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A Classification of Factorial Surfaces of Nongeneral Type

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1. Introduction

In this paper we deal exclusively with complex algebraic varieties.

An important invariant for a normal affine variety V is the logarithmic Kodaira dimension $\bar{\kappa}(V^\circ)$ as defined by S. Iitaka, where V° is the smooth locus of V. A rich structure theory of smooth quasiprojective surfaces has been developed by S. Iitaka, Y. Kawamata, T. Fujita, M. Miyanishi, T. Sugie, S. Tsunoda, R. Kobayashi, and other Japanese mathematicians (for an excellent exposition, see [7]). We will use this theory and standard algebraic topology to give a geometric description of all 2-dimensional affine UFDs (or, factorial) V such that $\bar{\kappa}(V^\circ)$ is at most 1. Many of the arguments in the proofs are by now standard (cf. [2, Sec. 3]).

This paper does not consider an algebraic description of the coordinate rings $\Gamma(V, \mathcal{O})$ of these unique factorization domains (UFDs). The multiplicative group of units in this ring will be denoted by $\Gamma(V, \mathcal{O})^*$.

We will prove the following four theorems.

Theorem 1. *Let V be a smooth, affine, factorial surface with $\Gamma(V, \mathcal{O})^* = \mathbb{C}^*$. Then we have the following assertions.*

- *If $\bar{\kappa}(V) = -\infty$, then $V \cong \mathbb{C}^2$.*
- *If $\bar{\kappa}(V) = 0$, then these surfaces are classified in* [3, Thm. 2] (*see Section 3*).
- *If $\bar{\kappa}(V) = 1$, then these surfaces are described in* [3] (*see Section 3*).

Remark. It is well known that any $\mathbb{Z}$-homology plane is factorial and has only trivial units.

Theorem 2. *Let V be an affine, factorial surface with at least one singular point and with $\Gamma(V, \mathcal{O})^* = \mathbb{C}^*$. Then we have the following assertions.*

- *If $\bar{\kappa}(V^\circ) = -\infty$ then $V \cong \mathbb{C}^2/\Gamma$, where Γ is the binary icosahedral group of order 120; hence V is the affine E_8-singularity $\{x^2 + y^3 + z^5 = 0\}$.*
- *If $\bar{\kappa}(V^\circ) = 0$, then V is obtained by one of the two constructions described in Section 4.*
- *If $\bar{\kappa}(V^\circ) = 1$, then V has a unique singular point with a good $\mathbb{C}^*$-action and $\Gamma(V, \mathcal{O})$ is a positively graded domain. These domains are all described by Mori in* [8] (*see Section 4*).

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THEOREM 3. *Let V be a smooth, affine, factorial surface with $\Gamma(V,\mathcal{O})^* \neq \mathbb{C}^*$. Then we have the following assertions.*

- *If $\bar{\kappa}(V) = -\infty$, then $V \cong \mathbb{C} \times (\mathbb{C} \setminus \{\text{at least one point}\})$.*
- *If $\bar{\kappa}(V) = 0$, then V is one of the surfaces $O(1,1,1)$ or $O(4,1)$ in Fujita's table in* [1, Sec. 8.64] *or is obtained from $O(1,1,1)$ as described in Section 5.*
- *If $\bar{\kappa}(V) = 1$ then V is obtained from a smooth $\mathbb{Z}$-homology plane with a $\mathbb{C}^*$-fibration by removing a certain number of regular fibers.*

THEOREM 4. *Let V be an affine, factorial surface with $\Gamma(V,\mathcal{O})^* \neq \mathbb{C}^*$. If $\bar{\kappa}(V^\circ) \leq 1$, then V has no singularities*; *hence V is one of the surfaces described in Theorem 3.*

In Section 7 we will use a result due to Parameswaran and Van Straten ([10]; see also [9]); this result states that any normal Gorenstein surface singularity can occur on a suitable 2-dimensional affine factorial surface V. It is easy to make sure that $\bar{\kappa}(V^\circ) = 2$. We can also give examples of affine factorial surfaces V with $\bar{\kappa}(V^\circ) = 2$ such that V has an arbitrarily large number of singular points. In view of these examples it seems impossible to classify factorial surfaces V with $\bar{\kappa}(V^\circ) = 2$. This explains our basic assumption in Theorems 1–4.

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2. Preliminaries

By an $\mathbb{A}^1$-fibration on a normal algebraic surface W we mean a morphism $f: W \to B$ onto a smooth algebraic curve B such that a general fiber of f is isomorphic to $\mathbb{A}^1$. Similarly, we can define a $\mathbb{P}^1$-fibration and a $\mathbb{C}^*$-fibration. Here $\mathbb{C}^*$ denotes the curve $\mathbb{A}^1 \setminus \{\text{one point}\}$. Such a $\mathbb{C}^*$-fibration is said to be *untwisted* if it is a trivial bundle on a Zariski-open dense subset of B. This is equivalent to the assertion that, in a suitable completion of W, there are two irreducible curves at infinity that are cross-sections for the corresponding $\mathbb{P}^1$-fibration on the completion. Otherwise the fibration is said to be *twisted*.

