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Proper Holomorphic Mappings on Flag Domains of SU(p,q) Type on Projective Spaces

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1. Introduction

The objects of study in this paper are the domains in $\mathbb{P}^n$ defined by

$$\mathbb{D}_n^\ell = \left\{ [z_0, \dots, z_n] \in \mathbb{P}^n : \sum_{j=0}^{\ell} |z_j|^2 > \sum_{j=\ell+1}^{n} |z_j|^2 \right\}$$

and the proper holomorphic mappings among them. They are examples of the so-called *flag domains* in $\mathbb{P}^n$ when the latter is regarded as a flag variety. More explicitly, they are open orbits of the real forms $\mathrm{SU}(\ell+1, n-\ell)$ of the complex simple Lie group $\mathrm{SL}(n+1,\mathbb{C})$ when both of which act on $\mathbb{P}^n$ as biholomorphisms.

The domain $\mathbb{D}_n^0$ is just the complex unit n-ball embedded in $\mathbb{P}^n$, and there has been an extensive literature in the study of their proper holomorphic mappings in the last couple of decades. For a survey, see [Fo]. In general, when the codimension is high, the set of proper holomorphic mappings between complex unit balls is large and difficult to determine. On the other hand, in [BH] and [BEH] the domains $\mathbb{D}_n^\ell$ with $\ell \geq 1$ and the associated holomorphic mappings are studied by methods in Cauchy–Riemann geometry. It appears that there is in general much more rigidity when $\ell \geq 1$. Indeed, there is one essential difference between the complex unit n-ball and the domains $\mathbb{D}_n^\ell$ with $\ell \geq 1$, for the latter contain linear subspaces of $\mathbb{P}^n$. Motivated by this, the author of this paper studied in [N] the domains $\mathbb{D}_n^\ell$, $\ell \geq 1$, and their generalizations in Grassmannians by exploiting the structure of the moduli spaces of compact complex analytic subvarieties. Rigidity results analogous to those of [BH] are obtained in a more geometric way.

We will follow the terminology in [BH; BEH] and call ℓ the *signature* of the domain $\mathbb{D}_n^\ell$. As far as rigidity of holomorphic mappings among those domains is concerned, the determining factor should be the difference in signatures rather than the codimension. This is illustrated in [BH], for instance, where it is shown that if the domain and target are of the same signature then any local proper holomorphic map is the restriction of a linear embedding between the ambient projective spaces. On the other hand, Baouendi, Ebenfelt, and Huang [BEH] studied the situations with a small signature difference. Together with other results, they proved that there is partial rigidity for local proper holomorphic mappings $h\colon U \subset \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$

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when $1 \le \ell < n/2$, $1 \le \ell' < m/2$, and $\ell' \le 2\ell - 1$. Furthermore, simple examples can be constructed explicitly to demonstrate that their partial rigidity is best possible and that, in particular, we cannot have full rigidity for local proper holomorphic mappings—in other words, there exist local proper holomorphic maps that are not the restrictions of linear embeddings between projective spaces.

The main purpose of the current paper is to prove the following theorem regarding the rigidity for *global* proper holomorphic mappings among $\mathbb{D}_n^\ell$ when the difference in signatures is small.

MAIN THEOREM. *Let $1 \le \ell < n/2$, let $1 \le \ell' < m/2$, and let $f : \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$ be a proper holomorphic map. If $\ell' \le 2\ell - 1$, then f extends to a linear embedding of $\mathbb{P}^n$ into $\mathbb{P}^m$.*

REMARKS. (1) In [BEH] it has been proved, under the same assumptions, that the image of f is contained in a projective linear subspace of dimension $n + (\ell' - \ell)$. Our proof is independent of this result.

(2) In the theorem, for $\ell = 1$ (which also forces $\ell' = 1$) the above result was obtained in [BH] and a more geometric proof was given in [N].

(3) Without further assumptions, the condition $\ell' \le 2\ell - 1$ is necessary to guarantee linearity. This is illustrated by the nonlinear mapping from $\mathbb{P}^3$ to $\mathbb{P}^5$ defined by

$$[z_0, z_1, z_2, z_3] \mapsto [z_0^2, \sqrt{2} z_0 z_1, z_1^2, z_2^2, \sqrt{2} z_2 z_3, z_3^2].$$

It is easy to see that this map restricts to a proper holomorphic map from $\mathbb{D}_3^1$ to $\mathbb{D}_5^2$. This example is taken from [BH].

