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Inversion Invariant Bilipschitz Homogeneity

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1. Introduction

This paper examines metric spaces that are bilipschitz homogeneous and remain so after they are inverted (see Section 2 for definitions). The general idea is that, in such spaces, the metric doubling property can be improved to Ahlfors Q-regularity and local connectedness can be improved to linear local connectedness.

Bilipschitz homogeneous Jordan curves have been well studied (see e.g. [Bi; GH2; HM; M1; R]). Progress has also been made in the study of (locally) bilipschitz homogeneous geodesic surfaces (see [L]). This paper focuses on the stronger assumption of inversion invariant bilipschitz homogeneity in the context of more general doubling metric spaces. Our main results are as follows.

Theorem 1.1. *Let $L, D \geq 1$. Suppose X is a proper, connected, and D-doubling metric space. If there exists a $p \in X$ such that both X and the inversion of X at p are L-bilipschitz homogeneous then X is Q-regular, with regularity constant depending only on D and L.*

Theorem 1.2. *Suppose X is a proper, connected, and locally connected doubling metric space. If there exists a $p \in X$ such that both X and the inversion of X at p are uniformly bilipschitz homogeneous, then X is LLC_1. If, in addition, we assume that X has no cut points, then X is also LLC_2.*

We remark that Theorem 1.2 is qualitative, not quantitative, in nature. It would be interesting to know if a quantitative result is possible.

Before proceeding into the body of the paper, we discuss a few immediate consequences of these two theorems. For one, these results allow us to recover a stronger version of [F1, Thm. 1.2] in which the LLC_1 condition (i.e., bounded turning) need not be assumed (see also [F1, Thm. 1.1]).

Corollary 1.3. *Let Γ denote a Jordan curve in $\mathbb{R}^n$. The curve Γ is an Ahlfors Q-regular quasicircle if and only if there exists a point $p \in \Gamma$ such that both Γ and the Euclidean inversion of Γ at p are uniformly bilipschitz homogeneous.*

The sufficiency follows from Theorem 1.1 and Theorem 1.2. The necessity follows from the fact that an LLC_1 and Alhfors Q-regular Jordan curve in $\mathbb{R}^n$ is bilipschitz

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homogeneous, and these two properties are preserved by Möbius maps (such as inversions; see [GH1, Thm. C]).

We also highlight the case in which X is homeomorphic to the unit 2-sphere $\mathbb{S}^2$. By a theorem of Bonk and Kleiner [BoK1, Thm. 1.1], it is known that a linearly locally connected and Ahlfors 2-regular metric space homeomorphic to $\mathbb{S}^2$ is in fact quasi-symmetrically homeomorphic to $\mathbb{S}^2$. Therefore, when the space X described in Theorem 1.2 is homeomorphic to $\mathbb{S}^2$ and has Hausdorff dimension 2, we find that X is quasi-symmetrically equivalent to $\mathbb{S}^2$. Note that a parallel result holds when X is homeomorphic to $\mathbb{R}^2$ (cf. [W, Thm. 1.2]). However, with our stronger assumption of inversion invariant bilipschitz homogeneity, it seems reasonable to expect a better parameterization of X (perhaps even a bilipschitz parameterization $f: \mathbb{R}^2 \to X$).

In Section 2 we provide relevant definitions and explain our notation. In Section 3 we discuss a generalization of Ahlfors regularity for bilipschitz homogeneous spaces. In Section 4 we prove Theorem 1.1 and Theorem 1.2. Section 5 concludes with a few simple examples and related questions.

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2. Preliminaries

Given a constant C, we write $C = C(A, B, \dots)$ to indicate that C is determined solely by the numbers $A, B, \dots$. Given two numbers A and B, we write $A \simeq_C B$ to indicate that $C^{-1}A \le B \le CA$, where C is typically independent of A and B. When the quantity C is understood, we simply write $A \simeq B$. Similarly, $A \lesssim B$ indicates that $A \le CB$.

An embedding $f: X \to Y$ is *L-bilipschitz* provided that, for all points $x_1, x_2 \in X$, we have

$$L^{-1} d_X(x_1, x_2) \le d_Y(f(x_1), f(x_2)) \le L d_X(x_1, x_2).$$

Two spaces X, Y are *L-bilipschitz equivalent* if there exists an L-bilipschitz homeomorphism f such that $f(X) = Y$. A space X is *bilipschitz homogeneous* if there exists a collection $\mathcal{F}$ of bilipschitz self-homeomorphisms of X such that, for every pair $x_1, x_2 \in X$, there exists a map $f \in \mathcal{F}$ with $f(x_1) = x_2$. When we can take every map in $\mathcal{F}$ to be L-bilipschitz, we say that X is L-bilipschitz homogeneous, or uniformly bilipschitz homogeneous when the particular constant is not important.

We use $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{S}$ to denote the natural numbers, the real line, and the unit circle, respectively. We write $X = (X, d)$ to denote a general metric space. When the distance d is understood, for two points $x, y \in X$ we write $|x - y|$ to denote $d(x, y)$. Open balls, spheres, and annuli are defined (respectively) as

$$B(x; r) := \{y \in X : |x - y| < r\},$$
$$S(x; r) := \{y \in X : |x - y| = r\}, \quad \text{and}$$
$$A(x; r, R) := \{y \in X : r < |x - y| < R\}.$$

We say that a space is *proper* if closed and bounded subsets of the space are compact.

For a set $E \subset X$ and $r > 0$, an *r-covering number* for E is given by

$$N(r; E) := \inf\{k \in \mathbb{N} : \exists \{x_i\}_{i=1}^k \subset X \text{ such that } E \subset \bigcup_{i=1}^k B(x_i; r)\},$$

where $0 < r < +\infty$. A metric space is *doubling* provided there exists some $0 < D < +\infty$ such that, for all $x \in X$ and $0 < r < \operatorname{diam}(X)$, we have $N(r; B(x; 2r)) \leq D$. If X is doubling, then there exists an increasing function $\nu\colon [1, +\infty) \to [1, +\infty)$ such that $N(r; E) \leq \nu(A) N(Ar; E)$ for each $A \geq 1$. Indeed, we may take $\nu(A) := DA^{\log_2(D)}$.

When a space is doubling, we may restrict ourselves to balls centered in a set E to find $N(r; E)$ simply by changing the resulting number by at most a factor of D^2. We also record the following information.

