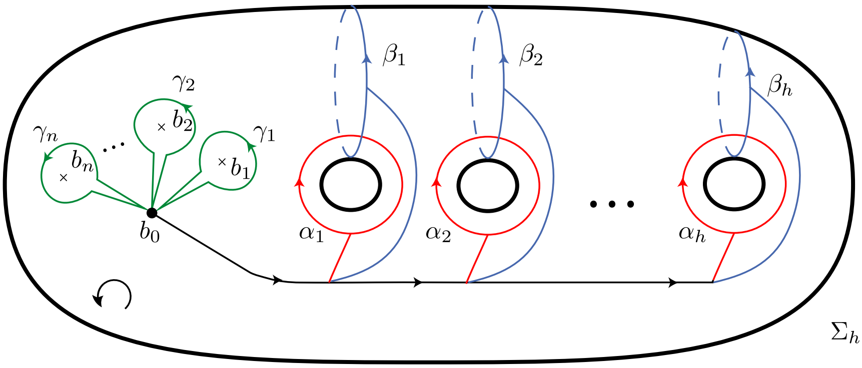

Figure 3 The generators of $\pi_1(\Sigma_h \setminus \{b_1, b_2, \ldots, b_n\}, b_0)$

and $t_{c_i} = \Phi(\gamma_i)$ are right-hand Dehn twists. We call this relation the *global monodromy* or the *monodromy factorization* of the Lefschetz fibration corresponding to Φ. Conversely, if there is a relation in the form that "a product of n right-hand Dehn twists is equal to a product of h commutators in $\mathcal{M}_g$,"

$$\prod_{j=1}^{h} [\phi_j, \psi_j] = \prod_{i=1}^{n} t_{c_i},$$

then we can define the antihomomorphism $\Phi : \pi_1(\Sigma_h \setminus \{b_1, b_2, \ldots, b_n\}, b_0) \to \mathcal{M}_g$ by setting $\Phi(\alpha_j) = \phi_j$, $\Phi(\beta_j) = \psi_j$, and $\Phi(\gamma_i) = t_{c_i}$. Consequently, we can construct a genus g Lefschetz fibration over Σ_h with n singular fibers such that its vanishing cycles are the simple closed curves c_i.

3. Mapping Class Groups of Holed Surfaces

We will use the following notation:

- $\Sigma = \Sigma_g^k$: the compact oriented surface of genus g with k boundary components,
- $\mathrm{Diff}^+(\Sigma, \partial\Sigma)$: the group of orientation-preserving self-diffeomorphisms of Σ that are the identity on the boundary,
- $\mathrm{Diff}_0^+(\Sigma, \partial\Sigma)$: the normal subgroup of $\mathrm{Diff}^+(\Sigma, \partial\Sigma)$ consisting of all elements isotopic to the identity relative to the boundary,
- $\mathcal{M}(\Sigma) = \mathrm{Diff}^+(\Sigma, \partial\Sigma)/\mathrm{Diff}_0^+(\Sigma, \partial\Sigma)$: the mapping class group of Σ,
- $\mathcal{M}_g^k = \mathcal{M}(\Sigma_g^k)$, $\quad \Sigma_g = \Sigma_g^0$, $\quad \mathcal{M}_g = \mathcal{M}_g^0$.

We will use the functional notation for the product of $\mathcal{M}_g^k$, namely, for two elements ϕ and ψ in $\mathcal{M}_g^k$, the product $\psi\phi$ means that we first apply ϕ and then ψ. In order to simplify the notation, we will denote a right-hand Dehn twist along a curve α also by α itself. A left-hand Dehn twist along α will be denoted by $\overline{\alpha}$. However, if we need to distinguish a Dehn twist from a curve or would like to emphasize a Dehn twist to be a mapping class, we will use the notation t_α.

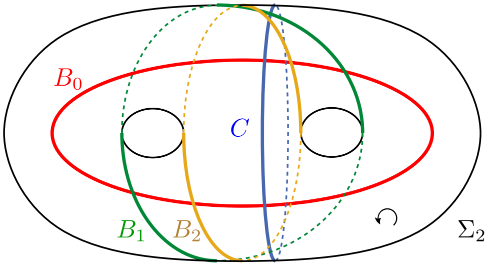

Figure 4 The curves for Matsumoto's relation

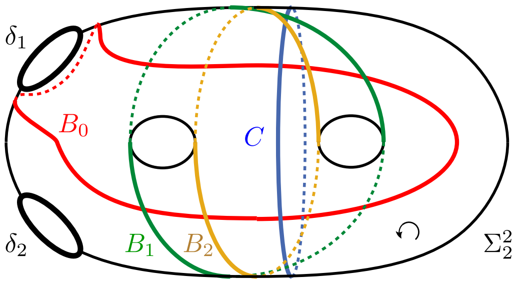

Figure 5 The curves for Matsumoto's relation with boundary

3.1. Matsumoto's Relation

Matsumoto [11] constructed a genus 2 Lefschetz fibration over the sphere on $(S^2 \times T^2)\sharp 4\overline{\mathbb{C}P^2}$ with eight singular fibers. The global monodromy of the Lefschetz fibration is

$$1 = (B_0 B_1 B_2 C)^2,$$

where the curves are as indicated in Figure 4.