We now discuss the scheme of proof for the Main Theorem. Our proof relies on the following linearity criterion of Feder [F].

FEDER'S THEOREM. *Let $h : \mathbb{P}^\ell \to \mathbb{P}^{\ell'}$ be a holomorphic immersion. If $\ell' \le 2\ell - 1$, then h is linear.*

In order to apply Feder's theorem, we have to show two things: (i) there exists some ℓ-dimensional projective linear subspace $L \subset \mathbb{D}_n^\ell$ on which the restriction of f is an immersion; (ii) the image $f(L)$ is contained in some ℓ'-dimensional projective linear subspace in $\mathbb{P}^m$.

We prove (i) by first showing that f extends to a rational map from $\mathbb{P}^n$ to $\mathbb{P}^m$; this is achieved by standard Hartogs' extension techniques in several complex variables. From that we can furthermore deduce the finiteness of f on $\mathbb{D}_n^\ell$. We then establish our key Proposition 3.4, which, roughly speaking, allows us to extract from a finite holomorphic mapping some holomorphic immersions of linear subspaces of sufficiently high dimension. The statement is obtained essentially by analyzing the kernel of the differential of f.

For (ii) we basically follow the same approach as in [N]. We first prove that ℓ-dimensional projective linear subspaces in the boundary $\partial\mathbb{D}_n^\ell$ are mapped to ℓ'-dimensional projective linear subspaces in the target space due to the properness of f. Then, by analyzing the moduli space of these projective linear subspaces, we prove that the boundary behavior can be carried over to the interior and hence (ii) follows.

Once we establish (i) and (ii), Feder's theorem says that the restriction of f on some ℓ-dimensional projective linear subspace is linear; then it is not difficult to deduce that $\deg(f) = 1$.

Remark. It has been pointed out to me by Professor Xiaojun Huang and the referee that the main theorem can also be obtained with the help of the results in [BEH].

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2. Linear Subspaces of $\mathbb{D}_n^\ell$ and Foliations

The structure of the set of projective linear subspaces contained in $\mathbb{D}_n^\ell$ is studied in [N], and it is crucial also to this paper. In order to make the paper more self-contained, we will briefly recall some relevant facts in this section.

For a point $[\mathbf{z}] = [z_0, \dots, z_n] \in \mathbb{P}^n$, we split its homogeneous coordinates as $[\mathbf{z}] = [\mathbf{z}', \mathbf{z}'']_\ell$, where $\mathbf{z}' = (z_0, \dots, z_\ell)$ and $\mathbf{z}'' = (z_{\ell+1}, \dots, z_n)$. We denote the closure of $\mathbb{D}_n^\ell$ in $\mathbb{P}^n$ by $\bar{\mathbb{D}}_n^\ell$. We first recall the definition of a type-I irreducible bounded symmetric domain and its compact dual, the complex Grassmannian.

Definition 2.1. Let $M(p, q; \mathbb{C})$ be the set of $p \times q$ complex matrices. We identify $M(p, q; \mathbb{C})$ as $\mathbb{C}^{pq}$. The type-I irreducible bounded symmetric domain $\Omega_{p,q}$ is the domain in $\mathbb{C}^{pq}$ defined by $\Omega_{p,q} = \{A \in M(p, q; \mathbb{C}) : I - AA^H > 0\}$, where A^H denotes the Hermitian transpose of A. As a Hermitian symmetric space, the compact dual of $\Omega_{p,q}$ is the complex Grassmannian of p-dimensional linear subspaces of $\mathbb{C}^{p+q}$ and we denote it by $G_{p,q}$.

Proposition 2.2. *$\mathbb{D}_n^\ell$ (resp. $\bar{\mathbb{D}}_n^\ell$) contains a family of ℓ-dimensional projective linear subspaces. They are maximal compact complex analytic subvarieties in $\mathbb{D}_n^\ell$ (resp. $\bar{\mathbb{D}}_n^\ell$). Moreover, the set of all such linear subspaces is parameterized by $\Omega_{\ell+1,n-\ell}$ (resp. $\bar{\Omega}_{\ell+1,n-\ell}$). Furthermore, if $\ell < n/2$, then the boundary $\partial\mathbb{D}_n^\ell$ also contains a family of ℓ-dimensional projective linear subspaces and the Shilov boundary of $\Omega_{\ell+1,n-\ell}$ parameterizes precisely those contained in the boundary.*

