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Hilbert Transform Characterization and Fefferman–Stein Decomposition of Triebel–Lizorkin Spaces

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1. Introduction

There are several equivalent definitions for $H^1(\mathbb{R}^n)$. One of these is the Riesz transforms characterization (cf. [10, p. 221]); that is, $H^1(\mathbb{R}^n)$ consists of the class of $L^1(\mathbb{R}^n)$ functions such that their Riesz transforms belong to $L^1(\mathbb{R}^n)$ as well. Furthermore,

$$\|f\|_{H^1} \approx \|f\|_{L^1} + \sum_{j=1}^{n} \|R_j(f)\|_{L^1},$$

where the R_j are the Riesz transforms. By the duality between H^1 and BMO, every $\varphi \in \mathrm{BMO}(\mathbb{R}^n)$ can be represented as

$$\varphi = \varphi_0 + \sum_{j=1}^{n} R_j(\varphi_j) \quad \text{(modulo constants)},$$

where $\varphi_0, \varphi_j \in L^\infty(\mathbb{R}^n)$ (see [4, Thm. 3]). This decomposition is widely known as the Fefferman–Stein decomposition.

Many authors (see e.g. [1; 2; 3; 7; 8; 11]) have generalized the Riesz transforms characterization and Fefferman–Stein decomposition to different variants of Hardy spaces and BMO spaces. Since both H^1 and BMO are special cases of Triebel–Lizorkin spaces, we seek to extend these two properties to (respectively) the Triebel–Lizorkin spaces $\dot{F}_1^{0,q}(\mathbb{R})$ $(2 \le q < \infty)$ and their duals $\dot{F}_\infty^{0,q'}(\mathbb{R})$.

Let $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ denote the Schwartz space and its dual, respectively. Choose a fixed function $\varphi \in \mathcal{S}(\mathbb{R})$ satisfying $\operatorname{supp}(\varphi) \subset \{\xi \in \mathbb{R} : 1/2 \le |\xi| \le 2\}$, $|\hat{\varphi}(\xi)| \ge C > 0$ for $3/5 \le |\xi| \le 5/3$, and $\sum_{j\in\mathbb{Z}} |\hat{\varphi}(2^j\xi)|^2 = 1$ if $\xi \ne 0$. Write $\varphi_j(x) = 2^j\varphi(2^jx)$, $j \in \mathbb{Z}$. For $1 < q < \infty$, the homogeneous Triebel–Lizorkin space $\dot{F}_1^{0,q}(\mathbb{R})$ is the collection of all $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$, the tempered distributions modulo polynomials, satisfying

$$\|f\|_{\dot{F}_1^{0,q}} := \left\| \left\{ \sum_{j\in\mathbb{Z}} (|\varphi_j * f|)^q \right\}^{1/q} \right\|_{L^1} < \infty.$$

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Its dual space $\dot{F}^{0,q'}_{\infty}(\mathbb{R})$, $1/q + 1/q' = 1$, is the collection of all $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$ with

$$\|f\|_{\dot{F}^{0,q'}_{\infty}} := \sup_{I \text{ dyadic}} \left\{ \frac{1}{|I|} \int_I \sum_{j=-\log_2|I|}^{\infty} (|\varphi_j * f(x)|)^{q'}\, dx \right\}^{1/q'} < \infty;$$

here the supremum is taken over all dyadic intervals $I \subset \mathbb{R}$ and $|I|$ denotes the length of I.

Denote by $\mathcal{H}(f)$ the Hilbert transform of f with Fourier transform

$$\widehat{\mathcal{H}(f)}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi).$$

As mentioned previously for the special case $n = 1$, the Hardy space $H^1(\mathbb{R})$ can be characterized by its Hilbert transform; that is, $f \in H^1(\mathbb{R})$ if and only if $f \in L^1(\mathbb{R})$ and $\mathcal{H}(f) \in L^1(\mathbb{R})$. By the Fefferman–Stein decomposition, a function $f \in \mathrm{BMO}(\mathbb{R})$ can be represented as the sum $f_0 + Hf_1$ for f_0 and f_1 in $L^\infty(\mathbb{R})$. Both properties play an important role in harmonic analysis. Since $H^1 = \dot{F}^{0,2}_1$, $\mathrm{BMO} = \dot{F}^{0,2}_{\infty}$, and $(H^1)' = \mathrm{BMO}$, it is natural to ask whether the Triebel–Lizorkin spaces $\dot{F}^{0,q}_1$ and their dual spaces $\dot{F}^{0,q'}_{\infty}$ have similar properties. In this paper, we give affirmative answers for both by using Meyer wavelets. The difficulties for both questions come from defining the suitable relative L^1 and L^∞ spaces and then proving the relative conclusions.

Wavelets greatly facilitate the study of function spaces. We recall the definition of Meyer's wavelets [9] as follows. Let $\Phi(\xi) \in C_0^\infty([-4\pi/3, 4\pi/3])$ be an even function satisfying $\Phi(\xi) \in [0,1]$ and $\Phi(\xi) = 1$ for $|\xi| \le 2\pi/3$. Set $\Psi(\xi) = \{\Phi(\xi/2)^2 - \Phi(\xi)^2\}^{1/2}$. Then $\Psi(\xi) \in C_0^\infty([-8\pi/3, 8\pi/3])$ is an even function and satisfies the following conditions:

(a) $\Psi(\xi) \in [0,1]$;
(b) $\Psi(\xi) = 0$ for $|\xi| \le 2\pi/3$;
(c) $\Psi^2(\xi) + \Psi^2(2\xi) = 1$ for $\xi \in [2\pi/3, 4\pi/3]$;
(d) $\Psi^2(\xi) + \Psi^2(4\pi - \xi) = 1$ for $\xi \in [4\pi/3, 8\pi/3]$.

Define the function $\phi(x)$ (father wavelet) and $\psi(x)$ (mother wavelet) by the Fourier transform $\hat{\phi}(\xi) = \Phi(\xi)$ and $\hat{\psi}(\xi) = \Psi(\xi)e^{-i\xi/2}$, respectively. For $j, k \in \mathbb{Z}$, we write $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. Then $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$.

For an arbitrary distribution $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$, define its wavelet coefficients by

$$a_{j,k} = \langle f, \psi_{j,k}\rangle \quad \text{for } j,k \in \mathbb{Z}.$$

Then the following wavelet expansion is true in the sense of distribution:

$$f(x) = \sum_{j\in\mathbb{Z}} \left\{ \sum_{k\in\mathbb{Z}} a_{j,k}\psi_{j,k}(x) \right\} := \sum_{j\in\mathbb{Z}} Q_j(f)(x).$$

Furthermore, Frazier and Jawerth [5, p. 47] proved that there is a one-to-one correspondence between a distribution f in $\dot{F}^{0,q}_1$ and a sequence of numbers $\{a_{j,k}\}_{j,k\in\mathbb{Z}}$.

