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A Geometric Criterion to Be Pseudo-Anosov

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Abstract. If S is a hyperbolic surface and $\mathring{S}$ the surface obtained from S by removing a point, the mapping class groups $\mathrm{Mod}(S)$ and $\mathrm{Mod}(\mathring{S})$ fit into a short exact sequence

$$1 \to \pi_1(S) \to \mathrm{Mod}(\mathring{S}) \to \mathrm{Mod}(S) \to 1.$$

We give a new criterion for mapping classes in the kernel to be pseudo-Anosov using the geometry of hyperbolic 3-manifolds. Namely, we show that if M is an ε-thick hyperbolic manifold homeomorphic to $S \times \mathbb{R}$, then an element of $\pi_1(M) \cong \pi_1(S)$ represents a pseudo-Anosov element of $\mathrm{Mod}(\mathring{S})$ if its geodesic representative is "wide." We establish similar criteria where M is replaced with a coarsely hyperbolic surface bundle coming from a δ-hyperbolic surface–group extension.

1. Introduction: Mapping Classes from Fibrations

If X is a surface, let $\mathrm{Mod}(X) = \pi_0(\mathrm{Homeo}^+(X))$ be its mapping class group, and let $\mathring{X}$ be the surface obtained from X by removing a point.

Surface bundles $X \to E \to B$ over a space B with fiber X are determined by homomorphisms $\pi_1(B) \to \mathrm{Mod}(X)$; see [21]. Thurston's geometrization theorem for fibered 3-manifolds opens the door to an investigation of the geometric behavior of such surface bundles. For instance, there are necessary and sufficient geometric conditions on $\pi_1(B) \to \mathrm{Mod}(X)$ that guarantee that $\pi_1(E)$ is word-hyperbolic; see [9; 10]. To verify these conditions, one is often faced with the problem of determining when a subgroup $G < \mathrm{Mod}(X)$ is purely pseudo-Anosov, a problem we take up here.

To describe our first result, let N be a closed hyperbolic 3-manifold that fibers over the circle with fiber a surface S, and let $N_{\mathbb{Z}} \to N$ be the corresponding infinite cyclic covering of N. The long exact sequence of the fibration is concentrated in a short exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \pi_1(N) \longrightarrow \mathbb{Z} \longrightarrow 1, \tag{1.1}$$

which injects into the Birman exact sequence [4]

$$1 \longrightarrow \pi_1(S) \longrightarrow \mathrm{Mod}(\mathring{S}) \longrightarrow \mathrm{Mod}(S) \longrightarrow 1.$$

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Choosing a lift t of the generator of $\mathbb{Z}$ to $\pi_1(N)$, any element of $\pi_1(N)$ may be written uniquely as a product gt^k, where g is an element of $\pi_1(S)$. When k is nonzero, this element represents a pseudo-Anosov mapping class in $\mathrm{Mod}(\mathring{S})$. When k is zero, this element lies in $\pi_1(S)$, and, by a theorem of Kra [15] (see also [14]), it is pseudo-Anosov in $\mathrm{Mod}(\mathring{S})$ if and only if it fills S. These observations were first made by Agol [2].

CRITERION 1 (Agol's criterion). *A subgroup H of $\pi_1(N)$ is a purely pseudo-Anosov subgroup of* $\mathrm{Mod}(\mathring{S})$ *if and only if every nontrivial element of $H \cap \pi_1(S)$ fills S.*

This topological criterion is very difficult to check. Our main theorem is a geometric criterion for an element of $\pi_1(N_{\mathbb{Z}})$ to be filling.

THEOREM 5. *Let S be a closed oriented surface of Euler characteristic $\chi = \chi(S) < 0$, and let ε and K be positive numbers. There is a $W = W(\chi, \varepsilon, K) > 0$ such that the following holds. Equip $M = S \times \mathbb{R}$ with any ε-thick hyperbolic structure, and let $\ell : M \to \mathbb{R}$ be a K-Lipschitz submersion. If Y is a proper incompressible subsurface of S and $\mathcal{C}_Y$ is the convex core of the corresponding cover of M, then the width* $\mathrm{diam}(\ell(\mathcal{C}_Y))$ *of $\mathcal{C}_Y$ is at most W. In particular, if γ is a geodesic loop in M such that* $\mathrm{diam}(\ell(\gamma)) > W$, *then γ fills S.*

If $\mathrm{diam}(\ell(\gamma)) > W$, then we say that γ is *wide.* Agol's criterion then becomes:

CRITERION 2 (width criterion). *A subgroup H of $\pi_1(N)$ is a purely pseudo-Anosov subgroup of* $\mathrm{Mod}(\mathring{S})$ *if every nontrivial element of $H \cap \pi_1(S)$ is wide.*

REMARKS. 1. The fact that geodesic representatives in N of elements of $\pi_1(S)$ realized by *simple* closed curves on S are not wide is fairly straightforward.
2. Filling elements need not be wide.

This criterion, and Theorem 5, arose out of the authors' attempts to find purely pseudo-Anosov surface subgroups of mapping class groups by exploiting the abundance of surface subgroups of hyperbolic 3-manifold groups (see [12]).

In Section 3, we prove a generalization of Theorem 5 to the case of punctured surfaces, Theorem 9. The authors and S. Dowdall use these theorems to prove the following.

THEOREM 3 (Dowdall–Kent–Leininger [8]). *Suppose N is a finite volume hyperbolic 3-manifold that fibers over the circle with fiber S and $G < \pi_1(N)$. As a subgroup of* $\mathrm{Mod}(\mathring{S})$, *$G$ is convex cocompact in the sense of Farb and Mosher* [9] *if and only if G is finitely generated and purely pseudo-Anosov.*

In particular, this answers a special case of Question 1.5 of [9] and generalizes Theorem 6.1 of [14].

In Section 4, we generalize Theorem 5 in a different direction by replacing M with a hyperbolic surface-group extension Γ.

THEOREM 11. *Let*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \xrightarrow{\ell} G \longrightarrow 1 \tag{1.2}$$

be a short exact sequence with Γ *a hyperbolic group, and equip* Γ *and* G *with word metrics on finite generating sets. There is a* $W > 0$ *such that, given any nonfilling* γ *in* $\pi_1(S)$ *and any* γ*-quasi-invariant geodesic* $\mathcal{G}$ *in* Γ, *we have* $\operatorname{diam}(\ell(\mathcal{G})) \leq W$.

Given an infinite cyclic subgroup of G, we obtain a short exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma_{\mathbb{Z}} \longrightarrow \mathbb{Z} \longrightarrow 1$$

that injects into (1.2), and one may be tempted to argue that Theorem 11 thus follows quickly from Criterion 2. This attack is thwarted by the fact that $\Gamma_{\mathbb{Z}}$ is wildly metrically distorted in Γ.

Again, the authors and S. Dowdall apply Theorem 11 to prove the following theorem.

THEOREM 4 (Dowdall–Kent–Leininger [8]). *Let*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

be a short exact sequence with Γ *hyperbolic. Any quasi-convex finitely generated purely pseudo-Anosov subgroup of* $\Gamma \subset \operatorname{Mod}(\mathring{S})$ *is convex cocompact.*

2. Criterion to Fill

If M is manifold, $\ell : M \to \mathbb{R}$ is a function, and X is a subset of M, we define the *width of* X *with respect to* ℓ (or simply the *width of* X) to be $\operatorname{diam}(\ell(X))$. If X is a subset of any covering space of $\Pi : N \to M$, we define the *width* of X to be $\operatorname{diam}(\ell(\Pi(X)))$.

Let S be a closed orientable hyperbolic surface. A closed curve in $S \times \mathbb{R}$ is *filling* if its projection to S is filling.

If $M = S \times \mathbb{R}$ is equipped with a hyperbolic metric and Y is an incompressible subsurface of S, then we let Γ_Y be the Kleinian group corresponding to $\pi_1(Y) \subset \pi_1(M)$, and $\Pi : M_Y = \mathbb{H}^3/\Gamma_Y \to M$ the corresponding cover. We let $\mathcal{C}_Y \subset M_Y$ denote the convex core. We say that this hyperbolic structure is ε-thick if the injectivity radius at every point is bounded below by ε.

THEOREM 5. *Let* S *be a closed oriented surface of Euler characteristic* $\chi = \chi(S) < 0$, *and let* ε *and* K *be positive numbers. There is a* $W = W(\chi, \varepsilon, K) > 0$ *such that the following holds. Equip* $M = S \times \mathbb{R}$ *with any* ε*-thick hyperbolic structure, and let* $\ell : M \to \mathbb{R}$ *be a* K*-Lipschitz submersion. If* Y *is a proper incompressible subsurface of* S, *then the width of* $\mathcal{C}_Y$ *is at most* W. *In particular, if* γ *is a geodesic loop in* M *such that* $\operatorname{diam}(\ell(\gamma)) > W$, *then* γ *fills* S.

When M is the cover of a fibered hyperbolic 3-manifold corresponding to the fiber, the following lemma follows from the main theorem of [24].

LEMMA 6. *If $M = S \times \mathbb{R}$ is equipped with a hyperbolic structure without parabolics, and Y is a proper incompressible subsurface of S, then the group Γ_Y is a Schottky group (a convex cocompact free Kleinian group).*

Proof. Suppose that Γ_Y is not a Schottky group.

Since M has no cusps, and Π is a covering, M_Y also has no cusps. So Γ_Y must be geometrically infinite.

If we let S_Y denote the covering of S corresponding to Y (which is homeomorphic to the interior of Y), then $M_Y \cong S_Y \times \mathbb{R}$ is homeomorphic to the interior of a handlebody. By Canary's covering theorem [6], there is a neighborhood $\mathcal{E}$ of the end of M_Y such that $\Pi|_{\mathcal{E}}$ is finite-to-one. Since Π is a covering map and $M_Y - \mathcal{E}$ is compact, we conclude that Π is finite-to-one. But M is homotopy equivalent to a closed surface, and Γ_Y is free. $\square$

Proof of Theorem 5. Let ∂Y^* be the geodesic representative of ∂Y in M.

The geodesic multicurve ∂Y^* is realized by a pleated surface $\mathcal{F} \to M$ (see Theorem 5.3.6 of [7]). Since M is ε-thick and $\mathcal{F} \to M$ is a 1-Lipschitz incompressible map, there is a number $B = B(\chi, \varepsilon)$ that bounds the diameter of (the image of) $\mathcal{F}$ in M. Since ℓ is K-Lipschitz, the width of $\mathcal{F}$ is at most KB, and hence so is the width of ∂Y^*.

If $\mathcal{C}_Y$ has no interior, we let $\partial\mathcal{C}_Y$ be the double $\mathfrak{D}\mathcal{C}_Y$, considered as a map $\mathfrak{D}\mathcal{C}_Y \to \mathcal{C}_Y \to M$. Note that since Γ_Y is Schottky, $\partial\mathcal{C}_Y$ is a nonempty, compact pleated surface.

LEMMA 7. *There is a number $W = W(\chi, \varepsilon, K)$ such that $\partial\mathcal{C}_Y$ has width less than W.*

Proof. Let δ be less than the minimum of ε and the two-dimensional Margulis constant.