LEMMA 2.1. *Let E and F be L-bilipschitz equivalent subsets of a D-doubling metric space. Then, for any $r > 0$, we have $N(r; E) \simeq N(r; F)$ up to the constant $D^3 L^{\log_2(D)}$.*

Proof. Assume E and F are bounded. Let $\{B_i\}_{i=1}^k$ be a minimal (with respect to cardinality) cover of F by balls $B_i := B(x_i, r)$, where $x_i \in F$. Given an L-bilipschitz map $f\colon E \to F$, we know that $\{B(f^{-1}(x_i); Lr)\}$ covers E. Therefore, $N(Lr; E) \leq k \leq D^2 N(r; F)$, where the factor of D^2 comes from the requirement that each $x_i \in F$. Using f^{-1}, we obtain $N(Lr; F) \leq D^2 N(r; E)$. The doubling condition then yields the desired conclusion. □

We write $\mathcal{H}^\alpha$ to denote the usual α-dimensional Hausdorff measure,

$$\mathcal{H}^\alpha(E) := \lim_{\varepsilon \to 0} \left[\inf \left\{ \sum_i (\operatorname{diam}(E_i))^\alpha : E \subset \bigcup_i E_i,\ \operatorname{diam}(E_i) \leq \varepsilon \right\} \right].$$

Given a nondecreasing function $\beta\colon (0, +\infty) \to (0, +\infty)$ for which $\beta(t) \to 0$ as $t \to 0$, we define the *Hausdorff β-measure* of a Borel set $E \subset X$ to be

$$\mathcal{G}^\beta(E) := \lim_{\varepsilon \to 0} \left[\inf \left\{ \sum_i \beta(\operatorname{diam}(E_i)) : E \subset \bigcup_i E_i,\ \operatorname{diam}(E_i) \leq \varepsilon \right\} \right].$$

We refer to such a function β as a *dimension gauge*. When there exists a constant D such that for all $0 < r < +\infty$ we have $\beta(2r) \leq D\beta(r)$, we say that β is a *doubling dimension gauge*. When β is D-doubling, it is straightforward to verify that

$$\mathcal{G}^\beta(E) \simeq_D \mathcal{S}^\beta(E) \tag{2.1}$$

where, given a set $E \subset X$,

$$\mathcal{S}^\beta(E) := \lim_{\varepsilon \to 0} \left[\inf \left\{ \sum_i \beta(r_i) : E \subset \bigcup_i B(x_i, r_i),\ x_i \in X,\ r_i \leq \varepsilon \right\} \right].$$

A space is *Ahlfors Q-regular* for $Q > 0$ provided that, for every $x \in X$ and $0 < r < \operatorname{diam}(X)$, we have $\mathcal{H}^Q(B(x; r)) \simeq r^Q$ up to some constant independent of r. Given a dimension gauge β, a space X is (A,β)-*regular* if for every $0 < r < \mathrm{diam}(X)$ we have $\mathcal{G}^{\beta}(B(x;r)) \simeq_A \beta(r)$. This generalization of Ahlfors regularity proves useful in the analysis of bilipschitz homogeneous spaces, as noted by Mayer in [M2, Chap. IV].

For $\lambda > 1$, we say that a space X is λ-*linearly locally connected* (or λ-LLC for short) provided that, for all $a \in X$ and $0 < r < \mathrm{diam}(X)$, the following statements hold:

(1) for each pair of distinct points $\{x, y\} \subset B(a;r)$ there exists a continuum $E \subset B(a;\lambda r)$ containing $\{x, y\}$;
(2) for each pair of distinct points $\{x, y\} \subset X \setminus B(a;r)$ there exists a continuum $E \subset X \setminus B(a;r/\lambda)$ containing $\{x, y\}$.

Recall that a *continuum* is a connected, compact set containing more than one point. The property described by (1) is referred to as the λ-LLC$_1$ property and (2) is the λ-LLC$_2$ property.

In [BoK2], Bonk and Kleiner generalized the notion of chordal distance on the Riemann sphere to unbounded locally compact metric spaces. In [BHX], Buckley, Herron, and Xie built on this notion to develop the concept of *metric inversions.* We record a few pertinent facts about such inversions. Define

$$\hat{X} := \begin{cases} X \cup \{\infty\} & \text{when } X \text{ is unbounded,} \\ X & \text{when } X \text{ is bounded.} \end{cases}$$

Given a basepoint $p \in X$ and any two points $x, y \in X_p := X \setminus \{p\}$, we define

$$i_p(x, y) := \frac{|x-y|}{|x-p||y-p|};$$

when X is unbounded, $i_p(x,\infty) := 1/|x-p|$. This does not define a distance function in general, but one can show (see [BHX, p. 843]) that

$$d_p := \inf\left\{ \sum_{i=0}^{k-1} i_p(x_i, x_{i+1}) : x = x_0, \ldots, x_k = y \in X_p \right\}$$

defines a distance on $\hat{X}_p = \hat{X} \setminus \{p\}$ such that, for all $x, y \in \hat{X}_p$,

$$\frac{1}{4} i_p(x, y) \le d_p(x, y) \le i_p(x, y).$$

We use the distance d_p to define the *inversion of X at p*, denoted by

$$\mathrm{Inv}_p(X) := (\hat{X}_p, d_p).$$

We often write $X^* := \mathrm{Inv}_p(X)$ when the basepoint is understood. The identity map from $(\hat{X}_p, d)$ to $X^* = (\hat{X}_p, d_p)$ is written as $\varphi_p \colon \hat{X}_p \to X^*$. When it is clear that we are working in X^* we simply write $|\cdot|$ to denote d_p, so for points $x, y \in \hat{X}_p$ we can write $|\varphi_p(x) - \varphi_p(y)|$ in place of $d_p(x, y)$. For points $x \in X_p$, it is sometimes convenient to write $x^* := \varphi_p(x)$. When X is unbounded, we write p^* to denote $\varphi_p(\infty)$. So for any $x \in X_p$ we have $1/(4|x-p|) \le |x^* - p^*| \le 1/|x-p|$.

In the proof of Theorem 1.1 it will be useful to consider the related notion of *metric sphericalization,* a concept that was originally defined and studied in [BoK2].

However, sphericalization can also be understood as a special case of metric inversion, and that viewpoint will streamline the proofs in this paper. Given a metric space (X, d), fix a point $p \in X$. Then define $X^q := X \sqcup \{q\}$, the disjoint union of X and some point q. We define a distance on X^q as

$$d^{p,q}(x, y) := d^{p,q}(y, x) := \begin{cases} 0 & \text{if } x = q = y, \\ d(x, y) & \text{if } x \neq q \neq y, \\ d(x, p) + 1 & \text{if } x \neq q = y. \end{cases}$$

Then we may define the sphericalization of X at p as

$$\mathrm{Sph}_p(X) := (\mathrm{Inv}_q(X^q), (d^{p,q})_q).$$

We remark that when X is unbounded, $1/4 \leq \mathrm{diam}(\mathrm{Sph}_p(X)) \leq 1$. We write ψ_p to denote the identity mapping $\hat{X} \to \mathrm{Sph}_p(X)$. We refer the reader to [BoK2] or [BHX] for more information on sphericalization.

The following estimates are utilized frequently (cf. [BHX, p. 848]).

FACT 2.2. *For* $0 < r < R < \mathrm{diam}(X)$ *and* $x, y \in A(p; r, R)$, *we have*:

$$\frac{|x - y|}{4R^2} \leq |\varphi_p(x) - \varphi_p(y)| \leq \frac{|x - y|}{r^2};$$
$$\frac{|x - y|}{4(1 + R)^2} \leq |\psi_p(x) - \psi_p(y)| \leq \frac{|x - y|}{(1 + r)^2}.$$

Having defined and discussed metric inversion, we can now make the following definition.