PROPOSITION 1.1. *Suppose* $1 < q < \infty$. *Then, for* $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$,

$$\|f\|_{\dot{F}_1^{0,q}} = \left\| \left\{ \sum_{j,k\in\mathbb{Z}} (2^{j/2}|a_{j,k}|\chi(2^j x - k))^q \right\}^{1/q} \right\|_{L^1};$$

here $\chi(x)$ *denotes the characteristic function of the interval* $[0,1)$.

We now proceed with the definition of the relative L^1 spaces. For any two integers N and N' with $N \le N'$, set

$$f_{N,N'}(x) := \sum_{N\le j\le N'} Q_j(f)(x).$$

We divide $f_{N,N'}$ into two parts, $f_{N,N'}^{N_1,1}$ and $f_{N,N'}^{N_1,2}$, which are given by

$$f_{N,N'}^{N_1,1}(x) := \begin{cases} f_{N,N'}(x) & \text{if } N_1 \le N, \\ \sum_{N_1\le j\le N'} Q_j(f)(x) & \text{if } N+1 \le N_1 \le N', \\ 0 & \text{if } N_1 \ge N'+1 \end{cases}$$

and $f_{N,N'}^{N_1,2}(x) := f_{N,N'}(x) - f_{N,N'}^{N_1,1}(x)$. For $2 \le q < \infty$, the space $L^{1,q}(\mathbb{R})$ is defined as the collection of all $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$ such that

$$\|f\|_{L^{1,q}} := \sup_{N\le N'} \min_{N\le N_1\le N'+1} \left(\|f_{N,N'}^{N_1,1}\|_{\dot{F}_1^{0,q}} + \|f_{N,N'}^{N_1,2}\|_{L^1} \right) < \infty.$$

It follows directly from the definition that $\|f\|_{L^{1,q}} \le \|f\|_{\dot{F}_1^{0,q}}$ and hence $\dot{F}_1^{0,q}(\mathbb{R}) \subset L^{1,q}(\mathbb{R})$. The orthogonality of the wavelet basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ clearly allows us to make the following statement.

REMARK 1.2. For $f_{N,N'} \in L^{1,q}(\mathbb{R})$,

$$\|f_{N,N'}\|_{L^{1,q}} = \max_{N\le\tilde{N}\le\tilde{N}'\le N'} \min_{\tilde{N}\le\tilde{N}_1\le\tilde{N}'+1} \left(\|f_{\tilde{N},\tilde{N}'}^{\tilde{N}_1,1}\|_{\dot{F}_1^{0,q}} + \|f_{\tilde{N},\tilde{N}'}^{\tilde{N}_1,2}\|_{L^1} \right).$$

The Hilbert transform characterization of $\dot{F}_1^{0,q}(\mathbb{R})$, $2 \le q < \infty$, can be stated as follows.

THEOREM 1.3. *For* $2 \le q < \infty$, *a distribution* $f \in \dot{F}_1^{0,q}(\mathbb{R})$ *if and only if* $f \in L^{1,q}(\mathbb{R})$ *and* $\mathcal{H}(f) \in L^{1,q}(\mathbb{R})$. *Moreover,* $\|f\|_{\dot{F}_1^{0,q}} \approx \|f\|_{L^{1,q}} + \|\mathcal{H}(f)\|_{L^{1,q}}$.

REMARK 1.4. The definition of $L^{1,q}(\mathbb{R})$ can be extended to $L^{1,q}(\mathbb{R}^n)$ for arbitrary dimension $n \in \mathbb{N}$ in a similar way, yet proving the result in dimension 1 requires only that we simultaneously control the norm of two functions. For $n \ge 2$ we must control the norm of several functions, which cannot be done at the same time. Hence we need to develop new skills for transferring the control over the norm of several functions to the case where we control the norm of two functions each time. Such a transformation may result in changes of norm each time. Therefore, the approach used here cannot be applied for $\mathbb{R}^n$ when $n \ge 2$.

REMARK 1.5. The proof of Theorem 1.3 depends on the following implication:

$$f, Hf \in L^1 \implies f \in H^1 \subset F_1^{0,q} \quad \text{for } q \geq 2.$$

This is why we require $q \geq 2$.

To obtain the Fefferman–Stein decomposition of $\dot{F}_\infty^{0,q}(\mathbb{R})$, $1 < q \leq 2$, we must deal with the dual space of $L^{1,q}(\mathbb{R})$. For $1 < q \leq 2$, define the space $L^{\infty,q}(\mathbb{R})$ as the collection of all $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$ satisfying

$$\|f\|_{L^{\infty,q}} := \sup_{N \leq N'} \max_{N \leq N_1 \leq N'+1} \left(\|f_{N,N'}^{N_1,1}\|_{\dot{F}_\infty^{0,q}} + \|f_{N,N'}^{N_1,2}\|_{L^\infty} \right) < \infty.$$

By definition, $\|f\|_{\dot{F}_\infty^{0,q}} \leq \|f\|_{L^{\infty,q}}$ and hence $L^{\infty,q}(\mathbb{R}) \subset \dot{F}_\infty^{0,q}(\mathbb{R})$. Let

$$\mathcal{S}_0(\mathbb{R}) := \{f \in \mathcal{S}(\mathbb{R}) : \hat{f} = 0 \text{ in a neighborhood of the origin}\}.$$

It is easy to see that $\mathcal{S}_0(\mathbb{R})$ is dense in $L^{1,q}(\mathbb{R})$. Using the dualities $(\dot{F}_1^{0,q})' = \dot{F}_\infty^{0,q'}$ and $(L^1)' = L^\infty$, we get the duality $(L^{1,q})' = L^{\infty,q'}$.

THEOREM 1.6. *Suppose* $2 \leq q < \infty$. *The dual space of* $L^{1,q}(\mathbb{R})$ *is* $L^{\infty,q'}(\mathbb{R})$ *in the following sense. If* $g \in L^{\infty,q'}(\mathbb{R})$ *then the map* $\mathcal{L}_g$ *given by* $\mathcal{L}_g(f) = \langle f, g\rangle$ *is a bounded linear functional on* $L^{1,q}(\mathbb{R})$; *conversely, if* $\mathcal{L} \in (L^{1,q}(\mathbb{R}))'$ *then there exists a* $g \in L^{\infty,q'}(\mathbb{R})$ *such that* $\mathcal{L} = \mathcal{L}_g$.

Applying the above results, we obtain the following Fefferman–Stein decomposition of $\dot{F}_\infty^{0,q}(\mathbb{R})$.

THEOREM 1.7. *Suppose* $1 < q \leq 2$. *Then* $f \in \dot{F}_\infty^{0,q}(\mathbb{R})$ *if and only if there exist* $f_0, f_1 \in L^{\infty,q}(\mathbb{R})$ *such that* $f = f_0 + \mathcal{H}(f_1)$.

We have seen that $\dot{F}_1^{0,q}(\mathbb{R}) \subset L^{1,q}(\mathbb{R})$ and $L^{\infty,q}(\mathbb{R}) \subset \dot{F}_\infty^{0,q}(\mathbb{R})$. As the final remark, we show at the end of this paper that both inclusions are proper.