By a $(-n)$-curve on a smooth algebraic surface we mean a smooth, projective, irreducible, rational curve C with $C^2 = -n$. For a possibly reducible curve C, by a component of C we mean an irreducible component of C. A simple normal crossing divisor on a smooth algebraic surface is called an SNC divisor. An SNC divisor D is said to be *minimal normal crossing* (MNC) if, for every (-1)-curve E in D, either (a) $E.(D-E) \geq 3$ or (b) $E.(D-E) = 2$ and E meets a unique irreducible component of $D-E$. By a *factorial* surface we mean an affine surface whose coordinate ring is a UFD.

Let X be a smooth projective surface. By Castelnuovo's criterion of rationality, X is rational if the linear system $|nK_X| = \emptyset$ for $n \geq 1$ and the irregularity $q_X = 0$. Let V be a normal affine surface with $\bar{\kappa}(V^\circ) \leq 1$. Let X be a smooth projective completion of V° such that $X \setminus V^\circ = D \cup E_1 \cup E_2 \cup \cdots$ is a simple normal crossing divisor; here D is the divisor at infinity for the affine surface V and the E_i are the exceptional divisors for the resolution of singularities of V. Then $\dim|n(K + D + E_1 + E_2 + \cdots)|$ grows at most linearly with $n \geq 1$. Since D supports a big and nef (numerically effective) divisor for X, it follows that $|nK| = \emptyset$ for $n \geq 1$. Therefore, X is birationally a ruled surface (it is actually ruled unless $X \cong \mathbb{P}^2$). Suppose further that V is factorial. Then the irreducible components of $D \cup E_1 \cup E_2 \cup \cdots$ generate the divisor class group of X. As a consequence, $q_X = 0$. Combining these observations yields the following useful result, which will be used often in later arguments.

LEMMA 1. *Let V be a 2-dimensional affine factorial surface with $\bar{\kappa}(V^\circ) \leq 1$. Then V is rational.*

We will use some standard terminology (e.g., branch point, rod, twig, tip) about dual graphs of SNC divisors on a smooth algebraic surface and then structure results for noncomplete algebraic surfaces (cf. [7, Chap. 2, Sec. 3]).

LEMMA 2. *Let W be a normal affine surface. Then we have the following assertions.*

(1) *If $\bar{\kappa}(W^\circ) = -\infty$, then either*: (a) *there is an $\mathbb{A}^1$-fibration $f: W^\circ \to B$, where B is a smooth curve*; *or* (b) *W° contains a Zariski-open subset U isomorphic to $(\mathbb{C}^2/\Gamma)^\circ$, where Γ is a finite noncyclic subgroup of* $\mathrm{GL}(2, \mathbb{C})$ *without nontrivial pseudo-reflections and $W^\circ \setminus U$ is a disjoint union of curves isomorphic to $\mathbb{A}^1$.*

(2) *If $\bar{\kappa}(W^\circ) = 1$, then there is a $\mathbb{C}^*$-fibration $f: W^\circ \to B$ onto a smooth curve B.*

The following properties of a singular fiber of a $\mathbb{P}^1$-fibration on a smooth algebraic surface will be used later on.

LEMMA 3. *Let $f: W \to B$ be a $\mathbb{P}^1$-fibration on a smooth projective surface W, and let F_0 be a scheme-theoretic singular fiber of f. Then we have the following assertions.*

(1) *F_0 is an SNC divisor of smooth rational curves that is a tree.*
(2) *F_0 contains at least one (-1)-curve.*
(3) *If a (-1)-curve in F_0 occurs with coefficient 1, then F_0 contains another (-1)-curve.*

In particular, by successively contracting (-1)-curves, F_0 can be contracted to a regular fiber of a $\mathbb{P}^1$-fibration on the new surface.

Now assume that W is a normal affine surface such that $\bar{\kappa}(W^\circ) = 0$. We will use some important results due to Fujita.

Let Y be a smooth completion of W° such that $\Delta := Y \setminus W^\circ$ is an MNC divisor. Recall that there is a Zariski–Fujita decomposition $K + \Delta \approx P + N$, where $\approx$ denotes numerical equivalence, P and N are $\mathbb{Q}$-divisors, P is nef, N is effective, and $P.\Delta_i = 0$ for every irreducible curve Δ_i in supp N. The intersection form on the irreducible components of supp N is negative definite. By Kawamata's result [5], $P \approx 0$. If N is supported on Δ then we say that (Y, Δ) is *NC-minimal*.

REMARK. In [1, Sec. 8.9], the definition of NC-minimality involves one more condition—namely, that any (-1)-curve on Y that does not occur in Δ must meet Δ in at least two points counted properly. When we use Fujita's classification from [1, Sec. 8.64] in the proof of Theorem 3 we will employ this more restrictive definition, but otherwise NC-minimal will have the meaning just defined.

Assume now that $\bar{\kappa}(W^\circ) = 0$ and (Y, Δ) is NC-minimal. Then the dual graph of any connected component of Δ is restricted by the following useful result [1, Cor. 8.8].

LEMMA 4. *With assumptions as before, any connected component Δ_i of Δ is of one of the following six types.*

- (Type I) *Δ_i is the exceptional divisor of a minimal resolution of a quotient singular point.*
- (Type O) *Δ_i is a simple loop of smooth rational curves.*
- (Type H) *Δ_i is an SNC divisor of smooth rational curves with exactly two branch points; there are exactly two twigs at each of the branch points that are single (-2)-curves.*
- (Type Y) *Δ_i is a tree of smooth rational curves and has exactly one branch point; there are exactly three twigs meeting the branch point and they are all admissible. The absolute values of the determinants d_j of the three twigs satisfy $\sum \frac{1}{d_j} = 1$.*
- (Type X) *Δ_i is a tree of smooth rational curves with exactly one branch point and with four twigs meeting the branch point such that each twig is a single (-2)-curve.*
- (Type $\star$) *Δ_i is a smooth elliptic curve.*

Now assume that (Y, Δ) is MNC but not NC-minimal. Then we have the following useful result [1, Lemma 6.20].

