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An Extension Theorem for Real Kähler Submanifolds in Codimension 4

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1. Introduction

Submanifold theory, and especially the study of Riemannian submanifolds in Euclidean spaces, are a classic subarea in differential geometry. The Nash embedding theorem [18] guarantees that any complete Riemannian manifold can be isometrically embedded into a Euclidean space. There are many important developments in submanifold theory, of which we mention just two. One is the work of Hongwei Xu and his collaborators [20; 21; 22] generalizing the differentiable sphere theorem of Brendle and Schoen [1; 2] to the submanifold case in order to obtain the optimal pinching constant. The other development we mention here is the work of Marques and Neves [17] in solving the long-standing Willmore conjecture.

Yet in the special case when the submanifold happens to be Kähler, the research is relatively sparse and sporadic, and we believe that the state of knowledge is still rather primitive. In this paper, we shall refer to a Kähler manifold that is isometrically embedded in a real Euclidean space as a *real Kähler Euclidean submanifold,* or *real Kähler submanifold* for short. That is, we have an isometric embedding $f: M^n \to \mathbb{R}^{2n+p}$ from a Kähler manifold M^n of complex dimension n into the real Euclidean space.

Because M^n is equipped with a complex structure, it would be ideal for the embedding f to be both isometric and holomorphic. However, the thesis of Calabi [3] established that very few Kähler metrics can be isometrically and holomorphically embedded in a complex Euclidean space or in other complex space forms. In fact, Calabi precisely characterized all such metrics. So to study generic Kähler manifolds in the extrinsic setting, one must abandon the holomorphicity assumption on the embedding and assume only that it is isometric.

For a real Kähler submanifold $f: M^n \to \mathbb{R}^{2n+p}$, the Kählerness of M^n imposes strong restrictions and makes M^n extremely sensitive to its codimension. For instance, when $p = 1$ (i.e., when M^n is a hypersurface) a result of Florit and Zhang [15] states that, if M^n is also assumed to be complete, then f must be the product of g and the identity map of $\mathbb{C}^{n-1}$; here $g: \Sigma \to \mathbb{R}^3$ is the isometric embedding of a complete surface, which is always Kähler. In other words, surfaces in $\mathbb{R}^3$ are essentially the only real Kähler submanifolds in codimension 1. In contrast, there are all kinds of real hypersurfaces in Euclidean spaces.

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In codimension 2, the situation is also well studied and fully understood. The minimal case was analyzed in detail by Dajczer and Gromoll (see [8] and [10] and the references therein); the nonminimal case was classified by Florit and Zhang [16]. In codimension 3, Dajczer and Gromoll [9] showed that, unless the submanifold M^n is a holomorphic hypersurface of a real Kähler submanifold of codimension 1, its rank must be less than or equal to 3 (the codimension of M^n).

Recall that the *rank* of a real Kähler submanifold $f\colon M^n \to \mathbb{R}^{2n+p}$ at $x \in M$ is defined as $n - \nu_0$ for ν_0 the complex dimension of $\Delta_0 = \Delta \cap J\Delta$, which is the J-invariant part of the kernel Δ of the second fundamental form of f. Of course, these spaces may not have constant dimensions on M. But if we let U be the open subset where Δ_0 takes the minimum (and thus constant) dimension, then r will be constant in U. Outside the closure of U, M will be a real Kähler submanifold with smaller rank. In general, by restricting to an open dense subset U' of M we can always assume that, in each connected component U of U', Δ and Δ_0 take constant dimensions and form distributions. Note that the leaves of Δ (Δ_0) are totally geodesic (complex) submanifolds in M^n; they are actually open subsets of (parallel translations of) linear subspaces in the ambient Euclidean space. We might later need to reduce U' further, but the conclusions we will draw will always be valid in each connected component of an open dense subset of M.

The main purpose of this paper is to show that the result of Dajczer and Gromoll in [9] can be extended to the codimension-4 case. In particular, we prove the following result.

MAIN THEOREM. *Let $f\colon M^n \to \mathbb{R}^{2n+4}$ be a real Kähler submanifold with rank $r > 4$ everywhere. Then there exists an open dense subset $U' \subset M$ such that, for each connected component U of U', the restriction $f|_U$ has a Kähler extension; namely, there exist a real Kähler submanifold $h\colon Q^{n+1} \to \mathbb{R}^{2n+4}$ of codimension 2 and a holomorphic embedding $\sigma\colon U \to Q^{n+1}$ such that $f|_U = h \circ \sigma$. Furthermore, when f is minimal, one can choose h to be minimal as well.*

Note that if h is minimal then f must be minimal. In general, the extension h might not be unique. But as we shall see from the proof, there is always a "canonical" extension unless f itself is a holomorphic isometric embedding into $\mathbb{C}^{n+2}$.

This result can be regarded as an extension of a phenomenon discovered by Dajczer [4] and Dajczer and Gromoll [9] in codimensions 2 and 3, respectively. In [4] Dajczer proved that, for any codimension-2 real Kähler submanifold of rank $>$ 2, in any connected component U of an open dense subset of M, the restriction $f|_U$ is a holomorphic embedding into $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. This is an important discovery. In codimension 3, Dajczer and Gromoll [9] proved that if a real Kähler submanifold of dimension 3 has rank > 3 then there exists an open dense subset $U' \subseteq M$ such that, in each connected component U of U', $f|_U$ has a Kähler extension into a real Kähler submanifold Q^{n+1} of codimension 1.

Note that these results in [4] and [9] employed assumptions about the *relative nullity* ν—namely, the (real) dimension of the kernel Δ of the second fundamental form α_f. Since $\Delta_0 \subseteq \Delta$, we have $2\nu_0 \le \nu$ and so $\nu \ge 2n - 2r$, where r is the rank. In [4] it was assumed that $\nu < 2n - 4$, which implies $r > 2$; in [9] it was assumed that $\nu < 2n - 6$, which implies $r > 3$. Even though these assumptions are slightly stronger than those employed here, it is easy to see that the arguments in [4] and [9] can be extended to cases where assumptions are made on the ranks.

We suspect that similar phenomenon will persist in higher codimensions as well. That is, the rank r should be controlled by the codimension p in a certain way unless the manifold is a complex submanifold of another real Kähler submanifold of a smaller codimension. We will explore the higher-codimensional cases elsewhere; in this paper we will merely conjecture that, for $p \leq 11$, the words "controlled by" in the preceding sentence should be interpreted as the rank being no greater than the codimension (i.e., $r \leq p$).

CONJECTURE. *Let $f: M^n \to \mathbb{R}^{2n+p}$ be a real Kähler submanifold with rank $r > p$ everywhere. If $p \leq 11$ then there exists an open dense subset $U' \subset M$ such that, for each connected component U of U', the restriction $f|_U$ has a Kähler extension; namely, there exist a real Kähler submanifold $h: Q^{n+s} \to \mathbb{R}^{2n+p}$ of codimension $p - 2s < p$ and a holomorphic embedding $\sigma: U \to Q^{n+s}$ such that $f|_U = h \circ \sigma$.*

Observe that the main theorem, together with results of [4] and [9], confirms the conjecture for $p \leq 4$. (When $p = 1$, one always has $r \leq 1$.)

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2. Preliminaries

In this section we collect some known results in the literature that will be needed in the proof of our theorem. We will also fix some notation and terminology that will be used.

Unless specified otherwise, we will always assume that M is a real Kähler submanifold of complex dimension n and codimension p, where f is the isometric embedding from M into $\mathbb{R}^{2n+p}$. At any $x \in M$, let Δ be the kernel of the second fundamental form α_f of f and let $\Delta_0 = \Delta \cap J\Delta$ be the J-invariant part of Δ. The rank r is defined as $n - \nu_0$ for $2\nu_0$ the real dimension of Δ_0. We always have $\nu \geq 2n - 2r$, where $\nu = \dim(\Delta)$ is the relative nullity.

The results in this paper are local in nature, and from time to time we will reduce from M into an open dense subset of it; in this way we create various subspaces in the tangent or normal bundle while taking constant dimensions and forming subbundles.

For $x \in M$, we denote by $T \cong \mathbb{R}^{2n}$ the real tangent space $T_x M$, by $N = T_x M^{\perp} \cong \mathbb{R}^p$ the normal space, and by $V \cong \mathbb{C}^n$ the space of all type-$(1,0)$ complex tangent vectors at x (viz., $V \oplus \bar{V} \cong T \otimes_{\mathbb{R}} \mathbb{C}$). Extending the second fundamental form $\alpha_f : T \times T \to N$ linearly over $\mathbb{C}$, we denote its $(1,1)$ and $(2,0)$ components by H and S, respectively: $H : V \otimes \bar{V} \to N_{\mathbb{C}}$ and $S : V \otimes V \to N_{\mathbb{C}}$, where $N_{\mathbb{C}} = N \otimes_{\mathbb{R}} \mathbb{C}$.

As observed in [11], the Kählerness of M implies that the Hermitian bilinear form H and the symmetric bilinear form S satisfy the symmetry conditions

$$\langle H_{X\bar{Y}}, H_{Z\overline{W}} \rangle = \langle H_{Z\bar{Y}}, H_{X\overline{W}} \rangle, \tag{2.1}$$

$$\langle H_{X\bar{Y}}, S_{ZW} \rangle = \langle H_{Z\bar{Y}}, S_{XW} \rangle, \tag{2.2}$$

$$\langle S_{XY}, S_{ZW} \rangle = \langle S_{ZY}, S_{XW} \rangle \tag{2.3}$$

for any $X, Y, Z, W \in V$. These are direct consequences of the Gauss equation.