DEFINITION 2.3. Given a metric space X, we use the term *inversion invariant bilipschitz homogeneity* to describe the situation in which both X and $\mathrm{Inv}_p(X)$ are uniformly bilipschitz homogeneous.

3. Generalized Ahlfors Regularity

The methods and results of this section closely resemble those found in [HM] and [M2, Chap. IV].

We now define a means of measuring the "thickness" of a space at a given scale. When X is bounded, for a scale $0 < r < \mathrm{diam}(X)$ we define

$$\delta(r) := N(r; X)^{-1}.$$

When X is unbounded, for a point $x \in X$ and scale $0 < r < +\infty$ we define

$$\delta(x; r) := \begin{cases} N(r; B(x; 1))^{-1} & \text{if } r \leq 1, \\ N(1; B(x; r)) & \text{if } r \geq 1. \end{cases}$$

We refer to δ as a *canonical dimension gauge* for the space X. When X is bilipschitz homogeneous, we shall demonstrate that (up to a multiplicative constant) Definition 2.3 does not depend on the basepoint x (used in the unbounded case). Therefore, we often write $\delta(r)$ to denote $\delta(x; r)$, suppressing our choice of a basepoint.

We say that X has the *weak bounded covering* property if there exists a constant $1 \le C < +\infty$ such that, for all points $x, y \in X$ and scales $0 < r < s < t < \operatorname{diam}(X)$, we have

$$N(r; B(x; s)) \le CN(r; B(y; t)).$$

We use the prefix "weak" because this condition is analogous to a stronger condition utilized when studying bilipschitz homogeneous Jordan curves (see [HM, p. 776]). This concept is also utilized in [M2, Prop. IV.5].

LEMMA 3.1. *Suppose a D-doubling metric space X is L-bilipschitz homogeneous. Then X has the C-weakly bounded covering property for some $C = C(D, L)$.*

Proof. Let $x, y \in X$ and $0 < r < s < t < \operatorname{diam}(X)$ be given. Let $\{B(y_i; r)\}_{i=1}^m$ denote a minimal covering of $B(y; t)$ by balls of radius r centered in $B(y; t)$, and let $\{B(x_j; t/L)\}_{j=1}^n$ denote a minimal covering of $B(x; s)$ by balls of radius t/L centered in $B(x; s)$. Note that

$$n \le D^2 N(t/L; B(x; s)) \le D^2 \nu(L) N(t; B(x; s)) \le D^2 \nu(L).$$

For $j = 1, \ldots, n$, Let $f_j : X \to X$ denote an L-bilipschitz homeomorphism such that $f_j(y) = x_j$. For each j, we have $B(x_j; t/L) \subset f_j(B(y; t))$. Since the balls $\{B(y_i; r)\}$ cover $B(y; t)$, we find that we can cover $B(x_j; t/L)$ by the sets $\{f_j(B(y_i; r))\}_{i=1}^m$. Since each of these sets has diameter no greater than $2Lr$, it follows that $N(2Lr; B(x_j; t/L)) \le m$. Therefore,

$$\begin{aligned} N(r; B(x; s)) &\le \sum_{j=1}^n N(r; B(x_j; t/L)) \le \nu(2L) \sum_{j=1}^n N(2Lr; B(x_j; t/L)) \\ &\le \nu(2L) nm \le D^4 \nu(L) \nu(2L) N(r; B(y; t)). \end{aligned}$$

□

COROLLARY 3.2. *Suppose X is unbounded, D-doubling, and L-bilipschitz homogeneous. Then there exists a constant $C = C(D, L)$ such that, for any $x, y \in X$ and $0 < r < +\infty$, we have $\delta(x; r) \simeq_C \delta(y; r)$.*

This corollary allows us to speak of "the" canonical dimension gauge for an unbounded space X. With this terminology we are actually describing an equivalence class of dimension gauges, all comparable up to a constant depending only on the doubling and homogeneity constants for X.

The following observation is similar to [M2, Lemme A.2].

LEMMA 3.3. *Suppose that X is L-bilipschitz homogeneous and D-doubling. Then there exists a constant $C = C(D, L)$ such that, for any $0 < r < s < t < \operatorname{diam}(X)$,*

$$N(r; B(x; t)) \simeq_C N(r; B(x; s)) N(s; B(x; t)).$$

In fact, we can take C to be the weak bounded covering constant for X.

Proof. Let $\{B(x_i; s)\}_{i=1}^n$ denote a minimal cover of $B(x; t)$ by balls of radius s. For each i, let $\{B(y_{i,j}; r)\}_{j=1}^{m_i}$ denote a minimal cover of $B(x_i; s)$ by balls of radius r. By Lemma 3.1 we know that there exists a $C = C(D, L)$ such that $m_i \simeq_C N(r; B(x; s))$ for each $i \in \{1, \ldots, n\}$. This yields

$$N(r;\, B(x;t)) \le \sum_{i=1}^{n} m_i \le CN(s;\, B(x;t))N(r;\, B(x;s)).$$

The reverse inequality follows in a similar manner. □

A metric space is (H,α)-*homogeneous* if for every $x \in X$ and numbers $0 < r \le s < \operatorname{diam}(X)$ we have $P(r;\, B(x;s)) \le H(s/r)^{\alpha}$. Here $P(r;\, E)$ denotes the maximal cardinality of an r-separated set contained in E and is referred to as a *packing number.* In a D-doubling metric space, given a bounded set E we have $N(r;\, E) \simeq_D P(r;\, E)$. Lemma 3.3, along with the easily verified fact that D-doubling metric spaces are $(D^2, \log_2(D))$-homogeneous, yields the following corollary. This, in particular, demonstrates that a canonical dimension gauge is doubling.

COROLLARY 3.4. *Suppose that X is connected, D-doubling, and L-bilipschitz homogeneous. Then there exist constants* $1 \le C < +\infty$ *and* $1 \le \alpha < +\infty$ *depending only on D and L and such that, for every* $x \in X$ *and* $0 < r < s < \operatorname{diam}(X)$, *we have*

$$C^{-1}(s/r)\delta(r) \le \delta(s) \le C(s/r)^{\alpha}\delta(r). \tag{3.1}$$

Observe that the lower bound in this corollary is a trivial consequence of the connectedness assumption. Without this assumption, the lower bound need not hold (consider $X = \mathbb{Z}$).