Proof. The complete proof is given in [N, Prop. 2.2, Prop. 2.3, Lemma 2.4]. Here we just give the explicit parameterization of the linear subspaces. Let $A \in M(\ell+1, n-\ell; \mathbb{C})$. Consider the ℓ-dimensional linear subspace

$$\{[\mathbf{z}', \mathbf{z}'']_\ell \in \mathbb{P}^n : \mathbf{z}'' = \mathbf{z}'A\} \cong \mathbb{P}^\ell \subset \mathbb{P}^n.$$

Then as $\mathbf{z}'\mathbf{z}'^H > \mathbf{z}'AA^H\mathbf{z}'^H$ for all $\mathbf{z}'$ if and only if $I - AA^H > 0$, we see that this linear subspace is contained in $\mathbb{D}_n^\ell$ if and only if $A \in \Omega_{\ell+1,n-\ell}$. The parameterization extends to the respective closures in the natural way. Note that when $\ell < n/2$, the Shilov boundary of $\Omega_{\ell+1,n-\ell}$ is just the set of all matrices A such that $AA^H = I$. For such an A, the above linear subspace will then be contained completely in $\partial\mathbb{D}_n^\ell$. □

In what follows, by the "ℓ-Grassmann bundle of a manifold M", denoted $G_\ell TM$, we mean the bundle of Grassmannians of the ℓ-planes in each tangent space on M. We denote the Grassmannian of ℓ-planes in the tangent space at $p \in M$ by $G_\ell T_p M$.

PROPOSITION 2.3. *Let $\pi: G_\ell T\mathbb{D}_n^\ell \to \mathbb{D}_n^\ell$ be the ℓ-Grassmann bundle of $\mathbb{D}_n^\ell$. There is an open set $V_n^\ell \subset G_\ell T\mathbb{D}_n^\ell$, $\pi(V_n^\ell) = \mathbb{D}_n^\ell$, such that V_n^ℓ is a trivial holomorphic $\mathbb{P}^\ell$-bundle over $\Omega_{\ell+1,n-\ell}$.*

Proof. Fix a point $p \in \mathbb{D}_n^\ell$. Since $\mathbb{P}^\ell$ is compact and $\mathbb{D}_n^\ell$ is open, we deduce that there is an open set $U_p \subset G_\ell T_p\mathbb{D}_n^\ell$ consisting of precisely all the tangent ℓ-planes that are tangent to some ℓ-dimensional projective linear subspace contained in $\mathbb{D}_n^\ell$. Since the tangent plane at a point uniquely determines the linear subspace, by Proposition 2.2 the statements in Proposition 2.3 are immediate. □

The $\mathbb{P}^\ell$-foliation of V_n^ℓ is just the universal family of ℓ-dimensional projective linear subspaces in $\mathbb{D}_n^\ell$, and we denote it by $\Pi: V_n^\ell \to \Omega_{\ell+1,n-\ell}$. Furthermore, it is simply the restriction of the standard universal family of ℓ-dimensional projective linear subspaces in $\mathbb{P}^n$, which we denote by $\Pi: G_\ell T\mathbb{P}^n \to G_{\ell+1,n-\ell}$.

LEMMA 2.4. *If $\ell < n/2$, then any germ of a complex submanifold in $\partial\mathbb{D}_n^\ell$ must lie in an ℓ-dimensional projective linear subspace contained in $\partial\mathbb{D}_n^\ell$.*

Proof. By [W], the ℓ-dimensional projective linear subspaces contained in $\partial\mathbb{D}_n^\ell$ are the *holomorphic arc components* or *boundary components* of $\partial\mathbb{D}_n^\ell$, whose defining properties imply the statement of the lemma. For a more elementary proof, see [N, Lemma 2.4]. □

3. Rationality, Finiteness, and Immersiveness

We will first prove that every proper holomorphic map $f: \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$, $\ell \geq 1$, extends to a finite rational map. We begin with an elementary lemma in algebraic geometry.

