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Enriques Surfaces: Brauer Groups and Kummer Structures

Alice Garbagnati & Matthias Schütt

1. Introduction

The Brauer group is an important but very subtle birational invariant of a projective surface. In [3], Beauville proved that generically the Brauer group of a complex Enriques surface injects into the Brauer group of the covering K3 surface. Subsequently Beauville asked for explicit examples where the Brauer groups pull back identically to zero. This problem has recently been solved in [6] (see also Section 7.10) and in [7], but only by isolated so-called singular K3 surfaces (Picard number 20). In this paper we develop methods to derive such surfaces in 1-dimensional families. Our results cover both the Kummer and the non-Kummer case. In Section 3, we construct for any integer $N > 1$ a 1-dimensional family $\mathcal{X}_N$ of complex K3 surfaces with Picard number $\rho \geq 19$ such that the general member admits an Enriques involution τ.

Theorem 1. *Let $N > 1$. Consider a general K3 surface $X_N \in \mathcal{X}_N$; that is, $\rho(X_N) = 19$. Denote the quotient by the Enriques involution τ by Z_N. Then*

$$\pi^* \operatorname{Br}(Z_N) = \begin{cases} \{0\} & \text{if } N \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } N \text{ is even.} \end{cases}$$

The K3 surfaces in the family $\mathcal{X}_N$ are generally not Kummer, but in Section 5 we exploit a geometric construction related to Kummer surfaces of N-isogenous elliptic curves. By similar methods, in Section 7 we derive for any $N \in \mathbb{N}$ a 1-dimensional family $\mathcal{Y}_N$ consisting of Kummer surfaces with Picard number $\rho \geq 19$ and Enriques involution τ.

Theorem 2. *Let $N \in \mathbb{N}$. Consider a Kummer surface $Y_N \in \mathcal{Y}_N$ with $\rho(Y_N) = 19$. Let τ denote the Enriques involution on Y_N. Then*

$$\pi^* \operatorname{Br}(Y_N/\tau) = \begin{cases} \{0\} & \text{if } N \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } N \text{ is even.} \end{cases}$$

The two theorems show that the Enriques surfaces in question come in families. We shall work out one family in detail in Theorem 14. The general assumption that the K3 surfaces have nonmaximal Picard number $\rho = 19$ is fairly mild and not strictly necessary (see Proposition 15 and Section 5.9).

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Our main construction uses elliptic fibrations and isogenies of K3 surfaces, sometimes in the context of Shioda–Inose structures. The aforementioned relation with Kummer surfaces of product type also give rises to applications of our techniques to string theory and Picard–Fuchs equations (see Section 4).

The paper is organized as follows. In the next section we review the basic properties of K3 surfaces and Enriques surfaces relevant to this work. We also discuss Beauville's result on the pull-back of the Brauer group and the subsequent problem posed by him (Section 2.6). In Section 3 we construct the families of K3 surfaces $\mathcal{X}_N$ with Enriques involution in Theorem 1 from a lattice-theoretic point of view. These families will be constructed as specializations of the Barth–Peters 2-dimensional family of K3 surfaces [2]. The geometric properties and the moduli space of the Barth–Peters family will be described in Section 4. We also point out relations to other K3 surfaces, Calabi–Yau threefolds, and their Picard–Fuchs equations investigated in string theory. Section 5 is the geometric center of the paper: the families $\mathcal{X}_N$ are reconsidered and constructed in a very geometric way. As an illustration, the family $\mathcal{X}_3$ is analyzed in Section 6. We give explicit equations over $\mathbb{Q}$ and describe the specializations of this family to singular K3 surfaces, relating them to the presence of complex multiplication on certain elliptic curves. In Section 7 we introduce the Kummer families $\mathcal{Y}_N$ and prove Theorem 2 by techniques similar to the ones applied in previous sections to the families $\mathcal{X}_N$.

2. Basic Properties

Throughout this paper we work over number fields (such as the field of rational numbers $\mathbb{Q}$) or over the field of complex numbers $\mathbb{C}$. For basic properties of K3 surfaces and Enriques surfaces relevant to our paper, we refer to [7] and the references therein, in particular [1]. Here we concentrate on the most important ingredients for this paper. The main motivation for this work is explained in Section 2.6, where we discuss Beauville's result on the pull-back of the Brauer group and the subsequent problem posed by him.