Korkmaz [7] mentioned without proof that the relation

$$\delta_1 \delta_2 = (B_0 B_1 B_2 C)^2$$

holds in $\mathcal{M}_2^2$, where δ_1 and δ_2 are as depicted in Figure 5. For completeness, we give a proof here. It is convenient to recall the following well-known lemma.

LEMMA 3.1 [3]. *Let α and β be simple closed curves in Σ_g^k.*

(1) *If α does not intersect with β, then $t_\alpha t_\beta = t_\beta t_\alpha$.*

(2) *If α intersects with β transversally at exactly one point, then $t_\beta t_\alpha t_\beta = t_\alpha t_\beta t_\alpha$.*

(3) *For any $\phi \in \mathcal{M}_g^k$, we have $t_{\phi(\alpha)} = \phi t_\alpha \phi^{-1}$.*

In particular, a conjugate of a right-hand Dehn twist is also a right-hand Dehn twist.

LEMMA 3.2. *We have $\delta_1 \delta_2 = (B_0 B_1 B_2 C)^2$ in $\mathcal{M}_2^2$.*

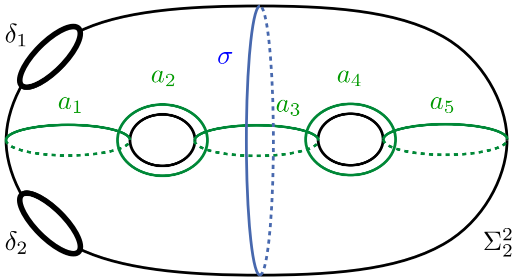

Figure 6 The curves for the chain relations

Proof. The argument here is based on the proof of Lemma 2.3 (3) in [5]. We start from the *chain relation*

$$\delta_1\delta_2 = (a_1a_2a_3a_4a_5)^6,$$

where the curves are as depicted in Figure 6 (cf. [3]). We consider this relation as $\delta_1\delta_2 = \{(a_1a_2a_3a_4a_5)^3\}^2$ and deform the part of $(a_1a_2a_3a_4a_5)^3$ as follows:

$$
\begin{aligned}
&(a_1a_2a_3a_4a_5)^3\\
&\quad = a_1a_2a_3a_4a_5a_1a_2a_3(a_4)(a_5)a_1a_2a_3a_4a_5\\
&\quad = a_1a_2a_3a_4a_5a_1a_2a_3(a_1)a_2a_4a_3a_5a_4a_5\\
&\quad = a_1a_2a_3a_4a_5(a_1a_2a_1)a_3a_2a_4a_3a_5a_4a_5\\
&\quad = a_1a_2a_3a_4a_5a_2a_1(a_2a_3a_2)a_4a_3a_5a_4a_5\\
&\quad = a_1a_2a_3a_4a_5a_2a_1a_3a_2(a_3a_4a_3)a_5a_4a_5\\
&\quad = a_1a_2a_3(a_4)(a_5)a_2a_1a_3a_2a_4a_3a_4a_5a_4a_5\\
&\quad = a_1(a_2a_3a_2)a_1a_4a_3a_2a_5a_4a_3(a_4a_5)^2 = a_1a_3a_2(a_3)a_1a_4a_3a_2a_5a_4a_3(a_4a_5)^2\\
&\quad = a_1a_3a_2a_1(a_3a_4a_3)a_2a_5a_4a_3(a_4a_5)^2 = a_1a_3a_2a_1a_4a_3(a_4)a_2a_5a_4a_3(a_4a_5)^2\\
&\quad = a_1a_3a_2a_1a_4a_3a_2(a_4a_5a_4)a_3(a_4a_5)^2 = a_1a_3a_2a_1a_4a_3a_2a_5a_4(a_5)a_3(a_4a_5)^2\\
&\quad = (a_1)a_3a_2a_1a_4a_3a_2a_5a_4a_3a_5(a_4a_5)^2 = a_3(a_1a_2a_1)a_4a_3a_2a_5a_4a_3a_5(a_4a_5)^2\\
&\quad = a_3a_2a_1(a_2)a_4a_3a_2a_5a_4a_3a_5(a_4a_5)^2 = a_3a_2a_1a_4(a_2a_3a_2)a_5a_4a_3a_5(a_4a_5)^2\\
&\quad = a_3a_2a_1a_4a_3a_2(a_3)a_5a_4a_3a_5(a_4a_5)^2 = a_3a_2a_1a_4a_3a_2a_5(a_3a_4a_3)a_5(a_4a_5)^2\\
&\quad = a_3a_2a_1a_4a_3a_2a_5a_4a_3a_4a_5(a_4a_5)^2 = a_3a_2a_1a_4a_3a_2a_5a_4a_3(a_4a_5)^3.
\end{aligned}
$$

