Cohomology of Local Systems on Loci of d-elliptic Abelian Surfaces

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1. Introduction

To an irreducible representation of Sp_{2g} with highest weight vector λ, one can associate in a natural way a local system $\mathbf{W}_\lambda$ on the moduli spaces $\mathcal{A}_g$ and hence also on $\mathcal{M}_g$. One reason for studying these local systems is that their complex (resp., ℓ-adic) cohomology groups will contain spaces of elliptic and Siegel modular forms (resp., their associated ℓ-adic Galois representations) as subquotients. In particular, one can study modular forms by looking at the cohomology of these local systems—and vice versa.

When $g = 1$ this is described by the Eichler–Shimura theory and in particular by its Hodge-theoretic/ℓ-adic interpretation [8], which expresses the cohomology of such a local system in terms of spaces of modular forms on the corresponding modular curve. See [12, Sec. 4] for a summary. For higher genera the situation is not as well understood. The (integer-valued) Euler characteristics of these local systems on $\mathcal{M}_2$ were calculated in [17]. Their Euler characteristics on $\mathcal{M}_g$ and $\mathcal{A}_g$ for $g = 2, 3$, now taken in the Grothendieck group of ℓ-adic Galois representations, have been investigated by means of point counting in the sequence of papers [10; 11; 2; 3].

Another reason to be interested in such local systems is that they arise when computing the cohomology of relative configuration spaces. For instance, in the case of $\mathcal{M}_g$, the results of [16] imply that calculating the Euler characteristics of all of these local systems on $\mathcal{M}_g$ is equivalent to calculating the $\mathbb{S}_n$-equivariant Euler characteristic of $\mathcal{M}_{g,n}$ for all n.

In this paper, we shall study the restriction of these local systems to certain loci in $\mathcal{A}_2$ of abelian surfaces with a degree d^2 isogeny to a product of elliptic curves. We call such surfaces d-elliptic and denote the (normalization of the) locus of d-elliptic surfaces by $\mathcal{E}_d$. A curve of genus 2 is d-elliptic in the usual sense—that is, it admits a degree d covering onto an elliptic curve if and only if its Jacobian is d-elliptic in this sense. Already the simplest case $d = 1$ (i.e., the locus $\mathcal{A}_2 \setminus \mathcal{M}_2$ of products of elliptic curves) is not entirely trivial, as one needs the branching formula of Section 3.

These loci of d-elliptic curves and surfaces are classically studied by algebraic geometers and number theorists. Biermann and Humbert showed that the locus of d-elliptic surfaces in $\mathcal{A}_2$ is exactly equal to the Humbert surface [25] of invariant d^2; see [21]. Moreover, a natural double cover of $\mathcal{E}_d$ can be described as a quotient $\Gamma \setminus (\mathfrak{H} \times \mathfrak{H})$ with a group Γ acting by a "twisted" diagonal action, which makes the double cover appear as a degenerate Hilbert modular surface. These degenerate Hilbert modular surfaces were studied in [19], and steps were taken in [6] toward studying modular forms on them. The latter work gives a concrete interpretation to the spaces of modular forms that we find to occur in the cohomology of these local systems.

We now give an outline of this paper. In Section 2, we define the spaces $\mathcal{E}_d$ via their modular interpretation and explain their description as quotients of products of modular curves. We also discuss how representations of $\mathrm{SL}_2(\mathbf{Z}/d)$ behave under conjugation by elements of $\mathrm{GL}_2(\mathbf{Z}/d)$. Section 3 proves branching formulas for $\mathrm{Sp}_2 \wr \mathbb{S}_2 \hookrightarrow \mathrm{Sp}_4$ and $\mathrm{Sp}_2 \times \mathbb{S}_2 \hookrightarrow \mathrm{Sp}_2 \wr \mathbb{S}_2$, which will later be used to determine how certain local systems behave under pullback between modular varieties. These local systems are introduced in Section 4. In Section 5 we recall the Eichler–Shimura theory, expressing the cohomology of such local systems on modular curves in terms of modular forms, and show how this leads to a description of the cohomology of local systems on $\mathcal{E}_d$. In Section 6, we specialize to $d = 2$ and show how the results of this paper can be used to compute the Euler characteristic of the space of n-pointed bi-elliptic genus 2 curves in the Grothendieck group of Hodge structures for any n.

I am grateful to my advisor Carel Faber for patient discussions.

REMARK 1.1.   Unless stated otherwise, all cohomology will be taken in the category of rational mixed Hodge structures. However, restricting to Hodge structures is not really necessary. We could, for instance, substitute "smooth ℓ-adic sheaf" for "local system" throughout and our computations would work equally well in the category of ℓ-adic Galois representations (hence also in positive characteristic, provided that the integer d is invertible on our base scheme). In fact, it is not so hard (although we shall not do so) to do everything motivically, using the results of [22] to construct these cohomology groups as Chow motives with coefficients in a number field.

2. The d-elliptic Loci

DEFINITION 2.1.   Let (A, Θ) be a principally polarized abelian surface. We say that A is *d-elliptic* if there is a (connected) curve E lying on A such that:

(1) E is a subgroup of A under the group law;
(2) the genus of E is 1; and
(3) $(E \cdot \Theta) = d$.

There are several equivalent characterizations of d-elliptic surfaces, which we now briefly recall. This material can be found in Frey and Kani [14], who write that

"the following construction appears to be known in principle". Much of this section is a special case of the general theory in [5, Chap. 12]; see also [21].

Let A be d-elliptic. Then $E \hookrightarrow A$ dualizes to a surjection $A \to E$ whose kernel is connected. Then this kernel, too, is an elliptic curve, which we denote by E' and refer to as the *conjugate* of E. In other words, E' is the "Prym variety" of $A \to E$. The curves E and E' intersect (inside A) exactly in their respective d-torsion points. The induced isomorphism $\phi: E[d] \to E'[d]$ of d-torsion groups inverts the Weil pairing; that is, $\langle x, y \rangle_E = \langle \phi x, \phi y \rangle_{E'}^{-1}$. It follows that the induced isogeny $E \times E' \to A$ has degree d^2 and that it is defined by "quotienting out" the subgroup defined by the graph of ϕ. Hence one could also define an abelian surface to be d-elliptic when it can be written as $E \times E' / \{(x, \phi x) \mid x \in E[d]\}$, where ϕ is an isomorphism that inverts the Weil pairing as before.