We remark that H and S together carry all the information of α_f. Also, by (2.1) we have

$$\left| \sum_{i=1}^{n} H_{i\bar{i}} \right|^2 = \sum_{i,j=1}^{n} |H_{i\bar{j}}|^2$$

for any unitary frame $\{e_1, \ldots, e_n\}$ of V. Here we wrote $H_{i\bar{j}}$ for $H_{e_i \overline{e_j}}$. So $H \equiv 0$ if and only if the trace of H, which is (a multiple of) the mean curvature of f, vanishes. Hence f is minimal if and only if $H = 0$.

Note that for $\Delta = \ker(\alpha_f)$, its J-invariant part $\Delta_0 = \Delta \cap J\Delta$ corresponds to a complex subspace $D \subseteq V$ with complex dimension ν_0, where D is exactly the intersection of the kernels of H and S. Let V' be the orthogonal complement of D in V. We have $V = D \oplus V'$ and $V' \cong \mathbb{C}^r$, where $r = n - \nu_0$ is the rank of M^n. Also, D (or Δ) is contained in the kernel of the curvature tensor of M, and the leaves of the foliation D are totally geodesic, flat complex submanifolds in M. They are actually open subsets of $\mathbb{C}^{n-r}$ embedded linearly (i.e., as parallel translations of linear subspace) in $\mathbb{R}^{2n+p}$. In a way, then, the rank r of M is like the essential (complex) dimension of M, even though M might not, in general, be isometric to the product space (i.e., the leaves of D might not be parallel to each other).

For any $\eta \in N$, the shape operator A_η is defined by $\langle A_\eta u, v \rangle = \langle \alpha_f(u,v), \eta \rangle$ for any $u, v \in T$ and is self-adjoint. For convenience, we will also denote by A^η the *shape form,* which is defined by $A^\eta_{uv} = \langle A_\eta(u), v \rangle = \langle \alpha_f(u,v), \eta \rangle$. It is the component of the second fundamental form in the η-direction.

Let $\{e_1, \ldots, e_n\}$ be a basis of V. For each $1 \leq i \leq n$, write

$$e_i = \frac{1}{\sqrt{2}} (\varepsilon_i - \sqrt{-1} \varepsilon_{n+i}).$$

Then under the basis $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$ of T, A^η will take the form

$$A^\eta = \begin{pmatrix} \mathrm{Re}(H^\eta) + \mathrm{Re}(S^\eta) & \mathrm{Im}(H^\eta) - \mathrm{Im}(S^\eta) \\ -\mathrm{Im}(H^\eta) - \mathrm{Im}(S^\eta) & \mathrm{Re}(H^\eta) - \mathrm{Re}(S^\eta) \end{pmatrix}, \tag{2.4}$$

where $H^\eta = \langle H_{i\bar{j}}, \eta\rangle$ and $S^\eta = \langle S_{ij}, \eta\rangle$. Note that, under any tangent frame $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$, the shape operator A_η and the shape form A^η are related by

$$A_\eta(\varepsilon_i) = \sum_{j=1}^{2n} (A^\eta g^{-1})_{ij} \varepsilon_j = \sum_{j,k=1}^{2n} A^\eta_{ik} g^{kj} \varepsilon_j;$$

here $A^\eta_{ij} = A^\eta_{\varepsilon_i \varepsilon_j}$, $g_{ij} = \langle \varepsilon_i, \varepsilon_j\rangle$, and (g^{ij}) is the inverse matrix of (g_{ij}).

Next we recall the Codazzi equation:

$$\nabla_u(A_\xi v) - \nabla_v(A_\xi u) - A_{\nabla^\perp_u \xi} v + A_{\nabla^\perp_v \xi} u - A_\xi[u, v] = 0 \tag{2.5}$$

for any vector fields u, v on M and normal section ξ. For any type-$(1,0)$ tangent vector X and any (possibly complexified) normal vector ξ, denote by

$$A_\xi X = H_\xi X + S_\xi X \tag{2.6}$$

the decomposition of $A_\xi X$ into its $(1,0)$ part and $(0,1)$ part. Thus we have the operators H_ξ and S_ξ, which are determined by

$$H_\xi X = \sum_{i=1}^{n} H^\xi_{X\bar{i}} e_i \quad \text{and} \quad S_\xi X = \sum_{i=1}^{n} S^\xi_{Xi} \overline{e_i}$$

under any unitary frame $\{e_1, \dots, e_n\}$ of V. Note that $H_\xi(V) \subseteq V$ and $H_\xi(\bar{V}) \subseteq \bar{V}$; at the same time, $S_\xi(V) \subseteq \bar{V}$ and $S_\xi(\bar{V}) \subseteq V$. If we extend the Codazzi equation linearly to all complexified tangent vectors and take the $(1,0)$ and $(0,1)$ parts in (2.5), then

$$\nabla_X(H_\xi Y) - \nabla_Y(H_\xi X) - H_{\nabla^\perp_X \xi} Y + H_{\nabla^\perp_Y \xi} X - H_\xi[X, Y] = 0, \tag{2.7}$$

$$\nabla_X(S_\xi Y) - \nabla_Y(S_\xi X) - S_{\nabla^\perp_X \xi} Y + S_{\nabla^\perp_Y \xi} X - S_\xi[X, Y] = 0, \tag{2.8}$$

and

$$\nabla_{\bar{Y}}(S_\xi X) - S_{\nabla^\perp_{\bar{Y}} \xi} X - S_\xi(\nabla_{\bar{Y}} X) = \nabla_X(H_\xi \bar{Y}) - H_{\nabla^\perp_X \xi} \bar{Y} - H_\xi(\nabla_X \bar{Y})$$

for any type-$(1,0)$ vector fields X, Y on M and any normal field ξ. In the minimal case—that is, when $H = 0$—we have

$$S_{\nabla^\perp_{\bar{Y}} \xi} X = \nabla_{\bar{Y}}(S_\xi X) - S_\xi(\nabla_{\bar{Y}} X) \tag{2.9}$$

for any ξ in N and any X, Y in V.

3. The Algebraic Lemma

In this paper we are primarily interested in the case when $p = 4$ and $r > 4$, although some of the arguments work also in more general cases. Our first objective is to show that, at a generic point x in M^n, the second fundamental form takes a rather special form. We shall begin with the following definition.

Definition. Let $V \cong \mathbb{C}^n$ and $N \cong \mathbb{R}^p$ be equipped with inner products, and let H (resp. S) be a Hermitian (resp., symmetric) bilinear map from V into $N_{\mathbb{C}} = N \otimes \mathbb{C}$ satisfying the symmetry conditions (2.1)–(2.3). Let E be a subspace of N.

A *compatible almost complex structure* J on E is an isometry from E onto itself such that $J^2 = -I$ and, for any $\eta \in E$, $H^\eta = 0$ and $S^{J\eta} = -\sqrt{-1}S^\eta$.

Here we have written $H^\eta = \langle H, \eta \rangle$ and $S^\eta = \langle S, \eta \rangle$. Note that E is necessarily even-dimensional and that the condition on J is equivalent to $A_{J\eta} = JA_\eta$ for any $\eta \in E$. Here A_η is the shape operator related to the shape form A^η by the metric on $T \cong V$, which in turn is related to H^η and S^η by (2.4).

We will assume that the dimension p of N is the smallest; thus, for any $\eta \neq 0$ in N, either H^η or S^η is not zero. This is equivalent to $A_\eta \neq 0$ for any $\eta \neq 0$ in N. Note that, under this assumption, the compatible almost complex structure on any subspace E of N (if it exists) must be unique. To see this, suppose J and J' are both compatible almost complex structures on $E \subseteq N$. Then for any $\eta \in E$ we have $H^\eta = 0$ and $S^{J\eta} = -\sqrt{-1}S^\eta = S^{J'\eta}$, so $S^{J\eta - J'\eta} = 0$. Therefore, if $J \neq J'$ then by (2.4) there is an $\eta \neq 0$ in E such that $A_\eta = 0$, which contradicts our assumption that p is the smallest.

As a consequence of this uniqueness, we know that if E_1 and E_2 are both subspaces of N admitting compatible almost complex structures, then both $E_1 \cap E_2$ and $E_1 + E_2$ also admit compatible almost complex structures. Hence there is always a (unique yet possibly trivial) maximal subspace E in N that is equipped with a compatible almost complex structure. We will call this subspace E the *complex part* of N.

Let E' be the orthogonal complement of the complex part E in N, and write $S' = \langle S, E' \rangle$. Then, by the definition of compatible almost complex structure, we know that S' again satisfies (2.3). Also, if S^η has rank at most 1 then, in $\{\eta\}^\perp$, S also satisfies (2.3). Our main goal in this section is to prove the following lemma.

ALGEBRAIC LEMMA. *Let $V \cong \mathbb{C}^r$ and $N \cong \mathbb{R}^4$ be equipped with inner products, and let H and S be* (*respectively*) *Hermitian and symmetric bilinear forms from V into $N_{\mathbb{C}}$ satisfying symmetry conditions* (2.1)–(2.3). *Suppose* $\ker(H) \cap \ker(S) = 0$ *and* $r > 4$. *Then N has nontrivial complex part. In other words, either N itself or a 2-dimensional subspace E in N admits a compatible almost complex structure. Furthermore, in the latter case we have*

$$\dim(\ker(H) \cap \ker(S')) \geq r - 2,$$

where $S' = \langle S, E' \rangle$ and E' is the orthogonal complement of E in N.