There is a number $D = D(\chi, \varepsilon)$ such that $\partial\mathcal{C}_Y$ lies in the D-neighborhood of ∂Y^*. To see this, let $\mathcal{P}(\delta)$ be the δ-thin part of $\partial\mathcal{C}_Y$, and note that the components of $\partial\mathcal{C}_Y - \mathcal{P}(\delta)$ have diameters bounded above by a constant $E = E(\chi, \delta)$. Since M is ε-thick, every loop in $\mathcal{P}(\delta)$ bounds a disk in M_Y. Moreover, every point in $\mathcal{P}(\delta)$ lies in a loop of length less than δ. Such a loop bounds a disk in M_Y of diameter at most δ, and since ∂Y^* is disk-busting, every point of $\mathcal{P}(\delta)$ is within δ of ∂Y^*. But every point of $\partial\mathcal{C}_Y - \mathcal{P}(\delta)$ is within E of $\mathcal{P}(\delta)$. Letting $D = E + \delta$, we have $\partial\mathcal{C}_Y$ contained in the D-neighborhood of ∂Y^*.

Since ∂Y^* has width at most KB, the width of $\partial\mathcal{C}_Y$ is at most $W = KB + 2KD$. $\square$

If $\partial\mathcal{C}_Y = \mathcal{C}_Y$, we are done by Lemma 7. So we assume that $\mathcal{C}_Y^\circ \neq \emptyset$. The map $\pi : \mathcal{C}_Y \to M$ is an immersion on $\mathcal{C}_Y^\circ$, and since ℓ is a submersion, the composition $\ell \circ \pi : \mathcal{C}_Y \to \mathbb{R}$ is a submersion on $\mathcal{C}_Y^\circ$ as well. It follows that $\ell \circ \pi$ achieves its extrema on $\partial\mathcal{C}_Y$. So the width of $\mathcal{C}_Y$ equals the width of $\partial\mathcal{C}_Y$, which is bounded by Lemma 7. □

3. The Cusped Case

Let S be a noncompact finite-volume hyperbolic surface with Euler characteristic $\chi < 0$, and let M be a hyperbolic manifold homeomorphic to $S \times \mathbb{R}$. Note that when M is the infinite cyclic cover of a 3-manifold fibering over the circle, the lift of the bundle projection is not a Lipschitz map to $\mathbb{R}$. Since such projections are natural for measuring width, we find the naive analog of Theorem 5 too restrictive. In this section, we discuss the correct analog, where one must first project onto the complement of a neighborhood of the cusps before taking a Lipschitz projection to $\mathbb{R}$ to compute widths.

Let $M = S \times \mathbb{R}$, and equip M with a type-preserving hyperbolic structure without accidental parabolics. Let $P \subset S$ denote a standard cusp neighborhood of the ends, so that $S^0 = S - P$ is a compact surface with boundary, and $S^0 \to S$ is a homotopy equivalence. Let $\mathbf{P} = P \times \mathbb{R} \subset M$ and set

$$M^0 = M - \mathbf{P} = S^0 \times \mathbb{R}.$$

We assume that the restriction of the hyperbolic metric to each component of $\mathbf{P}$ is isometric to a standard cusp neighborhood

$$\mathbf{P}_3(r) = \{(z,t) \in \mathbb{H}^3 \mid t > r\}/\langle (z,t) \mapsto (z+1,t)\rangle$$

for some r satisfying $\operatorname{arccosh}(1 + 1/2r^2) < \mu_3$, where μ_3 is the three-dimensional Margulis constant. We often write $\mathbf{P}(r) = \mathbf{P}$ when r is relevant.

Given an essential subsurface $Y \subset S$, let $M_Y \to M$ denote the cover corresponding to Y, and $\mathcal{C}_Y \subset M_Y$ its convex core. An argument similar to the proof of Lemma 6 shows that the Kleinian group Γ_Y corresponding to Y is geometrically finite without accidental parabolics. The boundary $\partial\mathcal{C}_Y$ is a locally convex pleated surface whose cusps are carried to cusps of M_Y (consequently, $\mathcal{C}_Y$ is bent along a compact geodesic lamination). Each cusp of $\partial\mathcal{C}_Y$ has a *standard neighborhood* $\mathcal{U}_r$ isometric to

$$\mathbf{P}_2(r) = \{(x,t) \in \mathbb{H}^2 \mid t > r\}/\langle (x,t) \mapsto (x+1,t)\rangle.$$

Note that there is a definite cusp neighborhood in any hyperbolic surface that misses every compact geodesic lamination. To see this, fix a cusp neighborhood and consider a sequence of leaves of compact laminations reaching deeper and deeper into the cusp neighborhood. By compactness, these leaves must be tangent to horocycles deeper and deeper in the cusp neighborhood. But these horocycles are getting shorter and shorter, from which it is apparent that the leaves must eventually have self-intersections, providing a contradiction. It follows that there is an $r_0 = r_0(\chi)$ such that $\mathcal{U}_r$ is disjoint from the pleating locus when $r \geq r_0$. It follows that, for $r \geq r_0$, our $\mathcal{U}_r$ is totally geodesic. We take $r \geq \max\{r_0, (2\cosh(\mu_3) - 1)^{-1/2}\}$, thus ensuring that $\mathcal{U}_r$ is totally geodesic and carried into $\mathbf{P}$.

PROPOSITION 8. *There is an $r = r(\chi)$ with the following property. Equip $M = S \times \mathbb{R}$ with a type-preserving hyperbolic metric without accidental parabolics, and suppose that each component of $\mathbf{P}$ is isometric to $\mathbf{P}_3(r)$. Let $Y \subset S$ be an essential subsurface whose corresponding cover $M_Y \to M$ has convex core $\mathcal{C}_Y$. Then each component of the intersection of $\mathcal{C}_Y$ and $\mathbf{P}$ is isometric to*

$$\mathbf{P}_3(r, R) = \{(z, t) \in \mathbb{H}^3 \mid t > r \text{ and } 0 \le \operatorname{Im}(z) \le R\}/\langle (z, t) \mapsto (z + 1, t)\rangle$$

for some $R > 0$.

Proof. An area argument shows that if $r > 0$ is sufficiently large (depending only on χ), any pleated surface representative of S meets $\mathbf{P}(r)$ only in its cusps. (To see this, note that if a pleated surface representative of S plunges deep into $\mathbf{P}(r)$, then its diameter would be large. This forces one of two alternatives: either an essential curve on S lies in $\mathbf{P}(r)$, violating our accidental parabolics hypothesis, or the pleated surface contains a large diameter disk, violating the Gauss–Bonnet theorem.) We assume that r is at least this large, in addition to the constraints already imposed on r.

Let Y be an essential subsurface of S. For a given $r > 0$, let $\mathcal{V}_r$ be the union of the cusp neighborhoods $\mathcal{U}_r \subset \partial\mathcal{C}_Y$ constructed above. If $r > 0$ is sufficiently large, and a point of $\partial\mathcal{C}_Y - \mathcal{V}_r$ is sufficiently deep in $\mathbf{P}(r)$, then area considerations again imply that $\partial\mathcal{C}_Y - \mathcal{V}_r$ must contain a compressible curve bounding a disk $\mathcal{D}$ contained in $\mathcal{C}_Y$ and some component of $\mathbf{P}(r)$. (As in the area argument above, the surface $\partial\mathcal{C}_Y - \mathcal{V}_r$ has bounded area, and, paired with the no accidental parabolics hypothesis, this guarantees that any essential curve in $\partial\mathcal{C}_Y - \mathcal{V}_r$ lying in $\mathbf{P}(r)$ must be nullhomotopic there. This produces the desired disk.) Since ∂Y is disk-busting in $\mathcal{C}_Y$, its geodesic representative $\partial Y^* \subset \mathcal{C}_Y$ must intersect $\mathcal{D}$, and hence $\mathbf{P}(r)$. But this means that if $\mathcal{F} \to M$ is any pleated surface representative of S realizing ∂Y^*, then the noncuspidal part of $\mathcal{F}$ must hit $\mathbf{P}(r)$, contradicting our choice of r. We find that $\partial\mathcal{C}_Y - \mathcal{V}_r$ is carried a uniformly bounded distance (depending only on χ) into $\mathbf{P}(r)$. Choosing a larger r, we assume that $\partial\mathcal{C}_Y$ hits $\mathbf{P}(r)$ only in the $\mathcal{U}_r$.

Let $\mathbf{P}_Y(r)$ be the preimage of $\mathbf{P}(r)$ in M_Y. Suppose $\mathcal{K}$ is a component of $\mathcal{C}_Y \cap \mathbf{P}_Y(r)$ which is not of the form $\mathbf{P}_3(r, R)$ for any $R > 0$. Then the closure of $\mathcal{K}$ must intersect $\partial\mathbf{P}_Y(r)$ in a locally convex (horospherical) surface $\mathcal{H}$. This surface lies in $\mathcal{C}_Y^\circ$ since $\partial\mathcal{C}_Y$ hits $\mathbf{P}_Y(r)$ only in the $\mathcal{U}_r$. Moreover, $\mathcal{H}$ is compact since $\mathcal{C}_Y$ is compact after its cuspidal thin part is thrown away. But this all implies that $\partial\mathbf{P}_Y(r)$ in M_Y has a compact component, namely $\mathcal{H}$, which is absurd. We conclude that every component of $\mathcal{C}_Y \cap \mathbf{P}_Y(r)$ has the form $\mathbf{P}_3(r, R)$. It follows that every component of $\mathcal{C}_Y \cap \mathbf{P}(r)$ has this form. □

We say that a hyperbolic structure on a noncompact manifold M is ε-**thick** if the length of its shortest geodesic loop is at least ε.

THEOREM 9. *Let S be a finite-type noncompact oriented surface of Euler characteristic $\chi < 0$. Let ε and K be positive numbers. Equip $M = S \times \mathbb{R}$ with an ε-thick hyperbolic metric, and let $r = r(\chi)$ be the number given by Proposition 8. There is a $W = W(\chi, \varepsilon, K) > 0$ such that the following holds. Let $\ell : M - \mathbf{P}(r) \longrightarrow \mathbb{R}$*

be a K-Lipschitz map, and let $\nu : M \longrightarrow M - \mathbf{P}(r)$ be the normal projection. If Y is a proper incompressible subsurface of S with convex core $\mathcal{C}_Y$ mapping to M via $\Pi : \mathcal{C}_Y \to M$, then $\operatorname{diam}(\ell(\nu(\Pi(\mathcal{C}_Y)))) \leq W$. *If γ is a geodesic loop in M with* $\operatorname{diam}(\ell(\nu(\gamma))) > W$, *then γ fills S.*

We define the *width* of a subset $X \subset M$ to be $\operatorname{diam}(\ell(\nu(X)))$, and of a subset $X \subset N$ of a covering space $\Pi : N \to M$ to be $\operatorname{diam}(\ell(\nu(\Pi(X))))$.

Lemma 10. *There is a constant $W = W(\chi, \varepsilon, K)$ such that $\partial\mathcal{C}_Y$ has width less than W. In particular, the boundary $\partial\mathcal{X}_Y$ of $\mathcal{X}_Y = \mathcal{C}_Y - \mathbf{P}(r)$ has width less than W.*

Proof. The proof is similar to the proof of Lemma 7.