REMARK 2.2. If $A = \mathrm{Jac}(C)$ then the composition $C \to A \to E$ is a covering of degree d, which is minimal in the sense that it does not factor through an isogeny of degree > 1; conversely, any such covering $C \to E$ induces a map $E \to \mathrm{Jac}(C)$ that makes the Jacobian d-elliptic.

DEFINITION 2.3. We denote by $\mathcal{E}_d$ the moduli stack of pairs $(A, \Theta, \{E, E'\})$, where (A, Θ) is a d-elliptic abelian surface and $\{E, E'\}$ is an unordered pair of conjugate elliptic subgroups of A. (To be more precise, one should not consider unordered pairs but rather the groupoid whose objects are 4-tuples (A, Θ, E, E') satisfying these conditions and whose isomorphisms are Cartesian diagrams that are allowed to switch E and E'.)

REMARK 2.4. The natural map from $\mathcal{E}_d$ to the locus of d-elliptic abelian surfaces in $\mathcal{A}_2$ exhibits $\mathcal{E}_d$ as the normalization of the d-elliptic locus [21, Cor. 3.10].

The characterization of d-elliptic surfaces as quotients of products of elliptic curves by the graph of an anti-isomorphism of the d-torsion implies an alternative description of $\mathcal{E}_d$. Let $Y(d)$ denote the open modular curve parameterizing elliptic curves with a symplectic basis of their d-torsion groups. Then clearly the space

$$Y(d) \times Y(d)/(\mathbb{S}_2 \times \mathrm{SL}_2(\mathbf{Z}/d)),$$

where $\mathbb{S}_2$ swaps the two factors and $\mathrm{SL}_2(\mathbf{Z}/d)$ acts diagonally, parameterizes unordered pairs of elliptic curves together with a symplectic isomorphism of their d-torsion groups. To invert the Weil pairing we need to consider instead an action of the semidirect product $\mathbb{S}_2 \ltimes \mathrm{SL}_2(\mathbf{Z}/d)$, where $\mathbb{S}_2$ acts on $\mathrm{SL}_2(\mathbf{Z}/d)$ by conjugation with an element $\varepsilon \in \mathrm{GL}_2(\mathbf{Z}/d)$ such that $\det(\varepsilon) = -1$. Then one has the following proposition.

PROPOSITION 2.5. *There is an isomorphism*

$$Y(d) \times Y(d)/(\mathbb{S}_2 \ltimes \mathrm{SL}_2(\mathbf{Z}/d)) \cong \mathcal{E}_d,$$

where $\mathrm{SL}_2(\mathbf{Z}/d)$ *acts normally on the first copy of* $Y(d)$ *and via the conjugated action on the second copy.*

Later, we shall need to see how the action of $\mathbb{S}_2$ on $\mathrm{SL}_2(\mathbf{Z}/d)$ just defined acts on representations of $\mathrm{SL}_2(\mathbf{Z}/d)$.

NOTATION 2.6.   If V is a representation of $\mathrm{SL}_2(\mathbf{Z}/d)$, then V^ε denotes the representation obtained by conjugation by $\varepsilon \in \mathrm{GL}_2(\mathbf{Z}/d)$ of determinant -1.

It is clear that $V^{\varepsilon\varepsilon} \cong V$ and that V^ε does not depend on the choice of ε up to isomorphism.

LEMMA 2.7.   *Let p be a prime and $\mathbf{Z}_p$ the p-adic integers, and choose $A \in \mathrm{SL}_2(\mathbf{Z}_p)$. There is always a matrix $\varepsilon \in \mathrm{GL}_2(\mathbf{Z}_p)$ with $\det(\varepsilon) = -1$ such that $A = \varepsilon A^{-1} \varepsilon^{-1}$.*

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and put $\varepsilon = \left(\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right)$. One checks that

$$\begin{pmatrix} x & y \\ z & -x \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$$

holds if and only if

$$(a - d)x + cy - bz = 0.$$

By multiplying this equation by an appropriate factor p^λ, we may assume that at least one of $(a - d)$, b, and c is a p-adic unit. (If not, A is $\pm\mathrm{Id}$ and we are done.) If b is a unit, we put $x = 1$, $y = 0$, and $z = b^{-1}(a - d)$. Then $\det(\varepsilon) = -1$ and we are done—and similarly if c is a unit. If $(a - d)$ is a unit and b and c are not, then substitute

$$x = (a - d)^{-1}(bz - cy)$$

into the equation

$$-x^2 - yz = -1$$

for the determinant. If $p > 2$, then reducing the resulting equation modulo p yields $-yz = -1$. For $p = 2$ we must reduce modulo 8, after which we have

$$4\delta_1 y^2 + 4\delta_2 z^2 - yz = -1$$

for some $\delta_1, \delta_2 \in \{0, 1\}$. Either way, one can now check that there is a solution in $\mathbf{Z}_p$ by a version of Hensel's lemma. $\square$

REMARK 2.8.   The proposition is false if we instead set, for example, the condition $\det(\varepsilon) = 1$. For instance, the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not conjugate in $\mathrm{SL}_2(\mathbf{Z}_p)$ to its inverse if $p \equiv 3 \pmod 4$.

PROPOSITION 2.9.   *Let V be any representation of $\mathrm{SL}_2(\mathbf{Z}/n)$. Then V^ε is isomorphic to the dual (contragredient) of V.*

Proof. By the Chinese remainder theorem, we may assume that $n = p^\lambda$ is a prime power. Let χ be the character of V. Then the character of its dual is $g \mapsto \chi(g^{-1})$ and the character of V^ε is $g \mapsto \chi(\varepsilon g \varepsilon^{-1})$. But g^{-1} and $\varepsilon g \varepsilon^{-1}$ lie in the same conjugacy class of $\mathrm{SL}_2(\mathbf{Z}/p^\lambda)$ by Lemma 2.7, so the two characters coincide. $\square$

EXAMPLE 2.10. Let p be an odd prime. In this case one can easily see Proposition 2.9 concretely from the character table of $SL_2(\mathbf{Z}/p)$. Note that an element $\varepsilon \in GL_2(\mathbf{Z}/p)$ can act nontrivially by conjugation on the representations of $SL_2(\mathbf{Z}/p)$ only if it is nonzero in $PGL_2(\mathbf{Z}/p)/PSL_2(\mathbf{Z}/p) \cong \mathbb{S}_2$—that is, when $\det(\varepsilon)$ is a nonsquare in $\mathbf{Z}/p$. The character table of $SL_2(\mathbf{Z}/p)$ is constructed in [15, Sec. 5]. That construction shows also that all but four exceptional irreducible representations are restrictions of representations from $GL_2(\mathbf{Z}/p)$ and hence are both invariant under conjugation and isomorphic to their duals (since every element of $GL_2(\mathbf{Z}/p)$ is conjugate to its inverse). The remaining four representations occur when restrictions from $GL_2(\mathbf{Z}/p)$ split into two irreducibles under restrictions, so they are pairwise switched by conjugation by ε precisely when $\det(\varepsilon)$ is a nonsquare. However, the entries in the character table for these latter representations contain a square root of the Legendre symbol $(-1/p)$ as their only potentially nonreal entries.