When X is bilipschitz homogeneous, the measure $\mathcal{G}^{\delta}$ takes on a particularly simple form. For a Borel set $E \subset X$, define

$$\mathcal{C}^{\delta}(E) := \lim_{\varepsilon \to 0}[\inf\{N(r;\, E)\delta(r) : r \le \varepsilon\}].$$

LEMMA 3.5. *Suppose X is a D-doubling and L-bilipschitz homogeneous metric space. Then, for a compact set* $E \subset X$, *we have* $\mathcal{G}^{\delta}(E) \simeq \mathcal{C}^{\delta}(E)$ *up to a constant depending only on D and L.*

Proof. From (2.1) it follows that $\mathcal{G}^{\delta}(E) \simeq \mathcal{S}^{\delta}(E)$. Clearly, $\mathcal{S}^{\delta} \le \mathcal{C}^{\delta}$; we verify that $\mathcal{C}^{\delta} \lesssim \mathcal{S}^{\delta}$ up to some constant depending only on D and L. Let $\{B(x_i;\, r_i)\}_{i=1}^{n}$ denote a finite open cover of a compact subset $E \subset X$. We may assume that

$$r_1 = \min\{r_i : i = 1, \dots, n\} \le \max\{r_i : i = 1, \dots, n\} < 1.$$

Then write $m_i := N(r_1;\, B(x_i;\, r_i))$. Since $\{B(x_i;\, r_i)\}_{i=1}^{n}$ covers E, we have $\sum_{i=1}^{n} m_i \ge N(r_1;\, E)$. If X is unbounded then—by Corollary 3.2, Lemma 3.3, and Lemma 3.1—we have

$$\sum_{i=1}^{n} \delta(r_i) \simeq \sum_{i=1}^{n} \frac{1}{N(r_i;\, B(x;1))} \simeq \sum_{i=1}^{n} \frac{N(r_1;\, B(x_i;\, r_i))}{N(r_1;\, B(x;1))}$$
$$= \frac{1}{N(r_1;\, B(x;1))} \sum_{i=1}^{n} m_i \ge \frac{N(r_1;\, E)}{N(r_1;\, B(x;1))} \simeq N(r_1;\, E)\delta(r_1).$$

The same sort of comparability holds when X is bounded. This allows us to conclude that $\mathcal{C}^{\delta}(E) \lesssim \mathcal{S}^{\delta}(E)$, and we are done. $\square$

We now treat the main result of this section. Recall that X is (A,β)-regular provided that, for all $0 < r < \operatorname{diam}(X)$ and $x \in X$, we have $\mathcal{G}^{\beta}(B(x;r)) \simeq_B \beta(r)$. For compact spaces X, this is [M2, Thm. 9].

THEOREM 3.6. *Suppose a proper metric space X is D-doubling and L-bilipschitz homogeneous. Then X is (A,δ)-regular, where δ is the canonical dimension gauge for X and $A = A(D,L)$.*

Before commencing with the proof, we observe that this result need not hold for spaces that are not proper. Indeed, $\mathbb{Q}$ (the set of rational numbers in $\mathbb{R}$) is doubling and 1-bilipschitz homogeneous. However, for the canonical dimension gauge δ we have $\mathcal{G}^{\delta} \simeq \mathcal{H}^1$, while $\dim_{\mathcal{H}}(\mathbb{Q}) = 0$.

Proof of Theorem 3.6. Suppose that for every closed ball $\bar{B}(x;r)$ we have $\mathcal{G}^{\delta}(\bar{B}(x;r)) \simeq \delta(r)$. Then, for any $B(x;s) \subset X$, we may use (3.1) to obtain

$$\delta(s) \simeq \mathcal{G}^{\delta}(\bar{B}(x;s/2)) \le \mathcal{G}^{\delta}(B(x;s)) \le \mathcal{G}^{\delta}(\bar{B}(x;s)) \simeq \delta(s).$$

Therefore, to prove our theorem it suffices to consider closed balls $\bar{B}(x;s)$.

Let $\bar{B}(x;s)$ denote a closed (thus compact) ball in X, and let $\{B(x_i;r)\}_{i=1}^n$ denote a cover of $\bar{B}(x;s)$ for $n := N(r;\bar{B}(x;s))$ and $r \le \min\{1,s\}$. Assume that X is unbounded. By Corollary 3.2 and Lemma 3.3,

$$N(r;\bar{B}(x;s))\delta(r) \simeq \frac{N(r;B(x;s))}{N(r;B(x;1))}.$$

When $s \le 1$, by Lemma 3.3 and Corollary 3.2 we have

$$\frac{N(r;B(x;s))}{N(r;B(x;1))} \simeq \frac{1}{N(s;B(x;1))} \simeq \delta(s).$$

When $s \ge 1$, again by Lemma 3.3 and Corollary 3.2 we have

$$\frac{N(r;B(x;s))}{N(r;B(x;1))} \simeq N(1;B(x;s)) \simeq \delta(s).$$

The same sort of comparability holds when X is bounded. All of these comparabilities depend only on D and L. By Lemma 3.5, we are done. $\square$

Given a metric space (X,d) and $s > 0$, define $sX := (X,sd)$. Thus sX is just a rescaling of the distance d by a factor of s. Note that if X is L-bilipschitz homogeneous then so is sX. It will be useful to know that δ-regularity is scale invariant in the following sense.

LEMMA 3.7. *Let X denote a proper, D-doubling, L-bilipschitz homogeneous metric space. For any $s > 0$, let δ_s denote the canonical dimension gauge for sX. Then sX is (A,δ_s)-regular, where $A = A(D,L)$.*

Proof. Let $B_s(x;r)$ denote a ball in sX and let $B(x;r)$ denote a ball in X centered at the same point x. Note that, as sets, $B_s(x;r) = B(x;sr)$. Assume that X is bounded. Then, by Lemma 3.5 and Lemma 3.3,

$$\begin{aligned}\mathcal{G}^{\delta_s}(\bar{B}_s(x;r)) &\simeq \lim_{\varepsilon\to 0}[\inf\{N(t;\,B_s(x;r))\delta_s(t) : t \le \varepsilon\}]\\ &= \lim_{\varepsilon\to 0}[\inf\{N(t/s;\,B(x;r/s))\delta_s(t) : t \le \varepsilon\}]\\ &= \lim_{\varepsilon\to 0}\left[\inf\left\{\frac{N(t/s;\,B(x;r/s))}{N(t/s;\,X)} : t \le \varepsilon\right\}\right]\\ &\simeq \lim_{\varepsilon\to 0}[\inf\{N(r/s;\,X)^{-1} : t \le \varepsilon\}]\\ &= N(r/s;\,X)^{-1} = \delta_s(r)\end{aligned}$$

As in the proof of Theorem 3.6, this is sufficient to establish that sX is δ_s-regular. The comparability constant depends only on D and L. □

4. Inversion Invariant Bilipschitz Homogeneity

In this section we prove Theorem 1.1 and Theorem 1.2. Before proving Theorem 1.1, we need the following two facts. The first is a straightforward modification of [GH1, Thm. 3.1]. Note that our assumption of connectedness avoids the use of modulus techniques that appear in the original proof. For a similar result in the case of metric sphericalization, see [W, Prop. 6.13].

FACT 4.1. *Suppose X is a connected Q-regular metric space. Then any inversion or sphericalization of X remains Q-regular, with regularity constant depending only on the original.*

The second fact is proved in Part 2 of the proof of [F1, Thm. 1.2].

FACT 4.2. *Suppose δ is a dimension gauge satisfying* (3.1) *with constant C. If there exists a constant $1 \le A < +\infty$ such that for all $s, r > 0$ we have $\delta(sr) \simeq_A \delta(s)\delta(r)$, then there exist constants $1 \le Q < +\infty$ and $1 \le B < +\infty$ such that, for all $t > 0$, we have $\delta(t) \simeq_B t^Q$. Here $B = B(A, C)$.*

Proof of Theorem 1.1. We follow the general method behind the proof of [F1, Thm. 1.2]. For now, we assume that X is unbounded (we will treat the case in which X is bounded a bit differently). Let δ denote the canonical dimension gauge for X, and let δ^* denote the canonical dimension gauge for $X^* := \mathrm{Inv}_p(X)$. We point out that the unboundedness of X^* is not relevant to the following argument; we only use the fact that $\mathrm{diam}(X^*) \ge 1$.