2.1. Lattices

A lattice is a pair (L, b) where L is a free $\mathbb{Z}$-module of finite dimension and b is a symmetric bilinear form defined over L and taking values in $\mathbb{Z}$. We often omit b when the bilinear form is clear from the context. If (L, b) is a lattice, we denote by $L(n)$, $n \in \mathbb{Z}$, the same $\mathbb{Z}$-module with the bilinear form multiplied by n. We will denote by nL the lattice obtained as a direct sum of L n times.

We denote by $L^\vee := \mathrm{Hom}(L, \mathbb{Z})$ the dual lattice of (L, b). The discriminant lattice of (L, b) is defined as $A_L := L^\vee/L$ and we denote by $l(A_L)$ the minimum number of its generators. The bilinear form b induces a bilinear form on $L^\vee/L$ taking values in $\mathbb{Q} \bmod \mathbb{Z}$, which is called discriminant form. The signature (s_+, s_-) of a lattice (L, b) is the signature of the $\mathbb{R}$-linear extension of the bilinear form b. A lattice (L, b) is said to be even if $b(l, l) \in 2\mathbb{Z}$ for each $l \in L$, and it is said to be unimodular if $L \simeq L^\vee$. We recall the following results, due to Nikulin, that will be used in the sequel.

PROPOSITION 3 [17, Cor. 1.13.3]. *Let L be an even lattice with signature (s_+, s_-) and discriminant form q_L. If $s_+ > 0$, $s_- > 0$, and $l(A_L) \le \operatorname{rank}(L) - 2$, then up to isometry L is the only lattice with these invariants.*

THEOREM 4 [17, Thm. 1.14.4]. *Let M be an even lattice with invariants (t_+, t_-, q_M) and let L be an even unimodular lattice of signature (s_+, s_-). Suppose that*

$$t_+ < s_+, \quad t_- < s_-, \quad l(A_M) \le \operatorname{rank}(L) - \operatorname{rank}(M) - 2.$$

Then there exists a unique primitive embedding of M in L.

The second cohomology group of a K3 surface W that has integer coefficients, $H^2(W, \mathbb{Z})$, with the pairing induced by the cup product is a lattice isometric to the even unimodular lattice $\Lambda_{K3} := 2E_8(-1) + 3U$ (the K3 lattice), where U is the hyperbolic rank-2 lattice with pairing $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $E_8(-1)$ is the rank-8 negative definite lattice associated to the Dynkin diagram E_8 (cf. [1]). The Néron–Severi group of W, $\mathrm{NS}(W) = H^{1,1}(W) \cap H^2(W, \mathbb{Z})$, is a sublattice of $H^2(W, \mathbb{Z})$ the rank of which is called the Picard number of W and denoted by $\rho(W)$. The transcendental lattice of W, denoted by $T(W)$, is the orthogonal complement of $\mathrm{NS}(W)$ in the lattice $H^2(W, \mathbb{Z})$.

2.2. Lattice Enhancements

We explain a lattice-theoretic method in order to determine certain subfamilies of a family of K3 surfaces with given transcendental lattice. The ideas are based on the theory of lattice-polarized K3 surfaces.

Let W be the generic member of a family of K3 surfaces with transcendental lattice $T(W)$. Here $\rho(W) < 20$ because otherwise the family would consist solely of W. Hence $T(W)$ is indefinite, and we can fix a vector v in $T(W)$ with negative self-intersection. By the surjectivity of the period map, there exists a K3 surface W_2 such that

$$T(W_2) \simeq v^{\perp_{T(W)}}.$$

The surface W_2 is the general member of a subfamily of the family of W of codimension 1. Clearly the Néron–Severi lattice of W_2 is an overlattice of finite index of $\mathrm{NS}(W) + \langle v \rangle$. This can be made precise as follows: $\mathrm{NS}(W_2)$ is the minimal primitive sublattice of $H^2(W, \mathbb{Z}) \simeq H^2(W_2, \mathbb{Z})$ containing $\mathrm{NS}(W) + \langle v \rangle$.