3. Branching Formulas

Recall that irreducible finite-dimensional representations of Sp_{2g} are indexed by their highest weight, which is a decreasing sequence $l_1 \geq \cdots \geq l_g \geq 0$ of integers. The corresponding irreducible representation appears for the "first" time inside

$$\operatorname{Sym}^{l_1-l_2}(\wedge^1 V) \otimes \operatorname{Sym}^{l_2-l_3}(\wedge^2 V) \otimes \cdots \otimes \operatorname{Sym}^{l_{g-1}-l_g}(\wedge^{g-1} V) \otimes \operatorname{Sym}^{l_g}(\wedge^g V),$$

where V is the defining $2g$-dimensional representation of Sp_{2g}. For example, the weight vector $l \geq 0 \geq \cdots \geq 0$ corresponds to the irreducible representation $\operatorname{Sym}^l V$. In particular, all irreducible representations of Sp_2 are symmetric powers of the defining one.

NOTATION 3.1. We denote the irreducible representation of Sp_4 with highest weight $l \geq m \geq 0$ by $W_{l,m}$. For integers l and m that do not satisfy $l \geq m \geq 0$, we put $W_{l,m} = 0$. We similarly index the irreducible representations of Sp_2 as V_a.

The wreath product $Sp_2 \wr \mathbb{S}_2 = (Sp_2 \times Sp_2) \rtimes \mathbb{S}_2$ embeds naturally in Sp_4 as the subgroup preserving a decomposition of a 4-dimensional symplectic vector space into a sum of two unordered symplectic subspaces. We now determine a branching rule for this inclusion. First we need a description of the irreducible representations of $Sp_2 \wr \mathbb{S}_2$.

DEFINITION 3.2. Let $a, b \in \mathbf{Z}$. First, define $U_{a,b}$ to be the representation of $Sp_2 \wr \mathbb{S}_2$ given by

$$V_a \otimes V_b \oplus V_b \otimes V_a$$

as a representation of $Sp_2 \times Sp_2$, with an $\mathbb{S}_2$-action given by

$$\sigma(x \otimes y, y' \otimes x') = (x' \otimes y', y \otimes x);$$

here $\sigma = (12) \in \mathbb{S}_2$. Second, for any $a \geq 0$, define two representations U_a^+ and U_a^- by giving the $(Sp_2 \times Sp_2)$-representation $V_a \otimes V_a$ the $\mathbb{S}_2$-actions

$$\sigma(x \otimes y) = y \otimes x \quad \text{and} \quad \sigma(x \otimes y) = -y \otimes x,$$

respectively.

PROPOSITION 3.3. *The representations $U_{a,b}$ ($a \neq b$) and $U_a^{\pm}$ are the only irreducible representations of $\mathrm{Sp}_2 \wr \mathbb{S}_2$.*

Proof. The representation theory of a semidirect product tells us that every irreducible representation of $\mathrm{Sp}_2 \wr \mathbb{S}_2$ occurs in a representation induced from an irreducible of $\mathrm{Sp}_2 \times \mathrm{Sp}_2$. The irreducible $V_a \otimes V_b$ induces to $U_{a,b}$ whereas the irreducible $V_a \otimes V_a$ induces to the sum $U_{a,a} = U_a^+ \oplus U_a^-$. □

PROPOSITION 3.4. *Assume that $l + m$ is even. Then the restriction of the representation $W_{l,m}$ of Sp_4 decomposes as*

$$\mathrm{Res}_{\mathrm{Sp}_2 \wr \mathbb{S}_2}^{\mathrm{Sp}_4} W_{l,m} = \bigoplus_{0 \le i \le m} \left(U_{(l-m)/2+i}^{(-1)^m} \oplus \bigoplus_{0 \le j < (l-m)/2} U_{l-m+i-j,i+j} \right),$$

where $(-1)^m$ denotes "+" if m is even and "−" if m is odd. If $l + m$ is odd, then

$$\mathrm{Res}_{\mathrm{Sp}_2 \wr \mathbb{S}_2}^{\mathrm{Sp}_4} W_{l,m} = \bigoplus_{0 \le i \le m} \bigoplus_{0 \le j < (l-m)/2} U_{l-m+i-j,i+j}.$$

Proof. We prove this by induction on m. Consider first $m = 0$. It is clear that the restriction of $W_{1,0} = W$ is the representation $U_{1,0} = V \oplus V$. The restriction of $W_{l,0}$ is then $\mathrm{Sym}^l(V \oplus V) = \bigoplus_{i+j=l} V_i \otimes V_j$, which agrees with the preceding formula.

For the induction step, we use the formula

$$W_{l,m} \otimes W_{1,0} = W_{l,m+1} \oplus W_{l+1,m} \oplus W_{l,m-1} \oplus W_{l-1,m};$$

this is a special case of Pieri's rule for the symplectic group. (The rule states that—instead of adding a horizontal k-strip in all possible ways, as one would do for GL_n—one should first remove a vertical i-strip and then add a horizontal $(k-i)$-strip for all $0 \le i \le k$.) One can then prove that the right-hand sides in the statement of Proposition 3.4 satisfy the same behavior upon tensoring with $U_{1,0}$, since it is easy to see that

$$U_{a,b} \otimes U_{1,0} = U_{a+1,b} \oplus U_{a,b+1} \oplus U_{a-1,b} \oplus U_{a,b-1}$$

(where $U_{a,a} = U_a^+ \oplus U_a^-$) and

$$U_a^{\pm} \otimes U_{1,0} = U_{a+1,a} \oplus U_{a,a-1}.$$

This completes the proof. □

REMARK 3.5. If we were not interested in a closed formula but only in being able to compute the branching algorithmically, then we could also have argued as Bergström and van der Geer do in [4, Sec. 7] in the case of $\mathrm{Sp}_2 \wr \mathbb{S}_3 \hookrightarrow \mathrm{Sp}_6$.