REMARK 1.8. (i) *For* $2 \leq q < \infty$, $\dot{F}_1^{0,q}(\mathbb{R}) \subsetneqq L^{1,q}(\mathbb{R})$.
(ii) *For* $1 < q \leq 2$, $L^{\infty,q}(\mathbb{R}) \subsetneqq \dot{F}_\infty^{0,q}(\mathbb{R})$.

The rest of the paper is organized as follows. In Section 2 we introduce a lemma and prove the Hilbert transform characterization of Triebel–Lizorkin spaces $\dot{F}_1^{0,q}(\mathbb{R})$, $2 \leq q < \infty$. In Section 3, the duality of $L^{1,q}(\mathbb{R})$ and $L^{\infty,q'}(\mathbb{R})$ is established. We show the Fefferman–Stein decomposition of $\dot{F}_\infty^{0,q}(\mathbb{R})$ in Section 4. Finally, in Section 5 we prove the proper inclusions $\dot{F}_1^{0,q} \subset L^{1,q}$ and $L^{\infty,q} \subset \dot{F}_\infty^{0,q}$.

2. Proof of the Hilbert Transform Characterization

The boundedness of the Hilbert transform acting on $\dot{F}_1^{0,q}$ was demonstrated in [6].

THEOREM 2.1. *Suppose* $1 < q < \infty$. *Then the Hilbert transform is bounded from the Triebel–Lizorkin spaces* $\dot{F}_1^{0,q}(\mathbb{R})$ *into itself.*

In order to prove Theorem 1.3, we need the following lemma.

LEMMA 2.2. *For* $2 \le q < \infty$, *if* $f(x) \in L^{1,q}(\mathbb{R})$ *then* $Q_j(f) \in H^1(\mathbb{R})$ *for all* $j \in \mathbb{Z}$. *Moreover, there exists a constant* C, *independent of* j *and* f, *such that* $\|Q_j(f)\|_{H^1} \le C\|f\|_{L^{1,q}}$.

Proof. For all $j \in \mathbb{Z}$ we have $Q_j f = f_{j,j}$. From the definition of $L^{1,q}(\mathbb{R})$ it follows that $\|f_{j,j}\|_{L^{1,q}} \le \|f\|_{L^{1,q}}$. So to prove Lemma 2.2, it suffices to prove that $\|Q_j(f)\|_{H^1} \le C\|Q_j(f)\|_{\dot F_1^{0,q}}$ and $\|Q_j(f)\|_{H^1} \le C\|Q_j(f)\|_{L^1}$ for all $j \in \mathbb{Z}$. The first inequality is a direct consequence of Proposition 1.1:

$$\begin{aligned}
&\|Q_j(f)\|_{\dot F_1^{0,2}} \\
&\quad = \left\| \left\{ \sum_{k\in\mathbb{Z}} (2^{j/2}|a_{j,k}|\chi(2^j x - k))^2 \right\}^{1/2} \right\|_{L^1} \\
&\quad = \left\| \left\{ \sum_{k\in\mathbb{Z}} (2^{j/2}|a_{j,k}|\chi(2^j x - k))^q \right\}^{1/q} \right\|_{L^1} = \|Q_j(f)\|_{\dot F_1^{0,q}} \quad \text{for } j \in \mathbb{Z}.
\end{aligned}$$

For the second inequality, we note that $Q_j(f) = Q_j(Q_j(f))$. Then, by Proposition 1.1,

$$\begin{aligned}
\|Q_j(f)\|_{\dot F_1^{0,2}} &= \left\| \left\{ \sum_{k\in\mathbb{Z}} (2^{j/2}|\langle Q_j(f), \psi_{j,k}\rangle|\chi(2^j x - k))^2 \right\}^{1/2} \right\|_{L^1} \\
&= \left\| \sum_{k\in\mathbb{Z}} 2^{j/2}|\langle Q_j(f), \psi_{j,k}\rangle|\chi(2^j x - k) \right\|_{L^1} \\
&\le \int_{\mathbb{R}} |Q_j(f)(y)| \sum_{k\in\mathbb{Z}} |\psi(2^j y - k)| \left\{ \int_{\mathbb{R}} 2^j \chi(2^j x - k)\, dx \right\} dy \\
&= \int_{\mathbb{R}} |Q_j(f)(y)| \sum_{k\in\mathbb{Z}} |\psi(2^j y - k)|\, dy \quad \text{for } j \in \mathbb{Z}.
\end{aligned}$$

Because $\psi(x)$ decreases rapidly, we have

$$\sum_{k\in\mathbb{Z}} |\psi(x-k)| \le C \quad \text{for } x \in \mathbb{R}.$$

As a result, $\|Q_j(f)\|_{H^1} \approx \|Q_j(f)\|_{\dot F_1^{0,2}} \le C\|Q_j(f)\|_{L^1}$, from which Lemma 2.2 follows. □

REMARK 2.3. The proof of Lemma 2.2 implies the following expression:

$$\|Q_j(f)\|_{H^1} \approx \|Q_j(f)\|_{L^1} \approx \|\mathcal{H}(Q_j(f))\|_{H^1}.$$

We now are ready to show the Hilbert transform characterization of $\dot F_1^{0,q}$.

Proof of Theorem 1.3. Suppose $f \in \dot F_1^{0,q}(\mathbb{R})$ for $2 \le q < \infty$. Then, by Theorem 2.1 and the inequality $\|f\|_{L^{1,q}} \le \|f\|_{\dot F_1^{0,q}}$, we immediately obtain $\|f\|_{L^{1,q}} + \|\mathcal{H}(f)\|_{L^{1,q}} \lesssim \|f\|_{\dot F_1^{0,q}}$.

To show the converse it is sufficient to prove that, for any two integers N and N' with $N \le N'$,

$$\|f_{N,N'}\|_{\dot F_1^{0,q}} \lesssim \|f_{N,N'}\|_{L^{1,q}} + \|\mathcal{H}(f_{N,N'})\|_{L^{1,q}},$$

where $C > 0$ is a constant independent of N and N'. By Remark 1.2,

$$\|f_{N,N'}\|_{L^{1,q}} = \max_{N \le \tilde N \le \tilde N' \le N'} \min_{\tilde N \le \tilde N_1 \le \tilde N'+1} \big(\|f_{\tilde N,\tilde N'}^{\tilde N_1,1}\|_{\dot F_1^{0,q}} + \|f_{\tilde N,\tilde N'}^{\tilde N_1,2}\|_{L^1}\big).$$

According to the support property of the Fourier transform of a Meyer wavelet, for any $k, k' \in \mathbb{Z}$ we have

$$\langle \mathcal{H}(\psi_{j',k'}), \psi_{j,k}\rangle = 0 \quad \text{if } |j - j'| \ge 2.$$

