Michigan Math. J. 62 (2013), 323–351

Four-manifolds Admitting Hyperelliptic Broken Lefschetz Fibrations

Kenta Hayano & Masatoshi Sato

1. Introduction

A broken Lefschetz fibration is a smooth map from a four-manifold to a surface that has at most two types of singularities: Lefschetz singularity and indefinite fold singularity. This fibration was introduced in [1] as a fibration structure compatible with near-symplectic structures.

A simplified broken Lefschetz fibration is a broken Lefschetz fibration over the sphere that satisfies several conditions on fibers and singularities. This fibration was first defined by Baykur [3]. Despite the strict conditions in the definition of this fibration, it is known that every closed oriented four-manifold admits a simplified broken Lefschetz fibration. For a simplified broken Lefschetz fibration, we can define a monodromy representation of this fibration as we do for a Lefschetz fibration. Thus we can define hyperelliptic simplified broken Lefschetz fibrations as a generalization of hyperelliptic Lefschetz fibrations. Hyperelliptic Lefschetz fibrations have been studied in many fields—for example, algebraic geometry and topology—and it has been shown that the total spaces of such fibrations satisfy strong conditions on the signature, the Euler characteristic, and so on (see e.g. [10]). Furthermore, we can obtain a signature formula of hyperelliptic simplified broken Lefschetz fibrations similar to that of hyperelliptic Lefschetz fibrations (see [13]). It is therefore natural to ask how far total spaces of hyperelliptic simplified broken Lefschetz fibrations are restricted as well as what conditions these spaces satisfy. The following result gives a partial answer.

Theorem 1.1. *Let $f: M \to S^2$ be a genus-g hyperelliptic simplified broken Lefschetz fibration. We assume that $g \geq 3$.*

(i) *Let s be the number of Lefschetz singularities of f whose vanishing cycles are separating. Then there exists an involution*

$$\omega: M \to M$$

such that the fixed point set of ω is the union of (possibly nonorientable) surfaces and s isolated points. Moreover, ω can be extended to an involution

Received January 5, 2012. Revision received November 1, 2012.

The first author was supported by Yoshida Scholarship "Master21" while this work was being carried out. The first author was supported by JSPS Research Fellowships for Young Scientists (24-993). The second author was supported by JSPS Research Fellowships for Young Scientists (22-2364).

$$\bar{\omega}\colon M\#s\overline{\mathbb{CP}^2} \to M\#s\overline{\mathbb{CP}^2}$$

such that $M\#s\overline{\mathbb{CP}^2}/\bar{\omega}$ is diffeomorphic to $S\#2s\overline{\mathbb{CP}^2}$ for S an S^2-bundle over S^2 and such that the quotient map

$$/\bar{\omega}\colon M\#s\overline{\mathbb{CP}^2} \to M\#s\overline{\mathbb{CP}^2}/\bar{\omega} \cong S\#2s\overline{\mathbb{CP}^2}$$

is a double branched covering.

(ii) *A regular fiber F of the fibration f represents a nontrivial rational homology class of M; that is, $[F] \neq 0$ in $H_2(M; \mathbb{Q})$.*

REMARK 1.2. As we will state in Theorem 4.7, Theorem 1.1 can be generalized to directed broken Lefschetz fibrations, which we will define later (see Definition 2.2).

REMARK 1.3. Auroux, Donaldson, and Katzarkov [1] gave a necessary and sufficient condition for a closed oriented four-manifold to admit a near-symplectic form. By using this result together with Theorem 1.1(i), we can prove that every total space of a hyperelliptic simplified broken Lefschetz fibration with genus $g \geq 3$ has a near-symplectic form. Moreover, we can take such a near-symplectic form so that all the fibers of the fibration are symplectic outside of the singularities.

Part (i) of Theorem 1.1 is a generalization of the results of Fuller [8] and of Siebert and Tian [17] on hyperelliptic Lefschetz fibrations. Indeed, they proved independently that, after blowing up s times, the total space of a hyperelliptic Lefschetz fibration (with arbitrary genus) is a double branched covering of a manifold obtained by blowing up a sphere bundle over the sphere $2s$ times, where s is the number of Lefschetz singularities of the fibration whose vanishing cycles are separating. Fuller proved this statement by using handle decompositions and Kirby diagrams, whereas Siebert and Tian did so using complex geometry. We also use handle decompositions to prove Theorem 1.1(i), but our method is slightly different from that of Fuller; we give an involution of the total space of a fibration explicitly, and that explicit description is used in the proof of Theorem 1.1(ii).

Since the self-intersection of a regular fiber of a broken Lefschetz fibration is equal to 0, we can obtain the following corollary immediately.

COROLLARY 1.4. *A closed oriented four-manifold with definite intersection form cannot admit any hyperelliptic simpified broken Lefschetz fibrations with genus $g \geq 3$.*

Note that the condition $g \geq 3$ is essential here. Indeed, it is proved in [1] that S^4 and $\#n\overline{\mathbb{CP}^2}$ ($n \geq 1$) admit genus-1 simplified broken Lefschetz fibrations. Since every simplified broken Lefschetz fibration with genus *less* than 3 is hyperelliptic, these examples mean that Corollary 1.4 does not hold without the assumption $g \geq 3$.

REMARK 1.5. It is shown in [11] that a simply connected four-manifold with a positive definite intersection form cannot admit any genus-1 simplified broken Lefschetz fibrations except S^4. In particular, $\#n\mathbb{CP}^2$ ($n \geq 1$) cannot admit any genus-1 simplified broken Lefschetz fibrations. However, it is proved in [4] that the manifold $\#n\mathbb{CP}^2$ admits a genus-2 simplified broken Lefschetz fibration for any $n \geq 0$ (see [4, Thm. 18]). In the argument in [4], Baykur constructed a genus-2 simplified broken Lefschetz fibration in an explicit way. He did not mention vanishing cycles of this fibration, yet we can easily obtain these cycles by using the result in [12].

This result also means that Corollary 1.4 does not hold without the assumption on genus. Moreover, it is easy to see that the genus-2 fibration on $\#n\mathbb{CP}^2$ cannot be compatible with any near-symplectic forms even though $\#n\mathbb{CP}^2$ $(n \geq 1)$ admits a near-symplectic form.

In general, a genus-g simplified broken Lefschetz fibration can be changed into a genus-$(g+1)$ simplified broken Lefschetz fibration by a certain homotopy of fibrations, called *flip and slip* (for the detail of this homotopy, see e.g. [2]). Hence for any $g \geq 3$ we can easily construct genus-g simplified broken Lefschetz fibrations on S^4 as well as on $\#n\mathbb{CP}^2$ and $\#n\overline{\mathbb{CP}^2}$ $(n \geq 1)$. However, these fibrations are not hyperelliptic by Corollary 1.4.

In Section 2, we review the definitions of broken Lefschetz fibrations and simplified ones. We also review the basic properties of monodromy representations of broken Lefschetz fibrations. After reviewing the hyperelliptic mapping class group, we give the definition of hyperelliptic simplified broken Lefschetz fibrations. In Section 3, we prove a certain lemma on the subgroup of the hyperelliptic mapping class group consisting of elements that preserve a simple closed curve c. This lemma plays a key role in the proof of Theorem 1.1. In Section 4, we prove Theorem 1.1.

2. Preliminaries

2.1. Broken Lefschetz Fibrations

We start by giving the precise definition of broken Lefschetz fibrations.

Definition 2.1. Let M and Σ be compact oriented smooth manifolds of dimension 4 and 2, respectively. A smooth map $f : M \to \Sigma$ is called a *broken Lefschetz fibration* (BLF) if it satisfies the following conditions.

(1) $f^{-1}(\partial\Sigma) = \partial M$.
(2) f has at most two types of singularities which is locally written as follows:
- $(z_1, z_2) \mapsto \xi = z_1 z_2$, where (z_1, z_2) (resp. ξ) is a complex local coordinate of M (resp. Σ) compatible with its orientation;
- $(t, x_1, x_2, x_3) \mapsto (y_1, y_2) = (t, {x_1}^2 + {x_2}^2 - {x_3}^2)$, where (t, x_1, x_2, x_3) (resp. (y_1, y_2)) is a real coordinate of M (resp. Σ).

The first singularity in condition (2) is called a *Lefschetz singularity* and the second is called an *indefinite fold singularity*. We denote by $\mathcal{C}_f$ the set of Lefschetz singularities of f and by Z_f the set of indefinite fold singularities of f. Note that a Lefschetz fibration is a BLF that has no indefinite fold singularities.

Let $f: M \to S^2$ be a BLF over the 2-sphere. Suppose that the restriction of f to the set of singularities is injective and that the image $f(Z_f)$ is the disjoint union of embedded circles parallel to the equator of S^2. We put $f(Z_f) = Z_1 \amalg \cdots \amalg Z_m$, where Z_i is the embedded circle in S^2. We choose a path $\alpha: [0,1] \to S^2$ that satisfies the following properties:

(1) $\operatorname{Im}\alpha$ is contained in the complement of $f(\mathcal{C}_f)$;

(2) α starts at the south pole $p_s \in S^2$ and connects the south pole to the north pole $p_n \in S^2$;

(3) α intersects each component of $f(Z_f)$ at a single point transversely.

We put $\{q_i\} = Z_i \cap \operatorname{Im}\alpha$ and $\alpha(t_i) = q_i$. We assume that $q_1, \dots, q_m$ appear in this order when we go along α from p_s to p_n (see Figure 1).

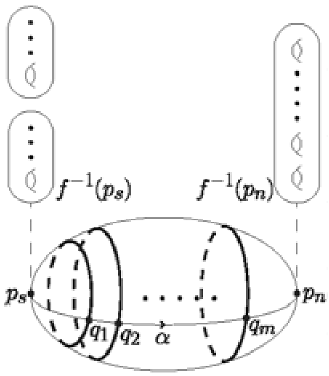

Figure 1 Example of the path α; the bold circles describe $f(Z_f)$

The preimage $f^{-1}(\operatorname{Im}\alpha)$ is a three-manifold that is a cobordism between $f^{-1}(p_s)$ and $f^{-1}(p_n)$. By the local coordinate description of the indefinite fold singularity, it is easy to see that $f^{-1}(\alpha([0, t_i + \varepsilon]))$ is obtained from $f^{-1}(\alpha([0, t_i - \varepsilon]))$ by either 1- or 2-handle attachment for each $i = 1, \dots, m$. In particular, we obtain a handle decomposition of the cobordism $f^{-1}(\operatorname{Im}\alpha)$.

DEFINITION 2.2. A BLF f is said to be *directed* if it satisfies the following conditions:

(1) the restriction of f to the set of singularities is injective and the image $f(Z_f)$ is the disjoint union of embedded circles parallel to the equator of S^2;

(2) all the handles in the handle decomposition of $f^{-1}(\operatorname{Im}\alpha)$ just described are of index 1; and

(3) all Lefschetz singularities of f are in the preimage of the component of $S^2 \setminus (Z_1 \amalg \cdots \amalg Z_m)$, which contains the point p_n.

Condition (3) is not essential. Indeed, we can change a BLF f by a homotopy that satisfies conditions (1) and (2) so that it satisfies condition (3) (cf. [3]).

For a directed BLF f, we assume that the set of indefinite fold singularities of f is connected and that all the fibers of f are connected. We call such a BLF a *simplified broken Lefschetz fibration* (SBLF). For an SBLF f, Z_f is empty set or an embedded circle in M. If Z_f is not empty, then the image $f(Z_f)$ is an embedded circle in S^2. Thus $S^2 \setminus \operatorname{Int} \nu f(Z_f)$ consists of two 2-disks, D_1 and D_2, and the genus of the regular fiber of the fibration res $f\colon f^{-1}(D_1) \to D_1$ is just 1 higher than that of the fibration res $f\colon f^{-1}(D_2) \to D_2$. We call $f^{-1}(D_1)$ (resp. $f^{-1}(D_2)$) the *higher side* (resp. *lower side*) of f and call $f^{-1}(\nu f(Z_f))$ the *round cobordism* of f. By our definition, all Lefschetz singularities of f are in the higher side of f. We call the genus of the regular fiber in the higher side the *genus* of f.