When studying bielliptic curves, we will need a second branching formula—now for $\mathrm{Sp}_2 \times \mathbb{S}_2$ sitting diagonally inside $\mathrm{Sp}_2 \wr \mathbb{S}_2$.

Branching for $U_{a,b}$ is easy. Because it consists of two isomorphic $\mathbb{S}_2$-invariant and anti-invariant parts, when restricted to the diagonal we find an invariant and an anti-invariant copy of $V_a \otimes V_b$.

To describe the $U_a^\pm$, we introduce some notation. Let V_a^+ and V_a^- denote the representation V_a tensored with the trivial and sign representations of $\mathbb{S}_2$, respectively. The representations V_a of Sp_2 are multiplied according to the rule

$$V_a \otimes V_b = V_{a+b} \oplus V_{a+b-2} \oplus \cdots \oplus V_{a-b}$$

if $a \geq b$ (cf. [15, Chap. 11]). When $a = b$, we find that the summands on the right-hand side alternate between the trivial and sign representations.

PROPOSITION 3.6.   *The representations $U_a^\pm$ decompose as*

$$\text{Res}_{\text{Sp}_2 \times \mathbb{S}_2}^{\text{Sp}_2 \wr \mathbb{S}_2} U_a^+ = \bigoplus_{k=0}^{a} V_{2a}^{(-1)^k}$$

and

$$\text{Res}_{\text{Sp}_2 \times \mathbb{S}_2}^{\text{Sp}_2 \wr \mathbb{S}_2} U_a^- = \bigoplus_{k=0}^{a} V_{2a}^{(-1)^{k+1}},$$

respectively, where again $(-1)^k$ denotes "+" if k is even and "−" if k is odd.

Proof. It suffices to consider U_a^+. We begin by decomposing the representation V_a into weight spaces. The case of Sp_2 is simple because we can replace our Cartan subalgebra with the single element

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sp}_2.$$

Decomposing V_a into 1-dimensional eigenspaces for H, by [15, Chap. 11] we have

$$V_a = E_a \oplus E_{a-2} \oplus \cdots \oplus E_{-a}.$$

Since $U_a^+ = V_a \otimes V_a$ as an Sp_2-representation, we obtain a similar eigenspace decomposition of U_a^+ into a sum of copies of $E_i \otimes E_j$. We see that $\mathbb{S}_2$ acts trivially on all eigenspaces of the form $E_i \otimes E_i$; in contrast, subspaces of the form $E_i \otimes E_j \oplus E_j \otimes E_i$ ($i \neq j$) split into two isomorphic subspaces with, respectively, trivial and sign representation of $\mathbb{S}_2$. This determines the decomposition of U_a^+ into 1-dimensional eigenspaces of H along with their $\mathbb{S}_2$-actions. Now one checks that the sum

$$\bigoplus_{k=0}^{a} V_{2(a-k)}^{(-1)^k}$$

has the same decomposition, which determines the representations uniquely. $\square$

4. The Relevant Local Systems

DEFINITION 4.1.   Let **W** denote the *standard local system* on $\mathcal{A}_2$, defined by

$$\mathbf{W} = \mathrm{R}^1 \pi_* \mathbf{Q};$$

here $\pi : \mathcal{X} \to \mathcal{A}_2$ is the universal abelian surface.

Since π is a smooth projective morphism, there is a natural variation of Hodge structure on $\mathbf{W}$. The fiber of $\mathbf{W}$ over a point $[(A, \Theta)]$ is canonically isomorphic to the 4-dimensional symplectic vector space $H^1(A)$. The local system $\mathbf{W}$ can also be defined via the inclusion

$$\pi_1^{\mathrm{orb}}(\mathcal{A}_2) = \mathrm{Sp}_4(\mathbf{Z}) \subset \mathrm{Sp}_4$$

and the natural action of Sp_4 on $H^1(A)$.

By pulling back $\mathbf{W}$ along the map $\mathcal{E}_d \to \mathcal{A}_2$, we get a local system on $\mathcal{E}_d$ that will also be denoted $\mathbf{W}$. There is a second natural way of writing down such a local system. We have the obvious forgetful map

$$\mathcal{E}_d \to \mathcal{A}_1 \times \mathcal{A}_1/\mathbb{S}_2$$

obtained by forgetting the isomorphism ϕ of d-torsion groups. Since $(\mathcal{A}_1 \times \mathcal{A}_1)/\mathbb{S}_2$ sits inside $\mathcal{A}_2$ as the complement of $\mathcal{M}_2$, we can pull back the local system $\mathbf{W}$ on $\mathcal{A}_2$ to $\mathcal{E}_d$ along this composition as well. Let us call the result $\widetilde{\mathbf{W}}$.

PROPOSITION 4.2.   There is a natural isomorphism $\widetilde{\mathbf{W}} \to \mathbf{W}$.

Proof. There exist two universal families $\mathcal{Y}$ and $\mathcal{X}$ over $\mathcal{E}_d$; the first is the universal product of two elliptic curves, and the second is the universal d-elliptic abelian surface. The graph of the isomorphism ϕ defines a finite flat group scheme $\mathcal{Z}$ in $\mathcal{Y}$ such that $\mathcal{Y}/\mathcal{Z} \cong \mathcal{X}$. The resulting map $\mathcal{Y} \to \mathcal{X}$ is fiberwise an isogeny and hence fiberwise an isomorphism on rational cohomology. Since $\widetilde{\mathbf{W}} = \mathrm{R}^1\pi_*\mathbf{Q}_{\mathcal{Y}}$ and $\mathbf{W} = \mathrm{R}^1\pi_*\mathbf{Q}_{\mathcal{X}}$, we conclude by base change. $\square$

Hereafter we shall not distinguish between $\mathbf{W}$ and $\widetilde{\mathbf{W}}$.

Every irreducible representation $W_{l,m}$ of Sp_4 naturally induces a local system $\mathbf{W}_{l,m}$ on $\mathcal{A}_2$. One way to see this is that we can apply the symplectic Schur functor corresponding to $W_{l,m}$ to the local system $\mathbf{W}$. The local system $\mathbf{W}_{l,m}$ is the same as the one obtained from the inclusion $\pi_1^{\mathrm{orb}}(\mathcal{A}_2) \subset \mathrm{Sp}_4$ and the representation $W_{l,m}$, but the construction with Schur functors shows that it carries a natural variation of Hodge structure. (See [13, Chap. VI, Sec. 5] for another approach to constructing the mixed Hodge structure on the cohomology of $\mathbf{W}_{l,m}$.)