Let δ be the minimum of ε and $\operatorname{arccosh}(1 + 1/2r^2) < \mu_3$.

We again let $\partial\mathcal{C}_Y$ be the double $\mathfrak{D}\mathcal{C}_Y$ when $\mathcal{C}_Y$ is two-dimensional, considered as a map $\mathfrak{D}\mathcal{C}_Y \to \mathcal{C}_Y \to M$.

There is a $D = D(\chi, \varepsilon)$ such that $\partial\mathcal{C}_Y - \mathbf{P}(r)$ lies in the D-neighborhood of ∂Y^*. To see this, let $\mathcal{P}(\delta)$ be the δ-thin part of $\partial\mathcal{C}_Y$. Note that, by our choice of δ, we have $\partial\mathcal{C}_Y - \mathbf{P}(r) \subset \partial\mathcal{C}_Y - \mathcal{P}(\delta)$.

The components of $\partial\mathcal{C}_Y - \mathcal{P}(\delta)$ have diameters uniformly bounded above by a constant $E = E(\chi, \delta)$.

The thin part $\mathcal{P}(\delta)$ is a union of cusp-neighborhoods and neighborhoods of short geodesics. The cusp neighborhoods lie in $\mathbf{P}(r)$. As before, the geodesic neighborhoods are within δ of the disk-busting ∂Y^*.

We conclude that $\partial\mathcal{C}_Y - \mathbf{P}(r)$ is contained in the D-neighborhood of ∂Y^* for $D = E + \delta$.

Since ∂Y^* has width at most KB, the width of $\partial\mathcal{C}_Y$, which is equal to the width of $\partial\mathcal{C}_Y - \mathbf{P}(r)$, is at most $W = KB + 2KD$. $\square$

Proof of Theorem 9. The proof is essentially the same as the proof of Theorem 5. If $\mathcal{C}_Y^\circ$ is empty, then $\mathcal{C}_Y = \partial\mathcal{C}_Y$, and the theorem follows immediately from Lemma 10. When $\mathcal{C}_Y^\circ \neq \emptyset$, we first observe that by the definition of ν and Proposition 8

$$\operatorname{diam}(\ell(\nu(\Pi(\mathcal{C}_Y)))) = \operatorname{diam}(\ell(\Pi(\mathcal{C}_Y - \mathbf{P}(r)))).$$

The composition $\ell \circ \Pi$ restricted to $\mathcal{C}_Y^\circ - \mathbf{P}(r)$ is a submersion and hence on $\mathcal{C}_Y - \mathbf{P}(r)$ attains its maximum and minimum values on $\partial\mathcal{X}_Y$. By Lemma 10, the width of $\mathcal{C}_Y$ is at most W. $\square$

4. General Surface Bundles

We again assume that S is a *closed* surface.

We assume that the reader is acquainted with the basic notions in the study of hyperbolic groups at the level of Chapters III.H and III.Γ of [5].

Consider a short exact sequence $1 \to \pi_1(S) \to \Gamma \to G \to 1$ where Γ is hyperbolic, which we call a *hyperbolic sequence.* We choose a finite generating set for Γ containing one for $\pi_1(S)$, which in turn provides one for G, and we let $X_{\pi_1(S)}$,

X_Γ, X_G be the corresponding Cayley graphs. Since X_G is of primary importance, we often write $X = X_G$. There are simplicial maps

$$X_{\pi_1(S)} \longrightarrow X_\Gamma \xrightarrow{\pi} X_G,$$

which induce our short exact sequence. For any γ in Γ, we let $\widetilde{\gamma}^*$ denote any geodesic in X_Γ whose endpoints are the ideal fixed points of γ. So $\widetilde{\gamma}^*$ is a γ-quasi-invariant geodesic.

THEOREM 11. *Given a hyperbolic sequence* $1 \to \pi_1(S) \to \Gamma \to G \to 1$, *there is a* $W > 0$ *such that, given any nonfilling* γ *in* $\pi_1(S)$ *and any* γ*-quasi-invariant geodesic* $\widetilde{\gamma}^*$, *we have* $\operatorname{diam}(\pi(\widetilde{\gamma}^*)) \le W$.

The statement needed in [8] is the following, which follows easily from Theorem 11. Given a hyperbolic sequence $1 \to \pi_1(S) \to \Gamma \to G \to 1$ and a proper subsurface $Y \subset S$ with associated subgroup $\Gamma_Y < \Gamma$, we let $\mathrm{WH}(\Gamma_Y)$ denote the union of all quasi-invariant geodesic axes of elements in Γ_Y, called the *weak hull of* $\mathrm{WH}(\Gamma_Y)$.

COROLLARY 12. *Given a hyperbolic sequence* $1 \to \pi_1(S) \to \Gamma \to G \to 1$, *there is a* $W' > 0$ *such that, given any proper subsurface* $Y \subset S$ *with corresponding subgroup* $\Gamma_Y < \Gamma$, *we have* $\operatorname{diam}(\pi(\mathrm{WH}(\Gamma_Y))) \le W'$.

Proof. Let W be as in Theorem 11, let δ be the hyperbolicity constant for Γ, and set $W' = W + 4\delta$. Given two elements γ_1 and γ_2 in Γ, let $\widetilde{\gamma}_1^*$ and $\widetilde{\gamma}_2^*$ be a pair of respective quasi-invariant geodesics. It suffices to show that $\operatorname{diam}(\pi(\widetilde{\gamma}_1^* \cup \widetilde{\gamma}_2^*)) \le W'$ since the diameter of $\pi(\mathrm{WH}(\Gamma_Y))$ is bounded by the supremum of such diameters over all pairs of quasi-invariant axes for all pairs of elements in Γ_Y.

We choose points x_i in $\widetilde{\gamma}_i^*$ with $\operatorname{diam}(\pi(x_1 \cup x_2)) = \operatorname{diam}(\pi(\widetilde{\gamma}_1^* \cup \widetilde{\gamma}_2^*))$. Applying γ_i to $\widetilde{\gamma}_i^*$ for $i = 1, 2$, we assume that x_1 and x_2 are far from $\widetilde{\gamma}_2$ and $\widetilde{\gamma}_1$, respectively. There is then a third element γ_3 in Γ_Y with a quasi-invariant geodesic $\widetilde{\gamma}_3^*$ that contains x_1 and x_2 in its 2δ-neighborhood $\mathcal{N}_{2\delta}(\widetilde{\gamma}_3^*)$. Since γ_3 is in Γ_Y and π is 1-Lipschitz, Theorem 11 gives us

$$\begin{aligned} \operatorname{diam}(\pi(\widetilde{\gamma}_1^* \cup \widetilde{\gamma}_2^*)) &= \operatorname{diam}(\pi(x_1 \cup x_2)) \le \operatorname{diam}(\mathcal{N}_{2\delta}(\widetilde{\gamma}_3^*)) \\ &\le W + 4\delta = W'. \end{aligned}$$ □

The short exact sequence $1 \to \pi_1(S) \to \Gamma \to G \to 1$ gives us a monodromy representation $\rho : G \to \mathrm{Mod}(S)$. By [9], hyperbolicity of the sequence implies that ρ has finite kernel and that $G_0 = \rho(G)$ is a *convex cocompact* subgroup of $\mathrm{Mod}(S)$, meaning that G_0 has a quasi-convex orbit in Teichmüller space.

The preimage of G_0 in $\mathrm{Mod}(\mathring{S})$ is an extension Γ_{G_0} of G_0 by $\pi_1(S)$, which is the homomorphic image of Γ, and we have the commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(S) \arrow[r] & \mathrm{Mod}(\mathring{S}) \arrow[r] & \mathrm{Mod}(S) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(S) \arrow[r] \arrow[u, equal] & \Gamma_{G_0} \arrow[r] \arrow[u, hook] & G_0 \arrow[r] \arrow[u, hook] & 1 \\
1 \arrow[r] & \pi_1(S) \arrow[r] \arrow[u, equal] & \Gamma \arrow[r] \arrow[u, two heads] & G \arrow[r] \arrow[u, two heads] & 1
\end{tikzcd}$$

The map $\Gamma \to \Gamma_{G_0}$ also has finite kernel and is thus a quasi-isometry. Using stability of geodesics in Gromov hyperbolic spaces (Theorem III.H.1.7 of [5]), one can easily check that it suffices to prove Theorem 11 when $\rho : G \to G_0$ is an isomorphism. We therefore assume that G is a convex cocompact subgroup of $\mathrm{Mod}(S)$ and that $\Gamma = \Gamma_G = \Gamma_{G_0}$.

There is a canonical S-bundle $\mathcal{S}(S)$ over Teichmüller space $\mathcal{T}(S)$ in which the fiber over $[m]$ in $\mathcal{T}(S)$ is identified with S endowed with the hyperbolic metric m. The universal cover of this space is a hyperbolic plane bundle $\mathcal{H}(S) \to \mathcal{T}(S)$. The Bers fibration [3] identifies $\mathcal{H}(S)$ and the Teichmüller space $\mathcal{T}(\mathring{S})$ of $\mathring{S}$, and we have the commutative diagram with equivariant actions

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(S) \arrow[r] & \mathrm{Mod}(\mathring{S}) \arrow[r] & \mathrm{Mod}(S) \arrow[r] & 1 \\
 & \circlearrowright & \circlearrowright & \circlearrowright & \\
 & \mathbb{H}^2 \arrow[r] \arrow[d] & \mathcal{H}(S) \arrow[r] \arrow[d] & \mathcal{T}(S) \arrow[d] & \\
 & S \arrow[r] & \mathcal{S}(S) \arrow[r] & \mathcal{T}(S) &
\end{tikzcd}$$

We fix a connection on $\mathcal{S}(S) \to \mathcal{T}(S)$, meaning that we choose smoothly varying direct-sum decomposition of each tangent space of $\mathcal{S}(S)$ into the tangent space of the fiber and a choice of *horizontal space.*

We pick a G-equivariant embedding $X = X_G \to \mathcal{T}(S)$ that sends edges to geodesics and is therefore Lipschitz. We have pullback bundles

$$\begin{tikzcd}
\mathbb{H}^2 \arrow[r] \arrow[d] & \mathcal{H}_X \arrow[r] \arrow[d] & X \arrow[d, equal] \\
S \arrow[r] & \mathcal{S}_X \arrow[r] & X
\end{tikzcd}$$

and we call $\mathbb{H}^2 \to \mathcal{H}_X \to X$ an *associated hyperbolic plane bundle.* For x in X, we let $\mathcal{H}_x$ denote the fiber of $\mathcal{H}_X \to X$ over x. We let π stand for any of the maps $\mathcal{H}_X \to X$, $\mathcal{S}_X \to X$, and $X_\Gamma \to X$, letting context determine which is meant.

Pulling our connection back to $\mathcal{S}_X$, we equip $\mathcal{S}_X$ with a piecewise Riemannian metric that locally splits as a product of the hyperbolic metric on the fibers and the metric lifted from X. We pull this metric back to $\mathcal{H}_X$.