We explain how to find a $\mathbb{Z}$-basis for $\mathrm{NS}(W_2)$. Recall the following connection between the discriminant forms of $T(W)$ and $\mathrm{NS}(W)$. Let $n_i \in \mathrm{NS}(W)$ and $d_i \in \mathbb{N}$ be such that n_i/d_i are generators of the discriminant form of $\mathrm{NS}(W)$. Then there exist $t_i \in T(W)$ such that t_i/d_i are generators of the discriminant form of $T(W)$ and $(n_i + t_i)/d_i$ are in $H^2(W, \mathbb{Z})$. In practice, we can always choose v primitive and set $v = t_1$ (possibly with $d_1 = 1$). Then $\mathrm{NS}(W) + \langle v \rangle$ has index d_1 in $\mathrm{NS}(W_2)$, and the full Néron–Severi lattice can be obtained from $\mathrm{NS}(W) + \langle v \rangle$ by adjoining the vector $(n_1 + t_1)/d_1 \in \mathrm{NS}(W_2)$.

2.3. Elliptic Fibrations

We will extensively use elliptic fibrations on K3 surfaces. To give an elliptic fibration, it suffices to exhibit a divisor D of Kodaira type, thus coinciding with one of the singular fibers. Then any irreducible curve meeting D transversally in exactly one point gives a section of the fibration. Elliptic fibrations with section are often called jacobian; they can be completely understood in terms of Mordell–Weil lattices [21]. A jacobian elliptic fibration on a K3 surface X is determined by a direct summand of the hyperbolic plane U in the Néron–Severi group:

$$\mathrm{NS}(X) = U + L.$$

Such decompositions can be generally classified by a gluing method going back to Kneser [11] and introduced to the K3 context by Nishiyama [18]. A singular fiber gives rise to a negative-definite root lattice of A-D-E type by omitting the identity component (met by the zero section O) and drawing the intersection graph of the remaining components (which are all (-2)-curves and thus yield roots). The trivial lattice of the fibration, generated by zero section and fiber components, thus takes the shape $U+$ (root lattices of A-D-E type). Conversely, the singular fibers are encoded in the roots of L (i.e., in the root lattice L_{root}) and the remainder of $\mathrm{NS}(X)$ comes from sections. In detail, let L'_{root} denote the primitive closure of L_{root} in L. Then the torsion in the Mordell–Weil group is given by

$$\mathrm{MW}(X)_{\text{tor}} \cong L'_{\text{root}}/L_{\text{root}},$$

and the Mordell–Weil lattice is $\mathrm{MWL}(X) = L/L'_{\text{root}}$. Here the orthogonal projection with respect to the trivial lattice in $\mathrm{NS}(X) \otimes \mathbb{Q}$ endows $\mathrm{MWL}(X)$ with the structure of a definite lattice (not necessarily integral) by [21].

Every jacobian elliptic fibration $X \to C$ corresponds to its generic fiber, an elliptic curve defined over the field of functions of C. In consequence, X inherits certain automorphisms from its generic fiber. Every elliptic curve admits a hyperelliptic involution that extends fiberwise to the elliptic surface. We denote the resulting involution by $-\mathrm{id}$. Moreover, on an elliptic curve there is translation by a point. Let P be a rational point on an elliptic curve; then the induced automorphism t_P sends a point Q to the point $Q + P$, where $+$ is the sum with respect to group law of the elliptic curve. Sections (i.e., points on the generic fiber) thus induce automorphisms on the elliptic surface.

Another fundamental construction used in the following is quadratic twisting, often also related to quadratic base change and the deck transformations. For background the reader is referred to [7, Sec. 3.3].

2.4. Picard Number and Shioda–Inose Structures

In general the Picard number is far from a birational invariant, since one can always consider blow-ups. In contrast, for complex K3 surfaces (which are by definition minimal) the Picard number is much more than that: by [8] it is preserved by rational dominant maps because the Hodge structure on the transcendental lattice is preserved. This is the main reason why K3 surfaces of high Picard number (at least 17) can often be studied through Kummer surfaces. Thus the K3 surfaces in Theorems 1 and 2 share the structure of Kummer surfaces to a strong extent.