There is similarly an inclusion

$$\pi_1^{\mathrm{orb}}((\mathcal{A}_1 \times \mathcal{A}_1)/\mathbb{S}_2) = \mathrm{Sp}_2(\mathbf{Z}) \wr \mathbb{S}_2 \subset \mathrm{Sp}_2 \wr \mathbb{S}_2;$$

hence, for each of the representations $U_{a,b}$ and $U_a^{\pm}$, one obtains the respective local systems $\mathbf{U}_{a,b}$ and $\mathbf{U}_a^{\pm}$ on $(\mathcal{A}_1 \times \mathcal{A}_1)/\mathbb{S}_2$. The pullback of $\mathbf{W}$ to $(\mathcal{A}_1 \times \mathcal{A}_1)/\mathbb{S}_2$ is exactly the local system $\mathbf{U}_{1,0}$, and the pullbacks of the local systems $\mathbf{W}_{l,m}$ are determined by the branching formulas of Proposition 3.4. If we wish to consider these also as variations of Hodge structure, then we need to add a Tate twist. For instance, the pullback of $\mathbf{W}_{2,1}$ is

$$\mathbf{U}_{2,1} \oplus \mathbf{U}_{1,0}(-1);$$

these twists are easily put in "by hand" so that the pulled-back expression is homogeneous. More conceptually, we could have worked with GSp_4 instead of Sp_4 from the beginning.

Of course, the preceding paragraph also describes the pullback of $\mathbf{W}$ to $\mathcal{E}_d$.

5. Cohomology of Local Systems

NOTATION 5.1. For a congruence subgroup Γ' of $SL_2(\mathbf{Z}) = \Gamma$ we denote by $E_k(\Gamma')$ and $S_k(\Gamma')$, respectively, the spaces of Eisenstein series and cusp forms of weight k for Γ'.

The cohomology of $\mathbf{V}_a$ on $Y(d)$ is described by the Eichler–Shimura isomorphism. For $a > 0$, $H^1(Y(d), \mathbf{V}_a)$ is the only nonzero cohomology group; it has a mixed Hodge structure whose nonzero bigraded pieces have Hodge numbers

$$(a+1, 0), \quad (0, a+1), \quad \text{and} \quad (a+1, a+1).$$

Under the Eichler–Shimura isomorphism, these subquotients are interpreted as spaces of modular forms for $\Gamma(d)$: they are (respectively) the holomorphic cusp forms of weight $a + 2$, their antiholomorphic complex conjugates, and the Eisenstein series of weight $a + 2$. Now define

$$\mathbf{S}_{a+2}(\Gamma(d)) = \mathrm{gr}^W_{a+1} H^1(Y(d), \mathbf{V}_a)$$

and

$$\mathbf{E}_{a+2}(\Gamma(d)) = \mathrm{gr}^W_{2a+2} H^1(Y(d), \mathbf{V}_a).$$

Tensoring with $\mathbf{C}$, one has that

$$\mathbf{S}_a(\Gamma(d))_\mathbf{C} = S_a(\Gamma(d)) \oplus \overline{S_a(\Gamma(d))}$$

and

$$\mathbf{E}_a(\Gamma(d))_\mathbf{C} = E_a(\Gamma(d)).$$

The extension is, in fact, trivial; that is,

$$H^1(Y(d), \mathbf{V}_a) = \mathbf{S}_{a+2}(\Gamma(d)) \oplus \mathbf{E}_{a+2}(\Gamma(d)).$$

These calculations are classical and can be found in [23; 26], although the mixed Hodge theory needed to state the result this way was not yet in place when these works were published. The cusp forms are the arithmetically interesting classes in that the Eisenstein series (when $d \geq 3$ and $a > 0$) are simply given by

$$\mathbf{E}_{a+2}(d) \cong H^0(X(d) \setminus Y(d))(-a-1),$$

an expression that relies on $\Gamma(d)$ having no irregular cusps. When $a = 0$ one must subtract a copy of the trivial representation from the right-hand side; when $d \leq 2$, the expression holds only when a is even. To eliminate the Eisenstein series, one can also consider the *inner* or *parabolic cohomology,* which is defined as the image of the compactly supported cohomology inside the ordinary cohomology and is denoted $H^*_!$. There one has that

$$H^*_!(Y(d), \mathbf{V}_a) = H^1_!(Y(d), \mathbf{V}_a) = \mathbf{S}_{a+2}(\Gamma(d))$$

for any $a \geq 0$.

NOTATION 5.2. If ρ is an irreducible representation of a group G and if V is any representation of G, then we put $V^{(\rho)} = \mathrm{Hom}_G(\rho, V)$. In other words, $V = \bigoplus_\rho \rho \otimes V^{(\rho)}$ is the decomposition of V into irreducibles.

THEOREM 5.3. *Let K be a splitting field of* $\mathrm{SL}_2(\mathbf{Z}/d)$ *over* $\mathbf{Q}$. *With notation as before, we then have*

$$H^2_!(\mathcal{E}_d, \mathbf{U}_{a,b})_K = \bigoplus_\rho \mathbf{S}_{a+2}(\Gamma(d))^{(\rho)}_K \otimes \mathbf{S}_{b+2}(\Gamma(d))^{(\rho)}_K,$$

$$H^2_!(\mathcal{E}_d, \mathbf{U}_a^+)_K = \bigoplus_\rho \wedge^2 \mathbf{S}_{a+2}(\Gamma(d))^{(\rho)}_K,$$

$$H^2_!(\mathcal{E}_d, \mathbf{U}_a^-)_K = \bigoplus_\rho \mathrm{Sym}^2 \mathbf{S}_{a+2}(\Gamma(d))^{(\rho)}_K;$$

here all sums are taken over the set of isomorphism classes of irreducible representations of $\mathrm{SL}_2(\mathbf{Z}/d)$.

Proof. We tensor with K only so that the resulting mixed K-Hodge structures admit decompositions into absolutely irreducible representations of $\mathrm{SL}_2(\mathbf{Z}/d)$, so from now on we shall omit K from the notation. Motivically, this corresponds to considering motives with coefficients in K.