Given two points x and y in X and a geodesic between them, there is a parallel transport map $\mathcal{H}_x \to \mathcal{H}_y$ defined by following the horizontal lines of the connection over the geodesic. Since G acts cocompactly on X, there is a $K_0 > 0$ such that for any two points x and y in X, this map is $K_0^{d(x,y)}$-bi-Lipschitz with respect to the hyperbolic metrics on the fibers.

There is a fiber-preserving Γ-equivariant quasi-isometry $X_\Gamma \to \mathcal{H}_X$ making the following diagram commute:

$$\begin{array}{ccccc} X_{\pi_1(S)} & \longrightarrow & X_\Gamma & \longrightarrow & X \\ \downarrow & & \downarrow & & \| \\ \mathbb{H}^2 & \longrightarrow & \mathcal{H}_X & \longrightarrow & X \end{array}$$

Given γ in $\pi_1(S)$, let $\mathcal{A}_x(\gamma)$ denote the axis of γ in the fiber $\mathcal{H}_x$ and define a subset $\mathcal{A}(\gamma)$ of $\mathcal{H}_X$ by

$$\mathcal{A}(\gamma) = \bigcup_{x \in X} \mathcal{A}_x(\gamma).$$

Let x_γ in X be a point for which the translation length of γ on $\mathcal{A}_{x_\gamma}(\gamma)$ is minimal over all $\mathcal{A}_x(\gamma)$. We endow $\mathcal{A}(\gamma)$ with the subspace metric coming from the path metric on the 1-neighborhood $\mathcal{N}_1(\mathcal{A}(\gamma))$ and denote both of these metrics by d_γ.

By the stability of geodesics in hyperbolic spaces (Theorem III.H.1.7 of [5]), the following theorem implies Theorem 11.

THEOREM 13. *Given a hyperbolic sequence $1 \to \pi_1(S) \to \Gamma \to G \to 1$ with associated hyperbolic plane bundle $\mathbb{H}^2 \to \mathcal{H}_X \to X$, there exist $K, C > 0$ such that if γ in $\pi_1(S)$ is a nonfilling loop in S, then $\mathcal{A}_{x_\gamma}(\gamma)$ is a (K, C)-quasi-geodesic in $\mathcal{H}_X$.*

Proof that Theorem 13 implies Theorem 11. The quasi-isometry $X_\Gamma \to \mathcal{H}_X$ sends $\widetilde{\gamma}^*$ to a γ-quasi-invariant uniform quasi-geodesic, which is therefore uniformly close to $\mathcal{A}_{x_\gamma}(\gamma)$. Since $\mathcal{A}_{x_\gamma}(\gamma)$ projects to the point x, and the projection to X is Lipschitz, the image of $\widetilde{\gamma}^*$ is within some uniform distance W of the image of x. □

The rest of the paper is devoted to the proof of Theorem 13, which is inspired by the ideas in [9], [10], [16], [17]. As the argument is somewhat involved, we pause to give a detailed sketch.

4.0. Outline of the Rest of the Paper

Sketch of the proof of Theorem 13. The basic idea is to construct a retraction $\mathcal{H}_X \to \mathcal{A}_{x_\gamma}(\gamma)$ that is uniformly coarsely Lipschitz. Being *coarsely Lipschitz* means that there are $K', C' > 0$ such that the distance in $\mathcal{A}_{x_\gamma}(\gamma)$ between the image of any two points is at most K' times their distance in $\mathcal{H}_X$, up to an additive error of C', and *uniformity* means that these constants do not depend on γ. The existence of such a map implies that $\mathcal{A}_{x_\gamma}(\gamma)$ is uniformly quasi-geodesic.

The map $\mathcal{H}_\Gamma \to \mathcal{A}_{x_\gamma}(\gamma)$ is a composition of two maps $\mathcal{H}_\Gamma \to \mathcal{A}(\gamma) \to \mathcal{A}_{x_\gamma}(\gamma)$.

The construction of the first map $\mathcal{H}_X \to \mathcal{A}(\gamma)$, and the fact that it is uniformly coarsely Lipschitz (see Lemma 14) is due to Mitra [20]. (This does not use the assumption that γ is nonfilling.) This first map is defined as the fiberwise closest-point projection: the restriction to a fiber $\mathcal{H}_x$ is the closest point projection to $\mathcal{A}_x(\gamma)$ with respect to the hyperbolic metric on the fiber. The details of this step are in Section 4.1.

The second map $\mathcal{A}(\gamma) \to \mathcal{A}_{x_\gamma}(\gamma)$ is defined using a collection $\{\Sigma_n\}_{n\in\mathbb{Z}}$ of sections $\Sigma_n \subset \mathcal{A}(\gamma)$ of the projection $\mathcal{A}(\gamma) \to X$ introduced in Section 4.2. These sections have the following properties (see Theorem 17):

1. The section map $X \to \Sigma_n \subset \mathcal{A}_{x_\gamma}(\gamma)$ is a uniform quasi-isometry.
2. For any x in X, the fiber $\mathcal{A}_x(\gamma) \cong \mathbb{R}$ intersects the set of sections in a biinfinite increasing sequence of points $\{\Sigma_n \cap \mathcal{A}_x(\gamma)\}_{n\in\mathbb{Z}}$. In other words, the sections intersect the fibers in order, escaping to the ends.
3. In the distinguished fiber $\mathcal{A}_{x_\gamma}(\gamma)$, the distance between consecutive points of $\{\Sigma_n \cap \mathcal{A}_{x_\gamma}(\gamma)\}_{n\in\mathbb{Z}}$ is constant.
4. The distance between consecutive points of $\{\Sigma_n \cap \mathcal{A}_x(\gamma)\}_{n\in\mathbb{Z}}$ is uniformly bounded below.

The existence of sections with the first and third properties is due to Mj and Sardar [16]. This is based on a result of Mosher [22] that provides uniform quasi-isometrically embedded sections through any point of $\mathcal{A}(\gamma)$ (see Lemma 16).

The second and fourth properties require some new ideas, explained below, and require the hypothesis that γ is nonfilling, unlike the first and third. Before this explanation, we describe the map $\mathcal{A}(\gamma) \to \mathcal{A}_{x_\gamma}(\gamma)$. Each of the fibers is isometric to $\mathbb{R}$, and, by the second property, the sections cut these fibers into intervals. The union of the intervals from Σ_n to Σ_{n+1} over all x forms a region $\mathcal{R}_n$, and the map $\mathcal{A}(\gamma) \to \mathcal{A}_{x_\gamma}(\gamma)$ is defined by sending this entire region to the point $\Sigma_n \cap \mathcal{A}_{x_\gamma}(\gamma)$. Uniform properness of the fibers implies that this map is uniformly coarsely Lipschitz as required. The detailed construction of this second map is in Section 4.2.1.

To establish the second and fourth properties of the sections, note that for any x in X, there is a uniform biinfinite quasi-geodesic g in X through x and x_γ. This quasi-geodesic is uniformly close to a Teichmüller geodesic τ in $\mathcal{T}(S)$. Moreover, the closest point projection from g to τ lifts to a fiber-preserving map between the corresponding hyperbolic plane bundles

$$\mathcal{H}_g \to \mathcal{H}_\tau,$$

and a result of Farb and Mosher [9] shows that this map may be taken a uniform quasi-isometry. To understand the sequence $\{\Sigma_n \cap \mathcal{A}_x(\gamma)\}_{n\in\mathbb{Z}}$, we analyze its image in $\mathcal{H}_g$. This lies in some fiber and is a biinfinite sequence uniformly close to the axis for γ in that fiber. As long as all estimates are uniform, it therefore suffices to consider a sequence of sections $\{\Sigma_n\}_{n\in\mathbb{Z}}$ of the axis bundle $\mathcal{A}_\tau(\gamma)$ over τ.

The Teichmüller geodesic τ is defined by a quadratic differential (see Section 4.4.1). It is therefore natural to replace the fiberwise hyperbolic metric on $\mathcal{H}_\tau$ with the singular Sol metric (see Section 4.5.1) for which the restriction to each fiber is the Euclidean cone metric defined by the quadratic differential (see Section 4.3). This is done at the expense of a uniform distortion in distances (see

Lemma 18), by a result of Minsky [18]. We thus reduce further to the axis bundle $\mathcal{A}_\tau(\gamma)^{\mathrm{SOL}}$ for γ with respect to the singular SOL metric and the attendant sections Σ_n, for which we prove properties 2 and 4.

The problem is now a technical one concerning geodesics in the Euclidean cone metrics of quadratic differentials. We refer the general audience to Section 4.4 for definitions and details and briefly sketch the key points for the expert.

The point x_γ is uniformly close to the *balance time* for γ along τ, which we take to be $\tau(0)$, so the role of x_γ in property 3 is taken by $\tau(0)$. Using arguments of Masur and Minsky [17], we prove that any segment of $\mathcal{A}_{\tau(0)}(\gamma)^{\mathrm{SOL}}$ of sufficient length (depending only on Γ) must increase in length exponentially in both forward and backward time along τ after a uniformly bounded amount of time (see Proposition 21). Taking the distance between consecutive points of the fiber to be sufficiently large, properties 2 and 4 follow.

To establish this exponential growth, we argue as follows. There is a simple closed curve α disjoint from γ since γ is nonfilling. From [17] we know that α becomes *mostly horizontal* and *mostly vertical,* respectively, after a uniformly bounded amount of time into the future and the past, respectively, measured from time zero at the balance point. We prove that after further uniform steps forward and backward in time, γ itself becomes mostly horizontal and vertical, respectively. There cannot be too many consecutive short saddle connections (by a compactness argument), and so, in the remote future and past, exponential growth kicks in for any sufficiently long segment. This is the last step and completes the proof.

We note that, due to certain logical dependencies, the description just given does not follow the sections below linearly. □

4.1. Fiberwise Projection

The following construction belongs to Mitra [20] and is used throughout his work. Consider the map $\mathfrak{p}_\gamma : \mathcal{H}_X \to \mathcal{A}(\gamma)$ obtained by fiberwise closest point projection to $\mathcal{A}(\gamma)$. That is, for z in $\mathcal{H}_x$, let $\mathfrak{p}_\gamma(z)$ be the point on $\mathcal{A}_x(\gamma)$ that is closest to z with respect to the hyperbolic metric on $\mathcal{H}_x$. The following lemma is a translation to our setting of the results in Section 3 of Mitra's paper [20]. We give the proof for the reader's convenience.