Therefore,

$$\mathcal{H}(f_{N,N'})(x) = \sum_{\substack{N-1 \le j \le N'+1 \\ k \in \mathbb{Z}}} \left\langle \sum_{\substack{N \le j' \le N' \\ k' \in \mathbb{Z}}} f_{j',k'} \mathcal{H}(\psi_{j',k'}), \psi_{j,k} \right\rangle \psi_{j,k}(x)$$

and

$$\begin{aligned} &\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}} \\ &\quad= \max_{N-1 \le \tilde N \le \tilde N' \le N'+1} \min_{\tilde N \le \tilde N_1 \le \tilde N'+1} \big(\|\mathcal{H}(f_{N,N'})_{\tilde N,\tilde N'}^{\tilde N_1,1}\|_{\dot F_1^{0,q}} + \|\mathcal{H}(f_{N,N'})_{\tilde N,\tilde N'}^{\tilde N_1,2}\|_{L^1}\big). \end{aligned}$$

For any fixed integers $N \le N'$, we choose an integer n_0 with $N \le n_0 \le N'+1$ such that

$$\|f_{N,N'}^{n_0,1}\|_{\dot F_1^{0,q}} + \|f_{N,N'}^{n_0,2}\|_{L^1} = \min_{N \le N_1 \le N'+1} \big(\|f_{N,N'}^{N_1,1}\|_{\dot F_1^{0,q}} + \|f_{N,N'}^{N_1,2}\|_{L^1}\big) \le \|f_{N,N'}\|_{L^{1,q}}$$

and then choose another integer n_1 with $N \le n_1 \le N'+1$ such that

$$\begin{aligned} &\|\mathcal{H}(f_{N,N'})_{N,N'}^{n_1,1}\|_{\dot F_1^{0,q}} + \|\mathcal{H}(f_{N,N'})_{N,N'}^{n_1,2}\|_{L^1} \\ &\quad= \min_{N \le N_1 \le N'+1} \big(\|\mathcal{H}(f_{N,N'})_{N,N'}^{N_1,1}\|_{\dot F_1^{0,q}} + \|\mathcal{H}(f_{N,N'})_{N,N'}^{N_1,2}\|_{L^1}\big) \le \|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}. \end{aligned}$$

Consider three cases: (1) $n_0 = n_1$; (2) $n_0 < n_1$; (3) $n_0 > n_1$. For $n_0 = n_1$, since $\|f_{N,N'}^{n_0,2}\|_{L^1} \le \|f_{N,N'}\|_{L^{1,q}}$ and both $\|Q_{n_0-1}(f_{N,N'})\|_{H^1}$ and $\|Q_N(f_{N,N'})\|_{H^1}$ are dominated by $\|f_{N,N'}\|_{L^{1,q}}$ by Lemma 2.2, it follows that

$$\|f_{N,N'}^{n_0,2} - \{Q_{n_0-1}(f_{N,N'}) + Q_N(f_{N,N'})\}\|_{L^1} \lesssim \|f_{N,N'}\|_{L^{1,q}}. \tag{2.1}$$

On the other hand,

$$\begin{aligned} \mathcal{H}(f_{N,N'})_{N,N'}^{n_1,2} &= \mathcal{H}\big(f_{N,N'}^{n_0,2} - \{Q_{n_0-1}(f_{N,N'}) + Q_N(f_{N,N'})\} \\ &\qquad + \{Q_{n_0-1}(f_{N,N'}) + Q_N(f_{N,N'})\}\big)_{N,N'}^{n_1,2} \\ &= \mathcal{H}\big(f_{N,N'}^{n_0,2} - \{Q_{n_0-1}(f_{N,N'}) + Q_N(f_{N,N'})\}\big) \\ &\qquad + \mathcal{H}(Q_{n_0-1}(f_{N,N'}) + Q_N(f_{N,N'}))_{N,N'}^{n_1,2}. \end{aligned}$$

In view of Proposition 1.1, Lemma 2.2, and the H^1-boundedness of the Hilbert transform, we have

$$\big\|\mathcal{H}(Q_{n_0-1}(f_{N,N'})+Q_N(f_{N,N'}))^{n_1,2}_{N,N'}\big\|_{H^1}\lesssim\|f_{N,N'}\|_{L^{1,q}};$$

this expression, when combined with the inequality

$$\big\|\mathcal{H}(f_{N,N'})^{n_1,2}_{N,N'}\big\|_{L^1}\le\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}},$$

implies that

$$\begin{aligned}\big\|\mathcal{H}\big(f^{n_0,2}_{N,N'}-\{Q_{n_0-1}(f_{N,N'})+Q_N(f_{N,N'})\}\big)\big\|_{L^1}&\\ \lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}.&\qquad(2.2)\end{aligned}$$

Both (2.1) and (2.2) yield

$$\begin{aligned}\big\|f^{n_0,2}_{N,N'}&-\{Q_{n_0-1}(f_{N,N'})+Q_N(f_{N,N'})\}\big\|_{H^1}\\ &\lesssim\big\|f^{n_0,2}_{N,N'}-\{Q_{n_0-1}(f_{N,N'})+Q_N(f_{N,N'})\}\big\|_{L^1}\\ &\quad+\big\|\mathcal{H}\big(f^{n_0,2}_{N,N'}-\{Q_{n_0-1}(f_{N,N'})+Q_N(f_{N,N'})\}\big)\big\|_{L^1}\\ &\lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}\end{aligned}$$

and so $\big\|f^{n_0,2}_{N,N'}\big\|_{\dot F^{0,q}_1}\lesssim\big\|f^{n_0,2}_{N,N'}\big\|_{H^1}\lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}$. The definition of n_0 implies that $\big\|f^{n_0,1}_{N,N'}\big\|_{\dot F^{0,q}_1}\le\|f_{N,N'}\|_{L^{1,q}}$. Therefore,

$$\|f_{N,N'}\|_{\dot F^{0,q}_1}\le\big\|f^{n_0,1}_{N,N'}\big\|_{\dot F^{0,q}_1}+\big\|f^{n_0,2}_{N,N'}\big\|_{\dot F^{0,q}_1}\lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}.$$

Now suppose $n_0<n_1$. Then, for $g(x)=\sum_{N\le j<n_0-1}\sum_{k\in\mathbb{Z}}\langle g,\psi_{j,k}\rangle\psi_{j,k}(x)$ with $\|g\|_{L^\infty}\le 1$,

$$\big\langle\mathcal{H}(f_{N,N'})^{n_1,2}_{N,N'},g\big\rangle=\big\langle\mathcal{H}\big(f^{n_0,2}_{N,N'}\big)^{n_1,2}_{N,N'},g\big\rangle.\qquad(2.3)$$

Similar to the case of $n_0=n_1$, we have

$$\big\|f^{n_0,2}_{N,N'}-\{Q_{n_0-1}(f_{N,N'})+Q_{n_0-2}(f_{N,N'})+Q_N(f_{N,N'})\}\big\|_{L^1}\lesssim\|f_{N,N'}\|_{L^{1,q}}.$$

