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Compact Subvarieties with Ample Normal Bundles, Algebraicity, and Cones of Cycles

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1. Introduction

In this paper we study two features of submanifolds (or possibly singular subvarieties) Z with ample normal bundle in a compact Kähler manifold X.

First we ask whether Z influences the algebraic dimension $a(X)$—that is, the maximal number of algebraically independent meromorphic functions. One expects the following conjecture to hold (for simplicity we shall assume Z smooth).

1.1. Conjecture. *Let X be a compact Kähler manifold containing a compact submanifold Z of dimension $d \geq 1$ with ample normal bundle. Then $a(X) \geq d+1$.*

For $d = \dim X - 1$ it is classically known that X is projective, but in higher codimensions there are only a few results [BaM; OP]. These results will be explicity discussed in Section 3. Here we remark that, for threefolds containing a curve with ample normal bundle, the conjecture holds up to a mysterious phenomenon concerning threefolds without meromorphic functions. Our results can be summarized as follows.

1.2. Theorem. *Conjecture 1.1 has a positive answer in any one of the following cases:*

(1) *Z moves in a family covering X;*
(2) *X is hyper-Kähler with $a(X) \geq 1$;*
(3) *Z is uniruled.*

In all cases, X is automatically projective.

Up to the standard conjecture that compact Kähler manifolds with non–pseudo-effective canonical bundles must be uniruled, assertion (3) holds even if Z is not of general type. These results suggest that Conjecture 1.1 may have a stronger version claiming that X must be projective, but this is very unlikely. We do exhibit (following [OP]) a candidate for a Kähler threefold X with $a(X) = 2$ containing a curve with ample normal bundle. However, a construction is still missing.

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The second part of the paper is concerned with *projective* manifolds X and curves $C \subset X$ with ample normal bundles. In the "dual" situation of a hypersurface Y with ample normal bundle, the line bundle $\mathcal{O}_X(Y)$ is big and is therefore in the interior of the pseudo-effective cone. We thus expect that the class $[C]$ is in the interior of the Mori cone $\overline{NE}(X)$ of curves, as follows.

1.3. CONJECTURE. *Let X be a projective manifold, and let $C \subset X$ be a curve with ample normal bundle. Then $[C]$ is in the interior of $\overline{NE}(X)$.*

Equivalently, if L is any nef line bundle such that $L \cdot C = 0$, then $L \equiv 0$. We prove the following theorem.

1.4. THEOREM. *Let X be a projective manifold, $C \subset X$ a smooth curve with ample normal bundle, and L a nef line bundle on X. If $H^0(X, mL) \neq 0$ for some $m > 0$ and if $L \cdot C = 0$, then $L \equiv 0$.*

The key is the fact (due to [S]) that the complement $X \setminus C$ is $(n-1)$-convex in the sense of Andreotti–Grauert, where $n = \dim X$. Hence we can prove the following more general result.

1.5. THEOREM. *Let Z be an $(n-1)$-convex manifold of dimension n. Let $Y \subset Z$ be a compact hypersurface (possibly reducible and nonreduced). Then the normal bundle $N_{Y/Z}$ cannot be nef.*

It is tempting to seek generalizations for q-convex manifolds and subvarieties of higher codimension; we discuss this in Section 4. We also prove some further results in the spirit of Theorem 1.4.

Finally, I would like to thank the referee for several valuable comments.

2. Preliminaries

We start by fixing some notation.

(1) Given a complex manifold X and a complex subspace $Y \subset X$ with defining ideal sheaf $\mathcal{I}$, the normal sheaf $\mathcal{N}_{Y/X}$ of Y is given by

$$\mathcal{N}_{Y/X} = \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) = (\mathcal{I}/\mathcal{I}^2)^*.$$

(2) A coherent sheaf $\mathcal{S}$ on a compact complex space X is *ample* if the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{S})$ is ample. Here the projectivization is taken in Grothendieck's sense (cf. e.g. [H2, II.7]). For details on ample sheaves we refer to [AT].

(3) The algebraic dimension of a compact manifold X (i.e., its transcendence degree over $\mathbb{C}$ in the field of meromorphic functions) will be denoted by $a(X)$.

(4) A compact Kähler manifold (or a manifold in class $\mathcal{C}$; i.e., bimeromorphic to a Kähler manifold) is called *simple* if there is no proper compact subvariety through a very general point of X. Equivalently, there is no family of proper subvarieties of X that cover X. In particular, $a(X) = 0$.