Since we work with rational coefficients, we may compute the cohomology of these local systems on $Y(d) \times Y(d)$ and take $(\mathrm{SL}_2(\mathbf{Z}/d) \rtimes \mathbb{S}_2)$-invariants. Consider first $\mathbf{U}_{a,b}$. One has

$$H^*(Y(d) \times Y(d), \mathbf{U}_{a,b}) = H^*(Y(d), \mathbf{V}_a) \otimes H^*(Y(d), \mathbf{V}_b)$$
$$\oplus H^*(Y(d), \mathbf{V}_b) \otimes H^*(Y(d), \mathbf{V}_a)$$

by the Künneth formula. Let us first take $\mathrm{SL}_2(\mathbf{Z}/d)$-invariants. Schur's lemma implies that, whenever V and W are irreducible representations of a group G, the trivial representation occurs with multiplicity 1 in $V \otimes W$ if V and W are duals and does not occur otherwise. It then follows from Proposition 2.9 that the $\mathrm{SL}_2(\mathbf{Z}/d)$-invariants of $H^*(Y(d)^{\times 2}, \mathbf{U}_{a,b})$ are given by

$$\bigoplus_\rho H^*(Y(d), \mathbf{V}_a)^{(\rho)} \otimes H^*(Y(d), \mathbf{V}_b)^{(\rho)} \oplus H^*(Y(d), \mathbf{V}_b)^{(\rho)} \otimes H^*(Y(d), \mathbf{V}_a)^{(\rho)},$$

since the action of $\mathrm{SL}_2(\mathbf{Z}/d)$ was twisted by ε on the second factor. This, in turn, splits into two isomorphic subspaces, one $\mathbb{S}_2$-invariant and one anti-invariant. Clearly the inner cohomology of $\mathbf{V}_a \otimes \mathbf{V}_b$ on $Y(d) \times Y(d)$ is the tensor product of the respective inner cohomologies, and the result follows.

For $\mathbf{U}_a^\pm$, one starts by arguing as before and then finds that it is necessary to determine the $\mathbb{S}_2$-invariant and anti-invariant subspaces of

$$\bigoplus_\rho H^1_!(Y(d), \mathbf{V}_a)^{(\rho)} \otimes H^1_!(Y(d), \mathbf{V}_a)^{(\rho)}.$$

Since these are odd cohomology classes, it follows from the presence of the Koszul sign rule in the Künneth isomorphism that the alternating tensors are $\mathbb{S}_2$-invariant and that the symmetric tensors are anti-invariant. This finishes the proof. □

REMARK 5.4. It is clear from the proof that we can also compute the nonparabolic cohomology of $\mathbf{U}_{a,b}$ and $\mathbf{U}_a^\pm$ in much the same way. One gets a statement of the

theorem that is only slightly more complicated; it involves both Eisenstein series and the "extra" nonzero cohomology group that one gets when $a = 0$, in which case one has $H^0(Y(d), \mathbf{V}_0) = H^0(Y(d)) = \mathbf{Q}$.

REMARK 5.5.   Using Proposition 2.9 and applying Schur's lemma as in the preceding proof, we can demonstrate the following claim. Let V be any representation of $\mathrm{SL}_2(\mathbf{Z}/d)$, and let V^A be the representation obtained by conjugation with an element $A \in \mathrm{GL}_2(\mathbf{Z}/d)$. Then the dimension of $(V \otimes V^A)^{\mathrm{SL}_2(\mathbf{Z}/d)}$ is maximized when $\det A = -1$. Indeed, $V \mapsto V^A$ is an involution switching the irreducible representations of $\mathrm{SL}_2(\mathbf{Z}/d)$ pairwise. Let W be an irreducible representation. If the representation W' is the sum of n copies of W and m copies of W^A, then $\dim(W' \otimes W'^A)^{\mathrm{SL}_2(\mathbf{Z}/d)} = n^2 + m^2$ if W and W^A are duals and is equal to $2nm$ otherwise. The dimension is therefore maximized when we choose A such that V and V^A are always duals, as occurs when $\det A = -1$. By taking $V = H^1(Y(d), \mathcal{O}_{Y(d)})$ one recovers a theorem of Carlton [6, Cor. 5.4] that the geometric genus of a quotient $Y(d) \times Y(d)/\mathrm{SL}_2(\mathbf{Z}/d)$, where $\mathrm{SL}_2(\mathbf{Z}/d)$ acts normally on the first factor and by a conjugated action on the second factor, is maximized when one conjugates with a matrix of determinant -1. Carlton's proof is rather different and uses an analogue of Atkin–Lehner theory on such quotients.

REMARK 5.6.   When a is even and $d \geq 3$, the dimension of each isotypical component of $S_a(\Gamma(d))$ can be found in [27, Thm. 3.4.3].

6. The Case $d = 2$; Pointed Bi-elliptic Curves

We now focus on the case $d = 2$ as a special case of the theory described previously. Strictly speaking, this case is a bit easier. The condition that the Weil pairing be inverted meant that we had to consider a conjugation action of $\mathbb{S}_2$ on $\mathrm{SL}_2(\mathbf{Z}/d)$ by a matrix ε of determinant -1. Over $\mathbf{Z}/2$, we can take $\varepsilon = \mathrm{Id}$, so the semidirect product is actually a direct product. Moreover, the isomorphism $\mathrm{SL}_2(\mathbf{Z}/2) \cong \mathbb{S}_3$ makes the representation theory very simple.

Another minor difference arises because $Y(2)$ is a stack and not a scheme: the elliptic involution fixes the 2-torsion of any elliptic curve. The elliptic involution acts as multiplication by $(-1)^a$ on the fibers of the local system $\mathbf{V}_a$ on $Y(2)$, so the cohomology of this local system vanishes unless a is even. Thus the local systems $\mathbf{U}_{a,b}$ have vanishing cohomology on $Y(2) \times Y(2)$, and hence also on $\mathcal{E}_2$, unless a and b are both even. Similarly, $\mathbf{U}_a^{\pm}$ has vanishing cohomology unless a is even.

Let us compute the cohomology groups $H^*(Y(2), \mathbf{V}_a)$ as $\mathbb{S}_3$-representations. The subgroups $\Gamma(2)$ and $\Gamma_0(4)$ are conjugate, so $Y(2) \cong Y_0(4)$. We prefer to work with $Y_0(4)$, essentially because of Atkin–Lehner theory.