2.2. *Monodromy Representations*

Let $f\colon M \to B$ be a genus-g Lefschetz fibration. We denote by $\mathcal{C}_f = \{z_1, \ldots, z_n\}$ the set of Lefschetz singularities of f and put $y_i = f(z_i)$. For a base point $y_0 \in B \setminus f(\mathcal{C}_f)$ we can define a homomorphism $\varrho_f\colon \pi_1(B \setminus f(\mathcal{C}_f), y_0) \to \mathcal{M}_g$, called a *monodromy representation* of f, where $\mathcal{M}_g = \operatorname{Diff}^+ \Sigma_g / \operatorname{Diff}_0^+ \Sigma_g$ is the mapping class group of the closed oriented surface Σ_g. We endow the C^∞ topology with $\operatorname{Diff}^+ \Sigma_g$ so that $\operatorname{Diff}_0^+ \Sigma_g$ is the component of $\operatorname{Diff}^+ \Sigma_g$ containing the identity map. (Readers are referred to [9] for the precise definition of monodromy representations.)

We examine the case $B = D^2$. For each $i = 1, \ldots, n$ we take embedded paths $\alpha_1, \ldots, \alpha_n \subset D^2$ that satisfy the following conditions:

- each α_i connects y_0 to y_i;
- $\alpha_i \cap f(\mathcal{C}_f) = \{y_i\}$;
- $\alpha_i \cap \alpha_j = \{y_0\}$ for all $i \neq j$; and
- $\alpha_1, \ldots, \alpha_n$ appear in this order when we travel counterclockwise around y_0.

For each $i = 1, \ldots, n$, denote by $a_i \in \pi_1(D^2 \setminus f(\mathcal{C}_f), y_0)$ the element represented by the loop obtained when we connect a counterclockwise circle around y_i to y_0 by using α_i. We put $W_f = (\varrho_f(a_1), \ldots, \varrho_f(a_n)) \in {\mathcal{M}_g}^n$; this sequence is called a *Hurwitz system* of f. By the conditions on paths $\alpha_1, \ldots, \alpha_n$, the product $\varrho_f(a_1) \cdots \varrho_f(a_n)$ is equal to the monodromy of the boundary of D^2. It is known that each $\varrho_f(a_i)$ is the right-handed Dehn twist along a simple closed curve c_i, called a *vanishing cycle* of the Lefschetz singularity z_i [14; 16].

REMARK 2.3. The sequence W_f is not unique for f. Indeed, it depends on the choice of paths $\alpha_1, \ldots, \alpha_n$ and the choice of the identification of $f^{-1}(y_0)$ with the closed oriented surface Σ_g. Yet it is known that another Hurwitz system, $\tilde{W}_f$, is obtained from W_f by successive application of the transformations

- $(\ldots, g_i, g_{i+1}, \ldots) \mapsto (\ldots, g_{i+1}, {g_{i+1}}^{-1} g_i g_{i+1}, \ldots)$ (and its inverse transformation) and
- $(g_1, \ldots, g_n) \mapsto (h^{-1} g_1 h, \ldots, h^{-1} g_n h)$,

where $g_i, h \in \mathcal{M}_g$ (cf. [9]). Two sequences of elements in $\mathcal{M}_g$ are said to be *Hurwitz equivalent* if one is obtained from the other by successive application of the transformations just described.

Let $\hat{f}\colon M \to S^2$ be a genus-g SBLF with nonempty indefinite fold singularities. We denote by M_h the higher side of $\hat{f}$. The restriction $\operatorname{res}\hat{f}\colon M_h \to D^2$ is a Lefschetz fibration over D^2. Thus, a monodromy representation and a Hurwitz system of $\operatorname{res}\hat{f}$ can be defined and are called (respectively) a *monodromy representation* and a *Hurwitz system* of $\hat{f}$, which we denote by $\varrho_{\hat{f}}$ and $W_{\hat{f}}$. For the Lefschetz fibration $\operatorname{res}\hat{f}\colon M_h \to D^2$, we choose a base point y_0 and paths $\alpha_1, \dots, \alpha_n$ as in the previous paragraph. We also take a path $\alpha\colon [0,1] \to S^2$ that satisfies the following conditions:

- α connects y_0 to a point in the image of the lower side of $\hat{f}$;
- $\alpha \cap \alpha_i = \{y_0\}$ for each $i = 1, \dots, n$;
- α intersects the image $\hat{f}(Z_{\hat{f}})$ at one point transversely; and
- $\alpha_1, \dots, \alpha_n, \alpha$ appear in this order when we travel counterclockwise around y_0.

We put $q = \alpha(t) \in \operatorname{Im}\alpha \cap \hat{f}(Z_{\hat{f}})$. The preimage $\hat{f}^{-1}(\alpha([0, t+\varepsilon]))$ is obtained from the preimage $\hat{f}^{-1}(\alpha([0, t-\varepsilon])) \cong \hat{f}^{-1}(p_0) \times [0, t-\varepsilon]$ by 2-handle attachment. We regard the attaching circle c of the 2-handle as a simple closed curve in $\hat{f}^{-1}(p_0) \cong \Sigma_g$, which we call a *vanishing cycle* of the indefinite fold singularity of $\hat{f}$.

LEMMA 2.4 (Auroux, Donaldson, and Katzarkov [1]; see also Baykur [3]). *The product $\varrho_{\hat{f}}(a_1)\cdots\varrho_{\hat{f}}(a_n)$ is contained in $\mathcal{M}_g(c)$, where $\mathcal{M}_g(c)$ is the subgroup of the group $\mathcal{M}_g$ consisting of elements represented by a map that preserves the curve c.*

For an element $\psi \in \mathcal{M}_g(c)$, we take a representative $T \in \psi$ such that T preserves the curve c. Then T induces the homeomorphism $T\colon \Sigma_g \setminus c \to \Sigma_g \setminus c$, and this homeomorphism can be extended to the homeomorphism $\hat{T}\colon \Sigma_{g-1} \to \Sigma_{g-1}$ by regarding $\Sigma_g \setminus c$ as the genus-$(g-1)$ surface with two punctures. Eventually, we can define the homomorphism Φ_c as follows:

$$\begin{array}{rccc} \Phi_c\colon & \mathcal{M}_g(c) & \longrightarrow & \mathcal{M}_{g-1} \\ & \Psi & & \Psi \\ & \psi = [T] & \longmapsto & [\hat{T}]. \end{array}$$

REMARK 2.5. Let $c \subset \Sigma_g$ be a separating simple closed curve. We can regard $\Sigma_g \setminus c$ as the disjoint union of the two once-punctured surfaces of genus h and $g-h$. Thus, we can define the homomorphism $\Phi_c\colon \mathcal{M}_g(c^{\mathrm{ori}}) \to \mathcal{M}_h \times \mathcal{M}_{g-h}$ as we define Φ_c for a nonseparating curve c, where $\mathcal{M}_g(c^{\mathrm{ori}})$ is the subgroup of $\mathcal{M}_g(c)$ consisting of elements represented by maps that preserve c and its orientation.

LEMMA 2.6 [3]. *The product $\varrho_{\hat{f}}(a_1)\cdots\varrho_{\hat{f}}(a_n)$ is contained in the kernel of Φ_c. Conversely, if simple closed curves $c, c_1, \dots, c_n \subset \Sigma_g$ satisfy the conditions*

- *c is nonseparating and*
- *$t_{c_1}\cdots t_{c_n} \in \operatorname{Ker}\Phi_c$,*

then there exists a genus-g SBLF $f\colon M \to S^2$ such that $W_f = (t_{c_1}, \dots, t_{c_n})$ and a vanishing cycle of the indefinite fold of f is c.

2.3. The Hyperelliptic Mapping Class Group

Let Σ_g be a closed oriented surface of genus $g \geq 1$. Denote by $\iota\colon \Sigma_g \to \Sigma_g$ the involution described in Figure 2.

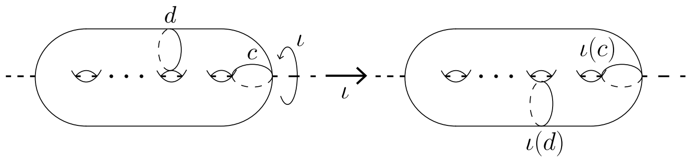

Figure 2 Hyperelliptic involution on the surface Σ_g

Let $C(\iota)$ denote the centralizer of ι in the diffeomorphism group $\mathrm{Diff}_+ \Sigma_g$, and endow $C(\iota) \subset \mathrm{Diff}_+ \Sigma_g$ with the relative topology. The inclusion homomorphism $C(\iota) \to \mathrm{Diff}_+ \Sigma_g$ induces a natural homomorphism $\pi_0 C(\iota) \to \mathcal{M}_g$ between their path-connected components.

THEOREM 2.7 (Birman and Hilden [6]). *When $g \geq 2$, the homomorphism $\pi_0 C(\iota) \to \mathcal{M}_g$ is injective.*

Denote the image of this homomorphism by $\mathcal{H}_g$ for $g \geq 1$. This group is called the *hyperelliptic mapping class group.* In fact, the authors also proved the same result in more general settings (see [6] for details).

A Lefschetz fibration is said to be *hyperelliptic* if we can take an identification of the fiber of a base point with the closed oriented surface so that the image of the monodromy representation of the fibration is contained in the hyperelliptic mapping class group. Thus, it is natural to generalize this definition to directed (and especially simplified) BLFs as follows. Let $f\colon M \to S^2$ be a directed BLF. We use the same notation as in the argument preceding Definition 2.2. We take a disk neighborhood $D \subset S^2 \setminus f(Z_f)$ of p_n so that $f(\mathcal{C}_f)$ is contained in D. We put

$$r_i = \alpha\left(\frac{t_i + t_{i+1}}{2}\right) \ (i = 1, \ldots, m-1) \quad \text{and} \quad r_m = p_n.$$

Let $d_i \subset f^{-1}(r_i)$ be the vanishing cycle of Z_i determined by α. After fixing an identification of $f^{-1}(r_m)$ with $\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_k}$, we obtain an involution ι_i on $f^{-1}(r_i)$ induced by the hyperelliptic involution on $f^{-1}(r_m)$ because we can use α to identify $f^{-1}(r_{i-1}) \setminus \{\text{two points}\}$ with $f^{-1}(r_i) \setminus d_i$. We say that f is *hyperelliptic* if it satisfies the following conditions for a suitable identification of $f^{-1}(r_m)$ with $\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_k}$:

- the image of the monodromy representation of the Lefschetz fibration $\mathrm{res}\, f\colon f^{-1}(D) \to D$ is contained in the group $\mathcal{H}_g$; and
- d_i is preserved by the involution ι_i up to isotopy.

We shall use HSBLF to denote a hyperelliptic simplified BLF.

REMARK 2.8. Every SBLF whose genus is no more than 2 is hyperelliptic since $\mathcal{H}_g = \mathcal{M}_g$ and since all simple closed curves in Σ_g are preserved by ι if $g \leq 2$.

2.4 Handle Decompositions

Let $f\colon M \to S^2$ be a genus-g SBLF, and let M_h (resp., M_r and M_l) be the higher side (resp., the round cobordism and the lower side) of f. The restriction $\operatorname{res} f\colon M_h \to D^2$ is a Lefschetz fibration over the disk. We choose $y_0 \in D^2$ and $\alpha_1, \dots, \alpha_n \subset D^2$ as in Section 2.2. Let $D \subset \operatorname{Int} D^2 \setminus \mathcal{C}_f$ be a disk whose boundary intersects each path α_i at one point transversely. Denote by $w_i \in \partial D$ the intersection between ∂D and α_i and by $c_i \subset f^{-1}(w_i)$ a vanishing cycle of the Lefschetz singularity in the fiber $f^{-1}(y_i)$.

THEOREM 2.9 (Kas [14]). *The higher side M_h is obtained by attaching n 2-handles to $f^{-1}(D) \cong D \times \Sigma_g$; the attaching circles are $c_1, \dots, c_n$, and the framings of these handles are -1 relative to the framing along the fiber.*

We call $R^\lambda = [0,1] \times D^\lambda \times D^{3-\lambda}/((1, x_1, x_2, x_3) \sim (0, \pm x_1, x_2, \pm x_3))$ a (4-dimensional) *round λ-handle* ($\lambda = 1, 2$); then $X^4 \cup_\varphi R^\lambda$ is a four-manifold obtained by attaching a round λ-handle to the four-manifold X^4, where $\varphi\colon [0,1] \times \partial D^\lambda \times D^{3-\lambda}/\sim \to \partial X$ is an embedding. A round handle R^λ is said to be *untwisted* if the sign in the equivalence relation is positive and is said to be *twisted* otherwise.