LEMMA 5. *There exists a (-1)-curve L in* supp N *that is not in Δ and satisfies one of the following conditions.*

(1) $L \cap \Delta = \emptyset$.
(2) *$L.\Delta = 1$ and L meets a twig of Δ.*
(3) *$L.\Delta = 2$ and L meets two connected components of Δ, one of which is a tip of a negative definite linear chain of smooth rational curves (i.e., a rod) while the other is an admissible rational twig of D.*

Furthermore, in this case $\bar{\kappa}(Y \setminus (\Delta \cup L)) = 0$.

The next result is well known to experts in the field. It uses standard properties of the Zariski–Fujita decomposition (called "theory of peeling" in [7, Chap. 2]).

LEMMA 6. *Let W be a normal affine surface with at worst rational double points as singularities. Then, for a resolution of singularities $\widetilde{W}$ of W, we have $\bar{\kappa}(W^{\circ}) = \bar{\kappa}(\widetilde{W})$.*

Recall that, for a normal affine variety V, the group $U_V := \Gamma(V, \mathcal{O})^*/\mathbb{C}^*$ is finitely generated because any nonconstant unit in the coordinate ring of V gives rise to a relation $a_1D_1 + \cdots + a_nD_n \sim 0$, where $D_1, \dots, D_n$ are the irreducible components at infinity in a normal projective completion of V and the a_i are integers. The next result will be needed later in the proofs.

LEMMA 7. *Let V be a normal affine factorial variety. Then, for any irreducible divisor D in V, we have* $\operatorname{rank} U_V < \operatorname{rank} U_{V-D}$.

We omit the easy proof.

The next result is probably well known to experts. However, we include it here for completeness.

LEMMA 8. *Let V be a smooth affine factorial surface with a $\mathbb{C}^*$-fibration $f: V \to B$. Then f is trivial on a nonempty Zariski-open subset of B.*

According to Lemma 8, the $\mathbb{C}^*$-fibration is untwisted.

Proof of Lemma 8. We briefly sketch the argument. The irregularity of V is zero, so the curve B is rational. Let $V \subset X$ be a smooth completion with a $\mathbb{P}^1$-fibration $\Phi: X \to \bar{B}$, where $\bar{B}$ is a smooth completion of B. Suppose the result is not true. Then $X - V$ contains a unique component D_h that is horizontal for Φ. By removing all the irreducible components of all the singular fibers of f from V, we obtain an affine factorial open subvariety $V_0 \subset V$ such that $X - V_0 = D_h \cup F_1 \cup F_2 \cup \cdots \cup F_m$, where the F_i are full fibers of Φ. After changing the compactification of V_0 (via Lemma 3), we may assume that the F_i are regular fibers of Φ. Since any two fibers of Φ are rationally equivalent, it follows that the curves D_h and F_1 generate $\operatorname{Pic}(X)$ freely. However, it is easy to see that the intersection form of these two curves is not unimodular (see Remark (1) to follow). This contradiction proves the result. □

REMARKS. (1) For a smooth, projective, rational surface X, the cohomology group $H^2(X; \mathbb{Z})$ is generated by finitely many cohomology classes corresponding to irreducible curves $C_1, C_2, \dots$. If these classes are independent, then the intersection matrix $(C_i.C_j)$ is unimodular by Poincaré duality. Even if these classes are dependent, we can find finitely many integral linear combinations of these classes that generate $H^2(X; \mathbb{Z})$ freely.

(2) Let V be a factorial affine surface. Then the canonical divisor of V is principal. Since a normal surface is Cohen–Macaulay, it follows that V has Gorenstein singularities. If, in addition, V has at most rational singularities, then these singularities are all rational double points. The only unimodular rational singularity is E_8.

3. Proof of Theorem 1: V Is Smooth and $\Gamma(V, \mathcal{O})$ Has No Nonconstant Units

Case 1: $\bar{\kappa}(V) = -\infty$. By Lemma 2, we have an $\mathbb{A}^1$-fibration on V. Now, by the well-known Fujita–Miyanishi–Sugie result (see [7, Chap. 4, Thm. 2.2]), $V \cong \mathbb{C}^2$.

Case 2: $\bar{\kappa}(V) = 0$. We will describe the two possible surfaces in this case (as proved in [3]).

(1) Let L_1, L_2, L_3 be three lines in $\mathbb{P}^2$ that do not pass through a common point, and let $p_1 \in L_1$ and $p_2 \in L_2$ be points that do not lie on the other L_i. Let $X \to \mathbb{P}^2$ be the blow-up at p_1, p_2. Then $V := X - (L_1' \cup L_2' \cup L_3')$ is a simply connected, factorial surface with $\bar{\kappa}(V) = 0$. Here L_i' is the proper transform of L_i.