The only known examples of simple manifolds are (up to bimeromorphic equivalence) general complex tori and "general" hyper-Kähler manifolds. In dimension 3, Brunella [Br] proved that the canonical bundle of a simple manifold X must be pseudo-effective. It is also known [DP] that, if a simple threefold X has a minimal model X' (i.e., X' is a normal Kähler space with only, say, terminal singularities and $K_{X'}$ is nef), then $\kappa(X) = 0$; yet it is very much an open question whether $K_{X'} \equiv 0$. Once this is known, it follows that X is bimeromorphic to a quotient of a torus by a finite group.

The following proposition will be used to establish projectivity in Section 3.

2.1. Proposition. *Let X be a compact Kähler manifold, and let $(Z_s)_{s\in S}$ be a covering family of subvarieties (with S irreducible). Assume that the general Z_s is irreducible and reduced and that some irreducible reduced member Z_0 has ample normal sheaf. Then the general member Z_s is Moishezon.*

Proof. Let $q: U \to S$ be the graph of the family with projection $p: U \to X$. We obtain an inclusion $p^*(\Omega^1_X) \to \Omega^1_U$ and, in combination with the canonical surjection $\Omega^1_U \to \Omega^1_{U/S}$, a map

$$\alpha: p^*(\Omega^1_X) \to \Omega^1_{U/S}.$$

Let $\mathcal{S} = \operatorname{Ker}(\alpha)$, a torsion-free sheaf of rank r, say. Take $s \in S$ such that Z_s is irreducible and reduced and consider the complex-analytic fiber $\tilde{Z}_s := q^{-1}(s)$. Then $\tilde{Z}_s$ is generically reduced and set-theoretically we have $\tilde{Z}_s = Z_s$; in other words, Z_s is the reduction of $\tilde{Z}_s$. It follows immediately that

$$(\mathcal{S}|Z_s)/\mathrm{tor} = p^*(\mathcal{N}^*_{Z_s/X})/\mathrm{tor}.$$

We essentially need this equation for $s = 0$. In particular, let $\mathcal{T} = \left(\bigwedge^r \mathcal{S}\right)^*$; then $\mathcal{T}$ is a torsion-free sheaf of rank 1 and, by our assumption, $(\mathcal{T}|Z_0)/\mathrm{tor}$ is ample. Now we take normalizations $\tilde{U} \to U$ and $\tilde{S} \to S$ followed by a desingularization $\hat{U} \to \tilde{U}$, thus inducing a projection $\hat{q}: \hat{U} \to \tilde{S}$. Let $s_0 \in \tilde{S}$ be a point over 0 and let $\hat{Z}_{s_0}$ be the set-theoretic fiber over s_0, which might be reducible. Let A_0 be the irreducible component of $\hat{Z}_{s_0}$ mapping onto Z_{s_0}. Then we have a birational map $A_0 \to Z_0$. We may choose π such that

$$\pi^*(\mathcal{T})/\mathrm{tor} =: \hat{\mathcal{T}}$$

is locally free. Since $\mathcal{T}|Z_0$ is ample, it follows that $\hat{\mathcal{T}}|A_0$ is big and nef. Because $\hat{U} \to U$ is a projective morphism, we find a line bundle $\mathcal{M}$ on $\hat{U}$ such that $L = (\hat{\mathcal{T}})^{\otimes N} \otimes \mathcal{M}$ is ample on every component of $\hat{Z}_{s_0}$. Hence $L|\hat{Z}_{s_0}$ is ample and so $L|\hat{Z}_s$ is ample for general s. Therefore, the general Z_s is Moishezon. $\square$

For the reader's benefit we recall some definitions from convexity theory. First, a C^∞-function φ on a complex manifold X of dimension n is strongly q-convex if $i\partial\bar{\partial}\varphi$ has at least $n - q + 1$ positive eigenvalues. Also, a function φ on an arbitrary complex space X is strongly q-convex if every point $x \in X$ admits an open neighborhood U that can be embedded as a closed subspace into an open set V in $\mathbb{C}^n$ such that there is a strongly q-convex function ψ on V satisfying $\psi|U = \varphi|U$.

2.2. DEFINITION. Let X be a complex space.

(1) X is *q-convex* if there exist a continous function $\varphi\colon X \to \mathbb{R}$ and a compact set $K \subset X$ such that $\varphi|(X \setminus K)$ is strongly pseudo-convex and such that, for all real numbers c, the sublevel sets $\{\varphi < c\}$ are relatively compact in X.
(2) A 1-convex space is called strongly pseudo-convex.

Note that some authors use a shift by 1 in the definition of q-convexity; however, we follow the classical notion.

The theorem of Andreotti and Grauert [AnG] states that, given a coherent sheaf $\mathcal{F}$ on a q-convex space X, the cohomology groups $H^j(X, \mathcal{F})$ are finite-dimensional for $k \geq q$.

3. The Algebraic Dimension

In this section we study the following conjecture.