The most prominent case of this situation consists of a Shioda–Inose structure. Like Morrison [14], we ask whether the K3 surface X admits a rational map of degree 2 to a Kummer surface $\mathrm{Km}(A)$ such that the intersection form on the transcendental lattice is multiplied by 2:

$$T(\mathrm{Km}(A)) = T(X)(2). \tag{1}$$

Equivalently one has $T(A) = T(X)$. In particular, $\mathrm{Km}(A)$ is the quotient of X by a Nikulin involution $\jmath$. An equivalent criterion for (1) is that $\jmath$ exchanges two perpendicular divisors of type $E_8(-1)$ (see [14]). Such involutions are called Morrison–Nikulin involutions.

2.5. *Enriques Involution*

Recall that an Enriques involution is a fixed point–free involution τ on a K3 surface X. The quotient X/τ is called an Enriques surface. Conversely, we recover X from Y through the universal cover

$$\pi : X \to Y.$$

The universal cover is directly related to the canonical divisor K_Y that gives the 2-torsion in $\mathrm{NS}(Y)$. Pulling back $\mathrm{Num}(Y) = \mathrm{NS}(X)/\langle K_Y\rangle \cong U + E_8(-1)$ via π^*, we obtain the primitive embedding

$$U(2) + E_8(-2) \hookrightarrow \mathrm{NS}(X). \tag{2}$$

By the Torelli theorem, Enriques involutions can be characterized by the lattice polarization (2) together with the additional assumption that the orthogonal complement of $U(2) + E_8(-2)$ in $\mathrm{NS}(X)$ does not contain any roots. (This ensures that the involution determined by (2) has no fixed points.) Thus we can study the moduli of K3 surfaces with Enriques involution through the lattice polarization (2).

For instance, any Kummer surface admits an Enriques involution by [9]. Contrary to this, Shioda–Inose structures generally do not accommodate Enriques involutions.

2.6. *Enriques Surfaces and Brauer Groups*

The Brauer group of a smooth projective surface can be defined in étale cohomology as $\mathrm{Br}(S) = H^2_{\text{ét}}(S, \mathbb{G}_m)$. The Brauer group is a birational invariant that encodes very subtle information. For instance, if S is a complex surface, then $\mathrm{Br}(S)$ contains a subgroup $\mathrm{Br}(S)'$ that is dual to the transcendental lattice in a suitable sense and the quotient $\mathrm{Br}(S)/\mathrm{Br}(S)'$ is isomorphic to the torsion in $H^3(S, \mathbb{Z})$.

For a complex Enriques surface Y it follows that

$$\mathrm{Br}(Y) = H^2_{\text{ét}}(Y, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}.$$

Consider the K3 surface X given by the universal cover $\pi : X \to Y$. The important problem of how $\mathrm{Br}(Y)$ pulls back to the K3 surface X via π^* was recently solved by Beauville.

THEOREM 5 (Beauville [3]). *Generally, the equation* $\pi^* \operatorname{Br}(Y) = \mathbb{Z}/2\mathbb{Z}$ *holds. One has* $\pi^* \operatorname{Br}(Y) = \{0\}$ *if and only if there is a divisor* D *on* X *such that* $\tau^* D = -D$ *in* $\operatorname{NS}(X)$ *and* $D^2 \equiv 2 \bmod 4$.

In other words, the Enriques surfaces with Brauer group pulling back identically to zero to the covering K3 surface lie on countably many hyperplanes in the moduli space of Enriques surfaces (cut out by the conditions of (3) to follow).

PROBLEM (Beauville).

(i) *Give explicit examples of Enriques surfaces such that* $\pi^*(\operatorname{Br}(Y)) = \{0\}$.
(ii) *Are there such surfaces defined over* $\mathbb{Q}$?
(iii) *If so, exhibit some.*

Note that the main problem in (ii) is the possibility that the countable number of Enriques surfaces in question might avoid the specific hyperplanes.