(2) Let C be a smooth conic and L a line meeting C transversally in $\mathbb{P}^2$. Let $p \in C$ be a general point, and let X be the blow-up of $\mathbb{P}^2$ at p. Then $V := X - (C' \cup L')$ is a simply connected, factorial surface with $\bar{\kappa}(V) = 0$.

It is easy to see that a simply connected normal affine variety cannot have nonconstant regular invertible functions.

Case 3: $\bar{\kappa}(V) = 1$. By Lemma 2, there is a $\mathbb{C}^*$-fibration $f: V \to B$. Since V is factorial, it follows from Lemma 8 that this is an untwisted fibration. The base $B \cong \mathbb{P}^1$ or $\mathbb{A}^1$. All such surfaces have been described in [3, Thm. 3, Thm. 4].

This completes the proof of Theorem 1.

4. Proof of Theorem 2: V Is Nonsmooth and Factorial with No Nonconstant Units

As before, let $V^\circ = V \setminus \operatorname{Sing} V$.

Let X be a suitable smooth compactification of V° such that $X \setminus V^\circ = D \cup E$ for D an MNC divisor of V at infinity and $E = \bigsqcup E_i$, where the E_i are the MNC exceptional divisors of the resolutions of singularities of V. The divisor class group of V° is also trivial and hence each singular point of V is unimodular—that is, the intersection form on each E_i is unimodular.

Case 1: $\bar{\kappa}(V^\circ) = -\infty$.

Subcase 1a: There exists an $\mathbb{A}^1$-fibration $\varphi: V^\circ \to B$. Suppose first that φ extends to an $\mathbb{A}^1$-fibration on V. Then, by [7, Chap. 3, Lemma 1.4.4], every singularity of V is a cyclic quotient singularity. Yet because cyclic quotient singularities are not unimodular, there cannot be any singularities in this case.

Now suppose that φ does not extend to a morphism on V. Then one of the singular points of V is a base point. This cannot happen, since the closure of an $\mathbb{A}^1$ is a complete curve and V is affine.

Subcase 1b: There exists no $\mathbb{A}^1$-fibration on V°. By Lemma 2, in this case V° contains an open subset U isomorphic to $\mathbb{C}^2/\Gamma \setminus \{p\}$, where Γ is a finite subgroup of automorphisms of $\mathbb{C}^2$ such that p is the singular point. Furthermore, $V^\circ \setminus U$ is a disjoint union of $\mathbb{A}^1$s.

Suppose $U \neq V^\circ$, and let $C \cong \mathbb{A}^1 \subset V^\circ \setminus U$. We know that C is a closed curve in V and, since V is factorial, that the prime ideal corresponding to C is principal (i.e., $C = (f)$). Because f is a regular function on V, its restriction to U is a unit that can be pulled back to a unit on $\mathbb{C}^2$ via the morphism $\mathbb{C}^2 \to \mathbb{C}^2/\Gamma$. But $\mathbb{C}^2$ has no nonconstant units. Hence f is constant on U and thus is also constant on V. This gives us a contradiction. Hence $V^\circ = U = \mathbb{C}^2/\Gamma \setminus \{p\}$ and so $V = \mathbb{C}^2/\Gamma$. Because V is factorial, Γ is the binary icosahedral group.

Case 2: $\bar{\kappa}(V^\circ) = 0$. We have the Zariski–Fujita decomposition of $K+D+E \sim P+N$. Since $\bar{\kappa}(V^\circ) = 0$, it follows that $P = 0$.

Subcase 2a: N is supported on the rational admissible twigs of $D+E$ (NC-minimal case). By Lemma 4 and unimodularity, it is clear that the connected components of D and E can be an elliptic curve, a loop of $\mathbb{P}^1$s, or a quotient singularity (which, because of unimodularity, must be E_8). Since D supports an ample divisor, it can only be an elliptic curve or a loop of $\mathbb{P}^1$s.

(1) Suppose first that D is an elliptic curve. Then, by unimodularity, $D^2 = 1$. Using this equality and that the surface is rational, by an easy application of the Riemann–Roch theorem we obtain $H^0(X, \mathcal{O}(D)) = 2$. Since D is smooth and irreducible, a general member of the linear system $|D|$ is smooth and irreducible and thus is an elliptic curve. This linear system has a base point because $D^2 = 1$.

Now consider the compactification Y of the normal factorial surface V by contracting E on X to normal singular points. Blowing up at the base point yields an elliptic fibration on the blown-up surface X_1. Also, $V = X \setminus D$ has a fibration over $\mathbb{A}^1$ whose general fiber F is an elliptic curve with one point missing. Therefore, any singular point of V lies in a fiber of the elliptic fibration on X_1. Let $D_1 := D' \cup L$, where L is the exceptional curve (which is a cross section of the elliptic fibration). Then the components of D_1 generate the divisor class group of X_1 freely, which implies that the fibration is relatively minimal.

In particular, we can use Kodaira classification of singular fibers of an elliptic fibration to show that any singular point of V is a rational double point. By unimodularity, any such point is analytically the E_8-singularity, which gives a fiber of type II^* in a relatively minimal elliptic fibration on a smooth surface.

Now we use the long exact sequence of cohomology for calculating the Euler characteristic of X_1, as in the smooth case. Since $H^1(D_1;\mathbb{Z}) = \mathbb{Z}^2$ and $H^2(X_1,\mathbb{Z}) \cong H^2(D_1,\mathbb{Z})$, we have $\chi(X_1) = 4$. Note that, since V is affine, it does not contain any complete curves. Because the fibration on X_1 has a cross section, every singular fiber of this fibration is irreducible.