3.1. CONJECTURE. *Let X be a compact Kähler manifold, and let $Z \subset X$ be an irreducible compact subvariety of dimension d. Assume that the normal sheaf $\mathcal{N}_{Z/X}$ is ample. Then $a(X) \geq d + 1$.*

If Z is a divisor then the line bundle $\mathcal{O}_X(Z)$ is big (and nef); hence X is Moishezon and thus projective. In higher codimensions, there are two main results confirming the conjecture.

3.2. THEOREM [OP]. *Let X be a smooth compact Kähler threefold, and let $C \subset X$ be an irreducible curve with ample normal sheaf. Then $a(X) \geq 2$ except possibly where X is a simple threefold that is not bimeromorphic to a quotient of a torus by a finite group.*

3.3. THEOREM [BaM]. *Let X be a compact Kähler manifold of dimension n, and let $Y \subset X$ be a locally complete intersection of dimension p with ample normal bundle. Assume there is a covering family $(Z_s)_{s\in S}$ of q-cycles with $p+q+1 = n$. Then either $a(X) \geq p + 1$ or the following statement holds: The compact irreducible parameter space S is simple with* $\dim S = p + 1$, *the set*

$$\Sigma = \{s \in S \mid Z_s \cap Y \neq \emptyset\}$$

has pure codimension 1 in S, and $S \setminus \Sigma$ is strongly pseudo-convex.

For the case $p = 1$, we can use Theorem 3.3 to obtain our next result.

3.4. COROLLARY. *Let X be a compact Kähler manifold of dimension n, and let $Y \subset X$ be a smooth curve (or 1-dimensional local complete intersection) with ample normal bundle. Suppose that X is covered by subvarieties of codimension 2. Then $a(X) \geq 2$.*

Proof. By our assumption, there is a covering family $(Z_s)_{s\in S}$ of $(n-2)$-cycles. We apply the theorem of Barlet and Magnusson but must exclude the second alternative in Theorem 3.3. So assume that $\dim S = 2$, that S is simple, and that set Σ

has dimension 1 with strongly pseudo-convex complement $S \setminus \Sigma$. Now normalize S and apply the following lemma to produce a contradiction. □

3.5. LEMMA. *Let S be a normal compact surface whose desingularization is Kähler, and assume there is an effective curve $\Sigma \subset S$ such that $S \setminus \Sigma$ is strongly pseudo-convex. Then $a(S) \geq 1$.*

Proof. Since S has only finitely many singularities, we may blow up and assume from the beginning that S is smooth. Let $\tau\colon S \to S_0$ be a minimal model. Then, arguing by way of contradiction, S is either a torus or a $K3$ surface with $a(S_0) = 0$. In the $K3$ case, we contract all (-2)-curves and call the result again S_0. So in both cases S_0 is a (normal) surface without any curves. By our assumption, we can find a nonconstant holomorphic function $f \in \mathcal{O}(S \setminus \Sigma)$. This function yields a nonconstant holomorphic function on S_0 outside a finite set, which extends to S—a contradiction. □

3.6. REMARK. Corollary 3.4 should hold for all $p = \dim Y$—that is, for any local complete intersection Y of any codimension p. For this to be true we must prove the following claim.

> *Let X be a normal compact Kähler space and $\Sigma \subset X$ purely 1-codimensional such that $X \setminus \Sigma$ is strongly pseudo-convex. Then X cannot be simple.*

Assume that X is simple and $\dim X = 3$. As explained in Section 2, X should be bimeromorphic to T/G, where T is a simple torus and G a finite group. We verify the claim in this case. So let $\Sigma \subset X$ be purely 1-codimensional such that $X \setminus \Sigma$ is strongly pseudo-convex. Let $\pi\colon \hat{X} \to X$ be bimeromorphic such that $\hat{X}$ admits a holomorphic bimeromorphic map $f\colon \hat{X} \to T/G$. Let $\hat{\Sigma}$ be the preimage of Σ. Then $\hat{X} \setminus \hat{\Sigma}$ carries nonconstant holomorphic functions. But since $\dim f(\hat{\Sigma}) = 0$, we have a contradiction.

Of course, this argument holds in all dimensions. That is to say, it works for any Kähler space X (of any dimension) that is bimeromorphic to a torus modulo a finite group.

Next we use ideas from [OP] to address the question of whether, in Theorem 3.2, the case $a(X) = 2$ can actually occur.

3.7. PSEUDO-EXAMPLE. Let X be a smooth compact Kähler threefold with $a(X) = 2$. Assume that we have a holomorphic algebraic reduction

$$f\colon X \to S$$

to a smooth projective surface S with the following properties:

(1) there is an irreducible curve $B \subset S$ with $B^2 > 0$ whose preimage $X_B = f^{-1}(B)$ is irreducible;
(2) the general fiber of $f|X_B$ is a singular rational curve (with a simple cusp or node);
(3) X_B is projective.