We begin by demonstrating that, for any positive numbers s, t, we have $\delta(st) \simeq \delta(s)\delta(t)$ up to a constant depending only on D and L.

Step 1. Let $0 < r \le 1$. We prove that $\delta(r) \simeq \delta^*(r)$, where the comparability depends only on D and L. Choose a basepoint x such that $x \in S(p; 2)$. Then $B(x; 1) \subset A(p; 1, 3)$ and so, by Fact 2.2, φ_p is a 27-bilipschitz map on $B(x; 1)$. By Corollary 3.2, Lemma 2.1, and Lemma 3.3 we have

$$\delta(r) \simeq N(r;\,B(x;1))^{-1} \simeq N(r;\,\varphi_p(B(x;1)))^{-1} \simeq N(r;\,B(x^*;1))^{-1} \simeq \delta^*(r).$$

Step 2. Let $0 < s \le 1$ and $0 < t \le 1$. We verify that $\delta(st) \simeq \delta(s)\delta(t)$. Again the comparability depends only on D and L. Begin by selecting a point x with

$|x-p| = 4s^{-1/2} \geq 4$. Therefore, any ball of radius t intersecting $B(x;1)$ must lie in the annulus $A(p;|x-p|/2, 2|x-p|)$. We assert that

$$N(t; B(x;1)) \simeq N(st; \varphi_p(B(x;1))). \tag{4.1}$$

Indeed, let $\{B(x_i;t)\}$ be a finite cover of $B(x;1)$. Then, by Fact 2.2,

$$B(x_i^*; st/256) \subset \varphi_p(B(x_i;t)) \subset B(x_i^*; st/4).$$

The assertion (4.1) then follows from the metric doubling property as in the proof of Lemma 2.1. Again using Fact 2.2, we have

$$B(x^*; s/256) \subset \varphi_p(B(x;1))) \subset B(x^*; s/4). \tag{4.2}$$

Therefore, by Corollary 3.2, (4.1), Corollary 3.4, and Lemma 3.3,

$$\begin{aligned} \frac{1}{\delta(t)} &\simeq N(t; B(x;1)) \simeq N(st; \varphi_p(B(x;1))) \simeq N(st; B(x^*; s)) \\ &\simeq \frac{N(st; B(x^*;1))}{N(s; B(x^*;1))} \simeq \frac{\delta^*(s)}{\delta^*(st)}. \end{aligned}$$

Using these calculations along with Step 1, we conclude that

$$\delta(st) \simeq \delta^*(st) \simeq \delta(t)\delta^*(s) \simeq \delta(t)\delta(s).$$

All comparability statements depend only on D and L.

Step 3. Let $1 \leq s \leq t$. We show that $\delta(s/t) \simeq \delta(s)/\delta(t)$, with comparability constant depending only on D and L. Choose $x \in X$ with $|x-p| = 4t$. By Corollary 3.2 and Lemma 3.3, we have

$$\delta(t) \simeq N(1; B(x;t)) \simeq N(1; B(x;s))N(s; B(x;t)) \simeq \delta(s)N(s; B(x;t)).$$

The comparability depends only on D and L.

Suppose $B(y;s) \cap B(x;t) \neq \emptyset$ for some $y \in X$. Since $s \leq t$ and $|x-p| = 4t$, we have $B(y;s) \subset A(p;|x-p|/2, 2|x-p|)$. Therefore, as in (4.1) and (4.2), we have

$$N(s; B(x;t)) \simeq N(s/|x-p|^2; \varphi_p(B(x;t))) \simeq N(s/t^2; B(x^*; 1/t)).$$

We can now use Lemma 3.3 and Corollary 3.2 to obtain

$$N(s/t^2; B(x^*; 1/t)) \simeq \frac{N(s/t^2; B(x^*;1))}{N(1/t; B(x^*;1))} \simeq \frac{\delta^*(1/t)}{\delta^*(s/t^2)}.$$

Finally, using Steps 1 and 2 leads to

$$\frac{\delta^*(1/t)}{\delta^*(s/t^2)} \simeq \frac{\delta(1/t)}{\delta(1/t)\delta(s/t)} = \frac{1}{\delta(s/t)}.$$

Stringing together the foregoing observations yields $\delta(s/t) \simeq \delta(s)/\delta(t)$. The comparability depends only on D and L.

Step 4. Let $s, t > 0$. We confirm that $\delta(st) \simeq \delta(s)\delta(t)$ up to a constant depending only on D and L. We perform a case analysis in order to prove the equivalent conclusion that, for every $s, t > 0$, we have $\delta(s/t) \simeq \delta(s)/\delta(t)$.

Case 1: $s \le 1$. Suppose first that $t \ge 1$. Then

$$\delta(s/t) \simeq \delta(s)\delta(1/t) \simeq \delta(s)\delta(1)/\delta(t) \simeq \delta(s)/\delta(t).$$

The first relation follows from Step 2 and the second from Step 3; the final relation follows from the definition of δ.

Suppose now that $t < 1$. If $s/t \le 1$, then by Step 2 we have

$$\delta(s) = \delta((s/t)t) \simeq \delta(s/t)\delta(t).$$

If $s/t > 1$, then from Step 3 it follows that

$$\delta(1/s)/\delta(1/t) \simeq \delta(t/s) = \delta(1/(s/t)) \simeq \delta(1)/\delta(s/t) \simeq 1/\delta(s/t). \tag{4.3}$$

Furthermore, since $s \le 1$, by Step 3 we have

$$\delta(s) = \delta(1/(1/s)) \simeq \delta(1)/\delta(1/s) \simeq 1/\delta(1/s).$$

Similarly, $\delta(t) \simeq 1/\delta(1/t)$. Putting this together yields $\delta(s/t) \simeq \delta(s)/\delta(t)$, where the comparability constant depends only on B, L, and n.

Case 2: $s > 1$. Suppose first that $t \ge 1$. If $s/t \le 1$ then, by Step 3, we have

$$\delta(s/t) \simeq \delta(s)/\delta(t).$$

If $s/t > 1$ then, again by Step 3,

$$\delta(s/t) \simeq 1/\delta(t/s) \simeq \delta(s)/\delta(t).$$

Now suppose that $t < 1$ (so $s/t > 1$). By the calculations in (4.3), $\delta(s/t) \simeq 1/\delta(t/s)$. By Step 2, $\delta(t/s) \simeq \delta(t)\delta(1/s)$; by Step 3, $\delta(1/s) \simeq \delta(1)/\delta(s)$. Putting this together yields $\delta(s/t) \simeq \delta(s)/\delta(t)$. The comparability depends only on D and L.