THEOREM 2.10 ([1]; cf. [3]). *The union $M_h \cup M_r$ is obtained by attaching a round 2-handle to M_h. Moreover, a circle $\{t\} \times \partial D^2 \times \{0\}$ in the attaching region of R^2 is attached along a vanishing cycle of indefinite fold singularities of f.*

Observe that the isotopy class of the attaching map $\varphi\colon [0,1] \times \partial D^2 \times D^1/\sim \to \partial M_h$ is uniquely determined by a vanishing cycle of an indefinite fold of f if the genus of f is no less than 2. In particular, if the genus of f is no less than 3, then the total space of f is uniquely determined by the vanishing cycle of an indefinite fold of f and those of Lefschetz singularities of f. However, there exist infinitely many SBLFs with genus $g \leq 2$ such that they have the same vanishing cycles despite each one's total space being mutually distinct (see [5] or [11]).

Round 2-handle attachment is given by 2-handle attachment followed by 3-handle attachment (cf. [3]). Thus, we obtain a handle decomposition of $M_h \cup M_r$ by the previous theorems. Since M_l contains no singularities of f, it follows that the map $\operatorname{res} f\colon M_l \to D^2$ is the trivial Σ_{g-1}-bundle. In particular, M_l is diffeomorphic to $D^2 \times \Sigma_{g-1}$ and we obtain a handle decomposition of $M = M_h \cup M_r \cup M_l$. Moreover, we can draw a Kirby diagram of M by the decomposition (see [3] for more details).

By the same argument we can also obtain a handle decomposition of the total space of a directed BLF $f\colon M \to S^2$. Indeed, we can decompose M into $D^2 \times (\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_m})$, n_1 2-handles, n_2 round 2-handles, and $D^2 \times (\Sigma_{h_1} \amalg \cdots \amalg \Sigma_{h_m})$; here n_1 is the number of the Lefschetz singularities of f, and n_2 is the number of the components of the set of indefinite fold singularities of f.

3. A Subgroup $\mathcal{H}_g(c)$ of the Hyperelliptic Mapping Class Group That Preserves a Curve c

Let c be an essential simple closed curve in the surface Σ_g that is preserved by the involution $\iota \in \mathrm{Diff}_+ \Sigma_g$ as a set. Let $\mathcal{H}_g(c)$ denote the subgroup of the hyperelliptic mapping class group defined by $\mathcal{H}_g(c) := \mathcal{H}_g \cap \mathcal{M}_g(c)$. As introduced in Theorem 2.7, the hyperelliptic mapping class group $\mathcal{H}_g$ is isomorphic to the group consisting of the path-connected components of $C(\iota)$. Hence the group $\mathcal{H}_g(c)$ consists of the mapping classes that can be represented not only by elements in the centralizer $C(\iota)$ but also by elements in $\mathrm{Diff}_+(\Sigma_g, c)$. Let $\mathcal{H}^s_g(c)$ denote the subgroup of $\pi_0 C(\iota)$ defined by $\mathcal{H}^s_g(c) := \{[T] \in \pi_0 C(\iota) \mid T(c) = c\}$. In this section we prove the following lemma.

Lemma 3.1. *Let $g \geq 2$. The natural isomorphism $\pi_0 C(\iota) \to \mathcal{H}_g$ in Theorem 2.7 restricts to an isomorphism between the groups $\mathcal{H}^s_g(c)$ and $\mathcal{H}_g(c)$.*

To prove the lemma, it is enough to show that the homomorphism maps $\mathcal{H}^s_g(c)$ onto $\mathcal{H}_g(c)$. Let $[T]$ be a mapping class in $\mathcal{H}_g(c)$. We can choose a representative $T\colon \Sigma_g \to \Sigma_g$ in the centralizer $C(\iota)$. Because this T is isotopic to some diffeomorphism on Σ_g that preserves the curve c, the curve $T(c)$ is isotopic to c.

We call an isotopy $L_0\colon \Sigma_g \times [0,1] \to \Sigma_g$ *symmetric* if and only if $L_0(*, t) \in C(\iota)$ for any $t \in [0,1]$. We shall construct a symmetric isotopy $L\colon \Sigma_g \times [0,1] \to \Sigma_g$ satisfying

$$L(*,0) = T \quad \text{and} \quad L(c,1) = c \subset \Sigma_g,$$

where $L(*,1)$ represents an element in $\mathcal{H}^s_g(c)$ and $[L(*,1)] = [T] \in \pi_0 C(\iota)$. Thus we see that the homomorphism $\mathcal{H}^s_g(c) \to \mathcal{H}_g(c)$ is surjective.

To construct the symmetric isotopy $L\colon \Sigma_g \times [0,1] \to \Sigma_g$, we need the following proposition, which gives the so-called bigon criterion.

Proposition 3.2 (Farb and Margalit [7, Prop. 1.7]). *Let S be a compact surface. The geometric intersection number of two transverse simple closed curves in S is minimal if and only if they do not form a bigon.*

We may assume that the curves c and $T(c)$ are transverse by changing the diffeomorphism T in terms of some symmetric isotopy. Since c and $T(c)$ are isotopic, their minimal intersection number is 0. Hence there exist bigons each of whose boundaries is the union of an arc of c and an arc of $T(c)$. Choose an innermost bigon Δ among them.

Let α be the arc $c \cap \partial\Delta$ and β the arc $T(c) \cap \partial\Delta$. Since Δ is a bigon, the endpoints of α and β coincide; denote these endpoints by $\{x_1, x_2\} \subset \partial\Delta$.

Lemma 3.3.

$$\mathrm{Int}\,\Delta \cap (T(c) \cup c) = \emptyset.$$

Proof. If the set $\mathrm{Int}\,\Delta \cap c$ is nonempty, then there exists an arc of c in Δ that forms a bigon with the arc β. Yet this is a contradiction because the bigon Δ is innermost. In the same way, we can show that $\mathrm{Int}\,\Delta \cap T(c) = \emptyset$. □

Note that the bigon $\iota(\Delta)$ is also innermost. By Lemma 3.3, we have $\Delta \cap \iota(\Delta) = \partial\Delta \cap \partial\iota(\Delta)$.

LEMMA 3.4.

$$\partial\Delta \cap \partial\iota(\Delta) \subset \{x_1, x_2\}.$$

Proof. Since $\partial\alpha = \partial\beta = \alpha \cap \beta = \{x_1, x_2\}$, it suffices to show that $\operatorname{Int}\alpha \cap \partial\iota(\Delta) = \operatorname{Int}\beta \cap \partial\iota(\Delta) = \emptyset$. Since $\alpha \cap T(c) = \{x_1, x_2\}$, we have $\operatorname{Int}\alpha \cap \iota(\beta) = \emptyset$. Next we show that $\operatorname{Int}\alpha \cap \operatorname{Int}\iota(\alpha) = \emptyset$. So assume by way of contradiction that $\operatorname{Int}\alpha \cap \operatorname{Int}\iota(\alpha) \neq \emptyset$. Since c is simple and contains α and $\iota(\alpha)$, it follows that α and $\iota(\alpha)$ must coincide. In particular, we have $\partial\alpha = \partial\iota(\alpha)$. Hence $\beta \cup \iota(\beta)$ forms a simple closed curve, and this curve is null-homotopic because both of the arcs β and $\iota(\beta)$ are homotopic to $\alpha = \iota(\alpha)$ relative to their boundaries. Since $T(c)$ is simple and contains both β and $\iota(\beta)$, $T(c)$ and $\beta \cup \iota(\beta)$ must coincide—contradicting the essentialness of $T(c)$. We can likewise show that $\operatorname{Int}\beta \cap \partial\iota(\Delta) = \emptyset$. □

Let Σ_g^ι denote the fixed point set of the involution ι on Σ_g.

LEMMA 3.5. *If c is nonseparating, then the set $c \cap \Sigma_g^\iota$ consists of two points and*

$$c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota;$$

if c is separating, then

$$c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset.$$

Proof. Endow the curves c and $T(c)$ with arbitrary orientations.

First we consider the case where c is a nonseparating simple closed curve. In this case the curve $T(c)$ is also nonseparating; c and $T(c)$ represent nontrivial homology classes in $H_1(\Sigma_g; \mathbf{Z})$. Because the involution ι acts on $H_1(\Sigma_g; \mathbf{Z})$ by -1, it changes the orientations of c and $T(c)$. Therefore, each of the sets $c \cap \Sigma_g^\iota$ and $T(c) \cap \Sigma_g^\iota$ consists of two points.

We will show that $T(c) \cap \Sigma_g^\iota = c \cap \Sigma_g^\iota$. Since c and $T(c)$ are isotopic, the Dehn twists t_c and $t_{T(c)}$ represent the same element in $\mathcal{H}_g$. The mapping classes $\Psi([t_c])$ and $\Psi([t_{T(c)}])$ in $\mathcal{M}_0^{2g+2}$ permute the branched points $p(c \cap \Sigma_g^\iota)$ and $p(T(c) \cap \Sigma_g^\iota)$, respectively; hence the sets $p(c \cap \Sigma_g^\iota)$ and $p(T(c) \cap \Sigma_g^\iota)$ coincide. This establishes that $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota$.

Next, let c be a separating simple closed curve. Since ι preserves the orientations of the subsurfaces bounded by c or $T(c)$, it also preserves the orientation of c and $T(c)$. In general, an involution that acts on a circle while preserving its orientation does *not* have a fixed point. Hence we have $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset$. □

Proof of Lemma 3.1. Let c be a nonseparating curve. By Lemma 3.5, the geometric intersection number of c and $T(c)$ is at least 2. Hence, there is an innermost bigon Δ. By Lemma 3.4, the cardinality $\sharp(\Delta \cap \iota(\Delta))$ is equal to 0, 1, or 2; see Figure 3.

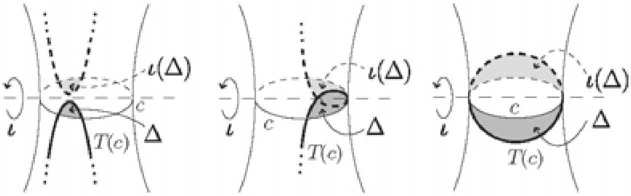

Figure 3 Cardinality of $\Delta \cap \iota(\Delta)$: 0 (left), 1 (center), 2 (right); the bold curves describe the curves $T(c)$

First assume that $\sharp(\Delta \cap \iota(\Delta)) = 0$. In this case, there is a symmetric isotopy $L_1\colon \Sigma_g \times [0,1] \to \Sigma_g$ such that $L_1(*,0)$ is the identity and $L_1(*,1)$ collapses the bigon Δ as in Figure 4. Therefore, the geometric intersection number of c and $T(c)$ is decreased by 4 when we replace the diffeomorphism T by $L_1(*,1)T$.

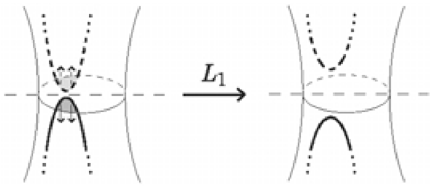

Figure 4

Now assume that $\sharp(\Delta \cap \iota(\Delta)) = 1$, in which case we also have a symmetric isotopy $L_2\colon \Sigma_g \times [0,1] \to \Sigma_g$; this isotopy decreases the geometric intersection number by 2 (see Figure 5). Note that $\Delta \cap \iota(\Delta)$ is a branched point and that $L_2(*,t)$ fixes it for any $t \in [0,1]$.

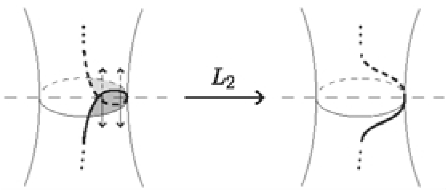

Figure 5

After replacing the diffeomorphism T in these two cases, the branch points $\{x_1, x_2\}$ remain in $c \cap T(c)$. So if we repeat the replacement of T, the case $\sharp(\Delta \cap \iota(\Delta)) = 2$ will definitely occur. Then there is a symmetric isotopy $L_3\colon \Sigma_g \times [0,1] \to \Sigma_g$ such that

- $L_3(*,0)$ is the identity map,
- $L_3(\beta,1) = \alpha$, and
- $L_3(\iota(\beta),1) = \iota(\alpha)$;

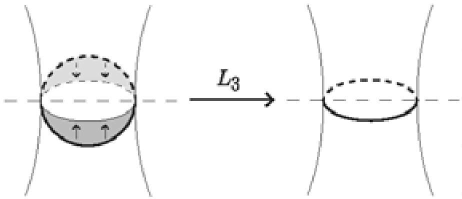

Figure 6

see Figure 6. This isotopy indicates that $L_3(*, 1)T$ preserves the curve c. Combining all three isotopies now yields the desired symmetric isotopy.