From Persson's [11] classification of the possible singular fibers of an elliptic fibration on a smooth rational surface, we see that there are two possibilities–namely, (II, II^*) and (I_1, I_1, II^*)—for the configuration of singular fibers on the minimal resolution of X_1. To get back the surface V: contract the E_8 configuration in the fiber (II^*); contract the exceptional curve, which is a cross section of the elliptic fibration to the base point; and remove D. Thus, when D is an elliptic curve, we have only one singularity (which is E_8) and V is obtained as just described.

(2) Now suppose that D is a loop of $\mathbb{P}^1$s.

We can assume that D is MNC. If *some* irreducible component of D is a (-1)-curve, then D has exactly two irreducible components (say, D_1 and D_2) such that $D_1^2 = -1$. By unimodularity, $D_2^2 = -3$. Let X_1 be obtained from X by contracting D_1 to a smooth point. Now the image of D_2 in X_1 (say, Δ_2) is an irreducible rational curve with exactly one singular point—which is an ordinary double point—and $\Delta_2^2 = 1$. We can argue as in the previous case. The linear system $|\Delta_2|$ has dimension 1 and one base point. Blowing up this base point yields an elliptic fibration, and we conclude as before that V has exactly one singularity (which is of E_8-type). In this case, the only possible configuration of singular fibers (on the minimal resolution of the blown-up surface) is of type (I_1, I_1, II^*).

Now assume that *no* irreducible component of D is a (-1)-curve. Because D supports a divisor with strictly positive intersection form, $D_i^2 \geq 0$ for some i. By blowing up at suitable points in D_i, if necessary, we assume that $D_i^2 = 0$. Then $|D_i|$ defines a $\mathbb{P}^1$-fibration φ on X, and the two components of D meeting D_i are cross sections for φ. It follows that E is contained in a finite union of singular fibers of φ. Hence E contracts to finitely many rational singular points, which (by unimodularity) are all E_8 singularities. Using Lemma 3, it is not difficult to see that E_8 cannot be a subgraph of a singular fiber of a $\mathbb{P}^1$-fibration. So in this case there are no singular points in V, which is a contradiction.

Subcase 2b: N is not supported on the rational admissible twigs of $D+E$ (non–NC-minimal case). Since $\bar{\kappa}(V^\circ) = 0$, by Lemma 5 there exists a (-1)-curve L on X that occurs in N, meets a rational twig of D transversally (and possibly a rational rod in E), and satisfies $\bar{\kappa}(V^\circ - L) = \bar{\kappa}(V^\circ) = 0$. By unimodularity of the connected components of E there cannot be a rational rod in E. Therefore, $L \not\subseteq D + E$ and $L.D = 1$.

Let Y be a suitable completion of V obtained from X by contracting E to normal singular points. Then $Y = V \sqcup D$. Since V is factorial, the class group of Y is freely generated by the components of D. Let the component of D that L intersects be D_1, and write $L \sim a_1 D_1 + a_2 D_2 + \cdots$. Since $L.D_1 = L.D = 1$, it follows that $L \sim -D_1 + a_2 D_2 + \cdots + a_r D_r$ in Y.

Let $\bar{Y}$ be obtained from Y by contracting L, and let $\bar{C}$ denote the image in $\bar{Y}$ of a curve C in Y. On $\bar{Y}$ we have $\bar{D}_1 \sim a_2 \bar{D}_2 + \cdots + a_r \bar{D}_r$, so $\bar{D}_2, \ldots, \bar{D}_r$ generate the class group of $\bar{Y}$ freely. Thus, $V_1 := \bar{Y} \setminus (\bar{D}_2 \cup \cdots \cup \bar{D}_r)$ is also factorial. Now $V_1 \setminus \bar{D}_1 \cong V \setminus L$, from which it follows that $0 = \bar{\kappa}(V^\circ \setminus L) = \bar{\kappa}(V_1^\circ \setminus \bar{D}_1)$.

If D_1 is not a tip of D, then we have a linear chain (viz., a rod) as a connected component in the infinity of V_1. Because V_1 is factorial, this chain must be unimodular. However, since this is not possible, we see that D_1 is a tip of D and hence $D_1^\circ := \bar{D}_1 \setminus \bar{D}_2$ is an $\mathbb{A}^1$ in the smooth part of V_1.

Since V_1 is factorial, we have $D_1^\circ = (f)$. Consider the map $f\colon V_1 \setminus D_1^\circ \to \mathbb{C}^*$. Using Kawamata's inequality [7, Chap. 2, Lemma 1.14.1] and the equality $\bar{\kappa}(V_1^\circ \setminus D_1^\circ) = 0$, we deduce that this map is a $\mathbb{C}^*$-fibration. However, V_1 has a singularity that must lie on some fiber and so must be E_8. As mentioned previously, by Lemma 3 this is not possible. Thus, we have a contradiction.

Case 3: $\bar{\kappa}(V^\circ) = 1$. By [7, Chap. 2, Thm. 6.1.5] there exists a $\mathbb{C}^*$-fibration $\pi : V^\circ \to B$. Since V° is rational, $B \cong \mathbb{A}^1$ or $\mathbb{P}^1$.