LEMMA 14 (Mitra [20]). *Given a hyperbolic sequence* $1 \to \pi_1(S) \to \Gamma \to G \to 1$ *with associated hyperbolic plane bundle* $\mathbb{H}^2 \to \mathcal{H}_X \to X$*, there are* $K_1, C_1 > 0$ *such that for any* γ *in* $\pi_1(S)$*, the projection* $\mathfrak{p}_\gamma : \mathcal{H}_X \to \mathcal{A}(\gamma)$ *is* (K_1, C_1)*-coarsely Lipschitz. Consequently,* $\mathcal{A}(\gamma)$ *is* (K_1, C_1)*-quasi-isometrically embedded in* $\mathcal{H}_X$*.*

Proof. We begin with a few observations about the metric d_γ. For any $0 < r < 1$ and x in X, consider the r-neighborhood of the fiber over x in X, $\mathcal{N}_r(\mathcal{H}_x) = \pi^{-1}(B(x,r))$. Because $r < 1$, $B(x,r)$ is a tree in X, and so there is a unique parallel transport to the fiber $\mathcal{H}_x$ for every point in $\mathcal{N}_r(\mathcal{H}_x)$. We denote this map

$$\mathfrak{f}_x : \mathcal{N}_r(\mathcal{H}_x) \to \mathcal{H}_x.$$

The map $\mathfrak{f}_x$ is K_0^r-Lipschitz and K_0^r–bi-Lipschitz when restricted to any fiber $\mathcal{H}_y$, for y in $B(x, r)$.

Choose $0 < r < 1$ so that the stability constant (see Theorem III.H.1.7 of [5]) for $(K_0^r, 0)$-quasi-geodesics in $\mathbb{H}^2$ is less than 1. For any x, y in X with $d(x, y) \leq r$, it follows that the parallel transport line from z in $\mathcal{A}_y(\gamma)$ to $\mathfrak{f}_x(z)$ in $\mathcal{H}_x$ is contained in $\mathcal{N}_1(\mathcal{A}(\gamma))$, and hence

$$d_\gamma(z, \mathfrak{f}_x(z)) = d(z, \mathfrak{f}_x(z)) = d(x, y) \leq r.$$

Let δ_h denote the hyperbolicity constant for $\mathbb{H}^2$.

Claim 15. *Given any two points w, z in $\mathcal{H}_X$ with $d(w, z) \leq r$, we have*

$$d_\gamma(\mathfrak{p}_\gamma(w), \mathfrak{p}_\gamma(z)) \leq K_0^r r + 2(1 + K_0^r \delta_h) + r.$$

Proof. Let w, z in X be any two points with $d(w, z) \leq r$, and let $x = \pi(w)$ and $y = \pi(z)$ such that $d(x, y) \leq r$.

Recall that for any $c \geq \delta_h$ and any geodesic triangle $\triangle \subset \mathbb{H}^2$, the set of points within a distance c of all three sides is nonempty and has diameter at most $2c$. The closest point projection of one vertex of $\triangle$ to the opposite side is such a point.

Inside $\mathcal{H}_y$, the point $\mathfrak{p}_\gamma(z)$ is within δ_h of all three sides of the geodesic triangle $\triangle$ having vertex z and opposite side $\mathcal{A}_y(\gamma)$. It follows that inside $\mathcal{H}_x$, the point $\mathfrak{f}_x\mathfrak{p}_\gamma(z)$ has distance at most $K_0^r\delta_h$ from all three sides of the $(K_0^r, 0)$-quasi-geodesic triangle $\mathfrak{f}_x(\triangle)$. Because the sides of this are within a distance 1 of the geodesics with the same endpoints, it follows that $\mathfrak{f}_x\mathfrak{p}_\gamma(z)$ is within a distance $1 + K_0^r\delta_h$ of all three sides of the geodesic triangle defined by $\mathfrak{f}_x(z)$ and $\mathcal{A}_x(\gamma)$. Since $\mathfrak{p}_\gamma\mathfrak{f}_x(z)$ has distance at most $\delta_h < 1 + K_0^r\delta_h$ from each of these sides, it follows that

$$d_x(\mathfrak{p}_\gamma\mathfrak{f}_x(z), \mathfrak{f}_x\mathfrak{p}_\gamma(z)) \leq 2(1 + K_0^r\delta_h).$$

Moreover, the path exhibiting this distance bound lies entirely inside $\mathcal{H}_x$, and the geodesic in $\mathcal{H}_x$ between these points lies within a distance 1 of $\mathcal{A}_x(\gamma)$. In particular, it follows that

$$d_\gamma(\mathfrak{p}_\gamma\mathfrak{f}_x(z), \mathfrak{f}_x\mathfrak{p}_\gamma(z)) \leq d_x(\mathfrak{p}_\gamma\mathfrak{f}_x(z), \mathfrak{f}_x\mathfrak{p}_\gamma(z)) \leq 2(1 + K_0^r\delta_h).$$

Applying the triangle inequality proves the claim since

$$\begin{aligned} d_\gamma(\mathfrak{p}_\gamma(w), \mathfrak{p}_\gamma(z)) &\leq d_\gamma(\mathfrak{p}_\gamma(w), \mathfrak{p}_\gamma\mathfrak{f}_x(z)) + d_\gamma(\mathfrak{p}_\gamma\mathfrak{f}_x(z), \mathfrak{f}_x\mathfrak{p}_\gamma(z)) \\ &\qquad + d_\gamma(\mathfrak{f}_x\mathfrak{p}_\gamma(z), \mathfrak{p}_\gamma(z)) &(4.1) \\ &\leq d_x(w, \mathfrak{f}_x(z)) + 2(1 + K_0^r\delta_h) + r &(4.2) \\ &\leq K_0^r d(w, z) + 2(1 + K_0^r\delta_h) + r &(4.3) \\ &\leq K_0^r r + 2(1 + K_0^r\delta_h) + r. &(4.4) \end{aligned}$$

In inequality (4.3), we have used the fact that $\mathfrak{f}_x$ is K_0^r-Lipschitz. □

From the claim we see that $\mathfrak{p}_\gamma$ is (K_1, C_1)-coarsely Lipschitz, where $K_1 = K_0^r + 2(1 + K_0^r \delta_h)/r + 1$ and $C_1 = K_0^r r + 2(1 + K_0^r \delta_h) + r$. Since the inclusion of $\mathcal{A}(\gamma)$ into $\mathcal{H}_X$ is 1-Lipschitz, it follows that $\mathcal{A}(\gamma)$ is (K_1, C_1)-quasi-isometrically embedded. $\square$

4.2. Quasi-isometric Sections

Let E and B be metric spaces, and let $\pi : E \to B$ be a 1-Lipschitz map. By a (k, c)-*quasi-isometric section* (or just (k, c)-*section*) of $\pi : E \to B$ we mean a subset $\Sigma \subset E$ that is the image of a (k, c)-coarsely Lipschitz map $\sigma : B \to E$ with $\pi \circ \sigma = id_B$. Since π is 1-Lipschitz, the map σ is a (k, c)-quasi-isometric embedding. In fact,

$$d(x, y) = d(\pi\sigma(x), \pi\sigma(y)) \leq d(\sigma(x), \sigma(y)) \leq kd(x, y) + c.$$

Mosher's quasiisometric section lemma [22] says that if $1 \to \pi_1(S) \to \Gamma \to G \to 1$ is hyperbolic, then there is a (k_0, c_0)-section of $\pi : X_\Gamma \to X$ for some k_0 and c_0. From this we obtain a (k_0, c_0)-section Σ of $\mathcal{H}_X \to X$ after enlarging k_0 and c_0. Using the fact that $\pi_1(S) < \Gamma$ acts cocompactly on the fibers, and by taking c_0 even larger, it follows that for any point z in $\mathcal{H}_X$, there is a (k_0, c_0)-section Σ for $\mathcal{H}_X \to X$ containing z; see also [16].

Given a (k_0, c_0)-section Σ of $\mathcal{H}_X \to X$, we have that $\mathfrak{p}_\gamma(\Sigma)$ is a (K_2, C_2)-section for $K_2 = k_0 K_1$ and $C_2 = K_1 c_0 + C_1$, by Lemma 14. We therefore have the following result of [16].

LEMMA 16 (Mj–Sardar [16]). *Given a hyperbolic sequence* $1 \to \pi_1(S) \to \Gamma \to G \to 1$ *with associated hyperbolic plane bundle* $\mathbb{H}^2 \to \mathcal{H}_X \to X$, *there are* K_2 *and* C_2 *with the following property. For all* γ *in* $\pi_1(S)$, *all* x *in* X, *and all* z *in* $\mathcal{A}_x(\gamma)$, *there exists a* (K_2, C_2)-*section* Σ *of* $\mathcal{H}_X \to X$ *with* $\Sigma \subset \mathcal{A}(\gamma)$ *and* $\Sigma \cap \mathcal{H}_x = \{z\}$.

A section Σ as in this lemma will be called a (K_2, C_2)-*section for* γ *(though* z*).* In the sequel, we are interested in collections of these. The leaf $\mathcal{A}_x(\gamma)$ is a line oriented by the action of γ and so possesses a well-defined order. We say that a collection $\{\Sigma_n\}_{n\in\mathbb{Z}}$ of (K_2, C_2)-sections for γ are *linearly ordered over* x if the assignment $n \mapsto \Sigma_n \cap \mathcal{A}_x(\gamma)$ is order preserving.

THEOREM 17. *Given a hyperbolic sequence* $1 \to \pi_1(S) \to \Gamma \to G \to 1$ *with associated hyperbolic plane bundle* $\mathbb{H}^2 \to \mathcal{H}_X \to X$, *there are* $D_1 > D_0 > 0$ *with the following property. If* γ *in* $\pi_1(S)$ *is nonfilling and* $\{\Sigma_n\}_{n\in\mathbb{Z}}$ *is a collection of* (K_2, C_2)-*sections for* γ *such that*

$$\{\Sigma_n\}_{n\in\mathbb{Z}} \textit{ is linearly ordered over } x_\gamma \quad \textit{and} \quad d_{x_\gamma}(\Sigma_n, \Sigma_{n+1}) = D_1,$$

then, for ***every*** x *in* X,

$$\{\Sigma_n\}_{n\in\mathbb{Z}} \textit{ is linearly ordered over } x \quad \textit{and} \quad d_x(\Sigma_n, \Sigma_{n+1}) \geq D_0.$$

4.2.1. Proof of Theorem 13 Assuming Theorem 17.

Proof of Theorem 13 assuming Theorem 17. Let γ be nonfilling. By Lemma 16, there are (K_2, C_2)-sections $\{\Sigma_n\}_{n\in\mathbb{Z}}$ for γ as in Theorem 17.

Let $\mathcal{R}_n$ denote the open region in $\mathcal{A}(\gamma)$ between Σ_n and Σ_{n+1}. By the conclusion of Theorem 17, each $\mathcal{R}_n$ is a union of intervals, one in each fiber. According to Theorem 3.2 of [16], there are constants K' and C' depending only the bundle $\mathbb{H}^2 \to \mathcal{H}_X \to X$ such that the fiberwise closest point projection

$$\mathfrak{p}_n : \mathcal{H}_X \to \mathcal{R}_n$$

is (K', C')-coarsely Lipschitz map (where $\mathcal{R}_n$ is given the metric inherited from the path metric on a sufficiently large neighborhood in $\mathcal{H}_X$). Theorem 3.2 of [16] is attributed to Mitra [20] since it is a direct translation of arguments there, much like the proof of Lemma 14.

Define

$$\eta_\gamma : \mathcal{A}(\gamma) \to \mathcal{A}_{x_\gamma}(\gamma)$$

by $\eta_\gamma(\mathcal{R}_n) = \eta_\gamma(\Sigma_n) = \Sigma_n \cap \mathcal{A}_{x_\gamma}(\gamma)$. We will show that η_γ is coarsely Lipschitz.