Proof. Since H is Hermitian, its image space is in the form $N'_{\mathbb{C}} = N' \otimes \mathbb{C}$ for some real linear subspace $N' \subseteq N$. Let $N = N' \oplus N''$ be the orthogonal decomposition and write $H = (H', H'')$ and $S = (S', S'')$ under this decomposition. We have $H'' = 0$ by definition. Denote by p' and $q = 4 - p'$ the respective dimensions of N' and N''.

Let V_0 be the kernel of H and $V = V_0 \oplus V_1$ the orthogonal decomposition. Write $r_i = \dim_{\mathbb{C}} V_i$ for $i = 0, 1$. Note that for any $X \in V_0$ we have $H_{X\bar{*}} = 0$ and so, by (2.2), $\langle S_{XY}, H_{*\bar{*}} \rangle = 0$ and thus $S'_{XY} = 0$ for any $Y \in V$. Hence $V_0 \subseteq \ker(S')$.

From the discussion in [11], we know that $r_1 \leq p'$ and that equality would imply that H' and S' can be simultaneously diagonalized. In particular, $p' = 4$ cannot occur because $r \geq 5$ and similarly for $p' = 3$. The reason is that, in this case, the rank of S' is at most $r_1 \leq 3$. The inequality $r \geq 5$ and the symmetry condition (2.3) imply that S'' (and thus S) has a zero eigenvector within V_0, in contradiction to $\ker(H) \cap \ker(S) = 0$ in V. Therefore, $p' \leq 2$.

If $p' = 2$, then r_1 is necessarily 2 and we are in the diagonal situation. That is, we will have orthonormal bases $\{\xi_1, \xi_2\}$ of N' and $\{e_1, e_2\}$ of V_1 such that $V_0 = \ker(H) \cap \ker(S')$ and such that, along V_1, we have the matrices

$$H^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad S^1 = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}.$$

Observe that both S^1 and S^2 have rank ≤ 1, so the symmetric bilinear form S'' from V into $N'' \cong \mathbb{R}^2$ satisfies (2.3) as well. The kernel of S'' cannot overlap with V_0, so its rank is at least 3. By Lemma 1 to follow, we know that N'' admits a compatible almost complex structure.

If $p' = 1$ then necessarily $r_1 = 1$, so V_1 is 1-dimensional and both H' and S' are zero in the codimension-1 subspace V_0 of V. Since S' is a matrix of rank ≤ 1, it follows that the remaining part S'' will satisfy (2.3) and that its rank is at least 4. So by Lemma 1, N'' contains a 2-dimensional subspace E that admits a compatible almost complex structure. Let $0 \neq \eta \in N''$ be perpendicular to E. Then S^η again satisfies (2.3), so its rank is at most 1. Putting η together with N' to form the space E', we know that the common kernel of H and S on E' has dimension at least $r - 2$.

Finally, when $p' = 0$, we are left with S from V into $N = \mathbb{R}^4$ satisfying (2.3) and with rank at least 5. So by Lemma 1 we know that either N itself admits a compatible almost complex structure or N contains a 2-dimensional subspace E that does. Let $E' = E^\perp$ in N. Since $S' = \langle S, E' \rangle$ also satisfies (2.3), if S does not admit an almost complex structure then (by Lemma 1) it must have rank ≤ 2; thus, $\dim(\ker(S)) \geq r - 2$. This completes the proof of the Algebraic Lemma. $\square$

LEMMA 1. *Let $V \cong \mathbb{C}^r$ and $N \cong \mathbb{R}^p$ be equipped with inner products. Write $N_{\mathbb{C}} = N \otimes \mathbb{C}$. Let $S \colon V \times V \to N_{\mathbb{C}}$ be a symmetric bilinear map satisfying* (2.3) *and with* $\ker(S) = 0$. *If $p \leq 4$ and $r > p$, then there exist $X, Y \in V$ such that $S_{XY} \neq 0$ and $\langle S_{XY}, S_{ZW} \rangle = 0$ for any $Z, W \in V$. In other words, N always has nontrivial complex part.*

Proof. The $p = 2$ case is due to Dajczer [4] and the $p = 3$ case is due to Dajczer and Gromoll [9] (although their notation is quite different from that used here). We shall prove only the $p = 4$ case because the same argument would work also for the $p = 2$ and $p = 3$ cases. Without loss of generality, we may assume that $r = 5$ (given that, when $r > 5$, we can simply apply the result to any 5-dimensional subspace of V).

For $X \in V$, consider the linear map $\phi_X \colon V \to N_{\mathbb{C}}$ sending Y to S_{XY}. Denote by K_X the kernel of ϕ_X and by k_X its complex dimension. Since $V \cong \mathbb{C}^5$, $N_{\mathbb{C}} \cong \mathbb{C}^4$, and $\ker(S) = 0$, we have $1 \leq k_X \leq 4$.

Let k be the minimum of k_X for all $X \in V$, and denote by V_0 the open dense subset of V consisting of all X with $k_X = k$. We will also write $m = 5 - k$, which is the dimension of the image of ϕ_X and ranges from 1 to 4. Note that the set $\Sigma = \{X \in V \mid S_{XX} = 0\}$ is the intersection of four quadratic hypersurfaces in V, so $V_0' = V_0 \setminus \Sigma$ is still open dense in V.

Fix any $X \in V_0'$. Let $\{e_1, \dots, e_5\}$ be a basis of V such that $e_1 = X$; $\{e_{m+1}, \dots, e_5\}$ forms a basis of K_X. Again we write S_{ij} for $S_{e_i e_j}$. The frame $\{S_{11}, \dots, S_{1m}\}$ forms a basis of the image space $P = \phi_X(V)$. We will denote by Q the subspace of $N_{\mathbb{C}}$ spanned by $S_{i\alpha}$ for all $1 \le i \le 5$ and all $m < \alpha \le 5$. That is, $Q = S(K_X \times V)$. Since $S_{1\alpha} = 0$, the symmetry condition (2.3) implies that $\langle P, Q \rangle = 0$.

We claim that $Q \subseteq P$, so assume the contrary. Then there will some $m < \alpha \le 5$ and some $1 \le i \le 5$ such that $S_{i\alpha}$ is not contained in P. Consider the vector $Y = e_1 + \lambda e_i$ for a sufficiently small λ. Then $S_{Y\alpha} = \lambda S_{i\alpha}$ and we have

$$S_{Y1} \wedge \cdots \wedge S_{Ym} \wedge S_{Y\alpha} = \lambda(S_{11} \wedge \cdots \wedge S_{1m} \wedge S_{i\alpha} + O(\lambda)),$$

whose leading term is not zero. So for a sufficiently small value of λ, the image of ϕ_Y has dimension exceeding m, a contradiction. This proves that $Q \subseteq P$, and we have $Q \neq 0$ because $\ker(S) = 0$.

If $m = 1$ then $Q = P$, so $0 \neq S_{11} \in P = Q$ satisfies $\langle S_{11}, S_{ij} \rangle = 0$ for any i, j. If $m = 2$ then, since we can take $e_2 \in V_0'$ also, both K_1 and K_2 are of codimension 2; thus there will be $0 \neq Z \in K_1 \cap K_2$. Take W such that $S_{ZW} \neq 0$. Then $S_{ZW} \in Q$ and $\langle S_{ZW}, S_{22} \rangle = 0$, so $\langle S_{ZW}, S_{ij} \rangle = 0$ for any i, j. On the other hand, since $\langle P, Q \rangle = 0$, we know that P is contained in the orthogonal complement of $\bar{Q}$ in $N_{\mathbb{C}}$; hence $m \le 3$. From now on, we will assume that $m = 3$.

Note that if there exist $\alpha, \beta \in \{4, 5\}$ such that $S_{\alpha\beta} \neq 0$ then, since $\langle Q, Q \rangle = 0$, by (2.3) it would follow that

$$\langle S_{\alpha\beta}, S_{ij} \rangle = \langle S_{\alpha i}, S_{\beta j} \rangle = 0$$

for any $i, j \le 3$. So $S_{\alpha\beta}$ will give us the proof of the lemma. In other words, if for some $X \in V_0'$ we have $S(K_X \times K_X) \neq 0$, then any nonzero element S_{ZW} in this subspace satisfies $\langle S_{ZW}, S_{ij} \rangle = 0$ for all i, j. Hence we may further assume that $S(K_X \times K_X) = 0$ for all $X \in V_0'$. We show that this will not be possible under any circumstances, thus completing the proof of the lemma.