Now we treat the case in which X is bounded. By Lemma 3.7, we may rescale so that $\operatorname{diam}(X) = 1$ without losing control of the regularity constant. We may also assume that there exists a point $q \in X$ such that $|p - q| \ge 1/2$. Write $X^* := \operatorname{Inv}_p(X)$ and set $q^* := \varphi_p(q) \in X^*$. Then X^* is unbounded and $X^{**} := \operatorname{Sph}_{q^*}(X^*)$ has diameter between $1/4$ and 1. By [BHX, Prop. 3.5] we know that X is 256-bilipschitz equivalent to X^{**}. Therefore, X^{**} is $L' := (256^2 L)$-bilipschitz homogeneous. We rescale so that $1 \le \operatorname{diam}(X^{**}) \le 4$. Such rescaling will only change the canonical dimension gauge for X^{**} by a factor that depends on the doubling constant.

We make the following observations: sphericalization is a special case of inversion; both X^* and X^{**} are L'-bilipschitz homogeneous; X^* is unbounded; and $\operatorname{diam}(X^{**}) \ge 1$. Therefore, up to minor adjustments, the arguments used in the case of unbounded X may be applied to conclude that, for all positive numbers s, t, we have $\delta^*(st) \simeq \delta^*(s)\delta^*(t)$. Here δ^* is the canonical dimension gauge for X^*, and comparability depends only on D and L.

By Corollary 3.4, we know that δ satisfies (3.1). Therefore, by the preceding portion of this proof and Fact 4.2, we conclude that there exist $1 \le B < +\infty$ and $1 \le Q < +\infty$ such that $\delta(t) \simeq_B t^Q$, where $B = B(D, L)$. When X is bounded, we reach the same conclusion for δ^*.

When X is unbounded, we use Theorem 3.6 to conclude that X is (C', Q)-regular for $C' = C'(D, L)$. When X is bounded, we use the same theorem to conclude that X^* is (C', Q)-regular for $C' = C'(D, L)$. By Fact 4.1, X is (C'', Q)-regular for $C'' = C''(D, L)$. □

Now we demonstrate that inversion invariant bilipschitz homogeneity implies the LLC condition when we assume a few additional conditions on the space X. We are currently unable to prove a quantitative implication as in Theorem 1.1 (except when $X \subset \mathbb{R}^2$ is an unbounded Jordan curve; see [F2, Thm. 1.1]).

Proof of Theorem 1.2. We proceed by way of contradiction, first for the LLC_1 condition and then for the LLC_2 condition. The two conditions require similar arguments. When X is bounded, we rescale so that $\operatorname{diam}(X) = 1$. Such rescaling does not affect the constants relevant to the LLC properties.

We first address the LLC_1 property. The main idea is to use bilipschitz homogeneity to demonstrate that X must be LLC_1 at fixed scales and then to use inversion invariance to show that the same LLC_1 constant must hold at all scales.

Let $\mathcal{T}_3 := \{(a, \lambda, r)\}$ denote a collection of triples such that there exists a pair of points $x, y \in B(a; r)$ that cannot be joined by a continuum in $B(a; \lambda r)$. Let $\mathcal{T}_2$ denote the pairs (λ, r) from the triples in $\mathcal{T}_3$. For $m \in \mathbb{N}$, we define

$$\mu_m := \sup\{\lambda : (\lambda, r) \in \mathcal{T}_2,\ 1/m \le \lambda r \le 1\}.$$

For each m, we claim that $1 \le \mu_m < +\infty$. The lower bound is trivial. To see that each μ_m is finite, suppose that $\{(a_n, \lambda_n, r_n)\}$ is a sequence of points from $\mathcal{T}_3$ for which $\lambda_n \to +\infty$ and $1/m \le \lambda_n r_n \le 1$. Then choose any point $a_0 \in X$. There exist L-bilipschitz homeomorphisms $f_n \colon X \to X$ with $f_n(a_n) = a_0$. Then, for each n, there exists a pair of points $x_n, y_n \in B(a_0; L r_n)$ that cannot be joined by a continuum in $B(a_0; \lambda_n r_n / L)$. Since $r_n \to 0$, this contradicts the assumption that X is locally connected at a_0. Therefore, we confirm that $\mu_m < +\infty$. This is what we mean by the phrase "X is LLC_1 at fixed scales."

Assume that X is not LLC_1. Then there exist arbitrarily large values for λ in triples from $\mathcal{T}_3$. We show that arbitrarily large values for λ correspond to arbitrarily small values for r. In other words, we show that $\mu_m \to +\infty$ as $m \to +\infty$. When X is bounded (and $\operatorname{diam}(X) = 1$), this is clear. However, when X is unbounded we proceed as follows. Assume there exists a constant $M < +\infty$ such that, for all m, $\mu_m \le M$. Since X is not LLC_1 (by assumption), there exists a sequence of points $\{(a_n, \lambda_n, r_n)\}$ from $\mathcal{T}_3$ such that $\lambda_n r_n \ge 1$ and $\lambda_n \to +\infty$. Choose n large enough to guarantee that $\lambda_n \ge 10^6 L^4 M$, and fix a basepoint $p \in X$. There exists an L-bilipschitz homeomorphism $f_n \colon X \to X$ such that $b_n := f_n(a_n) \in S(p; 2\lambda_n r_n)$. Let $b_n^* := \varphi_p(b_n)$; then, by Fact 2.2, we have

$$\begin{aligned}\varphi_p \circ f_n(B(a_n; r_n)) &\subset B(b_n^*; L/(\lambda_n^2 r_n)) \\ &\subset B(b_n^*; 1/(36 L \lambda_n r_n)) \subset \varphi_p \circ f_n(B(a_n; \lambda_n r_n)).\end{aligned}$$

Now we move b_n^* to a point $c_n^* \in S(p^*; 3/4) \subset X^*$ by an L-bilipschitz homeomorphism $g_n \colon X^* \to X^*$. Since $1/(36 L^2 \lambda_n r_n) < 1/4$, Fact 2.2 tells us that φ_{p^*} is 4-bilipschitz on $B(c_n^*; 1/(36 L^2 \lambda_n r_n))$. By [BHX, Prop. 3.3] we know that $\operatorname{Inv}_{p^*}(X^*)$

is 16-bilipschitz equivalent to the space X via some map denoted by h. Define $\Psi_n := h \circ \varphi_{p^*} \circ g_n \circ \varphi_p \circ f_n$. We now have

$$\Psi_n(B(a_n; r_n)) \subset B(c_n; 64L^2/(\lambda_n^2 r_n)) \subset B(c_n; 1/(2304L^2\lambda_n r_n)) \subset \Psi_n(B(a_n; \lambda_n r_n)).$$

Here $c_n := h \circ \varphi_{p^*}(c_n^*)$. By construction, there exists a pair of points in $B(c_n; 64L^2/(\lambda_n^2 r_n))$ that cannot be joined by a continuum in the larger ball $B(c_n; 1/(2304L^2\lambda_n r_n))$. Setting $r_n' := 64L^2/(\lambda_n^2 r_n)$ and $\lambda_n' := \lambda_n/(147456L^4)$, we find that $(\lambda_n', r_n') \in \mathcal{T}_2$ and $\lambda_n' r_n' \le 1$. Moreover, $\lambda_n' > M$. This contradicts the definition of M, so no such M can exist. We thus conclude that $\mu_m \to +\infty$ as $m \to +\infty$ (whether X is bounded or unbounded).