In the meantime, we have seen isolated examples as singular K3 surfaces answering all three questions (cf. [6; 7]) but no explicit families yet. Let us emphasize that here we ask for explicit defining equations as opposed to (moduli spaces of) K3 surfaces determined by a lattice polarization. We will make this difference clear in what follows. Single examples over $\mathbb{Q}$ have been exhibited independently by Garbagnati and van Geemen [6] (see Section 7.10) and by Hulek and Schütt [7]. Note that the first example is a Kummer surface whereas the second is not.

Our aim is to exhibit explicit families of K3 surfaces with Enriques involution as just described. Abstractly this is easily achieved lattice theoretically because we require only that the K3 surface X admit a primitive embedding,

$$U(2) + E_8(-2) + \langle -2N \rangle \hookrightarrow \operatorname{NS}(X), \tag{3}$$

for some odd $N > 1$ such that the orthogonal complement of $U(2) + E_8(-2)$ in $\operatorname{NS}(X)$ does not contain any (-2) vectors. However, it is nontrivial to exhibit explicit geometric constructions of such surfaces, let alone to find explicit equations.

3. Lattice Enhancements

In this section we will construct the surfaces X_N appearing in Theorem 1 using lattice theory and in particular the construction described in Section 2.2. The surfaces X_N will be the generic members of the family $\mathcal{X}_N$ of the $(U(2)+2E_8(-1)+\langle -2N \rangle)$-polarized K3 surfaces. Since this lattice admits a unique embedding into Λ_{K3} up to isometries (cf. Theorem 4), K3 surfaces with this polarization form a unique 1-dimensional family. It will be convenient to view $\mathcal{X}_N$ as subfamilies of the Barth–Peters family $\mathcal{X}$. This is a 2-dimensional family of K3 surfaces that admits an Enriques involution with exceptional properties [2] (see also [7; 16]). The Barth–Peters family $\mathcal{X}$ specializes to the 1-dimensional families $\mathcal{X}_N$ (cf. Section 3.2).

3.1. Barth–Peters Family $\mathcal{X}$

There is a unique 2-dimensional family of K3 surfaces $\mathcal{X}$ such that generally

$$\operatorname{NS}(\mathcal{X}) = U(2) + 2E_8(-1). \tag{4}$$

Here the primitive embedding (3) is achieved by realizing $E_8(-2)$ diagonally in the two copies of $E_8(-1)$. By construction, this induces an Enriques involution τ on the general member X of $\mathcal{X}$.

There are many ways to exhibit $\mathcal{X}$ geometrically (see [2; 16]). For instance, one can give it as a 2-dimensional family of elliptic fibrations

$$\mathcal{X}\colon\ y^2 = x(x^2 + a(t)x + 1),\ a(t) = a_0 + a_2t^2 + t^4 \in k[t]. \tag{5}$$

There is a 2-torsion section $(0,0)$ and a reducible fiber of Kodaira type I_{16} at ∞. The fibrations described by (5) are quadratic base changes from a 1-dimensional family $\mathcal{R}$ of rational elliptic surfaces that can be recovered as a quotient by the involution ι induced by $t \mapsto -t$. The composition of ι and translation by $(0,0)$ is an Enriques involution (the classical case of the more general construction from [7]).

In order to exhibit a basis of $\mathrm{NS}(\mathcal{X})$, we note that the rational elliptic surfaces in $\mathcal{R}$ generally have Mordell–Weil rank 1. Pulling back a generator, we obtain a section Q on $\mathcal{X}$ of height $h(Q) = 1$. Comparing discriminants, we find that the Mordell–Weil lattice of this elliptic fibration is generated by Q: $\mathrm{MWL}(\mathcal{X}) \simeq [1]$.

3.2. The Subfamilies $\mathcal{X}_N$ of $\mathcal{X}$

Starting from the Barth–Peters family $\mathcal{X}$, we want to describe the K3 surfaces with Picard number 19 and Néron–Severi lattice isometric to $U(2)+2E_8(-1)+\langle -2N\rangle$. Under a very mild condition, the Enriques involution specializes also from $\mathcal{X}$, as we will see in Section 3.3.