That

$$\big\|\mathcal{H}(Q_{n_0-1}(f_{N,N'})+Q_{n_0-2}(f_{N,N'})+Q_N(f_{N,N'}))^{n_1,2}_{N,N'}\big\|_{H^1}\lesssim\|f_{N,N'}\|_{L^{1,q}}$$

and $\big\|\mathcal{H}(f_{N,N'})^{n_1,2}_{N,N'}\big\|_{L^1}\le\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}$, together with (2.3), yield

$$\begin{aligned}\big\|\mathcal{H}\big(f^{n_0,2}_{N,N'}-\{Q_{n_0-1}(f_{N,N'})+Q_{n_0-2}(f_{N,N'})+Q_N(f_{N,N'})\}\big)\big\|_{L^1}&\\ \lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}.&\end{aligned}$$

As a result,

$$\begin{aligned}\big\|f^{n_0,2}_{N,N'}-\{Q_{n_0-1}(f_{N,N'})+Q_{n_0-2}(f_{N,N'})+Q_N(f_{N,N'})\}\big\|_{H^1}&\\ \lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}.&\end{aligned}$$

Then $\big\|f^{n_0,2}_{N,N'}\|_{\dot F^{0,q}_1}\lesssim\|f^{n_0,2}_{N,N'}\|_{H^1}\lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}$ and so

$$\|f_{N,N'}\|_{\dot F^{0,q}_1}\le\big\|f^{n_0,1}_{N,N'}\big\|_{\dot F^{0,q}_1}+\big\|f^{n_0,2}_{N,N'}\big\|_{\dot F^{0,q}_1}\lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}.$$

Finally, for the case $n_0>n_1$ we split $f_{N,N'}(x)$ into three parts: $f_{N,N'}(x)=f^{(1)}_{N,N'}(x)+f^{(2)}_{N,N'}(x)+f^{(3)}_{N,N'}(x)$, where

$$f^{(1)}_{N,N'}(x) = \sum_{n_0 \le j \le N'} Q_j(f_{N,N'})(x) = f^{n_0,1}_{N,N'}(x),$$
$$f^{(2)}_{N,N'}(x) = \sum_{n_1 \le j < n_0} Q_j(f_{N,N'})(x),$$
$$f^{(3)}_{N,N'}(x) = \sum_{N \le j < n_1} Q_j(f_{N,N'})(x).$$

The definition of n_0 implies that $\big\| f^{(2)}_{N,N'} + f^{(3)}_{N,N'} \big\|_{L^1} = \big\| f^{n_0,2}_{N,N'} \big\|_{L^1} \le \| f_{N,N'} \|_{L^{1,q}}$. For

$$h(x) = \sum_{N \le j < n_1} \sum_{k \in \mathbb{Z}} \langle h, \psi_{j,k} \rangle \psi_{j,k}(x) \text{ satisfying } \|h\|_{L^\infty} \le 1$$

we have $\big\langle f^{n_0,2}_{N,N'}, h \big\rangle = \big\langle f^{(3)}_{N,N'}, h \big\rangle$, which implies $\big\| f^{(3)}_{N,N'} \big\|_{L^1} \lesssim \| f_{N,N'} \|_{L^{1,q}}$. It follows from Lemma 2.2 that

$$\big\| f^{(3)}_{N,N'} - \{ Q_{n_1-1}(f_{N,N'}) + Q_N(f_{N,N'}) \} \big\|_{L^1} \lesssim \| f_{N,N'} \|_{L^{1,q}}. \tag{2.4}$$

We observe that

$$\begin{aligned} \mathcal{H}(f_{N,N'})^{n_1,2}_{N,N'} &= \mathcal{H}\big(f^{(3)}_{N,N'} + Q_{n_1}(f_{N,N'})\big)^{n_1,2}_{N,N'} \\ &= \mathcal{H}\big(f^{(3)}_{N,N'} - \{ Q_{n_1-1}(f_{N,N'}) + Q_N(f_{N,N'}) \}\big) \\ &\quad + \mathcal{H}(Q_{n_1-1}(f_{N,N'}) + Q_{n_1}(f_{N,N'}) + Q_N(f_{N,N'}))^{n_1,2}_{N,N'}. \end{aligned} \tag{2.5}$$

Applying Proposition 1.1, Lemma 2.2, and the H^1-boundedness of the Hilbert transform now yields

$$\big\| \mathcal{H}(Q_{n_1-1}(f_{N,N'}) + Q_{n_1}(f_{N,N'}) + Q_N(f_{N,N'}))^{n_1,2}_{N,N'} \big\|_{H^1} \lesssim \| f_{N,N'} \|_{L^{1,q}}. \tag{2.6}$$

Since $\big\| \mathcal{H}(f_{N,N'})^{n_1,2}_{N,N'} \big\|_{L^1} \le \| \mathcal{H}(f_{N,N'}) \|_{L^{1,q}}$, it follows from (2.5) and (2.6) that

$$\begin{aligned} &\big\| \mathcal{H}\big(f^{(3)}_{N,N'} - \{ Q_{n_1-1}(f_{N,N'}) + Q_N(f_{N,N'}) \}\big) \big\|_{L^1} \\ &\qquad \lesssim \| f_{N,N'} \|_{L^{1,q}} + \| \mathcal{H}(f_{N,N'}) \|_{L^{1,q}}. \end{aligned} \tag{2.7}$$

Combining (2.4) and (2.7), we obtain

$$\big\| f^{(3)}_{N,N'} - \{ Q_{n_1-1}(f_{N,N'}) + Q_N(f_{N,N'}) \} \big\|_{H^1} \lesssim \| f_{N,N'} \|_{L^{1,q}} + \| \mathcal{H}(f_{N,N'}) \|_{L^{1,q}};$$

hence

$$\big\| f^{(3)}_{N,N'} \big\|_{\dot{F}^{0,q}_1} \lesssim \big\| f^{(3)}_{N,N'} \big\|_{H^1} \lesssim \| f_{N,N'} \|_{L^{1,q}} + \| \mathcal{H}(f_{N,N'}) \|_{L^{1,q}}. \tag{2.8}$$

The definitions of n_0 and n_1 yield

$$\begin{aligned} &\big\| \mathcal{H}\big(f^{(1)}_{N,N'} + f^{(2)}_{N,N'} + Q_{n_1-1}(f_{N,N'})\big)^{n_1,1}_{N,N'} \big\|_{\dot{F}^{0,q}_1} \\ &\qquad = \big\| \mathcal{H}(f_{N,N'})^{n_1,1}_{N,N'} \big\|_{\dot{F}^{0,q}_1} \le \| \mathcal{H}(f_{N,N'}) \|_{L^{1,q}} \end{aligned}$$

and

$$\big\| f^{(1)}_{N,N'} \big\|_{\dot{F}^{0,q}_1} = \big\| f^{n_0,1}_{N,N'} \big\|_{\dot{F}^{0,q}_1} \le \| f_{N,N'} \|_{L^{1,q}}. \tag{2.9}$$

By Proposition 1.1 and Theorem 2.1, we have $\big\| \mathcal{H}\big(f^{(1)}_{N,N'}\big)^{n_1,1}_{N,N'} \big\|_{\dot{F}^{0,q}_1} \lesssim \| f_{N,N'} \|_{L^{1,q}}$. Therefore,

$$\big\|\mathcal{H}\big(f^{(2)}_{N,N'}+Q_{n_1-1}(f_{N,N'})\big)^{n_1,1}_{N,N'}\big\|_{\dot F^{0,q}_1}\lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}.$$