Notice that X_B is always Moishezon but that the projectivity is not automatic.

Given these data, we choose a general hyperplane section $C \subset X_B$; hence C is a local complete intersection in X. Furthermore we have an exact sequence of vector bundles:

$$0 \to N_{C/X_B} \to N_{C/X} \to N_{X_B/X}|C \to 0.$$

Since $N_{X_B/X}|C = f^*(N_{B/S})|C$ is ample, the bundle $N_{C/X}$ is ample, too. If B is smooth, then so is C.

Certainly a Kähler threefold X satisfying the first two conditions must exist, although an explicit construction seems difficult. It is also plausible that the third condition holds in certain cases.

It is easy to fulfill all three conditions by allowing $B^2 = 0$. Here are the details. We start with a Kähler surface S_1 of algebraic dimension 1 and with algebraic reduction $f_1\colon S_1 \to T = \mathbb{P}_1$. We may choose S_1 so that there is a point $x_0 \in T$ such that the fiber $f_1^{-1}(x_0)$ is an irreducible rational curve with a simple cusp or node. Let S_2 be $\mathbb{P}_2$ blown up in nine points so that there is an elliptic fibration $f_2\colon S_2 \to T$. Set

$$X = S_1 \times_T S_2.$$

Here we have managed for X to be smooth by arranging f_2 to be smooth over the singular set of f_1. The projection $h\colon X \to S_2$ is the algebraic reduction; in particular, $a(X) = 2$. Let $B = f_2^{-1}(x_0)$, an elliptic curve, and observe that $X_B \simeq f_1^{-1}(x_0) \times B$ (which is projective).

If Z is a subvariety with ample normal sheaf moving in a covering family, then matters become much easier.

3.8. THEOREM. *Let X be a compact Kähler manifold, and let $Z \subset X$ be an irreducible reduced subspace with ample normal sheaf. Assume that Z moves in a generically irreducible and reduced family $(Z_s)_{s \in S}$ that covers X. Then X is projective.*

Proof. By Proposition 2.1, the general Z_s is Moishezon. We may assume that the family (Z_s) is not connecting. That is, two general points cannot be connected by a chain of curves Z_s, for otherwise X is already projective by Campana [C1] since then X is *algebraically* connected. Hence we may consider the quotient of the family, which yields an almost holomorphic map $f\colon X \dashrightarrow W$ that contracts two general points to the same point in W if and only if they can be joined by a chain of members Z_s (cf. [C1; C2]). Because the family is not connecting, we have $\dim W > 0$. Now the general Z_s is contained in a compact fiber X_w. Thus we obtain a generically surjective map

$$\mathcal{N}_{Z_s/X} \to \mathcal{N}_{X_w/X}|Z_s \simeq \mathcal{O}_{Z_s}^{\oplus k},$$

which contradicts the ampleness of $\mathcal{N}_{Z_s/X}$. □

Notice that the ampleness of the normal sheaf of Z need not mean that some multiple of Z moves; see [FL] for a counterexample.

Finally, we address the interesting case where X is a hyper-Kähler manifold.

3.9. THEOREM. *Let X be a compact hyper-Kähler manifold, and let $C \subset X$ be an irreducible curve with ample normal sheaf. If $a(X) \geq 1$, then X is projective.*

Proof. Suppose X is not projective. Following arguments used in [COP, (3.4)], let $g: X \dashrightarrow B$ be an algebraic reduction and let $\pi: \hat{X} \to X$ be a bimeromorphic map from a compact Kähler manifold $\hat{X}$ such that the induced map $f: \hat{X} \to B$ is holomorphic. Now fix an ample line bundle A on B and set

$$\mathcal{L} = (\pi_* f^*(A))^{**}.$$

Then, by [COP, (3.4)], the line bundle $\mathcal{L}$ is nef with $\mathcal{L} \cdot C = 0$. Since $\mathcal{L}$ is effective, this contradicts Corollary 4.4 (to follow). □

3.10. REMARK. Theorem 3.9 should hold without the assumption that $a(X) \geq 1$. In other words, a hyper-Kähler manifold X of dimension $2n$ containing an irreducible curve C with ample normal sheaf should be projective. Let q_X be the Beauville form. Then we have an isomorphism,

$$\iota: H^{1,1}(X, \mathbb{Q}) \to H^{2n-1,2n-1}(X, \mathbb{Q})$$

(see [COP, p. 411] for details). In particular, there exists a $u \in H^{1,1}(X, \mathbb{Q})$ such that $\iota(u) = [C]$; that is,

$$a \cdot C = q_X(a, u)$$

for all $a \in H^{1,1}(X, \mathbb{Q})$. Since u is a rational class, there must exist a positive rational number λ and a line bundle L such that $u = \lambda c_1(L)$. The hope now is that the positivity of the normal sheaf of C implies that L is nef (and that L is semi-ample).