Subcase 3a: π extends to a $\mathbb{C}^*$-fibration on V. Since any singularity of V must lie on a fiber, it has to be both unimodular and a rational singular point (i.e., an E_8-singularity). As already shown, in this case V is smooth—a contradiction.

Subcase 3b: π does not extend to a $\mathbb{C}^*$-fibration on V. Then π has a base point at a singular point, say p, of V. In this case, the connected components of $D \cup E$ are each unimodular and hence are trees of smooth rational curves. Furthermore, the base B of the $\mathbb{C}^*$-fibration is isomorphic to $\mathbb{P}^1$.

We have the long exact sequence

$$\begin{aligned} H^1(X;\mathbb{Z}) &\to H^1(D\cup E;\mathbb{Z}) \to H^2(X, D\cup E;\mathbb{Z}) \to H^2(X;\mathbb{Z}) \\ &\to H^2(D\cup E;\mathbb{Z}) \to H^3(X, D\cup E;\mathbb{Z}) \to H^3(X;\mathbb{Z}). \end{aligned}$$

Because $H^3(X;\mathbb{Z}) = 0$ and $H^2(X;\mathbb{Z}) \cong H^2(D\cup E;\mathbb{Z})$ (since $\mathrm{Pic}(V) = 0$ and V has only trivial units), we get $H_1(V^\circ;\mathbb{Z}) = H^3(X, D\cup E;\mathbb{Z}) = 0$. Also, since $H^1(X;\mathbb{Z}) = 0$, we get $H^1(D\cup E;\mathbb{Z}) \cong H^2(X, D\cup E;\mathbb{Z})$. Then $H^1(D\cup E;\mathbb{Z}) = 0$ because $D\cup E$ is a tree of $\mathbb{P}^1$s. Therefore, $H_2(V^\circ;\mathbb{Z}) = H^2(X, D\cup E;\mathbb{Z}) = 0$.

Now, since E is a unimodular tree of $\mathbb{P}^1$s, an easy application of the Mayer–Vietoris sequence yields $H_1(V;\mathbb{Z}) = 0$ and $H_2(V;\mathbb{Z}) = 0$. Since V is affine, $H_3(V;\mathbb{Z}) = 0$; since V is open, $H_4(V;\mathbb{Z}) = 0$. Thus, V is a $\mathbb{Z}$-homology plane and hence $\chi(V) = 1$.

We have already seen that V has only one singular point (namely, p) and that $V^\circ = V \setminus \{p\}$, so $\chi(V^\circ) = 0$. Then, by the Suzuki–Zaidenberg formula, all the singular fibers of the $\mathbb{C}^*$-fibration $\pi : V^\circ \to B$ have Euler characteristic 0 and hence are irreducible.

It is proved in [4, Lemma 4.4] that there is a $\mathbb{C}^*$-action on the fibers of π giving rise to a $\mathbb{C}^*$-action on V. This action has p as the only fixed point, and the closure of every orbit passes through p. This can be seen as follows. Since $\bar{\kappa}(V^\circ) = 1$, it follows that π restricted to V° has at least three singular fibers that are multiple $\mathbb{C}^*$s (otherwise, $\mathbb{C}^* \times \mathbb{C}^*$ is contained in V°). By taking a suitable ramified cover $\tilde{B} \to B$ of B with prescribed ramification and normalized fiber product, we obtain a smooth surface $\tilde{V}^\circ$ with a $\mathbb{C}^*$-bundle $\tilde{V}^\circ \to \tilde{B}$. The $\mathbb{C}^*$-action on V° lifts to an action on $\tilde{V}^\circ$, which can be seen is fixed point free.

In short: the $\mathbb{C}^*$ action on V has a unique closed orbit, which is a point. It follows that $\Gamma(V, \mathcal{O})$ is a positively graded, 2-dimensional UFD. These domains have been classified by Mori [8] as complete intersections of hypersurfaces of the form $\{X_1^{a_1} + b_2 X_2^{a_2} + \cdots + b_n X_n^{a_n} = 0\}$.

This completes the proof of Theorem 2.

5. Proof of Theorem 3: V Is Affine, Smooth, and Factorial with Nonconstant Units

Case 1: $\bar{\kappa}(V) = -\infty$. Let $u \in \Gamma(V, \mathcal{O})^* \setminus \mathbb{C}^*$. Then u gives a dominant map $u : V \to \mathbb{C}^*$. Let $\varphi : V \to B$ be the Stein factorization of this map.

Since we have a dominant map $B \to \mathbb{C}^*$, it follows that $\bar{\kappa}(B) \geq \bar{\kappa}(\mathbb{C}^*) = 0$ and so $B \cong \mathbb{C} \setminus \{\text{at least one point}\}$. Since φ has irreducible general fibers, applying Kawamata's inequality to φ yields $\bar{\kappa}(V) \geq \bar{\kappa}(B) + \bar{\kappa}(F)$, where F is a general fiber. Since $\bar{\kappa}(V) = -\infty$ and $\bar{\kappa}(B) \geq 0$, we have $\bar{\kappa}(F) = -\infty$. Thus, φ is an $\mathbb{A}^1$-fibration.