Since V_0' is open dense in V, we may assume that e_2 and e_3 are also in V_0'. Consider their respective kernels K_2 and K_3. If they are both equal to K_1 then e_4 will be in the kernel of S, a contradiction. So we must have one of them, say K_2, not equal to K_1. Since Q has dimension 1, S_{24} and S_{25} must be proportional to each other. Replacing $\{e_4, e_5\}$ by another basis of K_1 if necessary, we may assume that $S_{24} = 0$. On the other hand, since $K_2 \neq K_1$, we may replace e_3 by another vector in K_2. So $K_2 = \text{span}\{e_3, e_4\}$. Since $e_2 \in V_0'$, we know that $S(K_2 \times K_2) = 0$ (unless the lemma holds). However, this means $S_{34} = S_{44} = 0$. But we already have $S_{14} = S_{54} = 0$ since $e_4 \in K_1$; hence $e_4 \in \ker(S)$, a contradiction once again. This completes the proof of Lemma 1. □

4. The Extension Theorems

Now we consider a real Kähler submanifold $f \colon M^n \to \mathbb{R}^{2n+4}$ of codimension 4. Reducing M to a connected component U of an open dense subset U' of M if necessary, we may assume that both Δ and Δ_0 are of constant dimensions and are distributions. We will also assume that, at any $x \in M$, the shape operator $A_\xi \neq 0$ for any $\xi \neq 0$. Note that the vanishing of some shape operator everywhere would mean that the codimension can be reduced. By the Algebraic Lemma proved in Section 3, either the entire normal bundle N or a rank-2 subbundle $E \subseteq N$ admits a compatible almost complex structure.

We will call a compatible almost complex structure J on E an *admissible almost complex structure* if

$$J(\nabla_v^\perp \xi)^E = (\nabla_v^\perp J\xi)^E \tag{4.1}$$

holds for any $\xi \in E$ and any vector field v in M. Here $(W)^E$ stands for the E component of W.

Notice that if E has rank 2 then any compatible almost complex structure J on E is automatically admissible. To show this, let $\{\xi_1, \xi_2\}$ be a local orthonormal frame of E with $\xi_2 = J\xi_1$. Equation (4.1) reduces to

$$J(\langle \nabla^\perp \xi_1, \xi_2 \rangle \xi_2) = \langle \nabla^\perp \xi_2, \xi_1 \rangle \xi_1$$

or, equivalently,

$$\langle \nabla^\perp \xi_1, \xi_2 \rangle = -\langle \nabla^\perp \xi_2, \xi_1 \rangle,$$

which always holds.

When N itself admits an admissible almost complex structure J, our goal is to show that M^n is actually a holomorphic submanifold in $\mathbb{C}^{n+2}$.

Proposition 1. *Let $f\colon M^n \to \mathbb{R}^{2n+4}$ be a real Kähler submanifold whose normal bundle admits an admissible almost complex structure. Then there exists an isometric identification $\sigma\colon \mathbb{R}^{2n+4} \cong \mathbb{C}^{n+2}$ such that $\sigma \circ f$ is a holomorphic isometric embedding.*

We will prove the proposition later in this section.

In the case of a rank-2 subbundle E of N admitting a compatible (and thus admissible) almost complex structure, we would like to show that M^n is a complex submanifold of another complex manifold Q^{n+1} and that this Q^{n+1} is a codimension-2 real Kähler submanifold of which M is the restriction. We will call such a Q^{n+1} a *Kähler extension* of M^n. To prove this extension theorem, we require more information about the behavior of the second fundamental form beyond the existence of the compatible almost complex structure on E. It turns out that what is needed here is the following data.

Definition. A *developable ruling* in $E \oplus T$ is a rank-2 subbundle L of $E \oplus T$ such that $L + T = E \oplus T$ and $\langle \tilde{\nabla} L, E' \rangle = 0$ along M. Here T is the tangent bundle of M, E' is the orthogonal complement of E in the normal bundle N, and $\tilde{\nabla}$ is the covariant differentiation of the ambient Euclidean metric.

Note that the subbundle L is necessarily transversal to T but is not, in general, contained in N. We will prove the following extension theorem.

PROPOSITION 2. *Let $f : M^n \to \mathbb{R}^{2n+4}$ be a real Kähler submanifold. If there exist a rank-2 subbundle E of the normal bundle N, a compatible almost complex structure J on E, and a developable ruling L in $E \oplus T$, then there exist a real Kähler submanifold $h : Q^{n+1} \to \mathbb{R}^{2n+4}$ and a holomorphic embedding $\sigma : M^n \to Q^{n+1}$ such that $f = h \circ \sigma$.*

Proof. Let $z = (z_1, \dots, z_n)$ be a local holomorphic coordinate in M and $\{\xi_1, \dots, \xi_4\}$ an orthonormal frame of N such that $\{\xi_1, \xi_2\}$ spans E' and $\{\xi_3, \xi_4\}$ spans E. Write $P = E \oplus T$. Because $L + T = P$, there will be a local frame of L given by

$$\eta_1 = \xi_3 - v_1 \quad \text{and} \quad \eta_2 = \xi_4 - v_2,$$

where v_1 and v_2 are real vector fields of M. Since $\langle \tilde{\nabla} L, E' \rangle = 0$, we know that

$$\tilde{\nabla}_v \eta_i \in P = L + T \tag{4.2}$$

for $i = 1, 2$ and for any vector field v in M.

Let $B \subseteq \mathbb{C}$ be a sufficiently small disc and let $t = t_1 + \sqrt{-1}t_2$ be the coordinate. Define a $(2n + 2)$-dimensional submanifold $h : Q \to \mathbb{R}^{2n+4}$ by

$$h(z, t) = f(z) + t_1\eta_1(z) + t_2\eta_2(z).$$

Since L is transversal to T, for sufficiently small values of $|t|$ the map h is an embedding. The manifold Q is ruled along the directions of L. By (4.2) the bundle E', which is the normal bundle of Q, is constant along each leave of L; thus Q is a developable submanifold (which means that its tangent space is constant along each ruling). Along the submanifold M of Q, the restriction of the tangent bundle $TQ|_M$ is simply $P = L + T$. Since $P = E \oplus T$ and since we have an almost complex structure J on both T and E, we can take their direct sum to get an almost complex structure on P. Now taking parallel translation along leaves of L yields an almost complex structure on TQ, which we also denote by J.

To show that Q is a Kähler manifold under the restriction of the Euclidean metric, it suffices to show that $\hat{\nabla} J = 0$ on Q for $\hat{\nabla}$ the connection on Q (viz., the Q component of $\tilde{\nabla}$). That is, we need only show that

$$\hat{\nabla}_Z(JW) = J(\hat{\nabla}_Z W) \tag{4.3}$$

for any two vector fields Z and W in Q. Since TQ is the parallel translation in $\mathbb{R}^{2n+4}$ of $TQ|_M = P$ along the leaves of L and since J is also defined by parallel translation along leaves of L, we just need to verify (4.3) at points in M and with Z tangent to M. If W is also tangent to M, then the equation holds in the tangential component of M because M is Kähler. For the normal components, we are concerned only with those within Q; hence we need only verify that, for the ξ_3 and ξ_4 directions,

$$\langle \hat{\nabla}_Z(JW), \xi_i \rangle = \langle J(\hat{\nabla}_Z W), \xi_i \rangle$$

for $i = 3, 4$, where Z and W are vector fields in M. This expression is equivalent to

$$JA_{\xi_i} = A_{J\xi_i} \tag{4.4}$$

for $i = 3,4$. Since $H^{\xi_3} = H^{\xi_4} = 0$, it follows that $S^{\xi_3} = \sqrt{-1}S^{\xi_4}$ and so, by (2.4),

$$JA_{\xi_3} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} R_3 & -I_3 \\ -I_3 & -R_3 \end{pmatrix} = \begin{pmatrix} I_3 & R_3 \\ R_3 & -I_3 \end{pmatrix} = A_{\xi_4}.$$

Here we have written $S^{\xi_3} = R_3 + \sqrt{-1}I_3$ and $S^{\xi_4} = R_4 + \sqrt{-1}I_4$, so $R_3 = -I_4$ and $I_3 = R_4$. Recall that we defined J on E by $J\xi_3 = \xi_4$ and $J\xi_4 = -\xi_3$. So (4.4) holds.

Now we are left with the case where Z is a tangent vector field of M and W is a section of E, since $P = E \oplus T$. By the linearity of J and the Leibniz formula, we need to check this just for $W = \xi_3$ and $W = \xi_4$:

$$\hat{\nabla}_Z(\xi_4) = J(\hat{\nabla}_Z\xi_3) \tag{4.5}$$

for any tangent vector field Z in M. First let us compare the tangential components on both sides. This expression reduces once again to (4.4). For the normal components in (4.5), observe that $\hat{\nabla}$ is just the TQ component of $\tilde{\nabla}$ and so

$$(\hat{\nabla}_Z\xi_3)^\perp = \langle\hat{\nabla}_Z\xi_3, \xi_4\rangle\xi_4 = \langle\nabla_Z^\perp\xi_3, \xi_4\rangle\xi_4 = -\langle\xi_3, \nabla_Z^\perp\xi_4\rangle\xi_4,$$

$$(\hat{\nabla}_Z\xi_4)^\perp = \langle\hat{\nabla}_Z\xi_4, \xi_3\rangle\xi_3 = \langle\nabla_Z^\perp\xi_4, \xi_3\rangle\xi_3.$$

Thus $(J\hat{\nabla}_Z\xi_3)^\perp = J((\hat{\nabla}_Z\xi_3)^\perp) = (\hat{\nabla}_Z\xi_4)^\perp$, which proves the Kählerness of the codimension-2 submanifold Q in the Euclidean space. The holomorphicity of M in Q is obvious, since we defined our J on Q in such a way that its restriction on M comes from the complex structure. This completes the proof of Proposition 2. □

For the Kähler extension h obtained in Proposition 2, it is clear that if h is minimal then f is necessarily minimal. Conversely, when f is minimal, we would like to know when h will be minimal.