Now, we use another chain relation

$$\sigma = (a_4a_5)^6,$$

where σ is as depicted in Figure 6. This relation can be changed into the form $(a_4a_5)^3 = (\overline{a}_5\overline{a}_4)^3\sigma$. Therefore, we have

$$
\begin{aligned}
(a_1a_2a_3a_4a_5)^3 &= a_3a_2a_1a_4a_3a_2a_5a_4a_3(\overline{a}_5\overline{a}_4)^3\sigma\\
&= a_3a_2a_1a_4a_3a_2a_5a_4a_3(\overline{a}_5)\overline{a}_4\overline{a}_5(\overline{a}_4\overline{a}_5\overline{a}_4)\sigma\\
&= a_3a_2a_1a_4a_3a_2(a_5a_4\overline{a}_5)a_3\overline{a}_4\overline{a}_5\overline{a}_5\overline{a}_4\overline{a}_5\sigma
\end{aligned}
$$

$$
\begin{aligned}
&= a_3a_2a_1a_4a_3a_2\overline{a}_4a_5(a_4a_3\overline{a}_4)\overline{a}_5\overline{a}_5\overline{a}_4\overline{a}_5\sigma\\
&= a_3a_2a_1a_4a_3a_2\overline{a}_4(a_5)\overline{a}_3a_4(a_3)\overline{a}_5\overline{a}_5\overline{a}_4\overline{a}_5\sigma\\
&= a_3a_2a_1a_4a_3a_2\overline{a}_4\overline{a}_3(a_5a_4\overline{a}_5)\overline{a}_5a_3\overline{a}_4\overline{a}_5\sigma\\
&= a_3a_2a_1a_4a_3a_2\overline{a}_4\overline{a}_3\overline{a}_4(a_5a_4\overline{a}_5)a_3\overline{a}_4\overline{a}_5\sigma\\
&= a_3a_2a_1a_4a_3a_2(\overline{a}_4\overline{a}_3\overline{a}_4)\overline{a}_4(a_5a_4a_3\overline{a}_4\overline{a}_5)\sigma\\
&= a_3a_2a_1a_4(a_3a_2\overline{a}_3)\overline{a}_4\overline{a}_3\overline{a}_4(a_5a_4a_3\overline{a}_4\overline{a}_5)\sigma\\
&= a_3a_2a_1(a_4)\overline{a}_2a_3a_2(\overline{a}_4)\overline{a}_3\overline{a}_4(a_5a_4a_3\overline{a}_4\overline{a}_5)\sigma\\
&= a_3a_2a_1\overline{a}_2(a_4a_3\overline{a}_4)a_2\overline{a}_3\overline{a}_4(a_5a_4a_3\overline{a}_4\overline{a}_5)\sigma\\
&= (a_3a_2a_1\overline{a}_2\overline{a}_3)(a_4a_3a_2\overline{a}_3\overline{a}_4)(a_5a_4a_3\overline{a}_4\overline{a}_5)\sigma.
\end{aligned}
$$

Thus, we have $\delta_1\delta_2 = \{(a_3a_2a_1\overline{a}_2\overline{a}_3)(a_4a_3a_2\overline{a}_3\overline{a}_4)(a_5a_4a_3\overline{a}_4\overline{a}_5)\sigma\}^2$. Finally, conjugating both sides of this equation by $a_4a_5a_4$, we obtain

$$
\begin{aligned}
\delta_1\delta_2 = \{&(a_4a_5a_4a_3a_2a_1\overline{a}_2\overline{a}_3\overline{a}_4\overline{a}_5\overline{a}_4)(a_4a_5a_4a_4a_3a_2\overline{a}_3\overline{a}_4\overline{a}_4\overline{a}_5\overline{a}_4)\\
&\cdot(a_4a_5a_4a_5a_4a_3\overline{a}_4\overline{a}_5\overline{a}_4\overline{a}_5\overline{a}_4)\sigma\}^2.
\end{aligned}
$$

Note that δ_1, δ_2, and σ do not intersect with a_4 nor a_5. By Lemma 3.1 (3), each factor of the right-hand side is the right-hand Dehn twist along the curve $t_{a_4}t_{a_5}t_{a_4}t_{a_3}t_{a_2}(a_1)$, $t_{a_4}t_{a_5}t_{a_4}t_{a_4}t_{a_3}(a_2)$, $t_{a_4}t_{a_5}t_{a_4}t_{a_5}t_{a_4}(a_3)$, and σ, respectively. Furthermore, we can observe that $t_{a_4}t_{a_5}t_{a_4}t_{a_3}t_{a_2}(a_1) = B_0$, $t_{a_4}t_{a_5}t_{a_4}t_{a_4}t_{a_3}(a_2) = B_1$, $t_{a_4}t_{a_5}t_{a_4}t_{a_5}t_{a_4}(a_3) = B_2$, and $\sigma = C$. This concludes the proof. □

3.2. *k-Holed Torus Relations*

Korkmaz and Ozbagci [9] systematically constructed *k-holed torus relations,* which represent the product of the right-hand Dehn twists along the simple closed curves δ_i parallel to the boundary components of the k-holed torus Σ_1^k as the product of twelve right-hand Dehn twists along certain essential simple closed curves α_j in the form

$$\delta_1\delta_2\cdots\delta_k = \alpha_1\alpha_2\cdots\alpha_{12} \quad \text{in } \mathcal{M}_1^k$$

for $1 \leq k \leq 9$. For example, they started with the well-known 1-holed torus relation

$$\delta_1 = (\alpha\beta)^6,$$

where the curves are as depicted in Figure 7. This relation is also known as the chain relation, which already appeared in the proof of Lemma 3.2. By combining the 1-holed torus relation with the *lantern relation* (see also [9]), which is a relation on the sphere with four boundary components, they obtained the 2-holed torus relation

$$\delta_1\delta_2 = (\alpha_1\alpha_2\beta)^4,$$

which is also well known, where the curves are as depicted in Figure 8. A general form of the lantern relation is