Next, let c be a separating curve. If the geometric intersection number of c and $T(c)$ is 0, then the curves c and $T(c)$ bound an annulus A. Since ι acts on A without fixed points, $A/\langle\iota\rangle$ is also an annulus. We can therefore make a symmetric isotopy that moves $T(c)$ to c.

Suppose the geometric intersection number is not 0. Since $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset$, it follows that the cardinality $\sharp(\Delta \cap \iota(\Delta)) \neq 1$. By Lemma 3.4, $\sharp(\Delta \cap \iota(\Delta)) = 0$ or 2. So by the same argument as for the case of nonseparating c, we can collapse the bigons Δ and $\iota(\Delta)$. □

4. An Involution on HSBLF

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1(i). Let $f\colon M \to S^2$ be an HSBLF of genus $g \geq 3$, let $c_1, \dots, c_n \subset \Sigma_g$ be vanishing cycles of Lefschetz singularities of f, and let $c \subset \Sigma_g$ be a vanishing cycle of indefinite fold singularities of f. We assume that $c_1, \dots, c_n$ and c are preserved by the involution $\iota\colon \Sigma_g \to \Sigma_g$. By the argument in Section 2.4, we can decompose M as follows:

$$M = D^2 \times \Sigma_g \cup (h_1^2 \amalg \cdots \amalg h_n^2) \cup R^2 \cup D^2 \times \Sigma_{g-1},$$

where $h_i^2 = D^2 \times D^2$ is the 2-handle attached along the simple closed curve $\{p_i\} \times c_i \in \partial D^2 \times \Sigma_g$ and R^2 is a round 2-handle. We first use this decomposition to prove the existence of an involution ω.

Step 1. We define an involution ω_1 on $D^2 \times \Sigma_g$ as follows:

$$\begin{array}{rccc} \omega_1 = \mathrm{id} \times \iota\colon & D^2 \times \Sigma_g & \longrightarrow & D^2 \times \Sigma_g \\ & \Psi & & \Psi \\ & (z, x) & \longmapsto & (z, \iota(x)). \end{array}$$

In the subsequent steps, we will define an involution on each component in the preceding decomposition of M that is compatible with the involution ω_1.

Step 2. We next define an involution $\omega_{2,i}$ on the 2-handle h_i^2 attached along $\{q_i\} \times c_i \subset \partial D^2 \times \Sigma_g$. We will abuse notation by denoting the attaching circle $\{q_i\} \times c_i$ simply by c_i.

We take a tubular neighborhood νc_i in $\{q_i\} \times \Sigma_g$ and an identification

$$\nu c_i \cong S^1 \times [-1,1]$$

such that c_i corresponds to the circle $S^1 \times \{0\}$ under the identification. We assume that the standard orientation of $S^1 \times [-1,1]$ coincides with that of $\{q_i\} \times \Sigma_g$. We take a sufficiently small neighborhood I_{q_i} of q_i in ∂D^2 as follows:

$$I_{q_i} = \{q_i \cdot \exp(\sqrt{-1}\theta) \in \partial D^2 \mid \theta \in [-\varepsilon_1, \varepsilon_1]\},$$

where $\varepsilon_1 > 0$ is a sufficiently small number. We further identify the neighborhood I_{q_i} with the unit interval $[-1,1]$ by using the following map:

$$\begin{array}{ccc} [-1,1] & \xrightarrow{\sim} & I_{q_i} \\ \Psi & & \Psi \\ s & \longmapsto & q_i \cdot \exp(\sqrt{-1}\varepsilon_1 s). \end{array}$$

We regard $I_{q_i} \times [-1,1]$ as the subset of $\mathbb{C}$ via the embedding

$$\begin{array}{ccc} I_{q_i} \times [-1,1] & \hookrightarrow & \{z \in \mathbb{C} \mid |\mathrm{Re}\, z| \le 1,\ |\mathrm{Im}\, z| \le 1\} \\ \Psi & & \Psi \\ (s,t) & \longmapsto & s + t\sqrt{-1}. \end{array}$$

We put $J = \{z \in \mathbb{C} \mid |\mathrm{Re}\, z| \le 1,\ |\mathrm{Im}\, z| \le 1\}$. The orientation of $\partial D^2 \times \Sigma_g$ is opposite to the natural orientation of $J \times S^1$. Thus, the attaching map of the 2-handle h_i^2 is described as

$$\begin{array}{ccc} \varphi_i \colon \partial D^2 \times D^2 & \longrightarrow & J \times S^1 \subset \partial D^2 \times \Sigma_g \\ \Psi & & \Psi \\ (w_1, w_2) & \longmapsto & (\varepsilon_2 w_2 w_1, w_1), \end{array}$$

where $\varepsilon_2 > 0$ is a sufficiently small number. Note that the map φ_i is orientation preserving if we give the natural orientation of $\partial D^2 \times D^2$.

Case 2.1. If c_i is nonseparating, then we can take a tubular neighborhood $\nu c_i \cong S^1 \times [-1,1]$ such that the involution ω_1 acts on νc_i as follows:

$$\begin{array}{ccc} \omega_1|_{\nu c_i} \colon S^1 \times [-1,1] & \longrightarrow & S^1 \times [-1,1] \\ \Psi & & \Psi \\ (z,t) & \longmapsto & (\bar{z}, -t). \end{array}$$

Since the involution $\omega_1\colon D^2 \times \Sigma_g \to D^2 \times \Sigma_g$ preserves the first component, ω_1 acts on $I_{q_i} \times \nu c_i \cong J \times S^1$ as follows:

$$\begin{array}{rccc} \omega_1|_{J\times S^1}\colon & J \times S^1 & \longrightarrow & J \times S^1 \\ & \Psi & & \Psi \\ & (z_1, z_2) & \longmapsto & (\bar{z}_1, \bar{z}_2). \end{array}$$

We define an involution $\omega_{2,i}$ on the 2-handle h_i^2 as

$$\begin{array}{rccc} \omega_{2,i}\colon & D^2 \times D^2 & \longrightarrow & D^2 \times D^2 \\ & \Psi & & \Psi \\ & (w_1, w_2) & \longmapsto & (\bar{w}_1, \bar{w}_2). \end{array}$$

Then the following diagram commutes:

$$\begin{tikzcd} \partial D^2 \times D^2 \arrow[r, "\omega_{2,i}"] \arrow[d, "\varphi_i"'] & \partial D^2 \times D^2 \arrow[d, "\varphi_i"] \\ J \times S^1 \arrow[r, "\omega_1"] & J \times S^1. \end{tikzcd}$$

Thus, we can define an involution $\omega_1 \cup \omega_{2,i}$ on the manifold $D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2$.

Case 2.2. If c_i is separating, we take a tubular neighborhood $\nu c_i \cong S^1 \times [-1,1]$ such that the involution ω_1 acts on νc_i as follows:

$$\begin{array}{rccc} \omega_1|_{\nu c_i}\colon & S^1 \times [-1,1] & \longrightarrow & S^1 \times [-1,1] \\ & \Psi & & \Psi \\ & (z, t) & \longmapsto & (-z, t). \end{array}$$

Then ω_1 acts on $I_{q_i} \times \nu c_i \cong J \times S^1$ as follows:

$$\begin{array}{rccc} \omega_1|_{J\times S^1}\colon & J \times S^1 & \longrightarrow & J \times S^1 \\ & \Psi & & \Psi \\ & (z_1, z_2) & \longmapsto & (z_1, -z_2). \end{array}$$

We define an involution $\omega_{2,i}$ on the 2-handle h_i^2 as

$$\begin{array}{rccc} \omega_{2,i}\colon & D^2 \times D^2 & \longrightarrow & D^2 \times D^2 \\ & \Psi & & \Psi \\ & (w_1, w_2) & \longmapsto & (-w_1, -w_2). \end{array}$$

Then the following diagram commutes:

$$\begin{tikzcd}
\partial D^2 \times D^2 \arrow[r, "\omega_{2,i}"] \arrow[d, "\varphi_i"'] & \partial D^2 \times D^2 \arrow[d, "\varphi_i"] \\
J \times S^1 \arrow[r, "\omega_1"] & J \times S^1.
\end{tikzcd}$$

Thus, we can define an involution $\omega_1 \cup \omega_{2,i}$ on the manifold $D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2$.

Combining Case 2.1 and Case 2.2, we can define the involution $\tilde{\omega}_2 = \omega_1 \cup (\omega_{2,1} \cup \cdots \cup \omega_{2,n})$ on the four-manifold $M_h = M \cup (h_1^2 \amalg \cdots \amalg h_n^2)$. Before giving an involution on the round 2-handle, we look at the Σ_g-bundle structure of ∂M_h. The projection $\pi_h \colon \partial M_h \to \partial D^2$ of this bundle is described as follows:

$$\pi_h(z, x) = z \quad ((z, x) \in \partial D^2 \times \Sigma_g \setminus (\amalg \operatorname{Int} \varphi_i(\partial D^2 \times D^2)),$$

$$\pi_h(w_1, w_2) = q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\operatorname{Re} w_1 \operatorname{Re} w_2 - \operatorname{Im} w_1 \operatorname{Im} w_2)) \qquad ((w_1, w_2) \in D^2 \times \partial D^2 \subset \partial h_i^2).$$

Indeed, the map π_h is well-defined. To see this, we need only verify that

$$q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\operatorname{Re} w_1 \operatorname{Re} w_2 - \operatorname{Im} w_1 \operatorname{Im} w_2)) = p_1 \circ \varphi_i(w_1, w_2),$$

where $(w_1, w_2) \in D^2 \times \partial D^2 \subset \partial h_i^2$ and $p_1 \colon J \times S^1 \to I_{q_i}$ is the projection. Now $p_1 \circ \varphi_i(w_1, w_2)$ is calculated as

$$\begin{aligned}
p_1 \circ \varphi_i(w_1, w_2) &= p_1(\varepsilon_2 w_2 w_1, w_1) \\
&= q_i \cdot \exp(\sqrt{-1}\varepsilon_1 \operatorname{Re}(\varepsilon_2 w_2 w_1)) \\
&= q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\operatorname{Re} w_1 \operatorname{Re} w_2 - \operatorname{Im} w_1 \operatorname{Im} w_2)).
\end{aligned}$$

This implies that our foregoing definition of π_k makes sense.

LEMMA 4.1. *The involution $\tilde{\omega}_2$ preserves the fibers of π_h. Moreover, there exists a lift X of the vector field $\frac{d}{d\theta}\exp(\sqrt{-1}\theta)$ by the map π_h, which is compatible with the involution $\tilde{\omega}_2$; that is,*

$$\tilde{\omega}_{2*}(X) = X.$$

Proof. It is easy to verify by direct calculation that $\tilde{\omega}_2$ preserves the fibers of π_h. The details are left to the reader.

To prove the existence of a lift X, we construct X explicitly. We define a vector field X_1 on $\partial D^2 \times \Sigma_g \setminus (\amalg \varphi_i(\partial D^2 \times D^2)$ as follows:

$$X_1(\exp(\sqrt{-1}\theta_0), x) = \frac{d}{d\theta}\exp(\sqrt{-1}\theta)\Big|_{\theta=\theta_0} \in T_{(\exp(\sqrt{-1}\theta_0), x)}(\partial D^2 \times \Sigma_g),$$

for a point $(\exp(\sqrt{-1}\theta_0), x) \in \partial D^2 \times \Sigma_g \setminus (\amalg \operatorname{Int} \varphi_i(\partial D^2 \times D^2))$. The vector field X_1 is described in $J \times S^1$ as

$$X_1(s + t\sqrt{-1}, z) = \frac{1}{\varepsilon_1}\frac{\partial}{\partial s}\Big|_s \in T_{(s+t\sqrt{-1}, z)}(J \times S^1).$$

We also define a vector field X_2 on $D^2 \times \partial D^2 \subset \partial h_i^2$ as

$$X_2(w_1,w_2)=\frac{\varrho(|w_1|^2)}{\varepsilon_1\varepsilon_2|w_1|^2}\left(x_1\frac{\partial}{\partial x_2}-y_1\frac{\partial}{\partial y_2}\right)+\frac{1-\varrho(|w_1|^2)}{\varepsilon_1\varepsilon_2}\left(x_2\frac{\partial}{\partial x_1}-y_2\frac{\partial}{\partial y_1}\right),$$

where $w_i=x_i+y_i\sqrt{-1}$ and $\varrho\colon[0,1]\to[0,1]$ is a monotone increasing smooth function that satisfies the following conditions:

- $\varrho(t)=0$ for $t\in\left[0,\frac{1}{3}\right]$;
- $\varrho(t)=1$ for $t\in\left[\frac{2}{3},1\right]$.