PROPOSITION 6. *Let X_N be the generic member of the subfamily of the Barth–Peters family $\mathcal{X}$ obtained as in Section 2.2 by choosing the vector v to be $v_N :=$ $(1,-N,0,0) \in U + U(2) \simeq T(X)$. Then* $\mathrm{NS}(X_N) \simeq \mathrm{NS}(X) + \langle -2N\rangle$, $T(X_N) \simeq \langle 2N\rangle + U(2)$.

(i) *If $N > 1$, then X_N admits an elliptic fibration with singular fibers $I_{16} + 8I_1$, Mordell–Weil group $(\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z}$, and Mordell–Weil lattice* $\mathrm{MWL} \simeq \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2N \end{smallmatrix}\right]$.

(ii) *If $N = 1$, then the induced elliptic fibration on X_1 has the singular fibers $I_{16} + I_2 + 6I_1$ and the Mordell–Weil group is $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. The transcendental lattice of X_N is the orthogonal complement in $T(X)$ to v_N, so it is isometric to $\langle 2N\rangle + U(2)$. In particular, the transcendental lattice has discriminant 2^3N. The Néron–Severi lattice is an overlattice of $\mathrm{NS}(X) + \langle -2N\rangle$ of finite index. Since the discriminant of $\mathrm{NS}(X) + \langle -2N\rangle$ is -2^3N, we conclude that $\mathrm{NS}(X_N) \simeq \mathrm{NS}(X) + \langle -2N\rangle$. We denote by F the class of the fiber of the elliptic fibration (5) on X and by O the class of the zero section. The elliptic fibration on X specializes to an elliptic fibration on X_N. If $N > 1$, then the class $u := NF + O + v_N$ corresponds to a section of infinite order on X_N. The section u meets the identity component C_0 of the I_{16} fiber. We have $u.O = N - 2$ and $u.Q = N$. This gives the Mordell–Weil lattice.

If $N = 1$ then the class v_N corresponds to a class with self-intersection -2 that is orthogonal to the class of the fiber and to the fiber components of I_{16}. Hence, on the fibration there is another reducible fiber that is of type I_2. So the elliptic fibration on X_1 has $I_{16} + I_2 + 6I_1$ as singular fibers and the Mordell–Weil group is $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as before. □

In Section 5 we work out an explicit geometric construction of X_N for odd N. Meanwhile, this section concludes with an investigation of how the Enriques involution τ on X specializes to X_N.

3.3. Enriques Involution on X_N

On K3 surfaces, specialization preserves many properties such as automorphisms. Along these lines, an Enriques involution will specialize to an involution, but it need not specialize to an Enriques involution. That is, the specialized involution need no longer be fixed point free. This subtlety is based on the fact that the moduli space of an Enriques surface is exactly the moduli space of $(U(2) + E_8(-2))$-polarized K3 surfaces with countably many hyperplanes removed. Whence one has to avoid the situation where the specialization hits (or even sits inside) those hyperplanes.

The hyperplanes correspond to the presence of some (-2)-curve in the orthogonal complement of $U(2) + E_8(-2)$ inside NS. In particular, we have seen an instance of an Enriques involution not specializing in Section 3.2. For $N = 1$, the singular fibers degenerate on X_N to form an I_2 fiber (where the corresponding base change ramifies). Naturally this gives a (-2)-curve in the specified orthogonal complement, so the family of X_1 lies completely in one such hyperplane. We will now check that this does not happen for $N > 1$.

PROPOSITION 7. *The Enriques involution τ on the Barth–Peters family $\mathcal{X}$ specializes to an Enriques involution on the subfamily $\mathcal{X}_N$ if and only if $N > 1$.*

Proof. We start with the primitive embedding of $U(2) + E_8(-2)$ in $\mathrm{NS}(X)$ given by the Enriques involution τ on $\mathcal{X}$. Clearly this induces a primitive embedding

$$U(2) + E_8(-2) \hookrightarrow \mathrm{NS}(X_N) \simeq \mathrm{NS}(X) + \langle -2N \rangle.$$

The orthogonal complement of $U(2) + E_8(-2)$ in $\mathrm{NS}(X_N)$ is thus isometric to

$$(U(2) + E_8(-2))^{\perp_{\mathrm{NS}(X)}} + \langle -2N \rangle \simeq E_8(-2) + \langle -2N \rangle. \tag{6}$$

Note that the orthogonal complement of $U(2) + E_8(-2)$ in $\mathrm{NS}(X_N)$ is negative definite. It contains no classes with self-intersection -2 if and only if $N > 1$. Hence it is in exactly the latter case where τ specializes to an Enriques involution on X_N.