Claim. *There is a $B_1 > 0$ depending only on the bundle $\mathbb{H}^2 \to \mathcal{H}_X \to X$ such that if w is in $\mathcal{R}_m \cup \Sigma_m$ and z is in $\mathcal{R}_n \cup \Sigma_n$ with $d(w,z) \le 1$, then $|m-n| \le B_1$.*

Proof. Assume that $m \le n$.

First assume that w and z are in the same fiber $\mathcal{A}_{\pi(w)}(\gamma) = \mathcal{A}_{\pi(z)}(\gamma)$. By Theorem 17 we have $d_{\pi(w)}(w,z) \ge D_0(n-m)$. Now, the fibers of $\mathcal{H}_X$ (in which the fibers of $\mathcal{A}(\gamma)$ are geodesic) are uniformly proper, and so there is a positive E_0 depending only on $\mathbb{H}^2 \to \mathcal{H}_X \to X$ such that $d(w,z) \ge E_0 d_{\pi(w)}(w,z)$. So

$$1 \ge d(w,z) \ge E_0 D_0 (n-m-1),$$

and we are done in this case with $B_1 = 1/E_0D_0 + 1$.

If w and z are in different fibers, we argue as follows. Let z' be a point in the fiber $\mathcal{H}_{\pi(w)}$ with

$$d(z,z') = d(z, \mathcal{H}_{\pi(w)}) \le d(z,w) \le 1.$$

We have $\mathfrak{p}_n(z) = z$ and $\mathfrak{p}_n(z') = z''$ for some z'' in $\mathcal{R}_n \cap \mathcal{H}_{\pi(w)}$. Since $\mathfrak{p}_n$ is (K', C')-coarsely Lipschitz, uniform properness gives us

$$\begin{aligned} 1 + K' + C' &\ge 1 + K'd(z,z') + C' \\ &\ge d(w,z) + d(z,z'') \\ &\ge d(w,z'') \\ &\ge E_0 D_0 (n-m-1), \end{aligned}$$

and the proof is complete with $B_1 = (1 + K' + C')/E_0D_0 + 1$. □

It follows from the claim that

$$d_{x_\gamma}(\eta_\gamma(z), \eta_\gamma(w)) \le B_1 D_1$$

if $d(z,w) \le 1$, and so η_γ is (B_1D_1, B_1D_1)-coarsely Lipschitz. It follows that $\mathcal{A}_{x_\gamma}(\gamma)$ is (B_1D_1, B_1D_1)-quasi-isometrically embedded in $\mathcal{A}(\gamma)$, and hence (K, C)-quasi-isometrically embedded in $\mathcal{H}_X$ for $K = K_1 B_1 D_1$ and $C = K_1 B_1 D_1 + C_1$, by Lemma 14.

This proves Theorem 13. $\square$

For x sufficiently far from x_γ, the distances $d_x(\Sigma_n, \Sigma_{n+1})$ are in fact much larger than the estimate in Theorem 17. As a function of $d(x, x_\gamma)$, they are exponentially larger than the distances $d_{x_\gamma}(\Sigma_n \cap \mathcal{A}_{x_\gamma}(\gamma), \Sigma_{n+1} \cap \mathcal{A}_{x_\gamma}(\gamma))$, due to *flaring*. For nonfilling γ, the exponential growth will kick in outside a ball about x_γ of a uniformly bounded radius.

The rest of the paper is devoted to the proof of Theorem 17, which requires a study of quadratic differentials, Teichmüller geodesics, and singular SOL metrics, taken up in the next section.

4.3. Quadratic Differentials and Flat Metrics

We refer the reader to [25] for a detailed treatment of quadratic differentials and their associated flat metrics.

Given a complex structure on S, a unit-norm holomorphic quadratic differential q on S both determines and is determined by a nonpositively curved Euclidean cone metric on S together with a pair of orthogonal singular foliations with geodesic leaves (called the *vertical* and *horizontal* foliations). Given q and a nonsingular point p, there is a *preferred coordinate* $\zeta = x + iy$ that carries a neighborhood of p isometrically into the plane such that the arcs of the horizontal and vertical foliations to horizontal and vertical segments, respectively.

We let $\mathcal{Q}^1(S)$ denote the space of all unit-norm holomorphic quadratic differentials on S, which forms the unit cotangent bundle over Teichmüller space $\mathcal{T}(S)$. We let $m = m(q)$ denote the hyperbolic metric in the conformal class of a quadratic differential q and write $q \mapsto m(q)$ for the map $\mathcal{Q}^1(S) \to \mathcal{T}(S)$.

Let $\widetilde{S} \to S$ be the universal covering. Given q in $\mathcal{Q}^1(S)$, we abuse notation and continue to refer to the pullback of q and m to $\widetilde{S}$ as q and m, respectively. The identity map $id_{\widetilde{S}} : \widetilde{S} \to \widetilde{S}$ is a quasi-isometry with respect to m and the singular flat metric for q. In fact, by Proposition 2.5 of [9] or Lemma 3.3 of [18], for example, we have the following lemma.

LEMMA 18 (Minsky [18]). *Given $r > 0$, there exist $K_3, C_3 > 0$ such that if q in $\mathcal{Q}^1(S)$ lies over the r-thick part of $\mathcal{T}(S)$, then*

$$id_{\widetilde{S}} : (\widetilde{S}, m) \to (\widetilde{S}, q)$$

is a (K_3, C_3)-quasi-isometry.

4.3.1. Geodesics and Straight Segments. Fix q in $\mathcal{Q}^1(S)$. Given γ in $\pi_1(S)$ a (nontrivial) element, we will let γ_0^* denote the q-geodesic representative in S and $\widetilde{\gamma}_0^*$ a lift of this geodesic to a biinfinite q-geodesic in $\widetilde{S}$. The geodesic γ_0^* should be considered a locally isometric map from a circle or interval of some length into S as the geodesic is not determined by its image.

The geodesics γ_0^* and $\widetilde{\gamma}_0^*$ are either Euclidean geodesics (geodesics in the complement of the singularities) or concatenations of *straight segments* (Euclidean geodesic segments connecting pairs of singular points with no singular points in their interior).

We let $\|\gamma\|_q$ denote the q-length of γ_0^* and $\|\gamma\|_{q,v}$ and $\|\gamma\|_{q,h}$ the vertical and horizontal lengths of γ_0^*, respectively. These are related by

$$\begin{aligned}\frac{1}{2}(\|\gamma\|_{q,v} + \|\gamma\|_{q,h}) &\leq \max\{\|\gamma\|_{q,v}, \|\gamma\|_{q,h}\} &(4.5)\\ &\leq \|\gamma\|_q &(4.6)\\ &\leq \|\gamma\|_{q,v} + \|\gamma\|_{q,h} &(4.7)\\ &\leq 2\max\{\|\gamma\|_{q,v}, \|\gamma\|_{q,h}\}. &(4.8)\end{aligned}$$

More generally, given a (local) q-geodesic $\delta : I \to S$ or $\delta : I \to \widetilde{S}$ defined on an interval $I \subset \mathbb{R}$, we let $\|\delta\|_q$, $\|\delta\|_{q,h}$, and $\|\delta\|_{q,v}$ denote the length, horizontal length, and vertical length, respectively.

We let $\|\gamma\|_m$ denote the length of the $m = m(q)$-geodesic representative. Given $r > 0$, if K_3, C_3 are as in Lemma 18, we have

$$\frac{1}{K_3}\|\gamma\|_q \leq \|\gamma\|_m \leq K_3\|\gamma\|_q. \tag{4.9}$$

Inequality (4.9) is free of the constant C_3 thanks to the fact that the length is equal to the asymptotic translation length.

More generally, given any geodesic metric m' on S for which the pullback to $\widetilde{S}$ makes $id_{\widetilde{S}} : (\widetilde{S}, m') \to (\widetilde{S}, q)$ a (K_6, C_6)-quasi-isometry, we have

$$\frac{1}{K_6}\|\gamma\|_q \leq \|\gamma\|_m \leq K_6\|\gamma\|_q. \tag{4.10}$$

From (4.9) we easily obtain the following.

LEMMA 19. *For any $r > 0$, there exists $\varepsilon > 0$ with the following property. Given any q in $\mathcal{Q}^1(S)$ lying over the r-thick part of $\mathcal{T}(S)$ and any (local) q-geodesic segment $\delta : [0, 1] \to S$ or $\delta : [0, 1] \to \widetilde{S}$, there is an arc of δ of length at least ε containing no singularities.*

Proof. We assume as we may that $r < 1$ and set $\varepsilon = r/(K_3(4g-2)) < 1/(4g-2)$.

Suppose that there is a q-geodesic segment $\delta : [0, 1] \to S$ such that every subsegment of length at least ε contains a singularity. This segment contains a concatenation δ' of at least $4g - 4$ straight segments of q-length less than ε, each connecting a pair of singularities. Since there are at most $4g - 4$ singularities of q, the segment δ' must visit some singularity more than once, thus forming a loop β of q-length less than $(4g - 4)\varepsilon < r/K_3$. Except at the basepoint, this loop β is locally geodesic and is therefore essential. By (4.9), the hyperbolic length of β is less than $K_3(r/K_3) = r$, which contradicts the fact that q lies over the r-thick part of $\mathcal{T}(S)$.

For $\delta : [0, 1] \to \widetilde{S}$, we push forward to S and appeal to the first case. $\square$

Applying the lemma to any closed geodesic γ_0, we have the following.

COROLLARY 20. *Let $r > 0$, and let ε be as in Lemma 19. If q in $\mathcal{Q}^1(S)$ lies over the r-thick part of $\mathcal{T}(S)$ and γ in $\pi_1(S)$, then γ_0 contains a straight segment of length at least ε.*

4.4. *Teichmüller Geodesics and Lengths*

We refer the reader to [1] and [11] for detailed treatments of Teichmüller theory.

4.4.1. Teichmüller Deformations. The Teichmüller deformation associated to a quadratic differential q in $\mathcal{Q}^1(S)$ determines a 1-parameter family of quadratic differentials q_t. More precisely, if q has preferred coordinate $\zeta = x + iy$, then q_t is determined by its preferred coordinate $\zeta_t = e^t x + ie^{-t} y$ (in particular, $q = q_0$). The map $\tau_q : \mathbb{R} \to \mathcal{T}(S)$ obtained by composing $t \mapsto q_t$ with the projection $\mathcal{Q}^1(S) \to \mathcal{T}(S)$, namely $\tau_q(t) = m_t = m(q_t)$, is a *Teichmüller geodesic.* Every geodesic in $\mathcal{T}(S)$ is of this form.

4.4.2. Balance Times. If $\delta : I : S$ or $\delta : I \to \widetilde{S}$ is a (local) q-geodesic, we can reparameterize δ to be a (local) q_t-geodesic for any t. In particular, straight segments can be linearly reparameterized to be (locally) geodesic. We denote the reparameterization by δ_t.