For $(w_1,w_2)\in\partial D^2\times\partial D^2$, we calculate $d\varphi_i(X_2(w_1,w_2))$ as follows:

$$\begin{aligned}
&d\varphi_i(X_2(w_1,w_2))\\
&=d\varphi_i\left(\frac{1}{\varepsilon_1\varepsilon_2}\left(x_1\frac{\partial}{\partial x_2}-y_1\frac{\partial}{\partial y_2}\right)\right) && (\because |w_1|=1)\\
&=\frac{1}{\varepsilon_1\varepsilon_2}x_1 d\varphi_i\left(\frac{\partial}{\partial x_2}\right)-\frac{1}{\varepsilon_1\varepsilon_2}y_1 d\varphi_i\left(\frac{\partial}{\partial y_2}\right)\\
&=\frac{1}{\varepsilon_1}x_1\left(x_1\frac{\partial}{\partial s}+y_1\frac{\partial}{\partial t}\right)-\frac{1}{\varepsilon_1}y_1\left(-y_1\frac{\partial}{\partial s}+x_1\frac{\partial}{\partial t}\right)\\
&=\frac{1}{\varepsilon_1}(x_1^2+y_1^2)\frac{\partial}{\partial s}\\
&=X_1(\varphi_i(w_1,w_2)).
\end{aligned}$$

Hence we can define a vector field $X=X_1\cup X_2$ on the manifold ∂M_h. Moreover, it can be shown that each of X_1 and X_2 is a lift of the vector field $\frac{d}{d\theta}\exp(\sqrt{-1}\theta)$ by the map π_h. Thus, the vector field X is a lift of $\frac{d}{d\theta}\exp(\sqrt{-1}\theta)$. We can show that the vector field X is compatible with the involution $\tilde{\omega}_2$ by direct calculation. This completes the proof of Lemma 4.1. □

We choose a base point $q_0\in\partial D^2\setminus(\amalg I_{q_i})$ and define a map $\Theta_X\colon f^{-1}(q_0)\to f^{-1}(q_0)$ as follows:

$$\Theta_X(x)=c_{X,x}(2\pi),$$

where $c_{X,x}$ is the integral curve of the vector field X constructed in Lemma 4.1 that satisfies $c_{X,x}(0)=x$. We identify $f^{-1}(q_0)$ with the surface Σ_g via the projection onto the second component. Then the map Θ_X is contained in the centralizer $C(\iota)\subset\mathrm{Diff}_+\Sigma_g$ because the vector field X is compatible with $\tilde{\omega}_2$. The isotopy class represented by Θ_X is the monodromy of the boundary of M_h. In particular, this class is contained in the group $\mathcal{H}_g(c)$. By Lemma 3.1, there exists an isotopy $H_t\colon\Sigma_g\to\Sigma_g$ that satisfies the following conditions:

- $H_0=\Theta_X$;
- H_1 preserves the curve c as a set;
- for each level t, H_t is in the centralizer $C(\iota)$.

We obtain the following isomorphism of Σ_g-bundles:

$$\partial M_h\cong[0,1]\times\Sigma_g/((1,x)\sim(0,H_1(x))).$$

We identify these Σ_g-bundles via the isomorphism. Under this identification, the involution $\tilde{\omega}_2$ acts on ∂M_h as

$$\tilde{\omega}_2(t,x) = (t, \iota(x)),$$

where (t,x) is an element in $[0,1] \times \Sigma_g/((1,x) \sim (0, H_1(x))) \cong \partial M_h$.

Step 3. In this step, we define an involution ω_3 on the round 2-handle R^2. Since c is nonseparating and since c is preserved by ι, it follows that c contains two fixed points of the involution ι; we denote these points by v_1 and v_2. We can take a tubular neighborhood $\nu c \cong S^1 \times [-1,1]$ in Σ_g such that the involution ι acts on νc as follows:

$$\iota(z,t) = (\bar{z}, -t).$$

By perturbing the map H_1, we can assume that H_1 preserves the neighborhood νc. Since the genus of the fibration f is not equal to 1, the attaching region of the round 2-handle R^2 must be $[0,1] \times \nu c/((1,x) \sim (0, H_1(x)))$.

Case 3.1. If H_1 preserves the orientation of c and of the two points v_1 and v_2, then the round handle R^2 is untwisted and the restriction $H_1|_{\nu c}$ is described as

$$H_1(z,t) = (z,t),$$

where $(z,t) \in S^1 \times [-1,1] \cong \nu c$. Moreover, the attaching map of the round handle is described as

$$\begin{array}{ccc} \varphi\colon [0,1] \times \partial D^2 \times D^1/\sim & \longrightarrow & [0,1] \times S^1 \times [-1,1]/\sim \\ \Psi & & \Psi \\ (s,z,t) & \longmapsto & (s,z,t), \end{array}$$

where $[0,1] \times \partial D^2 \times D^1$ is the attaching region of R^2 and $[0,1] \times S^1 \times [-1,1] \cong [0,1] \times \nu c$ is the subset of ∂M_h. We define an involution ω_3 on the round handle as follows:

$$\begin{array}{ccc} \omega_3\colon [0,1] \times D^2 \times D^1/\sim & \longrightarrow & [0,1] \times D^2 \times D^1/\sim \\ \Psi & & \Psi \\ (s,z,t) & \longmapsto & (s,\bar{z},-t). \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} [0,1] \times \partial D^2 \times D^1 & \xrightarrow{\omega_3} & [0,1] \times \partial D^2 \times D^1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ [0,1] \times S^1 \times [-1,1] & \xrightarrow{\tilde{\omega}_2} & [0,1] \times S^1 \times [-1,1]. \end{array}$$

Therefore, we obtain an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $M_h \cup M_r = M_h \cup R^2$.

Case 3.2. If H_1 preserves the orientation of c but does not preserve the points v_1 and v_2, then the round handle R^2 is untwisted and the restriction $H_1|_{\nu c}$ is described as

$$H_1(z,t) = (-z,t).$$

The attaching map of the round handle is described as

$$\begin{array}{cccc} \varphi\colon & [0,1] \times \partial D^2 \times D^1/\sim & \longrightarrow & [0,1] \times S^1 \times [-1,1]/\sim \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & (s,z,t) & \longmapsto & (s, \exp(\pi\sqrt{-1}s)z, t). \end{array}$$

We define an involution ω_3 on the round handle as follows:

$$\begin{array}{cccc} \omega_3\colon & [0,1] \times D^2 \times D^1/\sim & \longrightarrow & [0,1] \times D^2 \times D^1/\sim \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & (s,z,t) & \longmapsto & (s, \exp(-2\pi\sqrt{-1}s)\bar{z}, -t). \end{array}$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $M_h \cup M_r = M_h \cup R^2$ by the same reason as in Case 3.1.

Case 3.3. If H_1 does not preserve the orientation of c but preserves two points v_1 and v_2, then the round handle R^2 is twisted and the restriction $H_1|_{\nu c}$ is described as

$$H_1(z,t) = (\bar{z}, -t),$$

where $(z,t) \in S^1 \times [-1,1] \cong \nu c$. Moreover, the attaching map of the round handle is described as

$$\begin{array}{cccc} \varphi\colon & [0,1] \times \partial D^2 \times D^1/\sim & \longrightarrow & [0,1] \times S^1 \times [-1,1]/\sim \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & (s,z,t) & \longmapsto & (s,z,t). \end{array}$$

We define an involution ω_3 on the round handle as follows:

$$\begin{array}{cccc} \omega_3\colon & [0,1] \times D^2 \times D^1/\sim & \longrightarrow & [0,1] \times D^2 \times D^1/\sim \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & (s,z,t) & \longmapsto & (s,\bar{z},-t). \end{array}$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $M_h \cup M_r = M_h \cup R^2$.

Case 3.4. If H_1 preserves neither the orientation of c nor the points v_1 and v_2, then the round handle R^2 is twisted and the restriction $H_1|_{\nu c}$ is described as

$$H_1(z,t) = (-\bar{z}, -t),$$

where $(z,t) \in S^1 \times [-1,1] \cong \nu c$. Moreover, the attaching map of the round handle is described as

$$\begin{array}{ccc} \varphi\colon [0,1]\times\partial D^2\times D^1/\sim & \longrightarrow & [0,1]\times S^1\times[-1,1]/\sim \\ \Psi & & \Psi \\ (s,z,t) & \longmapsto & (s,\exp(\pi\sqrt{-1}s)z,t). \end{array}$$

We define an involution ω_3 on the round handle as follows:

$$\begin{array}{ccc} \omega_3\colon [0,1]\times D^2\times D^1/\sim & \longrightarrow & [0,1]\times D^2\times D^1/\sim \\ \Psi & & \Psi \\ (s,z,t) & \longmapsto & (s,\exp(-2\pi\sqrt{-1}s)\bar{z},-t). \end{array}$$

Then we can define an involution $\tilde{\omega}_3=\tilde{\omega}_2\cup\omega_3$ on $M_h\cup M_r=M_h\cup R^2$.

Eventually, we obtain the involution $\tilde{\omega}_3$ on $M_h\cup M_r$ in any case. We next look at Σ_{g-1}-bundle structure of $\partial(M_h\cup M_r)$. The projection $\pi_r\colon\partial(M_h\cup M_r)\to[0,1]/\{0,1\}$ of this bundle is described as

$$\pi_r(s,x)=s\quad\big((s,x)\in([0,1]\times\Sigma_g/(1,x)\sim(0,H_1(x)))\setminus([0,1]\times\nu c/\sim)\big);$$

$$\pi_r(s,z,t)=s\quad((s,z,t)\in[0,1]\times D^2\times\partial D^1).$$

Indeed, it is easy to show that π_r is well-defined.

LEMMA 4.2. *The involution $\tilde{\omega}_3$ preserves the fibers of π_r. Moreover, there exists a lift $\tilde{X}$ of the vector field $\frac{d}{ds}$ on $[0,1]/\{0,1\}$ by the map π_r that is compatible with the involution $\tilde{\omega}_3$.*

Proof. It is obvious that the involution $\tilde{\omega}_3$ preserves the fibers of π_r. We construct $\tilde{X}$ as in Lemma 4.1. Define a vector field $\tilde{X}_1$ on $([0,1]\times\Sigma_g/\sim)\setminus([0,1]\times\nu c/\sim)$ as follows:

$$\tilde{X}_1(s,x)=\frac{d}{ds}.$$

We first consider the case where H_1 preserves the points v_1 and v_2. In this case, we define a vector field $\tilde{X}_2$ on the round handle R^2 as

$$\tilde{X}_2(s,z,t)=\frac{d}{ds},$$

where $(s,z,t)\in[0,1]\times D^2\times\partial D^1\subset\partial R^2$. It is easy to verify the equality $d\varphi\big(\frac{d}{ds}\big)=\frac{d}{ds}$, so we can define a vector field $\tilde{X}=\tilde{X}_1\cup\tilde{X}_2$ on $\partial M_h\cup M_r$. It is obvious that $\tilde{X}$ is a lift of the vector field $\frac{d}{ds}$ on $[0,1]/\{0,1\}$ by π_r and is compatible with the involution $\tilde{\omega}_3$.