LEMMA 3.1. *Let $h: \mathbb{P}^n \to \mathbb{P}^m$ be a rational map. If $S \subset \mathbb{P}^n$ is a compact complex analytic subvariety in the domain of h and if h is constant on S, then S is a finite set of points.*

Proof. By composing h with a linear transformation, we may assume that $h(S) = [1,0,\dots,0]$. Let $h = [h_0,\dots,h_m]$, where all h_j are polynomials of the same degree. By assumption, for $1 \le j \le m$ we have $h_j|_S \equiv 0$. If S is of positive dimension, then the zero set of h_0 must intersect S and hence S intersects the set of indeterminacy of h, which violates our initial assumption. Thus, S is finite set of points. □

PROPOSITION 3.2. *The proper holomorphic map f extends to a rational map from $\mathbb{P}^n$ to $\mathbb{P}^m$. Furthermore, f is a finite map.*

Proof. For each $j \in \{0,\dots,n\}$, let $U_j \subset \mathbb{P}^n$ be the open set defined by $z_j \ne 0$. Note that the complement $\mathbb{P}^n \setminus \mathbb{D}_n^\ell$ is the domain defined by $\sum_{j=0}^{\ell}|z_j|^2 \le \sum_{j=\ell+1}^{n}|z_j|^2$. In particular, we have $\mathbb{P}^n \setminus \mathbb{D}_n^\ell \subset \bigcup_{j=\ell+1}^{n} U_j$.

Hence, it suffices to establish the meromorphic extension of the component functions of f (as meromorphic functions) on U_j for each $j \in \{\ell+1,\dots,n\}$. Now fix $j \in \{\ell+1,\dots,n\}$; then, in terms of the standard inhomogeneous coordinates $(w_1,\dots,w_n)$ on U_j, the domain $\mathbb{D}_n^\ell \cap U_j$ is defined by the inequality

$$\sum_{k=1}^{\ell+1}|w_k|^2 > \sum_{k=\ell+2}^{n}|w_k|^2 + 1.$$

If we decompose $U_j \cong \mathbb{C}^n = \mathbb{C}^{\ell+1} \times \mathbb{C}^{n-\ell-1}$ then, for every relatively compact open set $V \Subset \mathbb{C}^{n-\ell-1}$ containing the origin, the component functions of f extend meromorphically over $\mathbb{C}^{\ell+1} \times V \subset \mathbb{C}^n$ by Hartogs' extension [S] since $\ell+1 \ge 2$. In other words, f extends to a meromorphic map from U_j to $\mathbb{P}^m$. We have thereby established the meromorphic extension of f on each U_j, $j \in \{\ell+1,\dots,n\}$, so f extends to a meromorphic and hence rational map from $\mathbb{P}^n$ to $\mathbb{P}^m$.

Now, since $f : \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$ is proper and holomorphic, for every $p \in \mathbb{D}_m^{\ell'}$ it follows that the preimage $f^{-1}(p) \subset \mathbb{D}_n^\ell$ is a compact complex analytic subvariety in $\mathbb{P}^n$ and hence is a finite set (by Lemma 3.1). Thus, f is a finite map. □

In the remainder of this section we will prove the paper's key proposition. It is by virtue of this proposition that we can extract, from f, holomorphic immersions of projective spaces of sufficiently high dimension. We need the following dimension formula in its proof.

LEMMA 3.3. *Let V be an n-dimensional complex vector space, and let $G_V(\ell)$ be the Grassmannian of ℓ-dimensional vector subspaces of V. Fix a k-dimensional vector subspace $W \subset V$ and denote by $\mathcal{W} \subset G_V(\ell)$ the irreducible analytic subvariety consisting of elements having nontrivial intersection with W. Then*

$$\dim(\mathcal{W}) = \begin{cases} (k-1) + (\ell-1)(n-\ell) & \text{if } k \le n-\ell, \\ \ell(n-\ell) & \text{if } k > n-\ell. \end{cases}$$

Proof. If $k > n-\ell$ then the lemma is trivial, since in this case $\mathcal{W} = G_V(\ell)$ and $\dim(G_V(\ell)) = \ell(n-\ell)$.

So suppose $k \leq n - \ell$; in this case, $\mathcal{W}$ is simply the closure of a Schubert cell in $G_V(\ell)$ and one can follow the procedures in [GHa, Chap. 1, Sec. 5] to calculate its dimension. For the convenience of the reader, we provide an elementary proof here.