$$\delta_1\delta_2\delta_3\delta_4 = \alpha\beta\gamma,$$

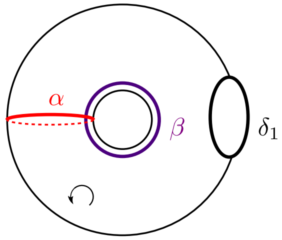

Figure 7 1-holed torus

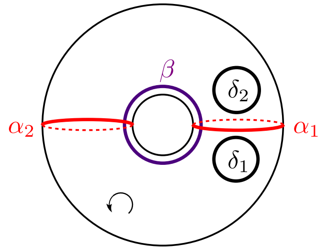

Figure 8 2-holed torus

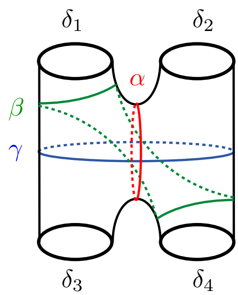

Figure 9 4-holed sphere

where the curves are as depicted in Figure 9. They successively constructed the $(k+1)$-holed torus relation by combining the k-holed torus relation with the lantern relation until they obtained the 9-holed torus relation.

However, we introduce an alternative version of the Korkmaz–Ozbagci 8-holed torus relation, which will be used to prove Theorem 1.1 (2).

LEMMA 3.3 (alternative 8-holed torus relation). *Let $\delta_1, \delta_2, \dots, \delta_7, \alpha_1, \alpha_2, \dots, \alpha_8, \beta, \sigma_1, \sigma_3, \sigma_5$, and σ_7 be the curves on Σ_1^8 as depicted in Figure 10. Then we have*

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 = \alpha_1\alpha_3\alpha_5\alpha_7\sigma_1\sigma_3\sigma_5\sigma_7\beta'''\beta''\beta'\beta$$

in $\mathcal{M}_1^8$, where $\beta' = \overline{\alpha}_6\overline{\alpha}_2\beta\alpha_2\alpha_6$, $\beta'' = \overline{\alpha}_8\overline{\alpha}_4\beta'\alpha_4\alpha_8$, and $\beta''' = \overline{\alpha}_6\overline{\alpha}_2\beta''\alpha_2\alpha_6$.

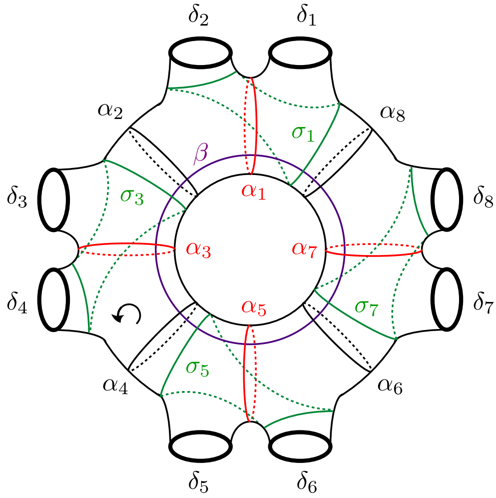

Figure 10 8-holed torus

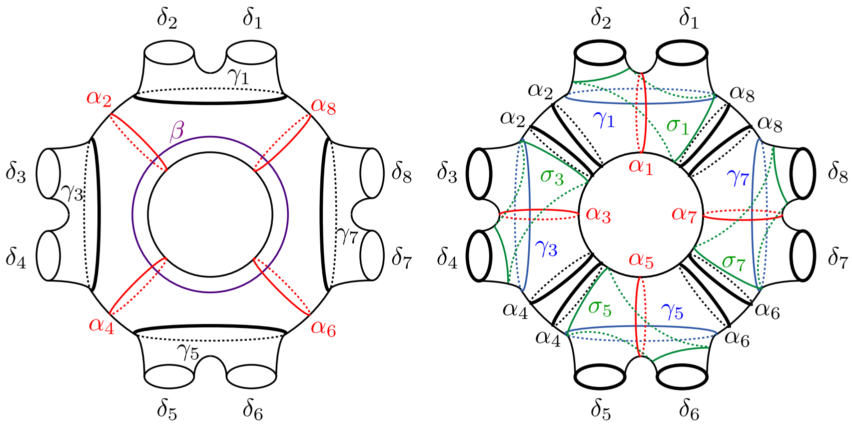

Figure 11 The 4-holed torus relation and the lantern relations

Proof. Consider the 8-holed torus in Figure 11. We combine the Korkmaz–Ozbagci [9] 4-holed torus relation with four lantern relations. The 4-holed torus relation on the subsurface bounded by $\{\gamma_1, \gamma_3, \gamma_5, \gamma_7\}$ is

$$\gamma_1\gamma_3\gamma_5\gamma_7 = (\alpha_8\alpha_4\beta\alpha_2\alpha_6\beta)^2.$$

The four lantern relations are

$$\delta_1\delta_2\alpha_2\alpha_8 = \alpha_1\sigma_1\gamma_1, \qquad \delta_3\delta_4\alpha_4\alpha_2 = \alpha_3\sigma_3\gamma_3,$$
$$\delta_5\delta_6\alpha_6\alpha_4 = \alpha_5\sigma_5\gamma_5, \qquad \delta_7\delta_8\alpha_8\alpha_6 = \alpha_7\sigma_7\gamma_7.$$