Let s_3, s_{21}, and s_{111} denote the representations corresponding to the respective partitions—that is, the trivial, standard, and sign representations, respectively. Given that

$$Y_0(4)/\mathbb{S}_3 \cong \mathcal{A}_1 \quad \text{and} \quad Y_0(4)/\mathbb{S}_2 \cong Y_0(2)$$

(where $\mathbb{S}_2$ is the subgroup generated by a transposition), it is not hard to see that decomposing the spaces of modular forms and cusp forms as $\mathbb{S}_3$-representations

is equivalent to determining which forms are newforms of the bigger groups $\Gamma_0(2)$ and the full modular group Γ. The s_3-part consists of exactly those that are modular forms for Γ; the s_{21}-part corresponds to those that are lifted from newforms for $\Gamma_0(2)$ (we get a 2-dimensional subspace of oldforms for $\Gamma_0(4)$ from a 1-dimensional space of newforms for $\Gamma_0(2)$ because there are two different liftings); and the s_{111}-part consists of the newforms. This decomposition is implicitly used in [2] and may be formally stated as follows.

PROPOSITION 6.1.   *We have $S_a(\Gamma(2))^{(s_3)} \cong S_a(\Gamma)$, $S_a(\Gamma(2))^{(s_{21})} \cong S_a^{\mathrm{new}}(\Gamma_0(2))$, and $S_a(\Gamma(2))^{(s_{111})} \cong S_a^{\mathrm{new}}(\Gamma_0(4))$.*

In particular, we can determine the decomposition of $S_a(\Gamma(2))$ from the respective dimension formulas for $\Gamma_0(4)$, $\Gamma_0(2)$, and Γ (see e.g. [9]). We omit the details.

REMARK 6.2.   This result corrects a minor error in [27, Thm. 3.4.3]. Weinstein gives a formula for how $S_a(\Gamma(d))$, $d \geq 2$ and a even, decomposes into irreducible representations. However, that formula is correct only for $d > 2$: Weinstein's calculation is an equivariant version of the usual derivation of the dimension formula for $\Gamma(d)$, which needs to be modified when $d = 2$ because $-1 \in \Gamma(2)$. The correct statement of Weinstein's result when $d = 2$ is easily deduced from Proposition 6.1.

If we are interested in the locus of d-elliptic curves in $\mathcal{M}_2$ instead of d-elliptic abelian surfaces in $\mathcal{A}_2$, we need to remove those pairs of elliptic curves that map into $(\mathcal{A}_1 \times \mathcal{A}_1)/\mathbb{S}_2$ inside $\mathcal{A}_2$. A description of this locus has been worked out by Frey and Kani [20], who show that it is a union of graphs of Hecke correspondences on $Y(d) \times Y(d)$. The special case of $d = 2$ becomes simple: here, we simply find the diagonal inside $Y(2) \times Y(2)$. So for bi-elliptic curves, we need to understand the cohomology of the local systems on the diagonal.

Let Δ denote the image in $\mathcal{E}_2$ of the diagonal substack of $Y(2) \times Y(2)$. Note that the diagonal is invariant under the action of $\mathrm{SL}_2(\mathbf{Z}/2) \times \mathbb{S}_2$.

PROPOSITION 6.3.   *Let a and b be even integers. Then*

$$H^*(\Delta, \mathbf{U}_{a,b}) = H^*(\mathcal{A}_1, \mathbf{V}_a \otimes \mathbf{V}_b),$$

and the cohomology vanishes if either a or b is odd. Similarly,

$$H^*(\Delta, \mathbf{U}_a^+) = \bigoplus_{k=0}^{a/2} H^*(\mathcal{A}_1, \mathbf{V}_{4k})$$

and

$$H^*(\Delta, \mathbf{U}_a^-) = \bigoplus_{k=1}^{a/2} H^*(\mathcal{A}_1, \mathbf{V}_{4k-2})$$

when a is even, and the cohomology vanishes otherwise.

Proof. We have already seen the vanishing part of the proposition. Hence we perform calculations on the diagonal inside $Y(2) \times Y(2)$ and take invariants. The pullback of $\mathbf{U}_{a,b}$ and $\mathbf{U}_a^\pm$ to the diagonal in $Y(2) \times Y(2)$ is determined by the branching

formulas in Proposition 3.6 and the remarks that precede it. It is also clear from these formulas what happens when we take $\mathbb{S}_2$-invariants. Taking $\mathrm{SL}_2(\mathbf{Z}/2)$-invariants simply corresponds to descending to level 1, which gives us the answer in terms of cohomology of local systems on $\mathcal{A}_1$. ☐

REMARK 6.4. Observe that Proposition 6.3 is false if we do not require a and b to be even: if a and b are odd, then $H^*(\Delta, \mathbf{U}_{a,b}) = 0$ but $H^*(\mathcal{A}_1, \mathbf{V}_a \otimes \mathbf{V}_b)$ is nonzero. These facts reflect that the diagonal embedding $Y(2) \to Y(2) \times Y(2)$ is not an isomorphism onto the diagonal substack—unlike the case when $d \geq 3$.

Let $\mathcal{B}_n$ be the moduli space parameterizing bi-elliptic curves of genus 2 with n distinct marked points. Here we define a bi-elliptic curve of genus 2 to be a curve C together with an unordered pair of conjugate double covers $C \to E$ and $C \to E'$ (cf. Definition 2.3).

As we shall now see, the knowledge of the cohomology of the local systems $\mathbf{W}_{l,m}$ can be used to compute the $\mathbb{S}_n$-equivariant Euler characteristic (in the Grothendieck group of mixed Hodge structures) of $\mathcal{B}_n$. It will be more convenient to switch to compactly supported cohomology at this point. We need some general results due to Getzler [16] on relative configuration spaces. In Getzler's setting, one considers a quasi-projective morphism of varieties $\pi \colon \mathcal{X} \to \mathcal{M}$ and the *relative configuration space* $F(\mathcal{X}/\mathcal{M}, n)$, which is the complement of the "big diagonal" in the nth fibered power of $\mathcal{X}$ over $\mathcal{M}$. Getzler proves the formula

$$\sum_{n \geq 0} \mathbf{e}_{\mathcal{M}}^{\mathbb{S}_n}(F(\mathcal{X}/\mathcal{M}, n), \mathbf{Q}) = \prod_{k \geq 0} (1 + p_k)^{(1/k) \sum_{d|k} \mu(k/d) \psi_d(\mathbf{e}_{\mathcal{M}}(\mathcal{X}, \mathbf{Q}))}.$$