We may simply take $V = \mathbb{C}^n$. Let $E_\ell = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_{\ell+1} = z_{\ell+2} = \cdots = z_n = 0\}$. Let $\pi : \mathbb{C}^n \to E_\ell$ be the canonical projection and put $U = \{Q \in G_{\mathbb{C}^n}(\ell) : \pi(Q) = E_\ell\}$. Then U is a standard Euclidean coordinate chart in $G_{\mathbb{C}^n}(\ell)$ and $U \cong \mathbb{C}^{\ell(n-\ell)}$. Note that every $Q \in U$ can be represented by an $\ell \times n$ matrix of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & z_{1,1} & \cdots & z_{1,(n-\ell)} \\ 0 & 1 & \cdots & 0 & z_{2,1} & \cdots & z_{2,(n-\ell)} \\ & & \ddots & & & \vdots & \\ 0 & 0 & \cdots & 1 & z_{\ell,1} & \cdots & z_{\ell,(n-\ell)} \end{bmatrix},$$

in which the rows constitute a basis of Q. The $z_{j,k}$ are precisely the standard Euclidean coordinates on U.

Adding a basis of W as rows to the preceding matrix yields a $(k+\ell) \times n$ matrix. Now it is easy to see that the condition $\dim(Q \cap W) \geq 1$ is given by the vanishing of $n - (k+\ell) + 1$ minors of size $(k+\ell) \times (k+\ell)$. Equivalently, $\mathcal{W} \cap U$ is defined by $n - (k+\ell) + 1$ independent algebraic equations and so

$$\begin{aligned} \dim(\mathcal{W}) &= \dim(G_{\mathbb{C}^n}(\ell)) - [n - (k+\ell) + 1] \\ &= \ell(n-\ell) - [n - (k+\ell) + 1] \\ &= (k-1) + (\ell-1)(n-\ell). \end{aligned}$$ □

PROPOSITION 3.4. *Let $g : \mathbb{P}^n \to \mathbb{P}^m$ be a finite rational map. Then, for $\ell < n/2$, the restriction of g on a general ℓ-dimensional projective linear subspace is a holomorphic immersion.*

Proof. Let $X \subset \mathbb{P}^n$ be the indeterminacy of g and let $U := \mathbb{P}^n \setminus X$. We still denote the restriction of g on U as g; thus $g : U \to \mathbb{P}^m$ is a finite holomorphic map.

Let $dg : TU \to T\mathbb{P}^m$ be the differential of g. Since g is finite—in particular, not totally degenerate—dg naturally induces a meromorphic map $[dg]$ from $G_\ell TU$ (the ℓ-Grassmann bundle of U) to $G_\ell T\mathbb{P}^m$. Let $Z \subset G_\ell TU$ be the set of indeterminacy of $[dg]$. We are going to show that the complex analytic subvariety Z is of dimension less than $(\ell+1)(n-\ell)$. Assume this dimension estimate for the moment. Now let $\Pi : G_\ell T\mathbb{P}^n \to G_{\ell+1,n-\ell}$ be the universal family of ℓ-dimensional projective linear subspaces in $\mathbb{P}^n$ (see Proposition 2.3 and the paragraph thereafter). Note that Π is proper and hence $\Pi(Z) \subset G_{\ell+1,n-\ell}$ is a locally closed complex analytic subvariety. But $\dim(G_{\ell+1,n-\ell}) = (\ell+1)(n-\ell)$ and so, by our dimension estimate, $\Pi(Z)$ is not dense in $G_{\ell+1,n-\ell}$. Thus, for a general point $q \in G_{\ell+1,n-\ell}$, the differential $[dg]$ is well defined on $\Pi^{-1}(q) \cong \mathbb{P}^\ell$. This is equivalent to saying that the restriction of g on the ℓ-dimensional projective linear subspace corresponding to $\Pi^{-1}(q)$ is an immersion, and the proof is complete.