For any γ in $\pi_1(S)$, we have

$$\|\gamma\|_{q_t,h} = \|\gamma\|_{q,h} e^t \quad \text{and} \quad \|\gamma\|_{q_t,v} = \|\gamma\|_{q,v} e^{-t}.$$

We let γ_t^* and $\widetilde{\gamma}_t^*$ denote the q_t-geodesic reparameterizations of the q_t-geodesics γ_0^* and $\widetilde{\gamma}_0^*$, respectively.

We say that γ is *balanced at time t* if $\|\gamma\|_{q_t,h} = \|\gamma\|_{q_t,v}$. If γ is balanced at time t_0, then for $b = \|\gamma\|_{q_{t_0},v} + \|\gamma\|_{q_{t_0},h}$, we have

$$b \cosh(t - t_0) \leq \|\gamma\|_{q_t} \leq 2b \cosh(t - t_0) \tag{4.11}$$

by (4.5). So $\|\gamma\|_{q_t}$ is minimized in the interval $[t_0 - \operatorname{arccosh}(2), t_0 + \operatorname{arccosh}^{-1}(2)]$ and grows exponentially in $|t|$.

Given any q, suppose m_t' is a 1-parameter family of hyperbolic metrics on S for which $id_{\widetilde{S}} : (\widetilde{S}, m_t') \to (\widetilde{S}, q_t)$ is a (K_6, C_6)-quasi-isometry. Then

$$\frac{b}{K_6} \cosh(t - t_0) \leq \|\gamma\|_{m_t} \leq 2bK_6 \cosh(t - t_0) \tag{4.12}$$

by (4.10) and (4.11). In particular, the m_t'-length along $\tau_q(t)$ is minimized in the interval $[t_0 - \operatorname{arccosh}(2{K_6}^2), t_0 + \operatorname{arccosh}(2{K_6}^2)]$.

As an example, we could take $m_t' = m_t = m(q_t)$ to be the underlying hyperbolic metric, and then $(K_6, C_6) = (K_3, C_3)$ by Lemma 18. However, Theorem 27 below provides our primary example of interest.

4.4.3. Vertical and Horizontal. Given $\varepsilon > 0$, $0 < \theta < \pi/4$, and q in $\mathcal{Q}^1(S)$, we say that a q-straight segment δ is *θ-almost vertical* (respectively, *θ-almost horizontal*) *with respect to q* if it makes an angle less than θ with the vertical (respectively, horizontal) direction. A closed geodesic γ_0^*, or its lift $\widetilde{\gamma}_0^*$, is called (ε, θ)-*almost vertical* (respectively, (ε, θ)-*almost horizontal*) *with respect to q,* provided that it is a concatenation of q-straight segments each of which is θ-almost vertical (respectively, θ-almost horizontal) or has length less than ε. Subject to certain constraints described below, the constants ε and θ will be fixed, and we will thus refer to segments and geodesics as simply almost vertical or almost horizontal. The discussion here differs from that of [17] in that the constraints we consider depend on the thickness constant $r > 0$.

4.4.4. Nonfilling Curves after Masur and Minsky. The next proposition relies heavily on the work of Masur and Minsky, specifically Sections 5 and 6 of [17]. In particular, Masur and Minsky place an upper bound on ε and θ, depending only on χ, that dictates, among other things, the amount of time it takes for a balanced geodesic to become almost horizontal. We henceforth assume that ε_0, θ_0 are less than this bound. For any fixed $r > 0$, we also assume that ε_0 is less than the constant ε coming from Lemma 19.

PROPOSITION 21. *Given $r > 0$, there is a $T_r > 0$ with the following property. Suppose that q in $\mathcal{Q}^1(S)$ defines an r-thick geodesic τ_q in $\mathcal{T}(S)$ and γ in $\pi_1(S)$ is nonfilling, balanced at time* 0 *in* $\mathbb{R}$. *For any geodesic subpath $\delta_0 \subset \widetilde{\gamma}_0^*$ with $\|\delta_0\|_q > e^{T_r}$, we have*

$$\|\delta_t\|_{q_t} > \frac{\varepsilon_0 e^{|t|-T_r}}{4}\|\delta_0\|_q = \frac{\varepsilon_0 e^{-T_r}}{4} e^{|t|}\|\delta_0\|_q$$

for any t.

We note the similarity between the conclusion of this proposition and (4.11). By comparison, (4.11) is a statement about the q_t-length of the entire curve γ, whereas this proposition provides information about the q_t-length of any definite length segment of γ_0^*. In particular, it also grows exponentially outside some neighborhood of the balance time. Furthermore, whereas (4.11) is true for any closed geodesic, Proposition 21 is false if we allow γ to be filling: there is no T making the proposition valid for all filling γ.

Proof of Proposition 21. In what follows, we appeal to Lemmas 6.4 and 6.5 of [17], which provide bounds on diameters of shadows in the curve complex $\mathcal{C}(S)$ of certain subsets of the Teichmüller geodesic τ_q. Since ours is an r-thick geodesic, the shadow is a uniform quasi-geodesic. This is Lemma 4.4 of [23]. It also follows quickly from the main theorem of [19] (see Section 7.4 of [13]). We may therefore turn bounds on diameters in $\mathcal{C}(S)$ into bounds on diameters in the domain $\mathbb{R}$ of τ_q, and we do so without further comment.

Since γ is nonfilling, there is an essential simple closed curve α disjoint from it. Let t_0 denote the balance time for α.

CLAIM 22. *There exists $T_0 > 0$, depending only on ε_0, θ_0, and r, such that γ_t^* is almost horizontal for all $t > T_0$ and is almost vertical for all $t < -T_0$.*

Proof. By Lemma 6.5 of [17], there is a $T_1 > t_0$ such that $T_1 - t_0$ is bounded by a constant $B(\varepsilon_0, \theta_0, r)$ and such that for all $t > T_1$, the geodesic α_t^* is almost horizontal. Since $i(\delta, \alpha) = 0$, no segment of γ_t^* intersects any segment of α_t^* away from the singularities. Pick a straight segment of α_t^* with length at least ε_0 (from Corollary 20). As in the last paragraph of the proof of Lemma 6.5 of [17], we can appeal to Lemma 6.4 of [17] to find a $T_2 > T_1$ such that, for all $t > T_2$, the geodesic γ_t^* is almost horizontal.[1] Moreover, the distance $T_2 - T_1$, and hence also $T_2 - t_0$, is bounded by a constant $C(\varepsilon_0, \theta_0, r)$.

Reversing the roles of horizontal and vertical, there is $T_3 < t_0$ such that γ_t^* is almost vertical for all $t < T_3$ and $t_0 - T_3$ is bounded by some $D(\varepsilon_0, \theta_0, r)$. The balance time 0 for γ must occur in the interval $[T_3, T_2]$ (since γ is neither almost vertical nor almost horizontal when it is balanced), and setting $T_0 = \max\{T_2, |T_3|\}$ proves the claim. $\square$

For all $t > 0$, we have

$$\|\delta_t\|_{q_t} \geq e^{-t} \|\delta_0\|_{q_0}. \tag{4.13}$$

For $t = T_0$, we have

$$\|\delta_{T_0}\|_{q_{T_0}} \geq e^{-T_0} \|\delta_0\|_{q_0},$$

and we set $T_r = 2T_0$.

Now, if $\delta_0 \subset \widetilde{\gamma}_{T_0}^*$ is a straight segment of length at least e^{T_r}, then we have

$$\|\delta_{T_0}\|_{q_{T_0}} \geq e^{-T_0} \|\delta_0\|_{q_0} \geq e^{-T_0} e^{T_r} > 1.$$

Therefore, by Lemma 19, the segment δ_{T_0} contains a segment δ'_{T_0} of length at least ε_0 contained in a straight segment. This segment δ'_{T_0} must be almost horizontal since $\gamma_{T_0}^*$ (and hence $\widetilde{\gamma}_{T_0}^*$) is almost horizontal. Therefore, for all $t \geq T_0$, we have

$$\|\delta'_t\|_{q_t} \geq \|\delta'_t\|_{q_t,h} \geq e^{t-T_0} \|\delta'_{T_0}\|_{q_{T_0},h} \geq \frac{e^{t-T_0}}{2} \|\delta'_{T_0}\|_{q_{T_0}} \geq \frac{\varepsilon_0 e^{t-T_0}}{2}.$$

There is such a segment δ'_{T_0} in each segment of length 1 in δ_{T_0}. By subdividing δ_{T_0} into a maximal number n of disjoint segments of length at least 1, so that $n \leq \|\delta_{T_0}\|_{q_{T_0}} < n + 1$, we have

$$\|\delta_t\|_{q_t} \geq \frac{n \varepsilon_0 e^{t-T_0}}{2} = \frac{n}{n+1} \frac{(n+1)\varepsilon_0 e^{t-T_0}}{2} \geq \frac{\varepsilon_0 e^{t-T_0}}{4} \|\delta_{T_0}\|_{q_{T_0}}.$$

Combining these strings of inequalities, we see that, for $t \geq T_0$, we have

$$\|\delta_t\|_{q_t} \geq \frac{\varepsilon_0 e^{t-T_0}}{4} e^{-T_0} \|\delta_0\|_{q_0} = \frac{\varepsilon_0 e^{t-T_r}}{4} \|\delta_0\|_{q_0}.$$

[1] The key to the proof of Lemma 6.5 of [17] is finding a disjoint almost horizontal straight segment. In our setting, this is provided by a segment of α_t^*.

On the other hand, if $0 \le t < T_0$, then $-t > t - T_r$. Since $\varepsilon_0/4 < 1$, we therefore have

$$\|\delta_t\|_{q_t} \ge e^{-t}\|\delta_0\|_{q_0} \ge e^{t-T_r}\|\delta_0\|_{q_0} \ge \frac{\varepsilon_0 e^{t-T_r}}{4}\|\delta_0\|_{q_0}$$

by (4.13). Thus, the proposition follows for $t \ge 0$. A symmetric argument proves the proposition for $t \le 0$. $\square$

4.5. *Surface Bundles over Teichmüller Geodesics*

4.5.1. Singular Sol *and Hyperbolic Metrics Are Uniformly Quasi-isometric.* Given q in $\mathcal{Q}^1(T)$ with Teichmüller geodesic τ_q, consider the pullback bundle

$$\mathbb{H}^2 \longrightarrow \mathcal{H}_{\tau_q} \longrightarrow \tau_q.$$

The lifted quadratic differential q_t defines a flat metric on the fiber $\mathcal{H}_{\tau_q(t)} \cong \mathbb{H}^2$. The lifted Teichmüller mapping identifies the fibers $\mathcal{H}_{\tau_q(t)}$ with $\mathcal{H}_{\tau_q(0)}$, determining a homeomorphism $\mathcal{H}_{\tau_q(t)} \cong \widetilde{S} \times \mathbb{R}$ so that $(z, 0) \mapsto (z, t)$ is the Teichmüller mapping. The coordinate t and preferred coordinates $\zeta = x + iy$ for q give local coordinates for $S \times \mathbb{R}$ away from $\{\text{singularities of } q\} \times \mathbb{R}$. We thus have the metric $e^{2t}\,dx^2 + e^{-2t}\,dy^2 + dt^2$ on $(S - \{\text{singularities of } q\}) \times \mathbb{R}$ whose metric completion is naturally identified with $\widetilde{S} \times \mathbb{R} \cong \mathcal{H}_{\tau_q}$ and whose restriction to each fiber is just the metric q_t. We let $\mathcal{H}^{\mathrm{SOL}}_{\tau_q}$ denote $\mathcal{H}_{\tau_q}$ with this metric. This is the *singular* Sol *metric associated to* q.