PROPOSITION 3. *Let f, (E, J), and L be as in Proposition 2, and let h be the Kähler extension of f obtained by L. If f is minimal, then h is minimal if and only if $(v_2 - Jv_1) \in \ker(A_{\xi_1}) \cap \ker(A_{\xi_2})$. Here $\{\xi_1, \dots, \xi_4\}$ is an orthonormal frame of N, $\{\xi_3, \xi_4\}$ is a frame of E, $\xi_4 = J\xi_3$, and $v_1, v_2 \in T$ are determined (uniquely) by the condition that $\{\xi_3 - v_1, \xi_4 - v_2\}$ spans L.*

Proof. Note that ξ_1 and ξ_2 span the normal bundle of Q in $\mathbb{R}^{2n+4}$ and that h is minimal if and only if its $H = 0$—or, equivalently, that $J\hat{A}_{\xi_\alpha} = \hat{A}_{\xi_\alpha}J$ for $\alpha = 1$ and 2, where J is the almost complex structure of Q and $\hat{A}$ is the shape operator of Q. That is, for $1 \le \alpha \le 2$ and any vector fields Z, W on Q, we have

$$\langle J\hat{A}_{\xi_\alpha}Z, W\rangle = \langle\hat{A}_{\xi_\alpha}JZ, W\rangle$$

or, equivalently,

$$-\langle\tilde{\nabla}_Z JW, \xi_\alpha\rangle = \langle\tilde{\nabla}_{JZ}W, \xi_\alpha\rangle. \tag{4.6}$$

By the construction of h, TQ is the parallel translate of $TQ|_M$ along the leaves of L, and J and both ξ_α are parallel along each leaf of L. Therefore, we need only check (4.6) at points in M and for Z a vector field in M.

Since $TQ|_M = E \oplus T$, we must verify (4.6) only for W a vector field in M and a section of E. In the former case, (4.6) is just the minimality of f. In the latter case, when W is a section of E, (4.6) becomes

$$\langle JW, \tilde{\nabla}_Z \xi_\alpha \rangle = -\langle W, \tilde{\nabla}_{JZ} \xi_\alpha \rangle \tag{4.7}$$

for each $\alpha = 1, 2$. Clearly, (4.7) must be verified only for $W = \xi_3$.

Now suppose that $\xi_3 - v_1$ and $\xi_4 - v_2$ span L and that $\xi_4 = J\xi_3$. Since L is transversal to T, the map $\pi|_L \colon L \to E$ is bijective; here π is the projection map from $E \oplus T$ onto E. Thus v_1, v_2 are uniquely determined by the choice of $\{\xi_3, \xi_4\}$. By the definition of developable ruling, we know that $\langle \tilde{\nabla} \xi_\alpha, L \rangle = 0$. Therefore,

$$\langle \xi_4, \tilde{\nabla}_Z \xi_\alpha \rangle = \langle v_2, \tilde{\nabla}_Z \xi_\alpha \rangle = \langle A_{\xi_\alpha}(v_2), Z \rangle \quad \text{and}$$
$$\langle \xi_3, \tilde{\nabla}_{JZ} \xi_\alpha \rangle = \langle v_1, \tilde{\nabla}_{JZ} \xi_\alpha \rangle = \langle A_{\xi_\alpha}(v_1), JZ \rangle = \langle A_{\xi_\alpha}(Jv_1), Z \rangle.$$

Note that in the last equality we used the minimality of M: we always have $JA = -AJ$. Plugging these two equalities into (4.7) for $W = \xi_3$, we get

$$\langle A_{\xi_\alpha}(v_2 - Jv_1), Z \rangle = 0$$

for any vector field Z in M; that is,

$$A_{\xi_\alpha}(v_2 - Jv_1) = 0, \quad \alpha = 1, 2. \tag{4.8}$$

So when f is minimal, h will be minimal if and only if $v_2 - Jv_1$ belongs to $\ker(A_{\xi_1}) \cap \ker(A_{\xi_2})$, which is the real subspace of T corresponding to $\ker(S')$ in V. Here $S' = (S^1, S^2)$. This completes the proof of Proposition 3. $\square$

REMARK. We denote by $\pi : E \oplus T \to E$ the projection map and by $\tau : E \to L$ the inverse of the restriction map $\pi|_L \colon L \to E$. Then the condition stated in Proposition 3 can be rephrased as

$$\tau(J\eta) - J\tau(\eta) \in \ker(A_{\xi_1}) \cap \ker(A_{\xi_2}) \tag{4.9}$$

for any η in E. Here $\{\xi_1, \xi_2\}$ is a basis of E', the orthogonal complement of E in N.

Proof of Proposition 1. Note that in this case the ambient Euclidean space is automatically a developable submanifold (of itself) over M, with fibers of the normal bundle N as rulings' leaves. Define an almost complex structure J on $T \oplus N$ by taking the direct sum of the almost complex structure of M with the given one on N, and use parallel translation along leaves of N to push it to a small tubular neighborhood Ω of M; the result is an almost complex structure J on the open subset Ω of $\mathbb{R}^{2n+4}$. Clearly, J is an isometry. One can see that $\tilde{\nabla} J = 0$, just as in the proof of Proposition 2, with the aid of (4.1). Thus J comes from an isometric identification $\mathbb{R}^{2n+4} \cong \mathbb{C}^{n+2}$ and M becomes a complex submanifold with complex codimension 2. This completes the proof of Proposition 1. $\square$

Next we show that, if the normal bundle N admits a compatible almost complex structure J, then N must be admissible.

PROPOSITION 4. *Let $f: M^n \to \mathbb{R}^{2n+4}$ be a real Kähler submanifold such that there is a compatible almost complex structure J on N. Assume that no shape operator vanishes and that the rank $r \geq 2$ everywhere. Then J is admissible; that is, for any tangent vector v and any normal field ξ,*

$$\nabla_v^{\perp} J\xi = J\nabla_v^{\perp}\xi. \tag{4.10}$$

Proof. Let us choose a local orthonormal frame $\{\xi_1, \dots, \xi_4\}$ for the normal bundle N, so that $\xi_3 = J\xi_1$ and $\xi_4 = J\xi_2$. For any $1 \leq \alpha, \beta \leq 4$, we denote by $\phi_{\alpha\beta}$ the real 1-form on M given by $\langle \nabla^{\perp}\xi_\alpha, \xi_\beta\rangle$. Write the 4×4 real, skew-symmetric matrix $\phi = (\phi_{\alpha\beta})$ in 2×2 blocks as

$$\phi = \begin{pmatrix} \phi^1 & \phi^2 \\ -{}^t\phi^2 & \phi^3 \end{pmatrix}.$$

It is easy to see that (4.10) is equivalent to $\phi^1 = \phi^3$ and ${}^t\phi^2 = \phi^2$. Write

$$(\phi^1 - \phi^3) + \sqrt{-1}({}^t\phi^2 - \phi^2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\lambda.$$

Then it suffices to show that $\lambda = 0$. Let $\{e_1, \dots, e_n\}$ be a unitary frame of V, and let $\{\varphi_1, \dots, \varphi_n\}$ be its dual coframe of $(1,0)$-forms on M. Write $\langle \tilde{\nabla} e_i, \xi_\alpha\rangle = \psi_i^\alpha$; then, since $H = 0$, it follows that each

$$\psi_i^\alpha = \sum_{j=1}^{n} S_{ij}^\alpha \varphi_j$$

is a $(1,0)$-form. We use ψ^α to denote the column vector ${}^t(\psi_1^\alpha, \dots, \psi_n^\alpha)$ and write

$$\psi = (\psi'; \psi'') = (\psi^1, \psi^2; \psi^3, \psi^4).$$

By our choice of the normal frame, we have $\psi'' = -\sqrt{-1}\psi'$ and so

$$\psi = (\psi', -\sqrt{-1}\psi'). \tag{4.11}$$

The connection matrix of $\tilde{\nabla}$ under the frame $\{e, \bar{e}, \xi\}$ is

$$\tilde{\theta} = \begin{pmatrix} \theta & 0 & \psi \\ 0 & \bar{\theta} & \bar{\psi} \\ -{}^t\bar{\psi} & -{}^t\psi & \phi \end{pmatrix}.$$

Applying (4.11) to the Codazzi equation $d\psi = \theta\psi + \psi\phi$ yields two equations. Multiplying the second equation by $\sqrt{-1}$, and taking its difference with respect to the first equation, we obtain

$$\psi'\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\lambda = 0$$

or, equivalently, $\psi^1 \wedge \lambda = \psi^2 \wedge \lambda = 0$. We claim that this will force $\lambda = 0$, thereby proving Proposition 4. Write $\lambda = \sum_k (a_k\varphi_k + b_k\overline{\varphi_k})$. By the equation just displayed, for each i and each α we have

$$\sum_{j,k=1}^{n} S^{\alpha}_{ij} a_k \varphi_j \wedge \varphi_k + \sum_{j,k=1}^{n} S^{\alpha}_{ij} b_k \varphi_j \wedge \overline{\varphi_k} = 0.$$