We next consider the case where H_1 does *not* preserve the points v_1 and v_2. In this case, we define a vector field $\tilde{X}_2$ on R^2 as

$$\tilde{X}_2(s,x+y\sqrt{-1},t)=\frac{d}{ds}+\pi y\frac{\partial}{\partial x}-\pi x\frac{\partial}{\partial y},$$

where

$$(s,x+y\sqrt{-1},t)\in[0,1]\times D^2\times\partial D^1\subset\partial R^2.$$

The differential $d\varphi(\tilde{X}_2(s, x+\sqrt{-1}y, t))$ is then calculated as follows:

$$
\begin{aligned}
&d\varphi(\tilde{X}_2(s, x+\sqrt{-1}y, t)) \\
&\quad = d\varphi\left(\frac{d}{ds} + \pi y \frac{\partial}{\partial x} - \pi x \frac{\partial}{\partial y}\right) \\
&\quad = \left(\frac{d}{ds} + \pi(-x\sin\pi s - y\cos\pi s)\frac{d}{dx} + \pi(x\cos\pi s - y\sin\pi s)\frac{d}{dy}\right) \\
&\qquad + \pi y\left(\cos\pi s\frac{d}{dx} + \sin\pi s\frac{d}{dy}\right) - \pi x\left(-\sin\pi s\frac{d}{dx} + \cos\pi s\frac{d}{dy}\right) \\
&\quad = \frac{d}{ds} \\
&\quad = \tilde{X}_1(\varphi(s, x+\sqrt{-1}y, t)).
\end{aligned}
$$

We can therefore define a vector field $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$ on $\partial(M_h \cup M_r)$. It is obvious that $\tilde{X}$ is a lift of the vector field $\frac{d}{ds}$ on $[0,1]/\{0,1\}$ by π_r. To verify that $\tilde{X}$ is compatible with the involution $\tilde{\omega}_3$, we need to prove that

$$
d\tilde{\omega}_3(\tilde{X}(x)) = \tilde{X}(\tilde{\omega}_3(x)) \quad \text{for any } \ x \in \partial(M_h \cup M_r).
$$

If x is contained in $[0,1]\times\Sigma_g/{\sim}\setminus([0,1]\times\nu c/{\sim})$, then this equation can be proved easily. If $x = (s, x+\sqrt{-1}y, t) \in [0,1]\times D^2 \times \partial D^1 \subset \partial R^2$, then $d\tilde{\omega}_3(\tilde{X}(x))$ is calculated as follows:

$$
\begin{aligned}
&d\tilde{\omega}_3(\tilde{X}(x)) \\
&= d\tilde{\omega}_3\left(\frac{d}{ds} + \pi y\frac{\partial}{\partial x} - \pi x\frac{\partial}{\partial y}\right) \\
&= \left(\frac{d}{ds} + 2\pi(-x\sin 2\pi s - y\cos 2\pi s)\frac{\partial}{\partial x} + 2\pi(-x\cos 2\pi s + y\sin 2\pi s)\frac{\partial}{\partial y}\right) \\
&\quad + \pi y\left(\cos 2\pi s\frac{\partial}{\partial x} - \sin 2\pi s\frac{\partial}{\partial y}\right) - \pi x\left(-\sin 2\pi s\frac{\partial}{\partial x} - \cos 2\pi s\frac{\partial}{\partial y}\right) \\
&= \frac{d}{ds} + \pi(-x\sin 2\pi s - y\cos 2\pi s)\frac{\partial}{\partial x} + \pi(-x\cos 2\pi s + y\sin 2\pi s)\frac{\partial}{\partial y} \\
&= \tilde{X}(\tilde{\omega}_3(x)).
\end{aligned}
$$

Thus, $\tilde{X}$ is compatible with the involution $\tilde{\omega}_3$. This completes the proof of Lemma 4.2. □

We now define the map $\Theta_{\tilde{X}} : \pi_r^{-1}(0) \to \pi_r^{-1}(0)$ as follows:

$$
\begin{array}{cccc}
\Theta_{\tilde{X}} : & \pi_r^{-1}(0) & \longrightarrow & \pi_r^{-1}(0) \\
& \uplus & & \uplus \\
& x & \longmapsto & c_{\tilde{X},x}(1);
\end{array}
$$

here $c_{\tilde{X},x}$ is the integral curve of $\tilde{X}$ starting at x. We identify the fiber $\pi_r^{-1}(0)$ with the surface Σ_{g-1}. The map $\Theta_{\tilde{X}}$ is contained in the centralizer $C(\iota)$ because $\tilde{X}$ is compatible with $\tilde{\omega}_3$. Furthermore, $\Theta_{\tilde{X}}$ is isotopic to the identity map. By Theorem 2.7, we can take an isotopy $\tilde{H}_t\colon \Sigma_{g-1} \to \Sigma_{g-1}$ that satisfies the following conditions:

- $\tilde{H}_0 = \Theta_{\tilde{X}}$;
- $\tilde{H}_1$ is the identity map;
- $\tilde{H}_t$ is contained in the centralizer $C(\iota)$.

Observe that such an isotopy may not be taken if the condition $g \geq 3$ is dropped. Indeed, the map $\pi_0 C(\iota) \to \mathcal{M}_1$ induced by the inclusion is not injective.

By using the isotopy $\tilde{H}_t$, we obtain the following isomorphism of a Σ_{g-1}-bundle:

$$\partial(M_h \cup M_r) \cong [0,1] \times \Sigma_{g-1}/(1,x) \sim (0,x).$$

The involution $\tilde{\omega}_3$ acts on $[0,1] \times \Sigma_{g-1}/(1,x) \sim (0,x)$ via that isomorphism as follows:

$$\tilde{\omega}_3(s,x) = (s,\iota(x)).$$

Step 4. We define an involution ω_4 on $D^2 \times \Sigma_{g-1}$ as

$$\omega_4(z,x) = (z,\iota(x)),$$

where $(z,x) \in D^2 \times \Sigma_{g-1}$. Let $\Phi\colon [0,1] \times \Sigma_{g-1}/\!\sim\ \to \partial D^2 \times \Sigma_{g-1}$ be the attaching map of the lower side. Since the genus of the fibration f is greater than 2, we can assume that Φ is given by $\Phi(s,x) = \big(\exp(2\pi\sqrt{-1}s), x\big)$. In particular, the following diagram commutes:

$$\begin{tikzcd}
{[0,1] \times \Sigma_{g-1}/\sim} \arrow[r, "\tilde{\omega}_3"] \arrow[d, "\Phi"'] & {[0,1] \times \Sigma_{g-1}/\sim} \arrow[d, "\Phi"] \\
\partial D^2 \times \Sigma_{g-1} \arrow[r, "\omega_4"] & \partial D^2 \times \Sigma_{g-1}.
\end{tikzcd}$$

Hence, we obtain an involution $\omega = \tilde{\omega}_3 \cup \omega_4$ on M.

We next look at the fixed point set of ω. The involution ω is equal to $\mathrm{id} \times \iota$ on $D^2 \times \Sigma_g$. Thus we obtain

$$M^\omega \cup D^2 \times \Sigma_g = D^2 \times \{v_1, \ldots, v_{2g+2}\},$$

where $v_1, \ldots, v_{2g+2} \in \Sigma_g$ are the fixed points of ι. Note that $M^\omega \cup D^2 \times \Sigma_g$ has the natural orientation derived from the orientation of D^2.

The involution ω acts on the 2-handle $h_i^2 = D^2 \times D^2$ as follows:

$$\omega(w_1,w_2) = \begin{cases} (\bar{w}_1, \bar{w}_2) & (c_i \text{ nonseparating}), \\ (-w_1, -w_2) & (c_i \text{ separating}); \end{cases}$$

here $(w_1,w_2) \in D^2 \times D^2$. Thus, the fixed point set $h_i^{2\,\omega}$ is equal to $(D^2 \cap \mathbb{R}) \times (D^2 \cap \mathbb{R})$ if c_i is nonseparating and is equal to $\{(0,0)\}$ if c_i is separating. Furthermore, if c_i is nonseparating then we can give an orientation to $(D^2 \cap \mathbb{R}) \times (D^2 \cap \mathbb{R})$

that is compatible with the orientation of $D^2 \times \{v_1, \dots, v_{2g+2}\}$. Hence, the fixed point set ${M_h}^\omega$ is the union of the oriented surfaces and the s points, where s is the number of Lefschetz singularities of f whose vanishing cycle is separating.

The involution ω acts on the round 2-handle R^2 as follows:

$$\omega(s,z,t) = \begin{cases} (s,\bar{z},-t) & \text{if } H_1 \text{ preserves the points } v_1 \text{ and } v_2, \\ (s, \exp(-2\pi\sqrt{-1}s)\bar{z}, -t) & \text{otherwise,} \end{cases}$$

where $(s,z,t) \in R^2 = [0,1] \times D^2 \times D^1/\sim$. Thus we obtain

$$R^{2\omega} = \begin{cases} [0,1] \times (D^2 \cap \mathbb{R}) \times \{0\}/\sim & \text{if } H_1 \text{ preserves the points } v_1 \text{ and } v_2, \\ \{(s,z,0) \in R^2 \mid \\ \quad z = r\exp(-\pi\sqrt{-1}s), r \in [-1,1]\} & \text{otherwise.} \end{cases}$$

Therefore, the fixed point set $R^{2\omega}$ is equal to the annulus or the Möbius band. As explained in the previous paragraph, we can give an orientation of the 2-dimensional part of ${M_h}^\omega$ in the canonical way. It is easy to see that any orientation of $R^{2\omega}$ is not compatible with this canonical orientation of ${M_h}^\omega$. In particular, even if $R^{2\omega}$ is the annulus, the 2-dimensional part of the fixed point set $(M_h \cup M_r)^\omega$ may not be orientable. Indeed, this part is orientable if and only if $R^{2\omega}$ is the annulus and there is a connected component in ${M_h}^\omega$ whose boundary contains only one component of $\partial R^{2\omega}$.

The involution ω is equal to $\mathrm{id} \times \iota$ on $D^2 \times \Sigma_{g-1}$. Thus, the fixed point set $(D^2 \times \Sigma_{g-1})^\omega$ is equal to $D^2 \times \{\tilde{v}_1, \dots, \tilde{v}_{2g}\}$, where $\{\tilde{v}_1, \dots, \tilde{v}_{2g}\}$ is the set of the fixed points of ι. Eventually, M^ω is the union of the closed surfaces and the s points. The 2-dimensional part of M^ω is orientable if and only if that part of $(M_h \cup M_r)^\omega$ is orientable. This completes the proof of Theorem 1.1's statement on the fixed point set of ω.

We next extend the involution ω to the manifold $M\#s\overline{\mathbb{CP}^2}$. We assume that the curves $c_{k_1}, \dots, c_{k_s}$ are separating. We construct the manifold $M\#s\overline{\mathbb{CP}^2}$ by blowing up M s times at $(0,0) \in h^2_{k_i}$ $(i = 1, \dots, s)$. We can obtain a natural decomposition of $M\#s\overline{\mathbb{CP}^2}$ as follows:

$$M\#s\overline{\mathbb{CP}^2} = D^2 \times \Sigma_g \cup (h^2_1 \amalg \overset{\hat{k}_1,\dots,\hat{k}_s}{\cdots} \amalg h^2_n) \cup (\tilde{h}_{k_1} \amalg \cdots \amalg \tilde{h}_{k_s}) \cup R^2 \cup D^2 \times \Sigma_{g-1},$$

where $\tilde{h}_{k_i} = \{((w_1,w_2),[l_1:l_2]) \in D^2 \times D^2 \times \mathbb{CP}^1 \mid w_1 l_2 - w_2 l_1 = 0\} \cong h_{k_i}\#\overline{\mathbb{CP}^2}$. We define an involution $\bar{\omega}$ on $M\#s\overline{\mathbb{CP}^2}$ as follows:

$$\bar{\omega}(x) = \omega(x) \quad (x \in M\#s\overline{\mathbb{CP}^2} \setminus (\tilde{h}_{k_1} \amalg \cdots \amalg \tilde{h}_{k_s})),$$

$$\bar{\omega}((w_1,w_2),[l_1:l_2]) = ((-w_1,-w_2),[l_1:l_2]) \quad (((w_1,w_2),[l_1:l_2]) \in \tilde{h}_{k_i}).$$

It is obvious that $\bar{\omega}$ is an extension of ω. The fixed point set of $\bar{\omega}$ is the union of the 2-dimensional part of M^ω and s 2-spheres.

We next prove that $M\#s\overline{\mathbb{CP}^2}/\bar{\omega}$ is diffeomorphic to $S\#2s\overline{\mathbb{CP}^2}$, where S is an S^2-bundle over S^2. Since Σ_g/ι is diffeomorphic to S^2, it is easy to see that $D^2 \times \Sigma_g/\bar{\omega}$

is diffeomorphic to $D^2 \times S^2$. Thus, the manifold $M \# s\overline{\mathbb{CP}^2}$ is obtained by attaching $h_j/\bar{\omega}$ ($j \neq k_1, \ldots, k_s$), $\tilde{h}_{k_i}/\bar{\omega}$, $R^2/\bar{\omega}$, and $D^2 \times \Sigma_{g-1}/\bar{\omega} \cong D^2 \times S^2$ to $D^2 \times S^2$.