We now prove the dimension estimate. For $k \in \{1, \dots, n\}$, let $I_k \subset U$ be the set of points for which the kernel of dg (as a linear map at each individual point) is of dimension at least k. Since g is finite, $I_k \subset U$ is a complex analytic subvariety of dimension at most $n-k$ and $I_n \subset \cdots \subset I_1 = \pi(Z)$, where $\pi : G_\ell TU \to U$ is the canonical projection. Now for every $p \in U$, the fibre of Z over p (i.e., $Z \cap G_\ell T_p U$) is the set of ℓ-planes in T_pU having a nontrivial intersection with the kernel of dg at p. By Lemma 3.3,

$$\begin{aligned} &\dim(Z \cap G_\ell T_p U) \\ &\quad = \begin{cases} (k-1)+(\ell-1)(n-\ell) & \text{if } p \in I_k \setminus I_{k+1},\ k \in \{1,\dots,n-\ell\}, \\ \ell(n-\ell) & \text{if } p \in I_{n-\ell+1}. \end{cases} \end{aligned} \tag{$*$}$$

Let $Z_k = \pi^{-1}(I_k) \subset Z$, where $1 \le k \le n$. It is clear that each Z_k is also a complex analytic subvariety of $G_\ell TU$ and that $Z_n \subset Z_{n-1} \subset \cdots \subset Z_1 = Z$. We start from Z_n. Considering the projection π, we deduce that

$$\dim(Z_n) \le \dim(\text{fibre}) + \dim(\text{base}) = \dim(Z \cap G_\ell T_p U) + \dim(I_n),$$

where $p \in I_n$ is arbitrary. Consequently, we have

$$\dim(Z_n) \le \ell(n-\ell) + 0 < (\ell+1)(n-\ell)$$

by $(*)$. Next, $Z_{n-1} \setminus Z_n$ is a locally closed complex analytic subvariety and its dimension, by similar reasoning, is at most equal to

$$\ell(n-\ell) + 1 < \ell(n-\ell) + (n-\ell) = (\ell+1)(n-\ell)$$

because $\ell < n/2$. Hence

$$\dim(Z_{n-1}) < (\ell+1)(n-\ell).$$

With $\ell < n/2$, for every $k \in \{1,\dots,\ell-1\}$ we analogously have

$$\ell(n-\ell) + k < \ell(n-\ell) + (n-\ell) = (\ell+1)(n-\ell).$$

Thus we can repeat the previous argument to conclude that

$$\dim(Z_n) \le \dim(Z_{n-1}) \le \cdots \le \dim(Z_{n-\ell+1}) < (\ell+1)(n-\ell).$$

Observe that $Z_{n-\ell} \setminus Z_{n-\ell+1}$ is a locally closed complex analytic subvariety and so

$$\begin{aligned} \dim(Z_{n-\ell} \setminus Z_{n-\ell+1}) &\le \dim(Z \cap G_\ell T_p U) + \dim(I_{n-\ell}) \quad (p \in I_{n-\ell}) \\ &\le [(n-\ell-1) + (\ell-1)(n-\ell)] + \ell \\ &= (n-1) + (\ell-1)(n-\ell) \\ &< (\ell+1)(n-\ell), \end{aligned}$$

where the term in brackets follows from $(*)$ and where the last inequality is again due to our assumption that $\ell < n/2$. Consequently,

$$\dim(Z_{n-\ell}) < (\ell+1)(n-\ell).$$

By repeating the argument, we get for every $k \in \{1,\dots,n-\ell\}$ that $\dim(Z_k) < (\ell+1)(n-\ell)$; thus we have established $\dim(Z) < (\ell+1)(n-\ell)$. $\square$

It is for the sake of notational simplicity that we work with global mappings in Proposition 3.4. Indeed, we can simply restrict the whole argument on open subsets and obtain the following local version.

PROPOSITION 3.5. *Let $D \subset \mathbb{P}^n$ be an open set, M a complex manifold, and $g\colon D \to M$ a finite holomorphic map. Let $\ell < n/2$ and let $L \subset \mathbb{P}^n$ be an arbitrary ℓ-dimensional projective linear subspace intersecting D. Then, for a general choice of L, the restriction of g on $L \cap D$ is a holomorphic immersion.*

4. Proof of the Main Theorem

Throughout this section we let $f\colon \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$ be a proper holomorphic map, $\ell \geq 1$.