The second summation implies that $S^{\alpha}_{ij} b_k = 0$ for any i, j, k, whence $b_k = 0$ for all k; the first implies that $S^{\alpha}_{ij} a_k = S^{\alpha}_{ik} a_j$ for any α and any i, j, k. Since M has rank $r \geq 2$, there will be some combination $S = \sum t_\alpha S^\alpha$ such that S is a complex symmetric matrix of rank ≥ 2. Take a unitary matrix P such that ${}^tP^{-1}SP^{-1} = D = \operatorname{diag}(d_1, \dots, d_n)$ is diagonal with $d_1 d_2 \neq 0$. Then $S = {}^tPDP$, and $S_{ij} a_k = S_{ik} a_j$ for any i, j, k becomes

$$d_l P_{lj} a_k = d_l P_{lk} a_j$$

for any l, j, k. Taking $l = 1$ and 2, we observe that if the a_k are not all zero then the first two rows of P will be proportional—a contradiction. Therefore, we must have $a_k = 0$ for all k. This completes the proof of Proposition 4. □

5. Proof of the Main Theorem

In this section, we will prove the main theorem. For $x \in M$, let us denote by $N_0(x)$ the subspace of N_x consisting of all η with $A_\eta = 0$. Note that the presence of normal directions in which the shape operator vanishes would mean that the codimension can be reduced (see [19, Prop. 24]). In the interior part U_0 of the set where $N_0 \neq 0$ there will be an open dense subset of U_0 such that, within each connected component of it, the submanifold M will be real Kähler submanifold with smaller codimensions. Since the main theorem is known for codimension 3 and less, hereafter we will assume that:

$N_0 = 0$ everywhere in M; that is, $A_\eta \neq 0$ for any $\eta \neq 0$.

First let us consider the nonminimal case. In other words, we restrict ourselves to the open subset of M in which $H \neq 0$ (if that set is nonempty). Since $r \geq 5$, we know that the image of H is either 1- or 2-dimensional. In the open subset U_2 where H has 2-dimensional image space E', there are exactly two directions, perpendicular to each other, in which H has rank 1. Let ξ_1 and ξ_2 be the unit vectors in those two directions; they are unique up to ± 1 and interchange. In this case, we can use (2.2) to diagonalize S^{ξ_1} and S^{ξ_2} accordingly.

In the open subset $M \setminus \overline{U_2}$, the image of H is 1-dimensional and we will let ξ_1 be the unit vector in this direction (unique up to a sign).

In both cases, by (2.4) and our discussion of the Algebraic Lemma, we know that locally there will be orthonormal frame $\{\xi_1, \dots, \xi_4\}$ such that A_{ξ_1} and A_{ξ_2} are both of rank ≤ 2 and $A_{\xi_4} = JA_{\xi_3}$ has rank ≥ 6. Furthermore, $E' = \operatorname{span}\{\xi_1, \xi_2\}$ because the set of all normal directions in which the shape operator has rank $\leq$ 4 is uniquely determined. Also, by restricting ourselves to a connected component U in an open dense subset of M, we may assume that in U the orthonormal frame $\{\xi_1, \xi_2\}$ of E' is also uniquely determined up to interchange and signs.

Letting $J\xi_3 = \xi_4$ and $J\xi_4 = -\xi_3$ allows us to obtain a compatible almost complex structure on E, the orthogonal complement of E' in N. So to prove the main theorem, it suffices (by Proposition 2) to find a developable ruling L for E. This will follow from the Codazzi equation (2.5) and a rather clever argument devised by Dajczer and Gromoll [9].

Consider $\eta = \xi_1$ or ξ_2, and recall that A_η has rank $q \leq 2$. Denote by Δ_η the kernel of A_η in T and by $\Delta_\eta^\perp$ its orthogonal complement in T; note that $\Delta_\eta^\perp$ is also the image space of A_η. First we make the following claim.

Claim. *For either $\eta = \xi_1$ or $\eta = \xi_2$, the E component of $\nabla_v^\perp \eta$, denoted by $(\nabla_v^\perp \eta)^E$, is always zero for all $v \in \Delta_\eta$. That is, for any $v \in \Delta_\eta$,*

$$\langle \nabla_v^\perp \eta, \xi_3 \rangle = \langle \nabla_v^\perp \eta, \xi_4 \rangle = 0. \tag{5.1}$$

To prove the claim, assume the contrary. Without loss of generality, we may assume that $\eta = \xi_1$ and there is a $v \in \Delta_\eta$ such that $\xi = (\nabla_v^\perp \eta)^E \neq 0$. By (2.5), since $A_\eta v = 0$ we have

$$A_{\nabla_v^\perp \eta} u = A_{\nabla_u^\perp \eta} v + \nabla_v (A_\eta u) + A_\eta [u, v] \tag{5.2}$$

for any $u \in T$. Let $T_\eta = \{u \in T \mid (\nabla_u^\perp \eta)^E = 0\}$. Since E is 2-dimensional, it follows that the codimension of T_η in T is at most 2.

Let $\{e_1, \ldots, e_n\}$ be a frame of V such that $\{e_3, \ldots, e_n\}$ is a unitary frame of $V_0 = \ker(H) \cap \ker(S')$ and is perpendicular to $\{e_1, e_2\}$. We will also assume that $\{e_{r+1}, \ldots, e_n\}$ is a unitary frame of $D \subseteq V$ corresponding to Δ_0, in which case $\{e_1, \ldots, e_r\}$ is a frame of $D^\perp$ that corresponds to $\Delta_0^\perp \cong \mathbb{R}^{2r}$.

Let $W \subseteq T$ be the subspace corresponding to V_0 under the identification $V \cong T$, and note that $W \subseteq \Delta_{\xi_1} \cap \Delta_{\xi_2}$. Now consider the space $W' = W \cap \Delta_0^\perp$. Its real dimension is $2r - 4 \geq 6$ (since $r \geq 5$), so the space $W'' = W' \cap T_\eta$ is at least 4-dimensional because T_η has codimension ≤ 2 in T.

By (5.2) we know that, for any $u \in W''$, $A_\xi u$ is contained in the space

$$\Delta_\eta^\perp + \operatorname{span}\{A_{\xi_2} v\},$$

which has dimension ≤ 3. Hence there will be $0 \neq u_0 \in W''$ such that $A_\xi u_0 = 0$. We have $A_{\xi_1} u_0 = A_{\xi_2} u_0 = 0$ since $u_0 \in W$. On the other hand, since $\xi \neq 0$, we have that $\{\xi, J\xi\}$ spans E; so given $A_{J\xi} = J A_\xi$, we obtain $A_{\eta'} u_0 = 0$ for any normal direction η'. This means that $\alpha_f(u_0, w) = 0$ for any $w \in T$.

If we write $u_0 = X + \bar{X}$ for (a unique) $X \in V$ then, for any $Y \in V$,

$$\alpha_f(u_0, Y) = S_{YX} + H_{Y\bar{X}} = 0 \quad \forall Y \in V.$$

Since $X \in W \subseteq \ker(H)$, it follows that $S_{YX} = 0$ for any Y and so $X \in \ker(S)$ as well. This will force $X = 0$ because we assumed that $u_0 \in \Delta_0^\perp$. Thus $u_0 = 0$, a contradiction, and we have completed the proof of the claim.

From our discussion of the Algebraic Lemma we know that there exists a local frame $\{e_1, \ldots, e_n\}$ of V such that $\{e_3, \ldots, e_n\}$ is a unitary frame of V_0 and is perpendicular to $\{e_1, e_2\}$. Under this local frame, we have

$$H^{\xi_1} = \mathrm{diag}(1, 0, 0, \dots, 0),$$
$$S^{\xi_1} = \mathrm{diag}(a, 0, 0, \dots, 0),$$
$$H^{\xi_2} = \mathrm{diag}(0, \delta, 0, \dots, 0),$$
$$S^{\xi_2} = \mathrm{diag}(0, b, 0, \dots, 0);$$

here $\delta = 0$ or 1 and both a and b are nonnegative. Write $e_i = \varepsilon_{2i-1} - \sqrt{-1}\varepsilon_{2i}$ for $1 \le i \le n$; then, under the real tangent frame $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$, the first two shape forms are given by

$$A^{\xi_1} = \mathrm{diag}(1 + a, 1 - a, 0, 0; 0, \dots, 0),$$
$$A^{\xi_2} = \mathrm{diag}(0, 0, \delta + b, \delta - b; 0, \dots, 0).$$

Our goal is to show that there exist vector fields v_1 and v_2 on M such that $L = \mathrm{span}\{\xi_3 - v_1, \xi_4 - v_2\}$ satisfies $\langle \tilde{\nabla} E', L\rangle = 0$. That is, for any $i, j = 1, 2$ we have

$$\langle \xi_{2+i} - v_i, \tilde{\nabla}\xi_j\rangle = 0$$

or, equivalently,

$$\langle \xi_{2+i}, \nabla_u^{\perp}\xi_1\rangle = \langle v_i, A_{\xi_1}u\rangle \tag{5.3}$$

and

$$\langle \xi_{2+i}, \nabla_u^{\perp}\xi_2\rangle = \langle v_i, A_{\xi_2}u\rangle \tag{5.4}$$

for each $i = 1, 2$ and any u in T.

By the Claim, both sides of (5.3) are zero if u is in the kernel space of A_{ξ_1}, which is spanned by ε_3 through ε_{2n} and also by ε_2 if $a = 1$. So (5.3) just needs to hold for all $u \in \Delta_{\xi_1}^{\perp} = \mathrm{Im}(A_{\xi_1})$.