Suppose $C \subset X$ is a smooth curve with small genus and ample normal bundle. Then we have the following algebraicity result (cf. [OP]).

3.11. PROPOSITION. *Let X be a compact Kähler manifold, and let $C \subset X$ be a smooth curve with ample normal bundle. If $g(C) \leq 1$, then the manifold X is projective.*

In fact, much more can be shown. If $g(C) = 0$, then X is rationally connected, and if $g(C) = 1$, then either X is rationally connected or the rational quotient has 1-dimensional image. In the latter case we have a holomorphic map $f: X \to W$ with rationally connected fiber to an elliptic curve W, and B is an étale multisection.

This result can be generalized to higher dimensions as follows.

3.12. THEOREM. *Let X be a compact Kähler manifold, and let $Z \subset X$ be a compact submanifold with ample normal bundle. If Z is uniruled, then X is uniruled and projective.*

Proof. Since Z is uniruled and since the normal bundle $N_{Z/X}$ is ample, it follows that X must be uniruled. In fact, since Z is uniruled, there exists a morphism $f: \mathbb{P}_1 \to Z$ such that $f^*(T_Z)$ is nef. The ampleness of $N_{Z/X}$ now implies that $f^*(T_X)$ is also nef. Therefore, X is uniruled.

Now let $f\colon X \dashrightarrow W$ be "the" rational quotient or MRC fibration. Then we apply [P2, Thm. 3.7] to conclude that Z dominates W. (We may apply that theorem because its essential ingredient—Lemma 3.6 in [P2]—works also in the Kähler case.) We know that W is projective (since it is dominated by Z). Moreover, the fibers of f are projective, so we may conclude that any two points of X can be joined by a chain of compact curves, whence X is projective (by [C1]). In other words, any two general points of X can be joined by a chain of algebraic subvarieties. □

3.13. REMARK. Theorem 3.12 should hold if we assume $\kappa(Z) < \dim Z$ instead of the uniruledness of Z. In fact, the Kähler version of [P2, Thm. 3.2] proves that at least K_X is not pseudo-effective. It is thus conjectured (although completely open in dimension ≥ 4) that X is uniruled. Once we know the uniruledness, we conclude as before.

We remark that one can also prove versions of Theorem 3.12 by weakening the ampleness condition.

4. Curves with Ample Normal Bundles and the Cone of Curves

If X is a projective manifold containing a hypersurface Y with ample normal bundle, then it follows (as we have already mentioned) that the line bundle $\mathcal{O}_X(Y)$ is big and hence that the class $[Y]$ is in the interior of the effective cone of X. We have the following dual expectation.

4.1. CONJECTURE. *Let X be a projective manifold and $C \subset X$ an irreducible curve. If the normal sheaf $N_{C/X}$ is ample, then $[C]$ is in the interior of the Mori cone $\overline{NE}(X)$.*

This conjecture can be restated as follows.

4.2. CONJECTURE. *Let X be a projective manifold, let L be a nef $\mathbb{R}$-line bundle, and let $C \subset X$ be an irreducible curve with ample normal bundle. If $L \cdot C = 0$, then $L \equiv 0$.*

It is noteworthy that, in codimensions other than 1 and $n-1$, the corresponding statement is false; this fact is demonstrated by an example of Voisin [V]. We consider first the case when C is smooth and L is an honest line bundle with section having smooth zero locus Y. In Theorem 4.5 we show that the smoothness assumption is not necessary, but it does make the argument a little easier. We will also see that Theorem 4.3 encodes a statement on convex manifolds, which we treat separately (Corollary 4.8) for the benefit of readers who are interested in only projective geometry.

4.3. THEOREM. *Let X be a projective manifold, and let $Y \subset X$ be a smooth hypersurface with nef normal bundle. Let $C \subset X$ be a smooth curve with ample normal bundle. Then $Y \cap C \neq \emptyset$.*

Proof. Let $n = \dim X$. Suppose to the contrary that $Y \cap C = \emptyset$ and set $Z = X \setminus C$. By [U], the normal bundle $N_{C/X}$ is "Griffiths positive" and so, by Schneider [S], Z is $(n-1)$-convex in the sense of Andreotti and Grauert. By their finiteness theorem [AnG],

$$\dim H^{n-1}(Z, \mathcal{F}) < \infty$$

for any coherent sheaf $\mathcal{F}$ on Z. Hereafter we shall use only line bundles $\mathcal{F}$. Let Y_k be the kth infinitesimal neighborhood Y; that is, Y is defined by the ideal $\mathcal{I}_Y^k$. Consider the exact sequence

$$H^{n-1}(Z, \mathcal{F}) \to H^{n-1}(Y_k, \mathcal{F}) \to H^n(Z, \mathcal{I}_Y^k \otimes \mathcal{F}).$$

Because the last group vanishes owing to the noncompactness of Z [Si], we conclude that $\dim H^{n-1}(Y_k, \mathcal{F})$ is bounded from above: there is a constant $M > 0$ such that

$$\dim H^{n-1}(Y_k, \mathcal{F}) \le M \tag{1}$$

for all positive k.