Here $\mathbf{e}_{\mathcal{M}}(\mathcal{X}, \mathbf{Q})$ denotes the compactly supported relative Euler characteristic obtained from $\mathrm{R}\pi_!\mathbf{Q}$ in the Grothendieck group of the bounded derived category of mixed Hodge modules on $\mathcal{M}$; $\mathbf{e}_{\mathcal{M}}^{\mathbb{S}_n}(F(\mathcal{X}/\mathcal{M}, n), \mathbf{Q})$ is similarly the $\mathbb{S}_n$-equivariant compactly supported Euler characteristic, taken in the same Grothendieck group but tensored with the ring Λ of symmetric functions (i.e., the sum of the representation rings of $\mathbb{S}_n$ for all n); the p_k are power sums; ψ_d denotes an Adams operation; μ denotes the Möbius function; and the factors on the right-hand side are meant to be expanded formally as a binomial series. In order to obtain the actual Euler characteristic from this formula, one must take the proper pushforward of both sides along $\mathcal{M} \to \mathrm{Spec}\,\mathbf{C}$, which produces an equality in the usual Grothendieck group of mixed Hodge structures tensored with Λ.

In our case we put $\mathcal{M} = \mathcal{E}_2 \times_{\mathcal{A}_2} \mathcal{M}_2$, the moduli space of bi-elliptic curves, and let $\pi \colon \mathcal{X} \to \mathcal{M}$ be its universal family of genus 2 curves. Therefore, $\mathcal{M} = \mathcal{B}_0$, $\mathcal{X} = \mathcal{B}_1$, and $F(\mathcal{X}/\mathcal{M}, n) = \mathcal{B}_n$. Then

$$\mathbf{e}_{\mathcal{M}}(\mathcal{X}, \mathbf{Q}) = \mathbf{Q} - \mathbf{W} + \mathbf{Q}(-1).$$

Expanding the right-hand side of Getzler's formula yields an expression in which the coefficient before each monomial in Λ is a formal sum of certain Schur functors applied to $\mathbf{W}$. Decomposing these Schur functors into irreducible representations of the symplectic group allows us to rewrite that expression as a sum of the local systems $\mathbf{W}_{l,m}$ with some Tate twists. It follows that the results presented so

Table 1 Compactly Supported Euler Characteristic of $\mathcal{B}_n$ in the
Grothendieck Group of Mixed Hodge Structures

n	$\mathbf{e}^{\mathbb{S}_n}(\mathcal{B}_n, \mathbf{Q})$
0	$\mathbf{L}^2 - \mathbf{L}$
1	$(\mathbf{L}^3 - \mathbf{L})s_1$
2	$(\mathbf{L}^4 - \mathbf{L}^2 + \mathbf{L})s_2 + (\mathbf{L}^3 - \mathbf{L}^2 - \mathbf{L} + 2)s_{11}$
3	$(\mathbf{L}^5 - 2\mathbf{L}^3 + 2\mathbf{L}^2 + \mathbf{L} - 2)s_3 + (\mathbf{L}^4 - \mathbf{L}^3 + 2\mathbf{L})s_{21} + (-\mathbf{L}^2 + \mathbf{L} + 2)s_{111}$
4	$(\mathbf{L}^6 - 2\mathbf{L}^4 + \mathbf{L}^3 + \mathbf{L}^2 - 3\mathbf{L})s_4 + (\mathbf{L}^5 - 2\mathbf{L}^4 + \mathbf{L}^3 + 3\mathbf{L}^2 - \mathbf{L} - 2)s_{31}$
	$+ (\mathbf{L}^4 - \mathbf{L}^2 - \mathbf{L} + 3)s_{22} + (-\mathbf{L}^3 + 5\mathbf{L} + 2)s_{211} + (-\mathbf{L}^3 - \mathbf{L}^2 + \mathbf{L} + 3)s_{1111}$

far allow us to compute the $\mathbb{S}_n$-equivariant Euler characteristic: one sees from the Gysin sequence that

$$\mathbf{e}(\mathcal{M}, \mathbf{W}_{l,m}) = \mathbf{e}(\mathcal{E}_2, \mathbf{W}_{l,m}) - \mathbf{e}(\Delta, \mathbf{W}_{l,m}),$$

and the right-hand side can be expressed in terms of the $\mathbb{S}_3$-equivariant Euler characteristics $\mathbf{e}^{\mathbb{S}_3}(Y(2), \mathbf{V}_a)$ by combining Proposition 3.4, Theorem 5.3, and Propositions 3.6 and 6.3. Finally, $\mathbf{e}^{\mathbb{S}_3}(Y(2), \mathbf{V}_a)$ can be computed from the Eichler–Shimura theory (quoted in Section 5) and Proposition 6.1.

From the foregoing discussion it is evident that one can calculate the $\mathbb{S}_n$-equivariant Euler characteristic $\mathbf{e}^{\mathbb{S}_n}(\mathcal{B}_n, \mathbf{Q})$ for any n. The first few results are given in Table 1. We put $\mathbf{L} = H^2(\mathbf{P}^1)$; polynomials in $\mathbf{L}$ with integer coefficients are interpreted in the natural way. The first occurrence of non-Tate cohomology is the s_{111111}-coefficient of $\mathbf{e}^{\mathbb{S}_6}(\mathcal{B}_6, \mathbf{Q})$, which is given by $\mathbf{S}_8(\Gamma_0(2)) - \mathbf{L}^4 + 3\mathbf{L} + 5$.

The problem of computing these Euler characteristics was studied by means of point counts over finite fields in the author's master's thesis by using techniques similar to those employed in [1]. It was proven in this thesis that, when $n \leq 5$ and q is odd, the number of $\mathbf{F}_q$-points of $\mathcal{B}_n$ is given by a polynomial in q. Moreover, the calculations are performed $\mathbb{S}_n$-equivariantly. One may thus obtain conjectural formulas for the $\mathbb{S}_n$-equivariant Euler characteristic of $\mathcal{B}_n$ in the Grothendieck group of ℓ-adic Galois representations when $n \leq 5$. (The results of [24] can not be applied in this case to yield an unconditional proof of these formulas for the Euler characteristics because of the restriction to odd q and, more seriously, because the natural compactification of $\mathcal{E}_2$ involves the modular curve $X(2)$—which is not smooth unless the prime 2 is invertible [7].) Needless to say, the results obtained there agree with those found here.

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