Similarly, both sides of (5.4) vanish if u is in the kernel of A_{ξ_2}, which is spanned by ε_1, ε_2, and ε_5 through ε_{2n} and also by ε_4 if $\delta = b$. So we just need (5.4) to hold for all $u \in \Delta_{\xi_2}^{\perp} = \mathrm{Im}(A_{\xi_2})$.

Since $\Delta_{\xi_1} + \Delta_{\xi_2} = T$, we must have $\Delta_{\xi_1}^{\perp} \cap \Delta_{\xi_2}^{\perp} = 0$. Hence there is a direct sum decomposition

$$T = (\Delta_{\xi_1} \cap \Delta_{\xi_2}) \oplus \Delta_{\xi_1}^{\perp} \oplus \Delta_{\xi_2}^{\perp},$$

and v_1, v_2 can be uniquely determined in $\Delta_{\xi_1}^{\perp} \oplus \Delta_{\xi_2}^{\perp}$ by (5.3) and (5.4). Yet adding any element of $\Delta_{\xi_1} \cap \Delta_{\xi_2}$ on to v_1 or v_2 would not affect (5.3) or (5.4). This fact establishes the existence of a developable ruling L for E, completing the proof of the main theorem in the nonminimal case.

Now we consider the minimal case, in which $H = 0$ everywhere. By our previous discussion on the Algebraic Lemma, we know that either (a) there exists a 2-dimensional subspace E' of N in which the kernel of S' has codimension $\le$ 2 and the orthogonal complement E admits a compatible almost complex structure J or (b) the entire normal bundle N admits a compatible almost complex structure J. In both cases, the compatible almost complex structure is unique because no shape operator is allowed to vanish.

When N itself is equipped with a compatible almost complex structure J, Proposition 4 states that J is admissible. So by Proposition 1 we know that there is an isometric identification $\mathbb{R}^{2n+4} \cong \mathbb{C}^{n+2}$ under which f becomes a holomorphic map. That is, $f: M^n \to \mathbb{C}^{n+2}$ is a holomorphic isometric embedding. Note that, in this case, any local piece of the holomorphic hypersurface Q^{n+1} containing (a piece of) M^n would be a Kähler extension of M. So the conclusion of the main theorem holds in this case.

We are left with the situation where (a) there exists an orthogonal decomposition $N = E' \oplus E$ such that E is equipped with a compatible almost complex structure J and (b) the kernel of S' is at most 2-dimensional. Here S' is the E' component of S. Write $V_0 = \ker(S')$ and denote by k its codimension; then k is either 1 or 2. Let $\{\xi_1, \dots, \xi_4\}$ be a local orthonormal frame of N such that $\{\xi_1, \xi_2\}$ is a frame of E'. We have $H = 0$ and $S^{\xi_3} = \sqrt{-1}S^{\xi_4}$.

By our previous discussion, we may exclude the possibility that E' is also equipped with an almost complex structure. In other words, we may assume that

$$S^{\xi_1} \neq \pm\sqrt{-1}S^{\xi_2}. \tag{5.5}$$

Also, the symmetry condition (2.3) holds for S' as well. Our goal is to establish the existence of a developable ruling L for E.

We will consider the case $k = 2$ first. Let $\{e_1, \dots, e_n\}$ be a unitary frame of V such that $\{e_3, \dots, e_n\}$ is a frame of $V_0 = \ker(S')$. As in the proof of Proposition 4, we will write

$$\psi_i^\alpha = \langle \tilde{\nabla} e_i, \xi_\alpha \rangle \quad \text{and} \quad \phi_{\alpha\beta} = \langle \nabla^\perp \xi_\alpha, \xi_\beta \rangle;$$

we denote by θ the connection matrix of M under e. We also let $\{\varphi_1, \dots, \varphi_n\}$ be the coframe of $(1,0)$-forms dual to e.

Note that since $\psi_i^\alpha = \sum_{j=1}^n S_{ij}^\alpha \varphi_j$ we have $\psi^3 = \sqrt{-1}\psi^4$, where ψ^α denotes the αth column of ψ. Also, $\psi_i^1 = \psi_i^2 = 0$ for each $i \geq 3$.

By the Codazzi equation $d\psi = \theta\psi + \psi\phi$, we get

$$d\psi^3 = \theta\psi^3 + \psi^1\phi_{13} + \psi^2\phi_{23} + \psi^4\phi_{43},$$
$$d\psi^4 = \theta\psi^4 + \psi^1\phi_{14} + \psi^2\phi_{24} + \psi^3\phi_{34}.$$

Multiplying $-\sqrt{-1}$ on the second line and then adding the result to the first line gives via $\psi^3 = \sqrt{-1}\psi^4$ that

$$0 = \psi^1(\phi_{13} - \sqrt{-1}\phi_{14}) + \psi^2(\phi_{23} - \sqrt{-1}\phi_{24}). \tag{5.6}$$

We put $\sigma_1 = \phi_{13} - \sqrt{-1}\phi_{14}$ and $\sigma_2 = \phi_{23} - \sqrt{-1}\phi_{24}$. Write

$$\psi_1^1 = a\varphi_1 + b\varphi_2, \qquad \psi_2^1 = b\varphi_1 + c\varphi_2,$$
$$\psi_1^2 = a'\varphi_1 + b'\varphi_2, \qquad \psi_2^2 = b'\varphi_1 + c'\varphi_2.$$

Since S' also satisfies the symmetry condition (2.3), it follows that

$$ac - b^2 + a'c' - b'^2 = 0. \tag{5.7}$$

We first claim that both σ_1 and σ_2 must be linear combinations of φ_1 and φ_2. Assume otherwise; then, by (5.6), we must have $\psi_1^1 \wedge \psi_1^2 = 0$ and $\psi_2^1 \wedge \psi_2^2 = 0$.

Hence (a,b) is proportional to (a',b') and (b,c) is proportional to (b',c'). The proportionality constants are equal, too, so we have $S^1 = \lambda S^2$ for some constant λ. Since S' satisfies (2.3), $\lambda^2 = -1$ because we assumed that $k = 2$ there. Then $S^1 = \pm\sqrt{-1}S^2$, in contradiction to (5.5). So the claim must hold, and we can write

$$\sigma_1 = \alpha\varphi_1 + \beta\varphi_2, \qquad \sigma_2 = \alpha'\varphi_1 + \beta'\varphi_2.$$

The first two rows of (5.6) become

$$a\beta - b\alpha + a'\beta' - b'\alpha' = 0, \tag{5.8}$$

$$b\beta - c\alpha + b'\beta' - c'\alpha' = 0. \tag{5.9}$$

We now claim that there exist w_1 and w_2 such that

$$(\alpha,\beta) = w_1(a,b) + w_2(b,c) \tag{5.10}$$

and

$$(\alpha',\beta') = w_1(a',b') + w_2(b',c') \tag{5.11}$$

hold simultaneously. First we assume that $ac - b^2 \neq 0$. Let w_1 and w_2 be uniquely determined by (5.10). In this case,

$$a\beta - b\alpha = w_2(ac - b^2), \qquad b\beta - c\alpha = w_1(b^2 - ac). \tag{5.12}$$

If we write

$$\delta_1 = \alpha' - (w_1a' + w_2b'), \qquad \delta_2 = \beta' - (w_1b' + w_2c'),$$

then

$$a'\beta' - b'\alpha' = w_2(a'c' - b'^2) + (a'\delta_2 - b'\delta_1),$$

$$b'\beta' - c'\alpha' = w_1(b'^2 - a'c') + (b'\delta_2 - c'\delta_1).$$

Adding with (5.12) and then using (5.7)–(5.9) allows us to derive that

$$\begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix}\begin{bmatrix} \delta_2 \\ -\delta_1 \end{bmatrix} = 0.$$

Since $a'c' - b'^2 = -(ac - b^2) \neq 0$, we get $\delta_1 = \delta_2 = 0$ and so both (5.10) and (5.11) hold.

If $ac - b^2 = 0$, then $a'c' - b'^2 = 0$ by (5.7). We claim that in this case (a,b) cannot be proportional to (a',b'). Assume otherwise—say, $(a,b) = \lambda(a',b')$. Since S^1 and S^2 have zero determinants, we also have $(b,c) = \lambda(b',c')$. Hence $S^1 = \lambda S^2$, contradicting $k = 2$, and so the claim holds. Note that the claim implies $\psi_1^1 \wedge \psi_1^2 \neq 0$. If we write $\psi_2^1 = \lambda_1\psi_1^1$ and $\psi_2^2 = \lambda_2\psi_1^2$ then, since $b = \lambda_1 a$ and $b' = \lambda_2 a'$, we know that $\lambda_1 \neq \lambda_2$ by our claim.

By (5.6), we have $\psi_1^1\sigma_1 + \psi_1^2\sigma_2 = 0$ and $\lambda_1\psi_1^1\sigma_1 + \lambda_2\psi_1^2\sigma_2 = 0$. Since $\psi_1^1 \wedge \psi_1^2 \neq 0$, the first equation implies that

$$\sigma_1 = x\psi_1^1 + y\psi_1^2, \qquad \sigma_2 = y\psi_1^1 + z\psi_1^2$$

for some scalar-valued functions x, y, and z. Plugging these sigmas into the second equation yields $y(\lambda_1 - \lambda_2) = 0$, so $y = 0$. Take $w_2 = (x - z)/(\lambda_2 - \lambda_1)$ and $w_1 = x - \lambda_1 w_2$. Then $x = w_1 + \lambda_1 w_2$ and $z = w_1 + \lambda_2 w_2$, so

$$\sigma_1 = w_1\psi_1^1 + w_2\psi_2^1 \quad \text{and} \quad \sigma_2 = w_1\psi_1^2 + w_2\psi_2^2$$

hold simultaneously. That is, (5.10) and (5.11) hold in this case as well.