Now choose $\mathcal{F}$ to be a negative line bundle on Y. Then, by Kodaira vanishing,

$$H^1(Y, K_Y \otimes N_Y^{\mu} \otimes \mathcal{F}^*) = 0 \tag{2}$$

for all $\mu \ge 0$. Dually,

$$H^{n-2}(Y, N_Y^{*\mu} \otimes \mathcal{F}) = 0.$$

We thus obtain an exact sequence

$$\begin{aligned} 0 \to H^{n-2}(Y_k, \mathcal{F}) &\xrightarrow{b_k} H^{n-2}(Y_{k-1}, \mathcal{F}) \to H^{n-1}(Y, (N_Y^*)^k \otimes \mathcal{F}) \\ \to H^{n-1}(Y_k, \mathcal{F}) &\xrightarrow{a_k} H^{n-1}(Y_{k-1}, \mathcal{F}) \to 0. \end{aligned}$$

Note that b_k is an isomorphism for large k. By the boundedness statement (1), a_k is an isomorphism for $k \gg 0$. Therefore,

$$H^{n-1}(Y, (N_Y^*)^k \otimes \mathcal{F}) = 0$$

for $k \gg 0$. Dualizing now yields

$$H^0(Y, K_Y \otimes N_Y^k \otimes \mathcal{F}^*) = 0$$

for all $k \ge k_0(\mathcal{F})$). Setting $B = N_Y$ and $A = \mathcal{F}^*$ (for simplicity), we are in the following situation:

> *B is a nef line bundle on Y such that, for all ample line bundles A, there is a number $k_0(A)$ such that*
>
> $$H^0(Y, K_Y \otimes kB \otimes A) = 0$$
>
> *for $k \ge k_0(A)$.*

Equivalently, by Kodaira vanishing we have

$$\chi(Y, K_Y \otimes kB \otimes A) = 0.$$

This is clearly impossible by Riemann–Roch—a contradiction. We actually do not need to consider all ample A here; it suffices to take for A the powers of a fixed ample line bundle. □

4.4. COROLLARY. *Let $L = \mathcal{O}_X(Y)$ be a nef line bundle with Y smooth, and let $C \subset X$ be a smooth curve with ample normal bundle. Then $L \cdot C > 0$.*

Proof. Assume $L \cdot C = 0$. Then, by Theorem 4.3, we have $C \subset Y$. Now the normal bundle sequence

$$0 \to N_{C/Y} \to N_{C/X} \to N_{Y/X}|C = \mathcal{O}_C \to 0$$

contradicts the ampleness of $N_{C/X}$. □

4.5. THEOREM. *Theorem 4.3 remains true for singular (and possibly nonreduced) reducible hypersurfaces Y.*

Proof. The proof in the smooth case basically goes over to this case with the following modifications. Of course, the use of Kodaira vanishing (2) is critical. We fix a negative line bundle $\mathcal{F}$, which we may choose as a restriction of a negative line bundle $\tilde{\mathcal{F}}$ on X. Then we can apply Kodaira vanishing on X to obtain the vanishing (2) and also for higher H^qs. Namely,

$$H^q(X, K_X \otimes \tilde{\mathcal{F}}^* \otimes \mathcal{O}_X((\mu+1)Y)) = H^{q+1}(X, K_X \otimes \tilde{\mathcal{F}}^* \otimes \mathcal{O}_X(\mu Y))$$

implies (via the adjunction formula) that

$$H^q(Y, K_Y \otimes \mathcal{F}^* \otimes N_Y^{\mu}) = 0.$$

At the end we compute $\chi(K_Y \otimes kB \otimes A)$ via Riemann–Roch on X. In fact, we obtain as before that

$$\chi(Y, K_Y \otimes N_Y^k \otimes A) = 0 \tag{3}$$

for all extendable ample line bundles A on Y and for $k \geq k_0(A)$. By "extendability" we mean that there is an ample line bundle $\tilde{A}$ on X such that $\tilde{A}|Y = A$. Then, by (3), $\chi(X, K_X \otimes \mathcal{O}(kY) \otimes \tilde{A})$ is constant for large (and hence all) k. By Riemann–Roch this immediately implies $Y \equiv 0$, which is absurd. □

4.6. COROLLARY. *Conjecture 4.2 holds if L is effective and C is smooth (with ample normal bundle).*

The claim is a consequence of Theorem 4.5 and the following lemma due to Fulton and Lazarsfeld.