Combining the four lantern relations, we have

$$\delta_1\delta_2\alpha_2\alpha_8\delta_3\delta_4\alpha_4\alpha_2\delta_5\delta_6\alpha_6\alpha_4\delta_7\delta_8\alpha_8\alpha_6 = \alpha_1\sigma_1\gamma_1\alpha_3\sigma_3\gamma_3\alpha_5\sigma_5\gamma_5\alpha_7\sigma_7\gamma_7.$$

Using the commutativity relation (Lemma 3.1 (1)) and multiplying both sides by $\overline{\alpha}_2^2\overline{\alpha}_4^2\overline{\alpha}_6^2\overline{\alpha}_8^2$, we obtain

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 &= \overline{\alpha}_2^2\overline{\alpha}_4^2\overline{\alpha}_6^2\overline{\alpha}_8^2\alpha_1\sigma_1\gamma_1\alpha_3\sigma_3\gamma_3\alpha_5\sigma_5\gamma_5\alpha_7\sigma_7\gamma_7\\
&= \overline{\alpha}_2^2\overline{\alpha}_4^2\overline{\alpha}_6^2\overline{\alpha}_8^2\alpha_1\alpha_3\alpha_5\alpha_7\sigma_1\sigma_3\sigma_5\sigma_7\gamma_1\gamma_3\gamma_5\gamma_7\\
&= \alpha_1\alpha_3\alpha_5\alpha_7\sigma_1\sigma_3\sigma_5\sigma_7\overline{\alpha}_2^2\overline{\alpha}_4^2\overline{\alpha}_6^2\overline{\alpha}_8^2\gamma_1\gamma_3\gamma_5\gamma_7.
\end{aligned}$$

Then we can substitute the 4-holed torus relation to obtain

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 &= \alpha_1\alpha_3\alpha_5\alpha_7\sigma_1\sigma_3\sigma_5\sigma_7\overline{\alpha}_2^2\overline{\alpha}_4^2\overline{\alpha}_6^2\overline{\alpha}_8^2(\alpha_8\alpha_4\beta\alpha_2\alpha_6\beta)^2\\
&= \alpha_1\alpha_3\alpha_5\alpha_7\sigma_1\sigma_3\sigma_5\sigma_7\overline{\alpha}_2^2\overline{\alpha}_4^2\overline{\alpha}_6^2\overline{\alpha}_8^2\\
&\quad\cdot\alpha_8\alpha_4(\alpha_2\alpha_6\alpha_8\alpha_4\alpha_2\alpha_6\overline{\alpha}_6\overline{\alpha}_2\overline{\alpha}_4\overline{\alpha}_8\overline{\alpha}_6\overline{\alpha}_2)\beta\\
&\quad\cdot\alpha_2\alpha_6(\alpha_8\alpha_4\alpha_2\alpha_6\overline{\alpha}_6\overline{\alpha}_2\overline{\alpha}_4\overline{\alpha}_8)\beta\\
&\quad\cdot\alpha_8\alpha_4(\alpha_2\alpha_6\overline{\alpha}_6\overline{\alpha}_2)\beta\\
&\quad\cdot\alpha_2\alpha_6\beta\\
&= \alpha_1\alpha_3\alpha_5\alpha_7\sigma_1\sigma_3\sigma_5\sigma_7\overline{\alpha}_2^2\overline{\alpha}_4^2\overline{\alpha}_6^2\overline{\alpha}_8^2\alpha_2^2\alpha_4^2\alpha_6^2\alpha_8^2\\
&\quad\cdot(\overline{\alpha}_6\overline{\alpha}_2\overline{\alpha}_4\overline{\alpha}_8\overline{\alpha}_6\overline{\alpha}_2\beta\alpha_2\alpha_6\alpha_8\alpha_4\alpha_2\alpha_6)\\
&\quad\cdot(\overline{\alpha}_6\overline{\alpha}_2\overline{\alpha}_4\overline{\alpha}_8\beta\alpha_8\alpha_4\alpha_2\alpha_6)\cdot(\overline{\alpha}_6\overline{\alpha}_2\beta\alpha_2\alpha_6)\cdot\beta\\
&= \alpha_1\alpha_3\alpha_5\alpha_7\sigma_1\sigma_3\sigma_5\sigma_7\beta'''\beta''\beta'\beta.
\end{aligned}$$

This gives the claimed relation. □

4. Proof of the Upper Bounds

Now, we prove Theorem 1.1 by constructing Lefschetz fibrations with the claimed number of singular fibers. Matsumoto's relation will be used for the proof of Theorem 1.1 (1), and the alternative 8-holed torus relation will be used for (2).