We note that

$$\begin{aligned}\mathcal{H}\big(f^{(2)}_{N,N'}&+Q_{n_1-1}(f_{N,N'})\big)^{n_1,1}_{N,N'}\\&=\mathcal{H}\big(f^{(2)}_{N,N'}-Q_{n_1}(f_{N,N'})-Q_{N'}(f^{(2)}_{N,N'})\big)\\&\quad+\mathcal{H}\big(Q_{n_1-1}(f_{N,N'})+Q_{n_1}(f_{N,N'})+Q_{N'}(f^{(2)}_{N,N'})\big)^{n_1,1}_{N,N'}.\end{aligned}$$

Again using Proposition 1.1, Lemma 2.2, and the H^1-boundedness of the Hilbert transform, we obtain

$$\begin{aligned}\big\|\mathcal{H}\big(Q_{n_1-1}(f_{N,N'})&+Q_{n_1}(f_{N,N'})+Q_{N'}\big(f^{(2)}_{N,N'}\big)\big)^{n_1,1}_{N,N'}\big\|_{\dot F^{0,q}_1}\\&\lesssim\big\|\mathcal{H}\big(Q_{n_1-1}(f_{N,N'})+Q_{n_1}(f_{N,N'})+Q_{N'}\big(f^{(2)}_{N,N'}\big)\big)^{n_1,1}_{N,N'}\big\|_{H^1}\\&\lesssim\|f_{N,N'}\|_{L^{1,q}},\end{aligned}$$

which implies

$$\big\|\mathcal{H}\big(f^{(2)}_{N,N'}-Q_{n_1}(f_{N,N'})-Q_{N'}\big(f^{(2)}_{N,N'}\big)\big)\big\|_{\dot F^{0,q}_1}\lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}.$$

Now, by Theorem 2.1,

$$\big\|f^{(2)}_{N,N'}-Q_{n_1}(f_{N,N'})-Q_{N'}\big(f^{(2)}_{N,N'}\big)\big\|_{\dot F^{0,q}_1}\lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}.$$

It then follows from Lemma 2.2 that

$$\big\|f^{(2)}_{N,N'}\big\|_{\dot F^{0,q}_1}\lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}}. \tag{2.10}$$

So from (2.8)–(2.10) we have

$$\|f_{N,N'}\|_{\dot F^{0,q}_1}\lesssim\|f_{N,N'}\|_{L^{1,q}}+\|\mathcal{H}(f_{N,N'})\|_{L^{1,q}},$$

and Theorem 1.3 follows. □

3. The Duality of $L^{1,q}(\mathbb{R})$ and $L^{\infty,q'}(\mathbb{R})$

For $f_{N,N'}\in L^{1,q}(\mathbb{R})$, by Remark 1.2 we may write

$$\|f_{N,N'}\|_{L^{1,q}}=\max_{N\le\tilde N\le\tilde N'\le N'}\ \min_{\tilde N\le\tilde N_1\le\tilde N'+1}\big(\big\|f^{\tilde N_1,1}_{\tilde N,\tilde N'}\big\|_{\dot F^{0,q}_1}+\big\|f^{\tilde N_1,2}_{\tilde N,\tilde N'}\big\|_{L^1}\big).$$

The maximum is actually attained when $\tilde N=N$ and $\tilde N'=N'$.

Lemma 3.1. *For $f_{N,N'}\in L^{1,q}(\mathbb{R})$,*

$$\|f_{N,N'}\|_{L^{1,q}}=\min_{N\le N_1\le N'+1}\big(\big\|f^{N_1,1}_{N,N'}\big\|_{\dot F^{0,q}_1}+\big\|f^{N_1,2}_{N,N'}\big\|_{L^1}\big).$$

Proof. We choose $N\le n_0\le N'+1$ such that

$$\big\|f^{n_0,1}_{N,N'}\big\|_{\dot F^{0,q}_1}+\big\|f^{n_0,2}_{N,N'}\big\|_{L^1}=\min_{N\le N_1\le N'+1}\big(\big\|f^{N_1,1}_{N,N'}\big\|_{\dot F^{0,q}_1}+\big\|f^{N_1,2}_{N,N'}\big\|_{L^1}\big).$$

For $N \le \tilde{N} \le \tilde{N}' \le N'$, Proposition 1.1 implies $\| f^{n_0,1}_{\tilde{N},\tilde{N}'}\|_{\dot{F}^{0,q}_1} \le \| f^{n_0,1}_{N,N'}\|_{\dot{F}^{0,q}_1}$. For $g(x) = \sum_{\tilde{N}\le j\le \tilde{N}'}\sum_{k\in\mathbb{Z}}\langle g,\psi_{j,k}\rangle\psi_{j,k}(x)$ with $\|g\|_{L^\infty} \le 1$, we have $\langle f^{n_0,2}_{N,N'}, g\rangle = \langle f^{n_0,2}_{\tilde{N},\tilde{N}'}, g\rangle$. The converse of Hölder's inequality gives $\| f^{n_0,2}_{\tilde{N},\tilde{N}'}\|_{L^1} \le \| f^{n_0,2}_{N,N'}\|_{L^1}$ and so the proof is completed. □

We can now apply Lemma 3.1 to prove the duality of $L^{1,q}(\mathbb{R})$ and $L^{\infty,q'}(\mathbb{R})$.

Proof of Theorem 1.6. For any fixed $N \le N'$, choose $N \le n_0 \le N'+1$ such that

$$\| f^{n_0,1}_{N,N'}\|_{\dot{F}^{0,q}_1} + \| f^{n_0,2}_{N,N'}\|_{L^1} = \min_{N\le N_1\le N'+1}\big(\| f^{N_1,1}_{N,N'}\|_{\dot{F}^{0,q}_1} + \| f^{N_1,2}_{N,N'}\|_{L^1}\big).$$

Since $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis, it follows that

$$\begin{aligned}\langle f_{N,N'}, g\rangle = \langle f_{N,N'}, g_{N,N'}\rangle &= \langle f^{n_0,1}_{N,N'}, g_{N,N'}\rangle + \langle f^{n_0,2}_{N,N'}, g_{N,N'}\rangle \\ &= \langle f^{n_0,1}_{N,N'}, g^{n_0,1}_{N,N'}\rangle + \langle f^{n_0,2}_{N,N'}, g^{n_0,2}_{N,N'}\rangle.\end{aligned}$$

Hence the dualities $(\dot{F}^{0,q}_1(\mathbb{R}))' = \dot{F}^{0,q'}_\infty(\mathbb{R})$ and $(L^1(\mathbb{R}))' = L^\infty(\mathbb{R})$ yield

$$\begin{aligned}|\langle f_{N,N'}, g\rangle| &\le C\big(\| f^{n_0,1}_{N,N'}\|_{\dot{F}^{0,q}_1}\|g^{n_0,1}_{N,N'}\|_{\dot{F}^{0,q'}_\infty} + \| f^{n_0,2}_{N,N'}\|_{L^1}\|g^{n_0,2}_{N,N'}\|_{L^\infty}\big) \\ &\le C\| f_{N,N'}\|_{L^{1,q}}\big(\|g^{n_0,1}_{N,N'}\|_{\dot{F}^{0,q'}_\infty} + \|g^{n_0,2}_{N,N'}\|_{L^\infty}\big).\end{aligned}$$

By the definition of $L^{\infty,q'}(\mathbb{R})$, we have $\|g_{N,N'}\|_{L^{\infty,q'}} \le \|g\|_{L^{\infty,q'}}$ for any $N \le N'$. Then

$$|\langle f_{N,N'}, g\rangle| \le C\| f_{N,N'}\|_{L^{1,q}}\|g\|_{L^{\infty,q'}}, \tag{3.1}$$

where $C > 0$ is a constant independent of N and N'.