Now we extract a subsequence (μ_{m_l}) that is strictly increasing; in particular, we may assume that $\mu_{m_l} > 2\mu_{m_l-1}$. Observe the difference between μ_{m_l-1} and $\mu_{m_{(l-1)}}$. For each l there exists a pair $(\lambda, r) \in \mathcal{T}_2$ such that $\mu_{m_l-1} < \lambda \le \mu_{m_l}$ and $1/m_l \le \lambda r \le 1$. Now, if $1/(m_l - 1) < \lambda r$ then we have contradicted the definition of μ_{m_l-1}. Therefore, $\lambda r \le 1/(m_l - 1) \le 2/m_l$ (here we assume that $m_l \ge 2$). Thus we have

$$\mu_{m_l} = \sup\{\lambda : (\lambda, r) \in \mathcal{T}_2,\ 1/m_l \le \lambda r \le 2/m_l\}.$$

To avoid nested subscripts, we write $m(l) := m_l$. Fix l_0 and l such that $m(l_0) > 16 \cdot 10^8 L^4$ and $\mu_{m(l)} > 2 \cdot 10^9 L^4 \mu_{m(l_0)} > 2 \cdot 10^{12} L^4$. We also want

$$\frac{1}{m(l)} < \frac{t_l}{4L}, \tag{4.4}$$

where

$$t_l := \frac{1}{10^4 L}\sqrt{\frac{m(l_0)}{m(l)}}.$$

For each $l \in \mathbb{N}$ there exists a triple $(a_l, \lambda_l, r_l) \in \mathcal{T}_3$ such that $1/m(l) \le \lambda_l r_l \le 2/m(l)$ and $\mu_{m(l)}/2 \le \lambda_l \le \mu_{m(l)}$. We send a_l to some point $b_l \in S(p; t_l)$ via an L-bilipschitz homeomorphism $f_l \colon X \to X$. By (4.4) we have

$$f_l(B(a_l; \lambda_l r_l)) \subset A(p; t_l/2, 2t_l).$$

By Fact 2.2, applying φ_p yields

$$B(b_l^*; \lambda_l r_l/(16Lt_l^2)) \subset \varphi_p(f(B(a_l; \lambda_l r_l))) \subset B(b_l^*; 4L\lambda_l r_l/t_l^2),$$

where $b_l^* := \varphi_p(b_l)$. Then we map b_l^* to a point $c_l^* \in S(p^*; 1)$ by an L-bilipschitz homeomorphism $g_l \colon X^* \to X^*$. Note that our choice of l_0 results in

$$\frac{4L^2\lambda_l r_l}{t_l^2} \le \frac{8 \cdot 10^8 L^4}{m(l_0)} < \frac{1}{2}.$$

Therefore,

$$g_l \circ \varphi_p \circ f_l(B(a_l; \lambda_l r_l)) \subset A(p^*; 1/2, 2).$$

When X is unbounded, we apply φ_{p^*} and then a 16-bilipschitz map h to get back into the original space X (such a map h exists by [BHX, Prop. 3.3]). For $\Phi_l := h \circ \varphi_{p^*} \circ g_l \circ \varphi_p \circ f_l$ we have

$$\Phi_l(B(a_l; r_l)) \subset B(c_l; 10^3L^2r_l/t_l^2) \\ \subset B(c_l; \lambda_l r_l/(10^6L^2t_l^2)) \subset \Phi_l(B(a_l; \lambda_l r_l)), \tag{4.5}$$

where $c_l := h \circ \varphi_{p^*}(c_l^*)$.

When X is bounded, we let $q \in X$ denote any point such that $|p - q| \geq 1/2$. Writing $q^* := \varphi_p(q)$, we use ψ_{q^*} to denote the identity map $X^* \to \mathrm{Sph}_{q^*}(X^*)$. By [BHX, Prop. 3.5] there exists a 256-bilipschitz homeomorphism h between X and $\psi_{q^*}(X^*)$. Writing $\Psi_l := h \circ \psi_{q^*} \circ g_l \circ \varphi_p \circ f_l$, we obtain the same inclusions using Ψ_l as when using Φ_l in (4.5).

Suppose that every pair of points in $B(c_l; 10^3L^2r_l/t_l^2)$ can be joined by a continuum in $B(c_l; \lambda_l r_l/(10^6L^2t_l^2))$. Then we pull back by Φ_l or Ψ_l to conclude that every pair of points in $B(a_l; r_l)$ can be joined by a continuum in $B(a_l; \lambda_l r_l)$. This would be a contradiction to our construction. Hence there exists a pair of points in $B(c_l; 10^3L^2r_l/t_l^2)$ that cannot be joined by a continuum in $B(c_l; \lambda_l r_l/(10^6L^2t_l^2))$.

Set $r' := 10^3L^2r_l/t_l^2$ and $\lambda' := \lambda_l/(10^9L^4)$. Then

$$\frac{1}{m(l_0)} < \lambda' r' \leq 1.$$

Therefore, we find that

$$\mu_{m(l_0)} \geq \lambda' = \frac{\lambda_l}{10^9L^4} \geq \frac{\mu_{m(l)}}{2 \cdot 10^9L^4} > \mu_{m(l_0)}.$$

This contradiction allows us to conclude that X must be LLC_1.

Now we turn our attention to the LLC_2 condition. Again we use (i) bilipschitz homogeneity to prove that X must be LLC_2 at fixed scales and (ii) inversion invariance to confirm that a single LLC_2 constant works at all scales.

Define $\mathcal{S}_3$ to be the collection of triples $\{(a, \lambda, r)\}$ for which there exist points $x, y \in X \setminus B(a; r)$ that cannot be joined by a continuum in $X \setminus B(a; r/\lambda)$. Let $\mathcal{S}_2$ denote the pairs (λ, r) from the triples in $\mathcal{S}_3$, and define

$$\rho_m := \sup\{\lambda : (\lambda, r) \in \mathcal{S}_2,\ 1/m \leq r \leq 1\}.$$

For each m we claim that $1 \leq \rho_m < +\infty$. The lower bound is trivial. To see that each ρ_m is finite, suppose that $\{(a_n, \lambda_n, r_n)\}$ is a sequence of points from $\mathcal{S}_3$ for which $\lambda_n \to +\infty$ and $1/m \leq r_n \leq 1$. Then choose any point $a_0 \in X$. There exist L-bilipschitz homeomorphisms $f_n\colon X \to X$ with $f_n(a_n) = a_0$. Then, for each n, there exists a pair of points $x_n, y_n \in X \setminus B(a_0; r_n/L)$ that cannot be joined by a continuum in $X \setminus B(a_0; Lr_n/\lambda_n)$. Note that we may assume x_n and y_n to be contained in the ball $B(a_0; 2r_n/L)$. By the properness of X, there exists a pair of points x_0, y_0 to which subsequences from (x_n) and (y_n) converge. For convenience, assume $x_n \to x_0$ and $y_n \to y_0$. Using properness along with local connectedness, we conclude that $x_0 \neq y_0$.