4.7. LEMMA. *Let $Y = \sum_{i=1}^N m_i Y_i$ be an effective nef divisor on the projective manifold X, and let $C \subset X$ be an irreducible curve with ample normal sheaf. Suppose that* $\operatorname{supp} Y \cap C \neq \emptyset$. *Then $Y \cdot C \neq 0$.*

Proof. We may assume that $C \subset \operatorname{supp}(Y)$. Let $Y_1, \ldots, Y_s$ be the components Y_j such that $C \subset Y_j$. We may also assume that $Y_j \cdot C \leq 0$ for some j, since otherwise we are already done. After renumbering, we have $j = 1$. Consider the canonical map

$$\kappa\colon N^*_{Y_1/X}|C \to N^*_{C/X}.$$

Since $N^*_{Y_1/X}|C$ is nef, it follows that $\kappa|C = 0$; that is, $C \subset \mathrm{Sing}(Y_1)$. Taking power series expansions of the local equation of Y_1, we obtain a number $k \geq 2$ and a nonzero map

$$N^*_{Y_1/X}|C \to S^k N^*_{C/X},$$

again contradicting the nefness of $N^*_{Y_1/X}|C$. □

It is interesting to note that the proof of Theorem 4.5 actually shows more—in particular, our next corollary.

4.8. Corollary. *Let Z be an $(n-1)$-convex complex manifold of dimension n, and let $Y \subset X$ be a compact hypersurface. Then the normal bundle $N_{Y/Z}$ cannot be nef.*

We are thus led to pose the following.

4.9. Question. Let X be a q-convex manifold, and let $Y \subset X$ be a compact subvariety with nef normal sheaf. Is then $\dim Y \leq q-1$?

Besides the case $q = n-1$, this question has a positive answer also for $q = 1$ because then there exists a proper modification $\phi\colon X \to W$ to a Stein space W. Thus $\dim \phi(C) = 0$, which easily contradicts the nefness of the normal sheaf of the curve Y.

Even if Y has ample normal sheaf, Question 4.9 remains open. In fact, a positive answer would imply a solution to this conjecture of Hartshorne [H1]:

> *Let Z be a projective manifold containing submanifolds X and Y with ample normal bundles. If* $\dim X + \dim Y \geq \dim Z$, *then* $X \cap Y \neq \emptyset$.

See [P3] for further information on this conjecture. The connection to Question 4.9 is provided by the convexity of $Z \setminus X$ (resp., of $Z \setminus Y$).

4.10. Remark. If C is a *singular* curve, then Theorem 4.5 should essentially remain valid. The only point that needs to be shown is the $(n-1)$-convexity of the complement $X \setminus C$, which is rather subtle. The results of Fritzsche [Fr1, Fr2] indicate that $X \setminus C$ is $(n-1)$-convex provided that the rank of the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ at every point is at most $n-1$—which means, of course, that $\mathcal{I}/\mathcal{I}^2$ is locally free and of rank $n-1$. This is true in particular when C is locally a complete intersection (see also [S] in this context).

Instead of making assumptions about the line bundle L, one might impose conditions on C.

4.11. Theorem. *Let X be a projective manifold, and let $C \subset X$ be an irreducible curve with ample normal bundle. Assume that C moves in a family (C_s) covering X. Let L be a nef line bundle with $L \cdot C = 0$. Then $L \equiv 0$.*

Proof. Let $f\colon X \dashrightarrow W$ be the nef reduction of L [Bau+]. The map f is almost holomorphic and, owing to the existence of the family (C_s), the map f is not trivial: $\dim W < \dim X$. A general member C_s still has ample normal bundle; in contrast, C_s is contained in a (compact) fiber of f. This is possible only when $\dim W = 0$. Hence, by [Bau+], we have $L \equiv 0$. □

5. Curves with Ample Normal Bundles: A Birational Point of View

A curve with ample normal bundle might not be in the interior of the movable cone $\overline{ME}(X)$. Simply start with a projective manifold Y containing a curve C with ample normal bundle and let $\pi : X \to Y$ be the blow-up of Y at a point $Y \notin C$. If $E \subset X$ is the exceptional divisor then $E \cdot C = 0$; hence $[C]$ is in the boundary of $\overline{ME}(X)$, since $\overline{ME}(X)$ is the dual cone of the pseudo-effective cone by [BDPaP]. However, there is a sense (albeit an imprecise one) in which this should be the only obstruction.