Proof of Theorem 1.1 (1). We embed the surface Σ_2^2 into Σ_{3+k} ($k \geq 0$) as in Figure 12. Then, by Lemma 3.2 we have

$$\begin{aligned}
\delta_1\delta_2 &= (B_0B_1B_2C)^2\\
&= B_0B_1B_2CB_0B_1B_2C.
\end{aligned}$$

Since B_0 commutes with δ_1 and δ_2, by conjugating both sides of the equation by $\overline{B}_0$ we get

$$\delta_1\delta_2 = B_1B_2CB_0B_1B_2CB_0.$$

Multiplying both sides by $\overline{B}_2\overline{B}_1$, we obtain

$$\overline{B}_2\overline{B}_1\delta_1\delta_2 = CB_0B_1B_2CB_0.$$

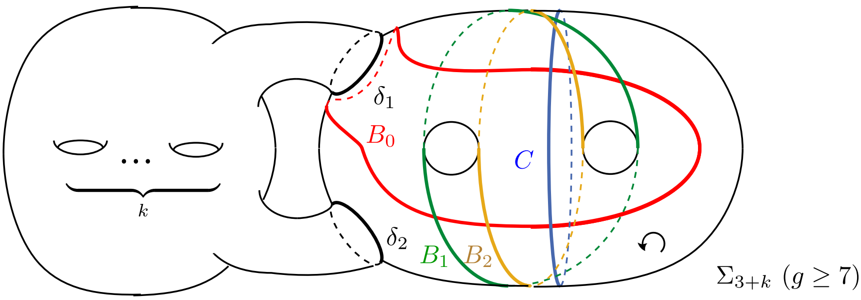

Figure 12 Embedding Σ_2^2 in Σ_{3+k}

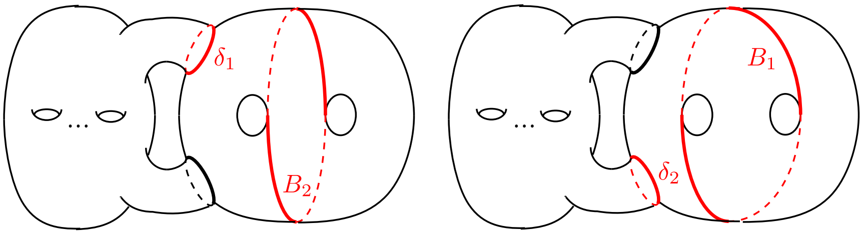

Figure 13 $\Sigma_{3+k} \setminus (B_2 \cup \delta_1)$ and $\Sigma_{3+k} \setminus (B_1 \cup \delta_2)$ are both connected

Then the term on the left-hand side is a commutator. To see this, by using the commutativity relation we first rearrange it as

$$\overline{B}_2\overline{B}_1\delta_1\delta_2 = (\overline{B}_2\delta_1)(\overline{B}_1\delta_2).$$

We observe that if one cuts Σ_{3+k} along B_2 and δ_1, then the resulting surface is still connected. If one cuts Σ_{3+k} along B_1 and δ_2, then the resulting surface is also connected (see Figure 13). By the classification of surfaces, this observation implies that there exists an element $\phi \in \mathcal{M}_{3+k}$ such that $\phi(\delta_1) = B_1$ and $\phi(B_2) = \delta_2$. Therefore, we have

$$\begin{aligned}(\overline{B}_2\delta_1)(\overline{B}_1\delta_2) &= \overline{B}_2\delta_1\overline{\phi(\delta_1)}\phi(B_2)\\ &= \overline{B}_2\delta_1\phi\overline{\delta}_1\phi^{-1}\phi B_2\phi^{-1}\\ &= (\overline{B}_2\delta_1)\phi(\overline{\delta}_1 B_2)\phi^{-1}\\ &= [\overline{B}_2\delta_1, \phi].\end{aligned}$$

Consequently, we obtain

$$[\overline{B}_2\delta_1, \phi] = \underbrace{C B_0 B_1 B_2 C B_0}_{6}.$$

As we have mentioned before, this relation enables us to construct, for all $g = k + 3 \geq 3$, a genus g Lefschetz fibration over the torus with six singular fibers, i.e., $N(g, 1) \leq 6$ for all $g \geq 3$. $\square$

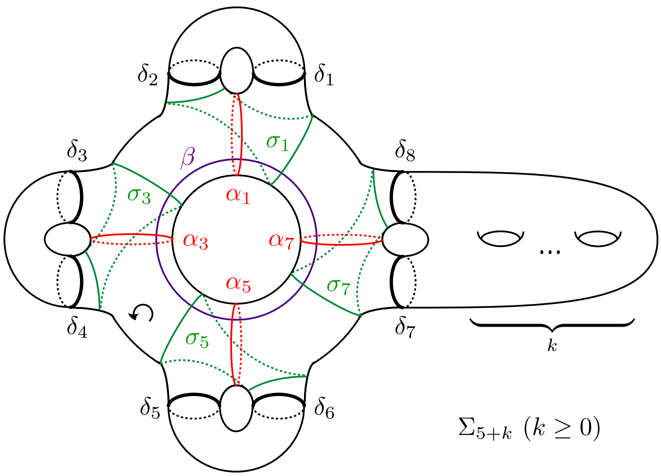

Figure 14 Embedding Σ_1^8 in Σ_{5+k}

Proof of Theorem 1.1 (2). For the alternative 8-holed torus relation, the similar procedure as above works well as follows. Embed the 8-holed torus Σ_1^8 into Σ_{5+k} $(k \geq 0)$ as in Figure 14. By Lemma 3.3 we have

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 = \alpha_1\alpha_3\alpha_5\alpha_7\sigma_1\sigma_3\sigma_5\sigma_7\beta'''\beta''\beta'\beta.$$