Note that we have proved, for $k = 2$ and E' not equipped with an almost complex structure, the existence of scalar-valued functions w_1 and w_2 such that $w = w_1e_1 + w_2e_2$ satisfies $\sigma_1 = \psi_w^1$ and $\sigma_2 = \psi_w^2$. In particular, for $\alpha = 1$ and 2 we have

$$\langle \nabla^\perp \xi_\alpha, \xi_3 - \sqrt{-1}\xi_4\rangle = \langle \tilde{\nabla} w, \xi_\alpha\rangle.$$

If we write $w = -v_1 + \sqrt{-1}v_2$ then the displayed equality simply means that $\langle \tilde{\nabla} E', L\rangle = 0$ for the rank-2 subbundle L in $T \oplus E$ spanned by $\{\xi_3 - v_1, \xi_4 - v_2\}$. In other words, L is a developable ruling of E. Thus, by Proposition 2 we obtain a Kähler extension h for f. Observe that, since w is a type-$(1,0)$ vector, we have $v_2 = Jv_1$ in this case. So h is minimal by Proposition 3.

Finally, we consider the $k = 1$ case—namely, when $V_0 = \ker(S')$ has codimension 1. Let $e = \{e_1, \dots, e_n\}$ be a unitary frame of V so that $\{e_2, \dots, e_n\}$ is a frame of V_0. Let φ be the dual coframe of e, and define ψ and ϕ as before. Then $\psi^3 = \sqrt{-1}\psi^4$ and $\psi_i^1 = \psi_i^2 = 0$ for all $i \geq 2$. We write $\psi_1^1 = a\varphi_1$ and $\psi_1^2 = \lambda a\varphi_1$. Then $a \neq 0$ and $\lambda \neq \pm\sqrt{-1}$, since we have excluded the case where S' admits an almost complex structure. By the Codazzi equation for ψ^3 and ψ^4, we again get

$$\psi^1(\phi_{13} - \sqrt{-1}\phi_{14}) + \psi^2(\phi_{23} - \sqrt{-1}\phi_{24}) = \psi^1\sigma_1 + \psi^2\sigma_2 = 0;$$

that is,

$$\varphi_1(\sigma_1 + \lambda\sigma_2) = 0. \tag{5.13}$$

On the other hand, $\psi^4 = -\sqrt{-1}\psi^3$ and so the Codazzi equation for ψ^1 and ψ^2 gives

$$d\psi^1 = \theta\psi^1 - \psi^2\phi_{12} - \psi^3\sigma_1,$$
$$d\psi^2 = \theta\psi^2 + \psi^1\phi_{12} - \psi^3\sigma_2.$$

Now using that $\psi^2 = \lambda\psi^1$, we obtain $d\psi^2 = d\lambda \wedge \psi^1 + \lambda d\psi^1$; hence the preceding two equations yield

$$d\lambda \wedge \psi^1 = (1 + \lambda^2)\psi^1\phi_{12} + \psi^3(\lambda\sigma_1 - \sigma_2).$$

Looking at the ith row of this equation, for any $i \geq 2$ we have

$$\psi_i^3(\lambda\sigma_1 - \sigma_2) = 0 \quad \text{for all } 2 \leq i \leq n.$$

If $\lambda\sigma_1 - \sigma_2 \neq 0$ then ψ_i^3 for all $2 \leq i \leq n$ are multiples of $\lambda\sigma_1 - \sigma_2$, which implies that the lower right $(n-1) \times (n-1)$ corner of S^{ξ_3} will have rank ≤ 1. This result, when combined with the equality $S^{\xi_4} = -\sqrt{-1}S^{\xi_3}$, shows that (S^{ξ_3}, S^{ξ_4}) and hence S must have nontrivial kernel in V_0 because the dimension of V_0 is greater than 2. This contradicts our assumption that the rank of M is at least 5, so we must have

$$\lambda\sigma_1 - \sigma_2 = 0. \tag{5.14}$$

Plugging this into (5.13) and using that $1 + \lambda^2 \neq 0$, we obtain $\varphi_1\sigma_1 = 0$; hence

$$\sigma_1 = w\psi_1^1 \quad \text{and} \quad \sigma_2 = \lambda\sigma_1 = w\psi_1^2$$

for some w. If we write $we_1 = -v_1 + \sqrt{-1}v_2$ for v_1 and v_2 real, then

$$\langle \tilde{\nabla} E', \xi_3 - v_1 \rangle = \langle \tilde{\nabla} E', \xi_4 - v_2 \rangle = 0.$$

In other words, $L = \operatorname{span}\{\xi_3 - v_1, \xi_4 - v_2\}$ gives a developable ruling for E. Note that, just as in the $k = 2$ case, here we have $v_2 = Jv_1$ and so h is minimal by Proposition 3. This finishes the proof of the $k = 1$ case, completing the proof of the main theorem.

REMARK. In both the minimal and nonminimal cases, the Kähler extension is not necessarily unique—at least as we have defined it—because one can add any vector fields in $\ker(A_{E'})$ onto v_1, v_2 and thereby obtain different developable rulings L. However, except when M^n is a complex submanifold of complex codimension 2 in $\mathbb{C}^{n+2}$, there is always a "canonical" way to choose the developable ruling L: take L such that v_1 and v_2 belong to the orthogonal complement of $\ker(A_{E'})$. This uniqueness of canonical extensions might become important in the discussion of global situations, when M is assumed to be complete.

References

[1] S. Brendle and R. Schoen, *Classification of manifolds with weakly 1/4-pinched curvatures,* Acta Math. 200 (2008), 1–13.

[2] ———, *Manifolds with 1/4-pinched curvature are space forms,* J. Amer. Math. Soc. 22 (2009), 287–307.

[3] E. Calabi, *Isometric imbedding of complex manifolds,* Ann. of Math. (2) 58 (1953), 1–23.

[4] M. Dajczer, *A characterization of complex hypersurfaces in* $\mathbb{C}^m$, Proc. Amer. Math. Soc. 105 (1989), 425–428.

[5] ———, *Submanifolds and isometric immersions,* Math. Lecture Ser., 13, Publish or Perish, Houston, TX, 1990.

[6] M. Dajczer and D. Gromoll, *Gauss parametrizations and rigidity aspects of submanifolds,* J. Differential Geom. 22 (1985), 1–12.

[7] ———, *Real Kähler submanifolds and uniqueness of the Gauss map,* J. Differential Geom. 22 (1985), 13–28.

[8] ———, *The Weierstrass representation for complete minimal real Kähler submanifolds of codimension two,* Invent. Math. 119 (1985), 235–242.

[9] ———, *Real Kähler submanifolds in low codimension,* Differential Geom. Appl. 7 (1997), 389–395.

[10] M. Dajczer and L. Rodriquez, *Rigidity of real Kähler submanifolds,* Duke Math. J. 53 (1986), 211–220.

[11] L. Florit, W.-S. Hui, and F. Zheng, *On real Kähler Euclidean submanifolds with non-negative Ricci curvature,* J. Eur. Math. Soc. (JEMS) 7 (2005), 1–11.

[12] L. Florit and F. Zheng, *On nonpositively curved Euclidean submanifolds: Splitting results,* Comment. Math. Helv. 74 (1999), 53–62.

[13] ———, *On nonpositively curved Euclidean submanifolds: Splitting results. II,* J. Reine Angew. Math. 508 (1999), 1–15.

[14] ———, *A local and global splitting result for real Kähler Euclidean submanifolds,* Arch. Math. (Basel) 84 (2005), 88–95.
[15] ———, *Complete real Kähler Euclidean hypersurfaces are cylinders,* Ann. Inst. Fourier (Grenoble) 57 (2007), 155–161.
[16] ———, *Complete real Kähler submanifolds in codimension two,* Math. Z. 258 (2008), 291–299.
[17] F. Marques and A. Neves, *Min-max theory and the Willmore conjecture,* preprint, arXiv:12026036.
[18] J. Nash, C^1 *isometric embeddings,* Ann. of Math. (2) 60 (1954), 383–396.
[19] M. Spivak, *A comprehensive introduction to differential geometry,* vol. 4, Publish or Perish, Wilmington, DE, 1979.
[20] H. W. Xu, *Recent developments in differentiable sphere theorems,* Fifth International Congress of Chinese Mathematicians, AMS/IP Stud. Adv. Math., 51, pp. 415–430, Amer. Math. Soc., Providence, RI, 2012.
[21] H. W. Xu and J. R. Gu, *An optimal differentiable sphere theorem for complete manifolds,* Math. Res. Lett. 17 (2010), 1111–1124.
[22] H. W. Xu and E. T. Zhao, *Topological and differentiable sphere theorems for complete submanifolds,* Comm. Anal. Geom. 17 (2009), 565–585.
[23] F. Zheng, *Isometric embedding of Kähler manifolds with nonpositive sectional curvature,* Math. Ann. 304 (1996), 769–784.
[24] ———, *Complex differential geometry,* AMS/IP Stud. Adv. Math., 18, Amer. Math. Soc., Providence, RI, 2000.

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