5.1. CONJECTURE. *Let X be a projective manifold and $C \subset X$ a curve with ample normal sheaf. Let L be a pseudo-effective line bundle with $L \cdot C = 0$. Then the numerical dimension $\nu(L) = 0$.*

This conjecture should be seen as a birational version of Conjecture 4.2. For the notion of the numerical dimension of a pseudo-effective line bundle, we refer to [B; BDPaP]. If L is nef then $\nu(L) = 0$ simply means that $L \equiv 0$, so Conjecture 5.1 implies Conjecture 4.2. (This is clear from the viewpoint of cones: $\overline{ME}(X) \subset \overline{NE}(X)$.)

Here is some evidence for Conjecture 5.1.

5.2. PROPOSITION. *Let L be a line bundle and $C \subset X$ an irreducible curve with ample normal sheaf. Assume that $L \cdot C = 0$. Then $\kappa(L) \le 0$.*

Proof. This is a direct consequence of [PSSo, Thm. 2.1]. □

5.3. REMARK. According to Boucksom [B], a pseudo-effective line bundle L admits a so-called divisorial Zariski decomposition:

$$L \equiv M + E,$$

where M is an $\mathbb{R}$-divisor that is nef in codimension 1 and E is an effective $\mathbb{R}$-divisor. Suppose $L \cdot C = 0$ for a curve C with ample normal sheaf. Since $[C] \in \overline{ME}(X)$, we have $E \cdot C \ge 0$ and $M \cdot C \ge 0$; hence

$$M \cdot C = 0.$$

Observe that $M \equiv 0$ is equivalent to $\nu(L) = 0$. As a result, Conjecture 5.1 is equivalent to the following.

5.4. CONJECTURE. *Let X be a projective manifold, let $C \subset X$ be an irreducible curve with ample normal sheaf, and let L be an $\mathbb{R}$-divisor that is nef in codimension 1. If $L \cdot C = 0$, then $L \equiv 0$.*

Showing that the class $[C]$ of an irreducible curve $C \subset X$ is in the interior of the movable cone requires a more global assumption than just the ampleness of the normal sheaf. We shall use the following notation introduced in [PSSo].

5.5. Definition. Let X be a projective manifold. A sequence

$$Y_q \subset Y_{q+1} \subset \cdots \subset Y_n = X$$

of k-dimensional irreducible subvarieties $Y_k \subset X$ is an ample q-flag if, for every $q \leq k \leq n-1$, there is an ample Cartier divisor D_k on the normalization $\eta_{k+1}\colon \tilde{Y}_{k+1} \to Y_{k+1}$ such that $Y_k = \eta_{k+1}(\operatorname{supp}(D_{k+1}))$.

5.6. Remark. The main result in [BDPaP] implies that the closed cone generated by the classes of curves appearing as the first member of an ample $(n-1)$-flag is the movable cone.

Our next theorem can now be easily shown.

5.7. Theorem. *Let X be a projective manifold, and let $C \subset X$ be an irreducible curve appearing in the ample flag*

$$C = Y_1 \subset \cdots \subset Y_q \subset \cdots \subset Y_n = X.$$

Then $[C]$ is in the interior of $\overline{ME}(X)$.

Proof. We must prove the following statement:

If L is a pseudo-effective line bundle with $L \cdot C = 0$, then $L \equiv 0$.

We prove inductively that $L|Y_i \equiv 0$ for all i.

The claim for $i = 1$ is our assumption $L \cdot C = 0$. So suppose the statement holds for i; that is, let $L|Y_i \equiv 0$. With notation as in Definition 5.5, we find an ample divisor D_{i+1} on $\tilde{Y}_{i+1}$ such that

$$\eta^*_{i+1}(L)|(\operatorname{supp} D_{i+1}) \equiv 0.$$

The proof is complete once we have shown the following proposition. □

5.8. Proposition. *Let X be a normal projective variety, let $D = \sum m_i D_i$ be an ample divisor, and let L be a pseudo-effective line bundle. If $L|D_i \equiv 0$ for all i, then $L \equiv 0$.*

Proof. By assumption, $L \cdot D_i = 0$ for all i; hence $L \cdot D = 0$. Let $H_1, \dots, H_{n-2}$ be arbitrary very ample divisors. Then

$$L \cdot H_1 \cdots H_{n-2} \in \overline{NE}(X)$$

because L is pseudo-effective. Hence the vanishing

$$L \cdot H_1 \cdots H_{n-2} \cdot D = 0,$$

together with the ampleness of D, implies that

$$L \cdot H_1 \cdots H_{n-2} = 0.$$

This equality holds only when $L \equiv 0$. □

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