If $f \in L^{1,q}(\mathbb{R})$ then, for any $0 < \varepsilon < 1$, there exist $N_\varepsilon, N'_\varepsilon$ such that, for all $N \le N_\varepsilon$ and $N' \ge N'_\varepsilon$, we have $\|f - f_{N,N'}\|_{L^{1,q}} \le \varepsilon\|f\|_{L^{1,q}}$. Because (3.1) holds for all $N \le N'$, we show that $\|\mathcal{L}_g\| \le C\|g\|_{L^{\infty,q'}}$.

Conversely, suppose $\mathcal{L} \in (L^{1,q}(\mathbb{R}))'$. We define g formally by

$$g(x) = \sum_{j,k\in\mathbb{R}}\mathcal{L}(\psi_{j,k})\psi_{j,k}(x).$$

It suffices to show that $g \in L^{\infty,q'}(\mathbb{R})$. For given $N \le N'$, we choose $N \le n_1 \le N'+1$ such that

$$\|g^{n_1,1}_{N,N'}\|_{\dot{F}^{0,q'}_\infty} + \|g^{n_1,2}_{N,N'}\|_{L^\infty} = \max_{N\le N_1\le N'+1}\big(\|g^{N_1,1}_{N,N'}\|_{\dot{F}^{0,q'}_\infty} + \|g^{N_1,2}_{N,N'}\|_{L^\infty}\big).$$

Let $V_{N,N'}$ be the closed subspace of $L^{1,q}(\mathbb{R})$ spanned by $\{\psi_{j,k} : N \le j \le N', k \in \mathbb{Z}\}$. We denote by $\bar{\mathcal{L}} := \mathcal{L}|_{V_{N,N'}}$, the restriction of $\mathcal{L}$ to the subspace $V_{N,N'}$. Then $\|\bar{\mathcal{L}}\| \le \|\mathcal{L}\|$ and

$$\bar{\mathcal{L}}(f) = \sum_{\substack{N\le j\le N' \\ k\in\mathbb{Z}}} a_{j,k}\mathcal{L}(\psi_{j,k}) = \langle f^{n_1,1}_{N,N'}, g^{n_1,1}_{N,N'}\rangle + \langle f^{n_1,2}_{N,N'}, g^{n_1,2}_{N,N'}\rangle \tag{3.2}$$

for all $f \in \mathcal{S}_0(\mathbb{R}) \cap V_{N,N'}$. Let $V^{(1)}_{N,N'}$ and $V^{(2)}_{N,N'}$ be the closed subspaces of $V_{N,N'}$ spanned by $\{\psi_{j,k} : n_1 \le j \le N', k \in \mathbb{Z}\}$ and $\{\psi_{j,k} : N \le j < n_1, k \in \mathbb{Z}\}$, respectively. It is easy to verify that $\mathcal{S}_0(\mathbb{R}) \cap V^{(1)}_{N,N'}$ is dense in $\big(V^{(1)}_{N,N'}, \|\cdot\|_{\dot F^{0,q}_1}\big)$ and that $\mathcal{S}_0(\mathbb{R}) \cap V^{(2)}_{N,N'}$ is dense in $\big(V^{(2)}_{N,N'}, \|\cdot\|_{L^1}\big)$. Thus there exist $\{h^{(1)}_m\}_{m\in\mathbb{N}} \subset \mathcal{S}_0(\mathbb{R}) \cap V^{(1)}_{N,N'}$ and $\{h^{(2)}_m\}_{m\in\mathbb{N}} \subset \mathcal{S}_0(\mathbb{R}) \cap V^{(2)}_{N,N'}$ with $\|h^{(1)}_m\|_{\dot F^{0,q}_1} \le 1$ and $\|h^{(2)}_m\|_{L^1} \le 1$ such that

$$\big\langle h^{(1)}_m, g^{n_1,1}_{N,N'}\big\rangle \to \big\|g^{n_1,1}_{N,N'}\big\|_{\dot F^{0,q'}_\infty} \quad \text{and} \quad \big\langle h^{(2)}_m, g^{n_1,2}_{N,N'}\big\rangle \to \big\|g^{n_1,2}_{N,N'}\big\|_{L^\infty} \quad \text{as } m \to \infty.$$

Let $h_m = h^{(1)}_m + h^{(2)}_m \in \mathcal{S}_0(\mathbb{R}) \cap V_{N,N'}$ for all $m \in \mathbb{N}$. Then $h^{(1)}_m = (h_m)^{n_1,1}_{N,N'}$ and $h^{(2)}_m = (h_m)^{n_1,2}_{N,N'}$. By Lemma 3.1, $\|h_m\|_{L^{1,q}} \le \|h^{(1)}_m\|_{\dot F^{0,q}_1} + \|h^{(2)}_m\|_{L^1} \le 2$. It follows from (3.2) that, for all $m \in \mathbb{N}$,

$$\big|\big\langle h^{(1)}_m, g^{n_1,1}_{N,N'}\big\rangle + \big\langle h^{(2)}_m, g^{n_1,2}_{N,N'}\big\rangle\big| = |\bar{\mathcal{L}}(h_m)| \le \|\bar{\mathcal{L}}\|\,\|h_m\|_{L^{1,q}} \le 2\|\mathcal{L}\|.$$

Taking $m \to \infty$, we get $\big\|g^{n_1,1}_{N,N'}\big\|_{\dot F^{0,q'}_\infty} + \big\|g^{n_1,2}_{N,N'}\big\|_{L^\infty} \le 2\|\mathcal{L}\|$. Therefore,

$$\max_{N\le N_1\le N'}\big(\big\|g^{N_1,1}_{N,N'}\big\|_{\dot F^{0,q'}_\infty} + \big\|g^{N_1,2}_{N,N'}\big\|_{L^\infty}\big) \le 2\|\mathcal{L}\| \quad \text{for all } N \le N'.$$

Taking the supremum over all $N \le N'$ on both sides now yields $\|g\|_{L^{\infty,q'}} \le 2\|\mathcal{L}\|$. □

4. Proof of the Fefferman–Stein Decomposition

In this section we prove the Fefferman–Stein decomposition of $\dot F^{0,q}_\infty(\mathbb{R})$, $1 < q \le 2$.