Let E denote a continuum joining x_0 and y_0 in X, and suppose that $a_0 \notin E$. Let $\varepsilon > 0$ be given such that $B(a_0; \varepsilon) \cap E = \emptyset$ and $\varepsilon < 1/2mL$, and take n large enough so that $L/\lambda_n < \varepsilon$. Since X is locally connected and proper, there exist arbitrarily small connected neighborhoods of x_0 and y_0 whose closures are compact. So for large enough n, we can join x_n to x_0 and y_n to y_0 by continua inside

$B(x_0; \varepsilon)$ and $B(y_0; \varepsilon)$, respectively. Let F_n and G_n denote these continua. Since $L/\lambda_n < \varepsilon$, the set $F_n \cup E \cup G_n$ is a continuum joining x_n to y_n that does not intersect $B(a_0; Lr_n/\lambda_n)$. This contradicts the construction of x_n and y_n, so we must have $a_0 \in E$. Thus any continuum containing $\{x_0, y_0\}$ must also contain a_0. By elementary topology, this means that a_0 is a cut point of X. This contradicts our assumption that X has no cut points, so we conclude that $\rho_m < +\infty$.

Furthermore, the same strategy used previously to show that $\mu_m \to +\infty$ as $m \to +\infty$ can be used to verify that $\rho_m \to +\infty$ as $m \to +\infty$. We extract (ρ_{m_l}), which is strictly increasing, so that $\rho_{m_l} > 2\rho_{m_l - 1}$. Hence for each l there exists a pair $(\lambda, r) \in \mathcal{S}_2$ such that $\rho_{m_l - 1} < \lambda \le \rho_{m_l}$. Now, if $r > 1/(m_l - 1)$ then we have contradicted the definition of $\rho_{m_l - 1}$. Therefore, $r \le 1/(m_l - 1) \le 2/m_l$. Thus we have

$$\rho_{m_l} = \sup\{\lambda : (\lambda, r) \in \mathcal{S}_2,\ 1/m_l \le r \le 2/m_l\}.$$

We proceed in close parallel to the preceding arguments to obtain an index l_0, a pair $(\lambda', r') \in \mathcal{S}_2$, and a point $c \in X$ such that there exists a pair of points in $X \setminus B(c; r')$ that cannot be joined by a continuum in $X \setminus B(c; r'/\lambda')$. However, we construct (λ', r') so that $\rho_{m(l_0)} < \lambda' \le \rho_{m(l_0)}$, reaching essentially the same contradiction that appeared in our proof of the LLC_1 condition. Therefore, X is LLC_2. □

5. Examples and Questions

Whereas inversion invariant bilipschitz homogeneity implies both Ahlfors Q-regularity and the LLC conditions for certain spaces, bilipschitz homogeneity alone implies neither. We say that X is a *surface* if X is homeomorphic to $\mathbb{R}^2$.

EXAMPLE 5.1. *There exists a proper surface $X \subset \mathbb{R}^4$ that is uniformly bilipschitz homogeneous but does not satisfy the LLC_1 condition.*

Proof. Let $\Gamma \subset \mathbb{R}^3$ denote the (nonbounded turning) helix-type curve constructed in [HM, Exm. 5.6]. Then define $S := \Gamma \times \mathbb{R} \subset \mathbb{R}^4$. Since Γ is a proper metric space homeomorphic to the real line, S is a proper metric space homeomorphic to $\mathbb{R}^2$. Since Γ is not LLC_1, it follows that S is not LLC_1. Since both Γ and $\mathbb{R}$ are uniformly bilipschitz homogeneous, so is S. □

EXAMPLE 5.2. *There exists a proper surface $X \subset \mathbb{R}^3$ that is uniformly bilipschitz homogeneous and LLC but not Ahlfors Q-regular for any Q.*

Proof. Let $\Gamma \subset \mathbb{R}^2$ denote the unbounded Jordan curve constructed in [F1, Exm. 7.1]. Nondegenerate compact subarcs of Γ have positive finite $\mathcal{H}^Q$ measure (for $Q := \log_3(4)$), but Γ is not Ahlfors Q-regular. Define $S := \Gamma \times \mathbb{R} \subset \mathbb{R}^3$. Then S has Hausdorff dimension $Q + 1$ but is not Ahlfors $(Q + 1)$-regular. □

These two examples motivate the following questions.

QUESTION 5.3. Does there exist a condition that, when coupled with bilipschitz homogeneity, would imply the LLC condition but not Ahlfors Q-regularity?

QUESTION 5.4. Does bilipschitz homogeneity imply the LLC condition when $X \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^{n-1}$?

Note that a positive answer to Question 5.4 would provide a positive answer to Question 5.3 and a higher-dimensional analogue to [Bi, Thm. 1.1].

References

[Bi] C. J. Bishop, *Bi-Lipschitz homogeneous curves in* $\mathbb{R}^2$ *are quasicircles,* Trans. Amer. Math. Soc. 353 (2001), 2655–2663.

[BoK1] M. Bonk and B. Kleiner, *Quasisymmetric parametrizations of two dimensional metric spheres,* Invent. Math. 150 (2002), 127–183.

[BoK2] ———, *Rigidity for quasi-Möbius group actions,* J. Differential Geom. 61 (2002), 81–106.

[BHX] S. M. Buckley, D. A. Herron, and X. Xie, *Metric space inversions, quasihyperbolic distance, and uniform spaces,* Indiana Univ. Math. J. 57 (2008), 837–890.

[F1] D. M. Freeman, *Bilipschitz homogeneous Jordan curves, Möbius maps, and dimension,* Illinois J. Math. 54 (2010), 753–770.

[F2] ———, *Unbounded bilipschitz homogeneous Jordan curves,* Ann. Acad. Sci. Fenn. Math. 36 (2010), 81–99.

[GH1] M. Ghamsari and D. A. Herron, *Higher dimensional Ahlfors regular sets and chordarc curves in* $\mathbf{R}^n$, Rocky Mountain J. Math. 28 (1998), 191–222.

[GH2] ———, *Bi-Lipschitz homogeneous Jordan curves,* Trans. Amer. Math. Soc. 351 (1999), 3197–3216.

[HM] D. A. Herron and V. Mayer, *Bi-Lipschitz group actions and homogeneous Jordan curves,* Illinois J. Math. 43 (1999), 770–792.

[L] E. Le Donne, *Doubling property for bi-lipschitz homogeneous geodesic surfaces,* J. Geom. Anal. 21 (2011), 783–806.

[M1] V. Mayer, *Trajectoires de groupes à 1-paramètre de quasi-isométries,* Rev. Mat. Iberoamericana 11 (1995), 143–164.

[M2] ———, *Phénomènes de rigidité en dynamique holomorphe et quasirégulière, ensembles Lip-homogènes,* Habilitation à Diriger des Recherches en Sciences Matématiques, 2000.

[R] S. Rohde, *Quasicircles modulo bilipschitz maps,* Rev. Mat. Iberoamericana 17 (2001), 643–659.

[W] K. Wildrick, *Quasisymmetric parametrizations of two-dimensional metric planes,* Proc. London Math. Soc. (3) 97 (2008), 783–812.

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