We rearrange this as follows:

$$\begin{aligned}\beta'''\beta''\beta'\beta &= \overline{\sigma}_7\overline{\sigma}_5\overline{\sigma}_3\overline{\sigma}_1\overline{\alpha}_7\overline{\alpha}_5\overline{\alpha}_3\overline{\alpha}_1\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\\ &= \delta_2\delta_4\delta_6\delta_8\overline{\sigma}_1\overline{\sigma}_3\overline{\sigma}_5\overline{\sigma}_7\cdot\delta_7\delta_5\delta_3\delta_1\overline{\alpha}_7\overline{\alpha}_5\overline{\alpha}_3\overline{\alpha}_1.\end{aligned}$$

Here let us assume that there is a mapping class $\phi \in \Sigma_{5+k}$ such that

$$\begin{aligned}\phi(\alpha_1) &= \delta_2, & \phi(\delta_1) &= \sigma_1,\\ \phi(\alpha_3) &= \delta_4, & \phi(\delta_3) &= \sigma_3,\\ \phi(\alpha_5) &= \delta_6, & \phi(\delta_5) &= \sigma_5,\\ \phi(\alpha_7) &= \delta_8, & \phi(\delta_7) &= \sigma_7,\end{aligned}$$

as described in Figure 15.

Then we have

$$\begin{aligned}&\delta_2\delta_4\delta_6\delta_8\overline{\sigma}_1\overline{\sigma}_3\overline{\sigma}_5\overline{\sigma}_7\cdot\delta_7\delta_5\delta_3\delta_1\overline{\alpha}_7\overline{\alpha}_5\overline{\alpha}_3\overline{\alpha}_1\\ &\quad= \phi(\alpha_1)\phi(\alpha_3)\phi(\alpha_5)\phi(\alpha_7)\overline{\phi(\delta_1)\phi(\delta_3)\phi(\delta_5)\phi(\delta_7)}\cdot\delta_7\delta_5\delta_3\delta_1\overline{\alpha}_7\overline{\alpha}_5\overline{\alpha}_3\overline{\alpha}_1\\ &\quad= \phi\alpha_1\phi^{-1}\phi\alpha_3\phi^{-1}\phi\alpha_5\phi^{-1}\phi\alpha_7\phi^{-1}\phi\overline{\delta}_1\phi^{-1}\phi\overline{\delta}_3\phi^{-1}\phi\overline{\delta}_5\phi^{-1}\phi\overline{\delta}_7\phi^{-1}\\ &\qquad\cdot\delta_7\delta_5\delta_3\delta_1\overline{\alpha}_7\overline{\alpha}_5\overline{\alpha}_3\overline{\alpha}_1\\ &\quad= \phi\alpha_1\alpha_3\alpha_5\alpha_7\overline{\delta}_1\overline{\delta}_3\overline{\delta}_5\overline{\delta}_7\phi^{-1}\delta_7\delta_5\delta_3\delta_1\overline{\alpha}_7\overline{\alpha}_5\overline{\alpha}_3\overline{\alpha}_1\\ &\quad= [\phi, \alpha_1\alpha_3\alpha_5\alpha_7\overline{\delta}_1\overline{\delta}_3\overline{\delta}_5\overline{\delta}_7].\end{aligned}$$

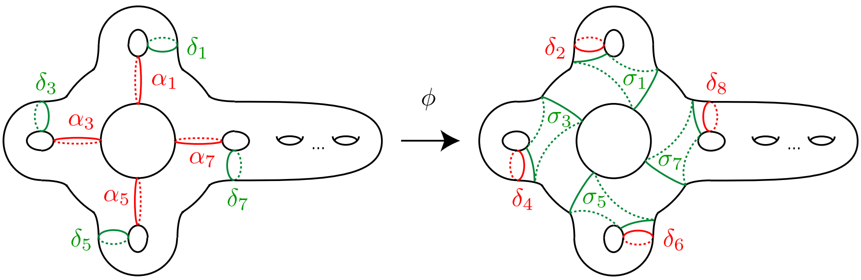

Figure 15 The action of ϕ on the curves

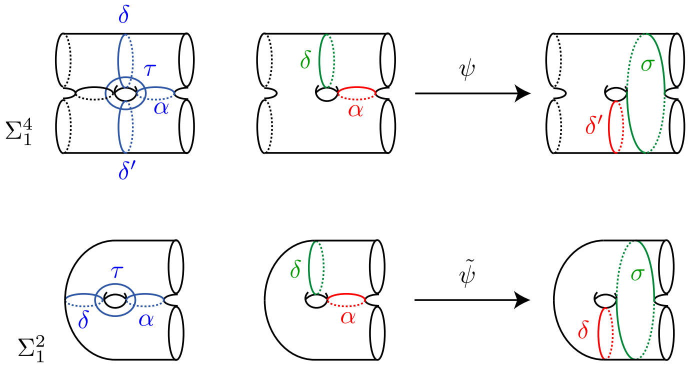

Figure 16 The action of ψ and $\tilde{\psi}$ on the curves

Therefore, we obtain

$$[\phi, \alpha_1\alpha_3\alpha_5\alpha_7\overline{\delta}_1\overline{\delta}_3\overline{\delta}_5\overline{\delta}_7] = \underbrace{\beta'''\beta''\beta'\beta}_{4},$$

which provides a genus g Lefschetz fibration over the torus with four singular fibers for $g = 5 + k \geq 5$; hence, we get that $N(g, 1) \leq 4$ for $g \geq 5$.