The rest of the argument is well known, so we cover it only briefly. Let X be a suitable smooth compactification of V such that φ extends to a $\mathbb{P}^1$-fibration Φ on X, and let $D = X \setminus V$. Over $\mathbb{Z}$, the generators of $\mathrm{Pic}(X)$ are obtained by taking one cross section, one general fiber, and all components of the singular fibers *except* any component of multiplicity 1. The factoriality of V now shows us that φ has all reduced and irreducible fibers, so V is a trivial $\mathbb{A}^1$-bundle over B. In this case, then, $V \cong \mathbb{C} \times \mathbb{C} \setminus \{\text{at least one point}\}$.

Case 2: $\bar{\kappa}(V) = 0$.

Subcase 2a. Suppose that V has an NC-minimal compactification (X, D) in Fujita's sense (see the Remark preceding Lemma 4). In this case we can use Fujita's result in [1, Sec. 8.64] to show that V is one of the surfaces $O(1,1,1)$, $O(4,1)$.

The surface $O(1,1,1)$ is the complement of the union of three general lines in $\mathbb{P}^2$; the surface $O(4,1)$ is the complement of the union of a smooth conic and a general line in $\mathbb{P}^2$. On the former surface, V has two independent units modulo $\mathbb{C}^*$. On the latter surface, V has one nonconstant unit that generates the group of units modulo $\mathbb{C}^*$.

Subcase 2b. Suppose that X is a smooth MNC completion of V and that $D = X - V$. Assume that (X, D) is not NC-minimal in Fujita's sense. Then (i) there is a (-1)-curve L in X such that L is not contained in $\operatorname{supp} D$ or in $\operatorname{supp} N$ and $L \cdot D = 1$; or (ii) by Lemma 5 there is a (-1)-curve L in X that is in $\operatorname{supp} N$ but not in D and that meets a twig of D and satisfies $L.D = 1$.

Now the surface $V - L \cap V$ is also factorial and has one new unit by Lemma 7. We get an NC-minimal open affine subvariety V_0 of V (in Fujita's sense) by successively removing such curves L from V. This NC-minimal subset V_0 is one of the surfaces in Subcase 2a.

Since V has a nonconstant unit, by Lemma 7 and Kojima's result [6, Thm. 3.1] we know that V_0 is $O(1,1,1)$ (i.e., $V_0 \cong \mathbb{C}^* \times \mathbb{C}^*$). We see that V_0 is obtained by removing exactly one such curve L from V. The MNC completion of V_0 is $\mathbb{P}^2$ and the infinity is the union of three lines $\{D_i : 1 \leq i \leq 3\}$.

If L does not occur in N, then L does not meet a twig of D. First we blow up at a point p on one of the lines, say D_1. If p does not lie on D_2, D_3, then we obtain X and L is the exceptional curve. If $p \in D_1 \cap D_2$ then let E be the exceptional curve obtained by blowing up p. In this case X is obtained by blowing up at a point in E that does not lie on the proper transforms of D_1, D_2 and L is the new exceptional curve.

If L does occur in N, then L meets a twig of D and X is obtained by blowing up at a point in, say, D_1 and then successively blowing up with centers on the previous (-1)-curve at a point that does not lie on any proper transform of the previous exceptional curves. The curve L is the final (-1)-curve in this sequence of blow-ups.

No subsequent blow-up is allowed on any point of intersection; otherwise, the (-1)-curve L would satisfy $D_1 \sim D_2 + \sum_{j>2} a_j D_j + 2L$, where $\{D_j : j > 2\}$ are the proper transforms of the exceptional curves occuring before L. This implies that L is not linearly equivalent to a divisor on D, which violates the factoriality of V.

Every blow-up has center on the previous (-1)-curve because D is MNC. Therefore, D is a union of the proper transforms of D_1, D_2, D_3 (and possibly the proper transform of the first exceptional curve) and a linear chain of (-2)-curves with one end intersecting D_1 (or a (-2)-curve if the first blow-up was a point of intersection of D_1 with either D_2 or D_3). The (-1)-curve L intersects the other tip of the chain.

Case 3: $\bar{\kappa}(V) = 1$. In this case there exists a $\mathbb{C}^*$-fibration $\varphi\colon V \to B$. By Lemma 8 this fibration has two cross sections—in other words, it is untwisted.

Let X be a suitable smooth compactification of V such that X has a $\mathbb{P}^1$-fibration Φ extending φ, and let $D = X \setminus V$. Over $\mathbb{Z}$, the generators of $\mathrm{Pic}(X)$ are obtained by taking one cross section, one general fiber, and all the components of the singular fibers except *one* component of multiplicity 1. We claim that at least one full fiber of Φ is contained in D. If not, then the number of irreducible components of D would be equal to the rank $\mathrm{Pic}(X)$. But then V could not have a nonconstant unit. A similar argument shows that every fiber of φ is irreducible.

Let $F_1, F_2, \dots, F_r$ be the complete fibers of Φ that are contained in D. We can assume (by Lemma 3, if necessary) that each F_i is a regular fiber of Φ. Let D' be the union of all the irreducible components of D except $F_2, \dots, F_r$. Then D' is connected and the irreducible components of D' generate $\mathrm{Pic}(X)$ freely. It follows that $X - D'$ is a $\mathbb{Z}$-homology plane. The surface V is obtained from $X - D'$ by removing the regular fibers $F_2, \dots, F_r$.

This completes the proof of Theorem 3.

6. Proof of Theorem 4: V Is Nonsmooth and Factorial with Nonconstant Units

We shall demonstrate that this combination cannot occur.