LEMMA 4.3. *Suppose that c_i is nonseparating. Then*

$$(D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2)/\bar{\omega} \cong D^2 \times S^2.$$

Proof. If we identify $h_i^2 = D^2 \times D^2$ with D^4, then $\bar{\omega}$ is equal to the covering transformation of the double covering $D^4 \to D^4$ branched at the unknotted 2-disk in D^4. In particular, we obtain that $h_i^2/\bar{\omega}$ is diffeomorphic to D^4. Moreover, the attaching region of h_i^2 corresponds to the 3-disk in ∂D^4 under the diffeomorphism. Denote by $\bar{\varphi}_i : h_i^2/\bar{\omega} \to \partial D^2 \times \Sigma_g/\bar{\omega}$ the embedding induced by φ_i. Then

$$\begin{aligned}(D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2)/\bar{\omega} &\cong (D^2 \times \Sigma_g/\bar{\omega}) \cup_{\bar{\varphi}_i} h_i^2/\bar{\omega} \\ &\cong D^2 \times S^2 \natural D^4 \\ &\cong D^2 \times S^2.\end{aligned}$$

This completes the proof of Lemma 4.3. □

LEMMA 4.4. *For each $i \in \{1, \ldots, s\}$, $(D^2 \times \Sigma_g \cup_{\varphi_i} \tilde{h}_{k_i}^2)/\bar{\omega} \cong D^2 \times S^2 \# 2\overline{\mathbb{CP}^2}$.*

Proof. By eliminating the corner of $D^2 \times D^2$, we identify $\tilde{h}_{k_i}^2$ with the manifold

$$H = \{((w_1, w_2), [l_1 : l_2]) \in D^4 \times \mathbb{CP}^1 \mid w_1 l_2 - w_2 l_1 = 0\}.$$

Under this identification, the attaching region of $\tilde{h}_{k_i}^2$ corresponds to the tubular neighborhood of the circle $\{((w_1, 0), [1 : 0]) \in \partial H \mid |w_1| = 1\}$ in ∂H. Let $p_2 : H \to \mathbb{CP}^1$ be the projection onto the second component. The map p_2 is the D^2-bundle over the 2-sphere with Euler number -1. We define $D_1, D_2 \subset \mathbb{CP}^1$ and also the local trivializations ψ_1 and ψ_2 of p_2 as follows:

$$D_1 = \{[l_1 : l_2] \in \mathbb{CP}^1 \mid |l_1| \geq |l_2|\},$$
$$D_2 = \{[l_1 : l_2] \in \mathbb{CP}^1 \mid |l_2| \geq |l_1|\};$$

$$\begin{array}{rccc} \psi_1 \colon & D^2 \times D^2 & \longrightarrow & p_2^{-1}(D_1) \\ & \Psi & & \Psi \\ & (w_1, w_2) & \longmapsto & \left(\frac{w_2}{\sqrt{1+|w_1|^2}}(1, w_1), [1, w_1]\right), \end{array}$$

$$\begin{array}{rccc} \psi_2 \colon & D^2 \times D^2 & \longrightarrow & p_2^{-1}(D_2) \\ & \Psi & & \Psi \\ & (w_1, w_2) & \longmapsto & \left(\frac{w_2}{\sqrt{1+|w_1|^2}}(w_1, 1), [w_1, 1]\right). \end{array}$$

Denote $p_2^{-1}(D_1)$ and $p_2^{-1}(D_2)$ by H_1 and H_2, respectively. We identify H_1 and H_2 with $D^2 \times D^2$ by the preceding trivializations. The manifold H can be identified with $D^2 \times D^2 \cup_\Psi D^2 \times D^2$, where $\Psi = \psi_1^{-1} \circ \psi_2 \colon (w_1, w_2) \longmapsto (1/w_1, w_1 w_2)$. Under the identification, the attaching region of H corresponds to $\partial D^2 \times D^2 \subset \partial H_1$.

We define $\tilde{H} = \tilde{H}_1 \cup_{\tilde{\Psi}} \tilde{H}_2$, where $\tilde{H}_i = D^2 \times D^2$ $(i = 1, 2)$ and $\tilde{\Psi} \colon \partial D^2 \times D^2 \to \partial D^2 \times D^2$ is a diffeomorphism defined as

$$\tilde{\Psi}(w_1, w_2) = (1/w_1, {w_1}^2 w_2).$$

We can define $\mathcal{P} \colon H \to \tilde{H}$ as follows:

$$\mathcal{P}(w_1, w_2) = \begin{cases} (w_1, {w_2}^2) \in \tilde{H}_1 & ((w_1, w_2) \in H_1), \\ (w_1, {w_2}^2) \in \tilde{H}_2 & ((w_1, w_2) \in H_2). \end{cases}$$

The map $\mathcal{P}$ is a double branched covering branched at the 0-section of $\tilde{H}$ as a D^2-bundle. Moreover, $\tilde{\omega}$ is the nontrivial covering transformation of $\mathcal{P}$. Thus we obtain that $H/\tilde{\omega}$ is diffeomorphic to $\tilde{H}$.

Since the attaching region of H is mapped by $\mathcal{P}$ to $D^2 \times \partial D^2 \subset \partial \tilde{H}_1$, we can regard $\tilde{H}_1$ and $\tilde{H}_2$ as 2-handles. Thus, $(D^2 \times \Sigma_g \cup_{\varphi_i} \tilde{h}^2_{k_i})/\bar{\omega}$ is obtained by attaching the 2-handles $\tilde{H}_1$ and $\tilde{H}_2$ to $D^2 \times S^2$. To prove the statement, we look at the attaching maps of $\tilde{H}_1$ and $\tilde{H}_2$.

Take an identification $\nu c_{k_i} \cong J \times S^1$ as in Step 2 of the construction of ω. The attaching map φ_{k_i} of the 2-handle $h^2_{k_i}$ satisfies $\varphi_{k_i}(w_1, w_2) = (\varepsilon_2 w_2 w_1, w_1)$. Since the manifold H is obtained by eliminating the corner of $\tilde{h}^2_{k_i}$, the attaching map of H_1 is described as

$$\begin{array}{rcl} \Phi \colon \partial H_1 \supset D^2 \times \partial D^2 & \longrightarrow & J \times S^1 \\ \cup\!\!\!\!\!\! \shortmid\;\; & & \cup\!\!\!\!\!\! \shortmid\;\; \\ (w_1, w_2) & \longmapsto & (\varepsilon_2 {w_2}^2 w_1, w_2). \end{array}$$

For an element $(z_1, z_2) \in J \times S^1$, the image $\bar{\omega}(z_1, z_2)$ is equal to $(z_1, -z_2)$. Thus, the manifold $J \times S^1/\bar{\omega}$ is diffeomorphic to $J \times S^1$ and the quotient map $/\bar{\omega} \colon J \times S^1 \to J \times S^1/\bar{\omega} \cong J \times S^1$ satisfies the equality $/\bar{\omega}(z_1, z_2) = (z_1, {z_2}^2)$. The attaching map $\tilde{\Phi} \colon D^2 \times \partial D^2 \to J \times S^1$ of $\tilde{H}_1$ satisfies the equality $\tilde{\Phi}(w_1, w_2) = (\varepsilon_2 w_2 w_1, w_2)$. It is easy to see that the attaching circle of $\tilde{H}_1$ is equal to the circle $c_{k_i}/\bar{\omega}$. Moreover, the framing of $\tilde{\Phi}$ is -1 relative to the framing along $\{*\} \times S^2 \subset \partial D^2 \times S^2$.

By the definition of $\tilde{\Psi}$, the attaching circle of $\tilde{H}_2$ is equal to the belt circle of $\tilde{H}_1$, which is isotopic to the meridian of the attaching circle of $\tilde{H}_1$. In particular, there exists the natural framing of the attaching circle of $\tilde{H}_2$ that is represented by the meridian of the attaching circle of $\tilde{H}_1$ parallel to the attaching circle of $\tilde{H}_2$. Since the Euler number of $\tilde{H}$ as a D^2-bundle is equal to -2, the framing of the attaching map $\tilde{\Psi}$ is equal to -2 relative to the natural framing. Therefore, we can draw a Kirby diagram of $(D^2 \times \Sigma_g \cup_{\varphi_i} \tilde{h}^2_{k_i})/\bar{\omega}$ as shown in Figure 7. It is obvious that this manifold is diffeomorphic to $D^2 \times S^2 \# 2\overline{\mathbb{CP}^2}$, and this completes the proof of Lemma 4.4. $\square$

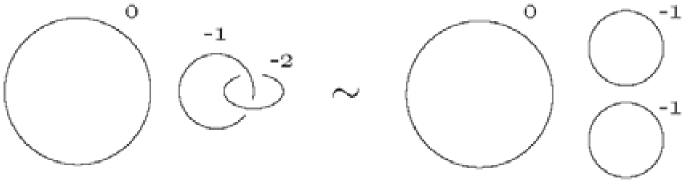

Figure 7 The (-1)-framed knot describes $\tilde{H}_1$; the (-2)-framed knot describes $\tilde{H}_2$

By applying the arguments in Lemma 4.3 and 4.4 successively, we can prove that $M_h \# s\overline{\mathbb{CP}^2}/\bar{\omega}$ is diffeomorphic to $D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$.

LEMMA 4.5. $((M_h \cup M_r)\# s\overline{\mathbb{CP}^2})/\bar{\omega} \cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$.

Proof. We can decompose R^2 into two components as follows:

$$R^2 = \left[0, \tfrac{1}{2}\right] \times D^2 \times D^1 \cup \left[\tfrac{1}{2}, 1\right] \times D^2 \times D^1.$$

Denote $\left[0, \frac{1}{2}\right] \times D^2 \times D^1$ and $\left[\frac{1}{2}, 1\right] \times D^2 \times D^1$ by R_1 and R_2, respectively. It is easy to see that $R_i/\bar{\omega}$ is diffeomorphic to D^4 and that R_i is the double covering of $D^4 \cong R_i/\bar{\omega}$ branched at the unknotted 2-disk.

The attaching region of R_1 is equal to $\left[0, \frac{1}{2}\right] \times \partial D^2 \times D^1$. The quotient $\left[0, \frac{1}{2}\right] \times \partial D^2 \times D^1/\bar{\omega}$ is a 3-ball in $\partial D^4 \cong \partial R_1$. Thus we obtain

$$\begin{aligned} (M_h \cup R_1)/\bar{\omega} &\cong M_h/\bar{\omega} \cup R_1/\bar{\omega} \\ &\cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2} \natural D^4 \\ &\cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}. \end{aligned}$$

The attaching region of R_2 is equal to $\left[\frac{1}{2}, 1\right] \times \partial D^2 \times D^1 \cup \left\{\frac{1}{2}, 1\right\} \times D^2 \times D^1$. The quotient $\left[\frac{1}{2}, 1\right] \times \partial D^2 \times D^1/\bar{\omega}$ is a 3-ball D_0 in $\partial D^4 \cong \partial R_2$, and $\left\{\frac{1}{2}, 1\right\} \times D^2 \times D^1/\bar{\omega}$ is a disjoint union of two 3-balls $D_1 \amalg D_2$ in ∂D^4. Both of the intersections $D_0 \cap D_1$ and $D_0 \cap D_2$ are 2-disks in ∂D_0. Eventually, the attaching region of R_2 is a 3-ball in ∂D^4. Thus, we can prove that $(M_h \cup R_1 \cup R_2)/\bar{\omega}$ is diffeomorphic to $D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$. This completes the proof of Lemma 4.5. □

It is easy to see that $D^2 \times \Sigma_{g-1}/\bar{\omega}$ is (a) diffeomorphic to $D^2 \times S^2$ and (b) attached to $(M_h \cup M_r)/\bar{\omega}$ such that the following diagram commutes:

$$\begin{array}{ccc} (M_h \cup M_r)/\bar{\omega} \supset S^1 \times S^2 & \longrightarrow & \partial D^2 \times S^2 \subset D^2 \times \Sigma_{g-1}/\bar{\omega} \\ \downarrow & & \downarrow \\ S^1 & \longrightarrow & \partial D^2; \end{array}$$

here the upper horizontal arrow in the diagram represents the attaching map, the lower horizontal arrow represents the identity map, and vertical arrows represent the projection onto the first component (in other words, the attaching map is a bundle map as a S^2-bundle over S^1). In particular, we obtain

$$M\#s\overline{\mathbb{CP}^2}/\bar{\omega} \cong S\#2s\overline{\mathbb{CP}^2}.$$

It is obvious that the quotient map $/\bar{\omega}\colon M\#s\overline{\mathbb{CP}^2} \to S\#2s\overline{\mathbb{CP}^2}$ is a double branched covering. This completes the proof of Theorem 1.1(i). $\square$

Proof of Theorem 1.1(ii). Let $F_h \subset M$ be a regular fiber in the higher side of f. It is easy to see that F_h represents the same rational homology class of M as that represented by F. Let $\omega\colon M \to M$ be the involution constructed in the proof of Theorem 1.1(i). If f has no indefinite fold singularities, then (a) the 2-dimensional part of the fixed point set M^{ω} of the involution ω is an orientable surface and (b) the algebraic intersection number between this part and F_h is equal to $2g+2$ and, especially, is nonzero. Then part (ii) of the theorem would hold.