Now we need to prove the existence of such a map ϕ. We construct ϕ explicitly. Consider the holed tori Σ_1^4 and Σ_1^2 in Figure 16. Keeping subsurfaces of Σ_{5+k} in mind, we first observe the following lemma.

LEMMA 4.1. (1) *In the mapping class group* $\mathcal{M}_1^4$, *set*

$$\psi = t_\tau t_\delta t_{\delta'} t_\tau t_\tau t_\alpha t_\delta t_\tau .$$

Then ψ *maps the pair of simple closed curves* (α, δ) *to* (δ', σ) *on* Σ_1^4.

(2) *In the mapping class group* $\mathcal{M}_1^2$, *the element*

$$\tilde{\psi} = t_\tau t_\delta t_\delta t_\tau t_\tau t_\alpha t_\delta t_\tau$$

maps the pair (α, δ) *to* (δ, σ) *on* Σ_1^2.

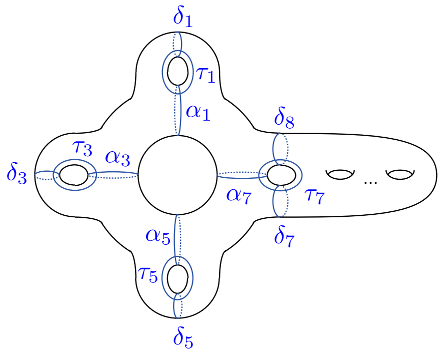

Figure 17 The Dehn twists for the definition of ϕ

Proof. (1) One can easily check that $t_\tau t_\alpha t_\delta t_\tau$ maps the pair (α, δ) to (δ, α) and $t_\tau t_\delta t_{\delta'} t_\tau$ maps the pair (δ, α) to (δ', σ). Then (2) immediately follows from (1). □

We continue the proof for Theorem 1.1 (2). Let us define the mapping class $\phi \in \mathcal{M}_{5+k}$ as follows:

$$\phi := \tau_1\delta_1\delta_1\tau_1\tau_1\alpha_1\delta_1\tau_1 \cdot \tau_3\delta_3\delta_3\tau_3\tau_3\alpha_3\delta_3\tau_3 \cdot \tau_5\delta_5\delta_5\tau_5\tau_5\alpha_5\delta_5\tau_5 \cdot \tau_7\delta_7\delta_8\tau_7\tau_7\alpha_7\delta_7\tau_7,$$

where the curves are as depicted in Figure 17. Then, by Lemma 4.1 we can easily see that ϕ satisfies the required conditions. This completes the proof. □

Remark 4.1. (1) We can apply the similar procedure as above to the (possibly, suitably modified) k-holed torus relation ($2 \leq k \leq 8$) to obtain a genus g Lefschetz fibration over the torus with $12 - k$ singular fibers for $g \geq 2$ ($k = 2$), $g \geq 3$ ($k = 3, 4$), $g \geq 4$ ($k = 5, 6$), or $g \geq 5$ ($k = 7, 8$), respectively. So it might be interesting to ask whether the number of singular fibers in a genus g Lefschetz fibration over the torus can be arbitrary. In other words, for any $n \geq N(g, 1)$, can we find a commutator in $\mathcal{M}_g$ which can be written as a product of n right-hand Dehn twists? We have eliminated the case $k = 9$ in the above argument even though we might have $N(g, 1) \leq 3$. It is because there is a difficulty to apply our technique to the 9-holed torus relation caused by the complexity of simple closed curves appearing in it. The author does not know whether an alternative version of the 9-holed torus relation can be applied.

(2) There is a possibility that the monodromy of Xiao's fibration [15] can be used to prove $N(g, 1) \leq 5$ for all $g \geq 3$. As we mentioned in Section 1, Xiao has discovered a genus 2 Lefschetz fibration over the sphere with seven singular fibers. Although the exact monodromy of Xiao's fibration has not been known yet, by the existence of such a Lefschetz fibration we know that there exists a relation in $\mathcal{M}_2$ such as

$$c_1c_2c_3c_4c_5c_6c_7 = 1,$$

where the curves c_i are essential simple closed curves in Σ_2. Moreover, from the information of the abelianization of $\mathcal{M}_2$ we can deduce that four of c_i are nonseparating and the other three are separating. In addition, let us assume that the relation can be lifted in $\mathcal{M}_2^2$ in the form

$$\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4\tilde{c}_5\tilde{c}_6\tilde{c}_7 = \delta_1\delta_2,$$

where the curves $\tilde{c}_i$ are simple closed curves in Σ_2^2 such that $P(\tilde{c}_i) = c_i$ for the natural homomorphism $P : \mathcal{M}_2^2 \to \mathcal{M}_2$, and δ_1 and δ_2 are the simple closed curves in Σ_2^2 parallel to the boundary components. Then our technique used to prove Theorem 1.1 can be applied to the above relation by embedding Σ_2^2 into Σ_{3+k} ($k \geq 0$). This argument would imply that $N(g, 1) \leq 5$ for all $g \geq 3$. Note that the assumption of the existence of $\{\tilde{c}_i\}$ is equivalent to the existence of a genus 2 Lefschetz fibration over the sphere with seven singular fibers and two disjoint (-1)-sections (cf. [2]).

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