We now note that Proposition 21 provides an "exponential growth" version of Theorem 17 for the singular Sol metric. Given γ in $\pi_1(S)$, define *isometric* sections $\{\Xi_n\}_{n\in\mathbb{Z}}$ of $\mathcal{H}^{\mathrm{SOL}}_{\tau_q} \to \tau_q$ by picking linearly ordered points $\{z_n\}_{n\in\mathbb{Z}} = \{(z_n, 0)\}_{n\in\mathbb{Z}} \subset \widetilde{\gamma}^*_0 \subset \widetilde{S} \times \{0\}$. Let $\Xi_n = \{(z_n, t) \mid t \in \mathbb{R}\} \subset \mathcal{H}^{\mathrm{SOL}}_{\tau_q} \cong \widetilde{S} \times \mathbb{R}$. By construction, the Ξ_n are linearly ordered over every $\tau_q(t)$. Let δ^n_0 denote the segment from z_n to z_{n+1} inside $\widetilde{\gamma}^*_0$, so that δ^n_t is the segment from Ξ_n to Ξ_{n+1} inside $\widetilde{\gamma}^*_t$. This gives us the following singular Sol variant of Theorem 17.

Proposition 23. *Given $r > 0$, let $T_r > 0$ be as in Proposition 21. Let q be a unit-norm quadratic differential defining an r-thick geodesic τ_q in $\mathcal{T}(S)$ and suppose that γ in $\pi_1(S)$ is nonfilling and balanced at time zero. Given isometric sections $\{\Xi_n\}_{n\in\mathbb{Z}}$ as above with*

$$d_{\tau_q(0)}(\Xi_n, \Xi_{n+1}) = \|\delta^n_0\|_{q_0} \ge e^{T_r},$$

we have

$$d_{\tau_q(t)}(\Xi_n, \Xi_{n+1}) \ge \frac{\varepsilon_0 e^{-T_r}}{4} e^{|t|} d_{\tau_q(0)}(\Xi_n, \Xi_{n+1}).$$

Given a unit-norm quadratic differential q defining an r-thick geodesic τ_q in $\mathcal{T}(S)$ and a nonfilling γ in $\pi_1(S)$, the space $\mathcal{A}^{\mathrm{SOL}}(\gamma) = \bigcup \widetilde{\gamma}^*_t$ is δ^{SOL}-hyperbolic for some $\delta^{\mathrm{SOL}} = \delta^{\mathrm{SOL}}(g, r)$. In fact, this space is quasi-isometric to the hyperbolic plane. Following the argument (in Section 4.2) that derives Theorem 13 from Theorem 17, we have the following corollary of Proposition 23.

If $[a,b]$ is an interval, we let

$$\mathcal{A}^{\mathrm{SOL}}_{[a,b]} = \bigcup_{a \le t \le b} \widetilde{\gamma}^*_t.$$

COROLLARY 24. *Let $r > 0$, and let T_r, q, and γ be as in Proposition 23. There are constants A_0, K_4, and C_4 depending only on r and the genus g of S such that the fiber $\widetilde{\gamma}_0$ is a (K_4, C_4)-quasi-geodesic in $\mathcal{A}^{\mathrm{SOL}}(\gamma)$ and $\mathcal{A}^{\mathrm{SOL}}_{[-a,a]}$ is A_0-quasi-convex for all a.*

Proposition 23 also has the following corollary.

COROLLARY 25. *Let $R, r > 0$, and let T_r, q, γ, and Ξ_n be as in Proposition 23. There is a $B_2 = B_2(R,r)$ such that if the R-neighborhood of Ξ_n intersects Ξ_m, then $|n-m| \le B_2$.*

We now promote Proposition 23 to a statement about arbitrary (k,c)-sections.

PROPOSITION 26. *Given $r, k, c > 0$, there exist $D_2 > D_3 > 0$ with the following property. Let q be a unit-norm quadratic differential defining an r-thick geodesic τ_q in $\mathcal{T}(S)$ and suppose that γ in $\pi_1(S)$ is nonfilling and balanced at time zero. Suppose that $\{\Sigma_n\}_{n\in\mathbb{Z}}$ are (k,c)-sections contained in $\mathcal{A}^{\mathrm{SOL}}(\gamma) = \bigcup_t \widetilde{\gamma}^*_t$ such that*

$$\{\Sigma_n\}_{n\in\mathbb{Z}} \text{ is linearly ordered over } \tau_q(0) \quad \text{and} \quad d_{\tau_q(0)}(\Sigma_n, \Sigma_{n+1}) \ge D_3.$$

Then

$$\{\Sigma_n\}_{n\in\mathbb{Z}} \text{ is linearly ordered over } \tau_q(t) \quad \text{and} \quad d_{\tau_q(t)}(\Sigma_n, \Sigma_{n+1}) \ge D_2 e^{|t|}$$

for every t in $\mathbb{R}$.

Proof. Let Ξ_n be the isometric sections as in Proposition 23. By Proposition 23, it suffices to show that there is a number B such that if Σ is a (k,c)-section contained in $\mathcal{A}^{\mathrm{SOL}}(\gamma)$, then there are numbers $n > m$ with $n - m \le B$ such that Σ lies in the region bounded by Ξ_m and Ξ_n.

Let Σ be a (k,c)-section contained in $\mathcal{A}^{\mathrm{SOL}}(\gamma)$. Let $n > m$ be such that Ξ_n and Ξ_m intersect Σ nontrivially.

Pick (z_m, t_m) in $\Xi_m \cap \Sigma$ and (z_n, t_n) in $\Xi_n \cap \Sigma$. Let (w_n, t_m) be the point in $\Xi_n \cap \widetilde{\gamma}^*_{t_m}$.

Assume that $0 \le t_m \le t_n$.

Let $\mathcal{G}_\Sigma : [0, j] \to \mathcal{A}^{\mathrm{SOL}}$ be a (k,c)-quasi-geodesic in Σ joining (z_m, t_m) and (z_n, t_n). Let $\mathcal{G}_\Xi$ be the geodesic in Ξ_n joining (w_n, t_m) and (z_n, t_n), and let $\mathcal{V}$ be a geodesic in $\mathcal{A}^{\mathrm{SOL}}(\gamma)$ joining (z_m, t_m) and (w_n, t_m).

By Corollary 24, the set $\mathcal{A}^{\mathrm{SOL}}_{[-t_m,t_m]}$ is A_0-quasi-convex. So $\mathcal{V}$ lies in an A_0-neighborhood of $\mathcal{A}^{\mathrm{SOL}}_{[-t_m,t_m]}$.

As the space $\mathcal{A}^{\mathrm{SOL}}(\gamma)$ is δ^{SOL}-hyperbolic, it follows that the quasi-geodesic triangle $\triangle = \mathcal{G}_\Sigma \cup \mathcal{G}_\Xi \cup \mathcal{V}$ is δ'-thin for some δ' depending only on δ^{SOL} and k and c.

Let $\delta'' = 3\max\{A_0, \delta'\}$. Since Σ is a (k, c)-section, there is an $i = i(k, c)$ such that

$$\mathcal{G}_\Sigma|_{[i,j]} \subset \mathcal{A}^{\mathrm{Sol}}_{[t_m+\delta'',\infty]}.$$

Since $\triangle$ is δ'-thin and $\mathcal{V}$ is contained in $\mathcal{A}^{\mathrm{Sol}}_{[-\infty,t_m+A_0]}$, the segment $\mathcal{G}_\Sigma|_{[i,j]}$ must lie in the δ'-neighborhood of $\mathcal{G}_\Xi$. So $\mathcal{G}_\Sigma$ lies in the $(ki + c + \delta')$-neighborhood of $\mathcal{G}_\Xi \subset \Xi_n$.

Corollary 25 now bounds $n - m$.

The cases $0 \le t_n \le t_m$, $t_m \le t_n \le 0$, and $t_n \le t_m \le 0$ are proven by essentially the same argument. The cases $t_n \le 0 \le t_m$ and $t_m \le 0 \le t_n$ are proven by breaking $\mathcal{G}_\Sigma$ into "positive" and "negative" segments and running the above argument on each half. □

The following theorem is due to Farb and Mosher (see Proposition 4.2 of [9] and its proof there) and is the final piece needed to prove Theorem 17.

THEOREM 27 (Farb–Mosher [9]). *Given $r, k, c > 0$, there exist K_5, C_5 with the following properties. Suppose that $g : \mathbb{R} \to \mathcal{T}(S)$ is a (k, c)-quasi-geodesic that stays a uniformly bounded distance from the r-thick Teichmüller geodesic τ_q and let $\nu : \mathbb{R} \to \mathbb{R}$ be a map such that $g(t) \mapsto \tau_q(\nu(t))$ is the closest point projection. Then this closest point projection is (K_5, C_5)-coarsely Lipschitz and lifts to a fiber-preserving (K_5, C_5)-quasi-isometry*

$$\mathcal{H}_g \to \mathcal{H}^{\mathrm{Sol}}_{\tau_q}$$

for which the maps on fibers $\mathbb{H}_{g(t)} \to (\widetilde{S}, q_{\nu(t)})$ are (K_5, C_5)-quasi-isometries.

Proof of Theorem 17. To simplify the discussion, we suppress many of the constants implicit in the proof and use "uniform" and "uniformly" to mean that the constants involved depend only on the sequence $1 \to \pi_1(S) \to \Gamma \to G \to 1$ and its associated bundle $\mathbb{H}^2 \to \mathcal{H}_X \to X$.

Let Σ_n be our (K_2, C_2)-sections of $\mathcal{H}_X \to X$.

For every x in X, take a biinfinite geodesic $\mathcal{G}_0$ in X through x and x_γ. Composing with $X \to \mathcal{T}(S)$, we get a uniformly quasi-geodesic $\mathcal{G}$ fellow traveling an r-thick Teichmüller geodesic τ_q for some $r = r(\Gamma)$. We apply Theorem 27 to produce a uniform fiber-preserving quasi-isometry $\mathcal{H}_\mathcal{G} \to \mathcal{H}^{\mathrm{Sol}}_{\tau_q}$. Pushing the $\Sigma_n|_\mathcal{G}$ over to $\mathcal{H}^{\mathrm{Sol}}_{\tau_q}$, we obtain uniformly quasi-isometric sections Σ'_n. We apply Proposition 26 and push the conclusion back to $\mathcal{H}_\mathcal{G}$. The result is a statement identical to that of Theorem 17 except that x_γ has been replaced with the pullback x_0 of the balance time $\tau_q(0)$. Setting $m'_t = g(t)$ and $\tau_q(\nu(t))$ (with the appropriate reparameterization) in the discussion at the end of Section 4.4.2, we have $(K_6, C_6) = (K_5, C_5)$, so that (4.12) implies that x_0 is uniformly close to x_γ, and this completes the proof. □

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