Proof of Theorem 1.7. To prove "if" part, it suffices to show that $\mathcal{H}(g) \in \dot F^{0,q}_\infty(\mathbb{R})$ for $g \in L^{\infty,q}(\mathbb{R})$. We observe that $\langle \mathcal{H}(g), h\rangle = -\langle g, \mathcal{H}(h)\rangle$ for all $h \in \mathcal{S}_0(\mathbb{R})$. By Theorems 1.6 and 2.1,

$$\begin{aligned} |\langle g, \mathcal{H}(h)\rangle| &\le \|g\|_{L^{\infty,q}}\|\mathcal{H}(h)\|_{L^{1,q'}} \\ &\le \|g\|_{L^{\infty,q}}\|\mathcal{H}(h)\|_{\dot F^{0,q'}_1} \le C\|g\|_{L^{\infty,q}}\|h\|_{\dot F^{0,q'}_1}. \end{aligned} \tag{4.1}$$

Since (4.1) holds for all $h \in \mathcal{S}_0(\mathbb{R})$, which is dense in $\dot F^{0,q'}_1(\mathbb{R})$, we get

$$\|\mathcal{H}(g)\|_{\dot F^{0,q}_\infty} \le C\|g\|_{L^{\infty,q}}.$$

Next we consider the "only if" part. By Theorem 1.3, $\dot F^{0,q'}_1(\mathbb{R})$ can be identified with a closed subspace of $L^{1,q'}(\mathbb{R}) \oplus L^{1,q'}(\mathbb{R})$ if we identify g with $(g, \mathcal{H}(g))$. The Hahn–Banach theorem states that any bounded linear functional on $\dot F^{0,q'}_1(\mathbb{R})$ extends to a bounded linear functional on $L^{1,q'}(\mathbb{R}) \oplus L^{1,q'}(\mathbb{R})$. For $f \in \dot F^{0,q}_\infty(\mathbb{R})$, let $\mathcal{L} \in \big(\dot F^{0,q'}_1(\mathbb{R})\big)'$ be defined by

$$\mathcal{L}(g) = \int_{\mathbb{R}} f(x)g(x)\,dx.$$

Since the dual space of $L^{1,q'}(\mathbb{R}) \oplus L^{1,q'}(\mathbb{R})$ is equivalent to $L^{\infty,q}(\mathbb{R}) \oplus L^{\infty,q}(\mathbb{R})$, there exist $f_0, f_1 \in L^{\infty,q}(\mathbb{R})$ such that

$$\begin{aligned} \mathcal{L}(g) &= \int_{\mathbb{R}} g(x)f_0(x)\,dx + \int_{\mathbb{R}} \mathcal{H}(g)(x)f_1(x)\,dx \\ &= \int_{\mathbb{R}} g(x)(f_0(x) - \mathcal{H}(f_1)(x))\,dx, \end{aligned}$$

which implies that f can be written as $f = f_0 - \mathcal{H}(f_1)$ with $f_0, f_1 \in L^{\infty,q}(\mathbb{R})$. □

5. Proof of Remark 1.8

We first recall some properties of Meyer's wavelets (see [9]). Let $\phi(x)$ denote the father wavelet given in Section 1. Since $\phi(x) \in \mathcal{S}(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} |\phi(x-k)| \le C \quad \text{for } x \in \mathbb{R}. \tag{5.1}$$

For $j, k \in \mathbb{Z}$, we write $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$. For $j \in \mathbb{Z}$, define

$$P_j(f)(x) = \sum_{k\in\mathbb{Z}} \langle f, \phi_{j,k}\rangle \phi_{j,k}(x).$$

Then $f_{N,N'} = P_{N'}(f) - P_N(f)$ for any $N \le N'$ and, by (5.1),

$$\begin{aligned} \|P_j(f)\|_{L^1} &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |f(y)| \sum_{k\in\mathbb{Z}} |\phi(2^j y - k)||2^j \phi(2^j x - k)| \, dx\, dy \\ &\le C \int_{\mathbb{R}} |f(y)| \sum_{k\in\mathbb{Z}} |\phi(2^j y - k)| \, dy \\ &\le C\|f\|_{L^1}. \end{aligned} \tag{5.2}$$

It suffices to prove assertion (i) of the remark because the dualities $(\dot{F}_1^{0,q}(\mathbb{R}))' = \dot{F}_\infty^{0,q'}(\mathbb{R})$ and $(L^{1,q}(\mathbb{R}))' = L^{\infty,q'}(\mathbb{R})$ for $2 \le q < \infty$ (by Theorem 1.7) imply assertion (ii).

Note that $\phi \in L^1(\mathbb{R})$. Then (5.2) gives

$$\min_{N\le N_1 \le N'+1} \left(\|\phi_{N,N'}^{N_1,1}\|_{\dot{F}_1^{0,q}} + \|\phi_{N,N'}^{N_1,2}\|_{L^1}\right) \le \|\phi_{N,N'}\|_{L^1} \le C\|\phi\|_{L^1} \quad \text{for any } N \le N'.$$

Taking the supremum over all $N \le N'$ on both sides, we have $\|\phi\|_{L^{1,q}} \le C\|\phi\|_{L^1}$. To show that $\phi \notin \dot{F}_1^{0,q}(\mathbb{R})$, let $a_{j,k} = \langle \phi, \psi_{j,k}\rangle$ for $j, k \in \mathbb{Z}$. Then

$$\begin{aligned} |a_{j,0}| &= \left| \int_{\mathbb{R}} \Phi(\xi) 2^{-j/2} \{\Phi(2^{-(j+1)}\xi)^2 - \Phi(2^{-j}\xi)^2\}^{1/2} e^{-i2^{-(j+1)}\xi} \, d\xi \right| \\ &= 2^{j/2} \left| \int_{[-2^{j+3}\pi/3,\ 2^{j+3}\pi/3]} \{\Phi(\eta/2)^2 - \Phi(\eta)^2\}^{1/2} e^{-i\eta/2} \, d\eta \right| \\ &\ge C 2^{j/2} \quad \text{if } j < -M \end{aligned}$$

for some $M > 0$ large enough. Therefore,

$$\begin{aligned} \int_{\mathbb{R}} \left\{ \sum_{j,k\in\mathbb{Z}} (2^{j/2}|a_{j,k}|\chi(2^j x - k))^q \right\}^{1/q} dx &\ge \int_{\mathbb{R}} \left\{ \sum_{j<-M} 2^{jq/2} |a_{j,0}|^q \chi(2^j x) \right\}^{1/q} dx \\ &\ge C \int_{\mathbb{R}} \left\{ \sum_{j<-M} 2^{jq} \chi(2^j x) \right\}^{1/q} dx \\ &\ge C \sum_{m=M}^{\infty} \int_{2^m}^{2^{m+1}} \left\{ \sum_{j=-\infty}^{-m} 2^{jq} \right\}^{1/q} dx = \infty. \end{aligned}$$

It now follows from Proposition 1.1 that $\phi \notin \dot{F}_1^{0,q}(\mathbb{R})$.

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