PROPOSITION 4.1. *If $\ell < n/2$ and $\ell' < m/2$, then for each ℓ-dimensional projective linear subspace $L \subset \mathbb{D}_n^\ell$ (as described in Proposition 2.2) we have $f(L) \subset L'$, where L' is some ℓ'-dimensional linear subspace in the target $\mathbb{P}^m$.*

Proof. By Proposition 3.2, f extends as a rational map and hence the induced meromorphic map $[df]\colon G_\ell T\mathbb{D}_n^\ell \to G_\ell T\mathbb{P}^m$ extends to an open neighborhood of $G_\ell T\mathbb{D}_n^\ell$ and, in particular, to an open neighborhood of the universal family $\Pi\colon V_n^\ell \to \Omega_{\ell+1,n-\ell}$ (see the paragraph after Proposition 2.3). More precisely, we mean that $[df]$ extends to an open neighborhood $W \supset \bar{V}_n^\ell$ and $\Pi(W) = U$ is some open neighborhood of $\bar{\Omega}_{\ell+1,n-\ell}$.

Now we consider the composition $f^\sharp := \pi \circ [df]$, where $\pi : G_\ell T\mathbb{P}^m \to \mathbb{P}^m$ is the canonical projection. Take a general point b in the Shilov boundary of $\Omega_{\ell+1,n-\ell}$ such that $[df]$ and hence $f^\sharp$ is defined on the ℓ-dimensional projective linear subspace over the point b (i.e., $\Pi^{-1}(b)$). By Lemma 2.4 and the properness of f, we have $f^\sharp(\Pi^{-1}(b)) \subset \partial\mathbb{D}_m^\ell$ and hence $f^\sharp(\Pi^{-1}(b)) \subset L'_b$ for some ℓ'-dimensional projective linear subspace $L'_b \subset \mathbb{P}^m$. In other words, on the holomorphic $\mathbb{P}^\ell$-bundle $\Pi\colon W \to U \supset \bar{\Omega}_{\ell+1,n-\ell}$, the map $f^\sharp$ maps the general fibres over the Shilov boundary of $\Omega_{\ell+1,n-\ell}$ to ℓ'-dimensional projective linear subspaces in $\mathbb{P}^m$. Note that this is an analytic condition—in other words, it can be expressed in terms of the vanishing of a set of holomorphic functions in local coordinates (e.g., some degeneracy conditions on a set of vertical derivatives on the base). Now we have a set of holomorphic functions that vanish on the intersection of an open set and the Shilov boundary of $\Omega_{\ell+1,n-\ell}$ and therefore must vanish on the entire open set. (For a proof of this, see [N, Lemma 2.9].) Hence, we conclude that this degeneracy property holds also for the general fibres in the interior; that is, $f^\sharp$ maps general fibres to ℓ'-dimensional projective linear subspaces in $\mathbb{P}^m$. This precisely means that f maps general and hence all ℓ-dimensional projective linear subspaces into ℓ'-dimensional projective linear subspaces. □

We are now ready to prove the Main Theorem.

Proof of the Main Theorem. By Proposition 3.2 together with Proposition 3.4, there exists an ℓ-dimensional projective linear subspace $L_0 \subset \mathbb{D}_n^\ell$ on which the restriction of f is a holomorphic immersion. However, by Proposition 4.1, $f(L_0)$ is contained in some ℓ'-dimensional projective linear subspace in $\mathbb{P}^m$. Since $\ell' \leq 2\ell - 1$, we have by Feder's theorem that the restriction of f on L_0 is linear. We are going to show that this implies that f itself is linear.

By Proposition 3.2, there exists a $d \in \mathbb{N}^+$ such that we can write $f = [p_0, \dots, p_m]$, where the p_j are relatively prime homogeneous polynomials of degree d in the homogeneous coordinates $z_0, \dots, z_n$ of $\mathbb{P}^n$. By composing with an automorphism of $\mathbb{P}^n$, we may assume that $L_0 = \{[z_0, \dots, z_\ell, 0, \dots, 0] \in \mathbb{P}^n : [z_0, \dots, z_\ell] \in \mathbb{P}^\ell\}$. We can write the restriction as $f|_{L_0} = [\tilde{p}_0, \dots, \tilde{p}_m]$, where each $\tilde{p}_j$ is obtained by setting $z_{\ell+1} = \cdots = z_m = 0$ in p_j. Thus, $\tilde{p}_j$ is either zero or a degree-d homogeneous polynomial in $z_0, \dots, z_\ell$. The polynomials $\tilde{p}_j$ are not all zero and are also relatively prime, for otherwise the indeterminacy of f would intersect L_0 and thus contradict our assumption that f is holomorphic on $\mathbb{D}_n^\ell$ (in particular, on L_0). But we also know that $f|_{L_0}$ is linear, so we must have $d = 1$. Hence, f is linear. □

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