Case 1: $\bar{\kappa}(V^\circ) = -\infty$. We showed in the proof of Theorem 3 that, since V has nonconstant units, Kawamata's inequality implies that V has an $\mathbb{A}^1$-fibration $\varphi\colon V \to B$. Hence, by Miyanishi's result used earlier, V has only cyclic quotient singularities. Yet this is impossible because cyclic quotient singularities are not unimodular.

Case 2: $\bar{\kappa}(V^\circ) = 0$. By Kawamata's inequality, a similar argument as in Case 1 of the proof of Theorem 3 shows that there is a $\mathbb{C}^*$-fibration $\varphi\colon V^\circ \to B$.

Subcase 2a: φ does extend to a C^*-fibration on V. Arguing as in Case 1, we see that the singularities must be unimodular as well as cyclic quotients. This gives us a contradiction.

Subcase 2b: φ does not extend to a C^*-fibration on V. Then there is a singular point (say, p) of V through which closures of the general fibers pass—that is, a base point for the fibration. Any unit of V must be constant on the closures of these fibers (since their normalizations are $\mathbb{A}^1$s). Because the closures of general fibers meet at p, we see that V has only constant units.

Case 3: $\bar{\kappa}(V^\circ) = 1$. In this case V° has a $\mathbb{C}^*$-fibration and so, as in Case 2, V has only constant units.

This completes the proof of Theorem 4.

7. Examples

(1) Let (W, q) be a germ of a Gorenstein normal surface singularity. In [10] it is proved that there exists an affine factorial surface V with a unique singular point p such that the germ (V, p) is analytically isomorphic to (W, q). The proof also shows that we can ensure $\bar{\kappa}(V \setminus \{p\}) = 2$. (For complete intersection singularities, see [9].)

(2) We shall construct affine factorial surfaces V such that $\bar{\kappa}(V^\circ) = 2$ and V can have arbitrarily large number of singular points.

As in the proof of Theorem 2 when $\bar{\kappa}(V^\circ) = 0$, let $\varphi\colon X \to \mathbb{P}^1$ be an elliptic fibration on a smooth projective rational surface that has only two singular fibers (of type II and II^*). Then there is a cross section S for φ with $S^2 = -1$. Let $F_1, F_2, \dots, F_r$ be r general fibers of φ. Blow up the r points $F_i \cap S$, and let F_i' be the proper transform of F_i in the new surface. We can show that all the F_i' can be contracted to normal singular points on a projective surface Y. Let S_0 and II_0^* be (respectively) the images of S and II^* in Y. Then we can show that $V := Y \setminus (S_0 \cup II_0^*)$ is affine and factorial with r singular points and that $\bar{\kappa}(V^\circ) = 2$ (cf. [2, Prop. 3.10]).

(3) We list some examples of smooth factorial surfaces with $\bar{\kappa} = 2$.

- Any smooth $\mathbb{Z}$-homology plane V with $\bar{\kappa}(V) = 2$ is factorial with trivial units.
- Let X be a general hypersurface of degree ≥ 5 in $\mathbb{P}^3$, and let H be a hyperplane section of X; then $V := X \setminus H$ is factorial with $\bar{\kappa} = 2$.
- More generally, from any smooth projective surface X with $H^1(X, \mathcal{O}) = (0)$ we can obtain a factorial surface by taking the complement of a union of finitely many irreducible curves that generate Pic X.

In view of these examples, a classification of factorial surfaces with $\bar{\kappa} = 2$ does not appear to be possible.

References

[1] T. Fujita, *On the topology of noncomplete algebraic surfaces,* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 29 (1982), 503–566.

[2] R. V. Gurjar, M. Koras, M. Miyanishi, and P. Russell, *Affine normal surfaces with simply-connected smooth locus,* Math. Ann. (to appear).

[3] R. V. Gurjar and M. Miyanishi, *Affine surfaces with* $\bar{\kappa} \leq 1$, Algebraic geometry and commutative algebra, vol. I (H. Hijikata et al., eds.), pp. 99–124, Kinokuniya, Tokyo, 1988.

[4] ———, *On the Jacobian conjecture for* $\mathbb{Q}$*-homology planes,* J. Reine Angew. Math. 516 (1999), 115–132.

[5] Y. Kawamata, *On the classification of noncomplete algebraic surfaces,* Algebraic geometry (Copenhagen, 1978), Lecture Notes in Math., 732, pp. 215–232, Springer-Verlag, Berlin, 1979.

[6] H. Kojima, *Open surfaces of logarithmic Kodaira dimension zero in arbitrary characteristic,* J. Math. Soc. Japan 53 (2001), 933–955.

[7] M. Miyanishi, *Open algebraic surfaces,* CRM Monogr. Ser., 12, Amer. Math. Soc., Providence, RI, 2001.

[8] S. Mori, *Graded factorial domains,* Japan J. Math. 3 (1977), 223–238.

[9] A. J. Parameswaran and V. Srinivas, *A variant of the Noether–Lefschetz theorem: Some new examples of unique factorization domains,* J. Algebraic Geom. 3 (1994), 81–115.

[10] A. J. Parameswaran and D. van Straten, *Algebraization with minimal class group,* Internat. J. Math. 4 (1993), 989–996.

[11] U. Persson, *Configurations of Kodaira fibers on rational elliptic surfaces,* Math. Z. 205 (1990), 1–47.

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