So suppose that f has indefinite fold singularities. We first prove that F_h represents a nontrivial rational homology class of $M_h \cup M_r$. To prove this, we construct an element $\mathcal{S}$ in the group $H_2(M_h \cup M_r, \partial(M_h \cup M_r); \mathbb{Q})$ such that $[F_h]\cdot\mathcal{S} \neq 0$. Let $\tilde{S}$ be the intersection between the 2-dimensional part of M^{ω} and M_h, which is the union of compact oriented surfaces. We use the notation H_1, c, v_1, v_2, and R^2 as in the proof of Theorem 1.1(i).

Case 1. If the map H_1 preserves the orientation of c and the points v_1 and v_2, then R^2 is untwisted and $\tilde{S} \cap R^2 = \{(s,\pm 1,0) \in R^2 \mid s \in [0,1]\}$ is a disjoint union of two circles. We define four annuli A_1, A_2, A_3, and A_4 as follows:

$$A_1 = \{(s,t,0) \in R^2 \mid s \in [0,1],\ t \in [0,1]\},$$

$$A_2 = \{(s,t,0) \in R^2 \mid s \in [0,1],\ t \in [-1,0]\},$$

$$A_3 = \{(s,0,t) \in R^2 \mid s \in [0,1],\ t \in [0,1]\},$$

$$A_4 = \{(s,0,t) \in R^2 \mid s \in [0,1],\ t \in [-1,0]\}.$$

The union $S = \tilde{S} \cup A_1 \cup A_2 \cup A_3 \cup A_4$ represents the homology class of the pair $(M_h \cup M_r, \partial(M_h \cup M_r))$ after giving suitable orientations to the annuli A_1, A_2, A_3, and A_4. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S}\cdot[F_h]$ is nonzero and equal to $2g+2$.

Case 2. If the map H_1 preserves the orientation of c but does not preserve the points v_1 and v_2, then R^2 is untwisted and $\tilde{S} \cap R^2 = \{(s, \pm\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1]\}$ is a circle. We define three annuli A_5, A_6, and A_7 as follows:

$$\begin{aligned} A_5 = {} & \{(s, t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1],\ t \in [0,1]\} \\ & \cup \{(s, -t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1],\ t \in [0,1]\}, \end{aligned}$$

$$A_6 = \{(s,0,t) \in R^2 \mid s \in [0,1],\ t \in [0,1]\},$$

$$A_7 = \{(s,0,t) \in R^2 \mid s \in [0,1],\ t \in [-1,0]\}.$$

The union $S = \tilde{S} \cup A_5 \cup A_6 \cup A_7$ represents the homology class of the pair $(M_h \cup M_r, \partial(M_h \cup M_r))$ after giving suitable orientations to the annuli A_5, A_6, and A_7. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S}\cdot[F_h]$ is nonzero and equal to $2g+2$.

Case 3. If the map H_1 does not preserve the orientation of c but does preserve the points v_1 and v_2, then R^2 is twisted and $\tilde{S} \cap R^2 = \{(s, \pm 1, 0) \in R^2 \mid s \in [0,1]\}$ is a disjoint union of two circles. We define three annuli A_8, A_9, and A_{10} as follows:

$$\begin{aligned} A_8 &= \{(s,t,0) \in R^2 \mid s \in [0,1],\ t \in [0,1]\}, \\ A_9 &= \{(s,t,0) \in R^2 \mid s \in [0,1],\ t \in [-1,0]\}, \\ A_{10} &= \{(s,0,t) \in R^2 \mid s \in [0,1],\ t \in [0,1]\} \\ &\qquad \cup \{(s,0,t) \in R^2 \mid s \in [0,1],\ t \in [-1,0]\}. \end{aligned}$$

The union $S = \tilde{S} \cup A_8 \cup A_9 \cup A_{10}$ represents the homology class of the pair $(M_h \cup M_r, \partial(M_h \cup M_r))$ after giving suitable orientations to the annuli A_8, A_9, and A_{10}. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S} \cdot [F_h]$ is nonzero and equal to $2g+2$.

Case 4. If the map H_1 preserves neither the orientation of c nor the points v_1 and v_2, then R^2 is twisted and $\tilde{S} \cap R^2 = \{(s, \pm\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1]\}$ is a circle. We define two annuli A_{11} and A_{12} as follows:

$$\begin{aligned} A_{11} &= \{(s, t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1],\ t \in [0,1]\} \\ &\qquad \cup \{(s, -t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1],\ t \in [0,1]\}, \\ A_{12} &= \{(s,0,t) \in R^2 \mid s \in [0,1],\ t \in [0,1]\} \\ &\qquad \cup \{(s,0,t) \in R^2 \mid s \in [0,1],\ t \in [-1,0]\}. \end{aligned}$$

The union $S = \tilde{S} \cup A_{11} \cup A_{12}$ represents the homology class of the pair $(M_h \cup M_r, \partial(M_h \cup M_r))$ after giving suitable orientations to the annuli A_{11} and A_{12}. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S} \cdot [F_h]$ is nonzero and equal to $2g+2$.

Eventually, we can construct the element $\mathcal{S}$ satisfying the desired condition in any case. We have therefore established that $[F_h]$ is nontrivial in $H_2(M_h \cup M_r; \mathbb{Q})$.

We can now complete the proof of Theorem 1.1(ii). There exists the following exact sequence, which is the part of the Meyer–Vietoris exact sequence:

$$H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \xrightarrow{i_1 \oplus i_2} H_2(M_h \cup M_r; \mathbb{Q}) \oplus H_2(D^2 \times \Sigma_{g-1}; \mathbb{Q}) \xrightarrow{j_1 - j_2} H_2(M; \mathbb{Q}).$$

Suppose that $(j_1 - j_2)([F_h], 0) = [F_h] = 0$. Then there exists an element $\mu \in H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q})$ that satisfies the equality $(i_1 \oplus i_2)(\mu) = ([F_h], 0)$. By a Künneth formula, we obtain the isomorphism

$$H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \cong H_2(\Sigma_{g-1}; \mathbb{Q}) \oplus (H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})).$$

Since the map $i_2 \colon H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \to H_2(D^2 \times \Sigma_{g-1}; \mathbb{Q}) \cong H_2(\Sigma_{g-1}; \mathbb{Q})$ can be viewed as the projection onto the first component via the preceding isomorphism, it follows that the element μ is contained in $H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})$. The involution ω acts on the component $H_2(\Sigma_{g-1}; \mathbb{Q})$ trivially and on the component $H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})$ via multiplication by -1. Thus we obtain

$$\omega_*(\mu) = -\mu.$$

The composition $i_1 \circ \omega_*$ is equal to $\omega_* \circ i_1$ because i_1 is induced by the inclusion map. Therefore,

$$\begin{aligned}[F_h] &= \omega_*([F_h]) \\ &= \omega_* \circ i_1(\mu) \\ &= i_1 \circ \omega_*(\mu) \\ &= i_1 \circ (-\mu) = -[F_h].\end{aligned}$$

This means that $2[F_h] = 0$ in $H_2(M_h \cup M_r; \mathbb{Q})$, which contradicts $[F_h] \neq 0$. Thus we obtain $[F_h] \neq 0$ in $H_2(M; \mathbb{Q})$, and this completes the proof of part (ii). □

REMARK 4.6. By an argument similar to the one used in the proof of Theorem 1.1, we can generalize that theorem to directed BLFs as follows.

THEOREM 4.7. *Let $f: M \to S^2$ be a hyperelliptic directed BLF. Suppose that the genus of every connected component of fiber of f is greater than or equal to* 2.

(i) *Let s_1 be the number of Lefschetz singularities of f whose vanishing cycles are separating. We define*

$$s_2 = \max\{s \in \mathbb{N} \mid f^{-1}(x) \textit{ has } s \textit{ components, } x \in S^2\}.$$

Then there exists an involution

$$\omega: M \to M$$

such that the fixed point set of ω is the union of (possibly nonorientable) surfaces and s_1 isolated points. Moreover, the involution ω can be extended to an involution

$$\bar{\omega}: M\#s_1\overline{\mathbb{CP}^2} \to M\#s_1\overline{\mathbb{CP}^2}$$

such that $M\#s_1\overline{\mathbb{CP}^2}/\bar{\omega}$ is diffeomorphic to $\#s_2 S\#2s_1\overline{\mathbb{CP}^2}$; here S is an S^2-bundle over S^2, and the quotient map

$$/\bar{\omega}: M\#s_1\overline{\mathbb{CP}^2} \to M\#s_1\overline{\mathbb{CP}^2}/\bar{\omega} \cong \#s_2 S\#2s_1\overline{\mathbb{CP}^2}$$

is the double branched covering.

(ii) *Let $F \in M$ be a regular fiber of f. Then F represents a nontrivial rational homology class of M.*

We leave the details of the proof of Theorem 4.7 to the reader.

ACKNOWLEDGMENTS. The authors would like to express their gratitude to Hisaaki Endo for his continuous support during the course of this work and also to Nariya Kawazumi for his helpful comments on an earlier draft of the paper.

References

[1] D. Auroux, S. K. Donaldson, and L. Katzarkov, *Singular Lefschetz pencils,* Geom. Topol. 9 (2005), 1043–1114.

[2] R. İ. Baykur, *Existence of broken Lefschetz fibrations,* Int. Math. Res. Not. IMRN, 2008.

[3] ———, *Topology of broken Lefschetz fibrations and near-symplectic four-manifolds,* Pacific J. Math. 240 (2009), 201–230.
[4] ———, *Broken Lefschetz fibrations and smooth structures on 4-manifolds,* preprint, arXiv:math.GT/1205.5439.
[5] R. İ. Baykur and S. Kamada, *Classification of broken Lefschetz fibrations with small fiber genera,* preprint, arXiv:math.GT/1010.5814.
[6] J. S. Birman and H. M. Hilden, *On the mapping class groups of closed surfaces as covering spaces,* Advances in the theory of Riemann surfaces (Stony Brook, 1969), Ann. of Math. Stud., 66, pp. 85–115, Princeton Univ. Press, Princeton, NJ, 1971.
[7] B. Farb and D. Margalit, *A primer on mapping class groups,* Princeton Math. Ser., 49, Princeton Univ. Press, Princeton, NJ, 2012.
[8] T. Fuller, *Hyperelliptic Lefschetz fibrations and branched covering spaces,* Pacific J. Math. 196 (2000), 369–393.
[9] R. E. Gompf and A. I. Stipsicz, *4-Manifolds and Kirby calculus,* Grad. Stud. Math., 20, Amer. Math. Soc., Providence, RI, 1999.
[10] Y. Z. Gurtas, *On the slope of hyperelliptic Lefschetz fibrations and the number of separating vanishing cycles,* preprint, arXiv:math.SG/0810.1088.
[11] K. Hayano, *On genus-1 simplified broken Lefschetz fibrations,* Algebr. Geom. Topol. 11 (2011), 1267–1322.
[12] K. Hayano, *Modification rule of monodromies in R_2-move,* preprint, arXiv:math.GT/1203.4299.
[13] K. Hayano and M. Sato, *A signature formula for hyperelliptic broken Lefschetz fibrations,* preprint, arXiv:math.GT/1110.5286.
[14] A. Kas, *On the handlebody decomposition associated to a Lefschetz fibration,* Pacific J. Math. 89 (1980), 89–104.
[15] Y. Lekili, *Wrinkled fibrations on near-symplectic manifolds,* Geom. Topol. 13 (2009), 277–318.
[16] Y. Matsumoto, *Lefschetz fibrations of genus two—A topological approach,* Topology and Teichmüller spaces (Katinkulta, 1995), pp. 123–148, World Science, River Edge, NJ, 1996.
[17] B. Siebert and G. Tian, *On hyperelliptic C^∞ Lefschetz fibrations of four-manifolds,* Commun. Contemp. Math. 1 (1999), 255–280.
[18] J. D. Williams, *The h-principle for broken Lefschetz fibrations,* Geom. Topol. 14 (2010), 1015–1063.

K. Hayano
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka
Osaka 560-0043
Japan

k-hayano@cr.math.sci.osaka-u.ac.jp

M. Sato
Department of Mathematics Education
Faculty of Education
Gifu University
1-1 Yanagido, Gifu
Gifu 501-1193
Japan

msato@gifu-u.ac.jp