3.4. Abstract Proof of Theorem 1

Thanks to the specific form of our K3 surfaces and the Enriques involution, we can determine explicitly how the Brauer group pulls back from the Enriques quotient. This enables us to prove Theorem 1.

Recall the setup with $N > 1$ and X_N a general member of the K3 family $\mathcal{X}_N$. Let τ denote the Enriques involution induced from $\mathcal{X}$ and let $Z_N = X_N/\tau$.

We have computed the orthogonal complement of $U(2) + E_8(-2)$ in $\mathrm{NS}(X_N)$ in (6). Clearly this gives exactly those divisors that are anti-invariant for τ^*. The lattice in (6) represents only 4-divisible integers if and only if N is even. Thus we deduce from Theorem 5 that

$$\tau^* \operatorname{Br}(Z_N) = \begin{cases} \{0\} & \text{if } N \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } N \text{ is even.} \end{cases}$$

This proves Theorem 1.

REMARK 8. The same argument applies to the Barth–Peters family $\mathcal{X}$ to show that, for a general member, the second alternative (injectivity of the Brauer group under pull-back) holds true.

4. Barth–Peters Family: Elliptic Fibrations and Moduli

In this section and in the next one we will describe geometric properties and elliptic fibrations of the families introduced in Section 3 in order to describe their moduli spaces and to exhibit a geometric proof of Theorem 1. In particular, we will associate to the Barth–Peters family $\mathcal{X}$ a family of Kummer surfaces and hence of abelian surfaces. Using the relations between these families, one can easily describe the moduli and the Picard–Fuchs equation of the Barth–Peters family. This answers a problem on Enriques Calabi–Yau threefolds originating from string theory.

4.1. The Elliptic Fibration $[2III^*, 2I_2]$ *on* $\mathcal{X}$

We choose another convenient model of the Barth–Peters family $\mathcal{X}$ of K3 surfaces following [7] and [16]. It is defined as jacobian elliptic fibration through a family of quadratic base changes of $\mathbb{P}^1$,

$$f : t \mapsto \frac{(t-a)(t-b)}{t}, \quad ab \neq 0,$$

over the unique rational elliptic surface R with singular fibers III^*, I_2, I_1 and $\mathrm{MW} = \mathbb{Z}/2\mathbb{Z}$. One finds the model

$$\mathcal{X}\colon\ y^2 = x(x^2 + t^2 x + t^3(t-a)(t-b)) \tag{7}$$

generally with reducible fibers of type III^* at $0, \infty$ and of type I_2 at a, b. Here the 2-torsion section is given by $(0,0)$. Despite the symmetry in a, b, it is natural to study the family in the parameters a, b (as opposed to $a+b, ab$) because we want to parameterize K3 surfaces with $\rho = 19$ over a given field (say over $\mathbb{Q}$)—in other words, without Galois action on the two I_2 fibers.

4.2. Enriques Involution on $\mathcal{X}$

On $\mathcal{X}$ we have several interesting involutions. We will need the following:

- the deck transformation corresponding to f;
- translation by the 2-torsion section $(0,0)$;
- the hyperelliptic involution $-\mathrm{id}$.

As in [7], the composition of the first two involutions defines an Enriques involution τ on a general member of the family $\mathcal{X}$. It was checked in [7] that this is exactly the involution induced by the decomposition (4) of $\mathrm{NS}(\mathcal{X})$ and the specified embedding of the Enriques lattice. Denote the quotient family by $\mathcal{Y} = \mathcal{X}/\tau$. Then the hyperelliptic involution $-\mathrm{id}$ induces an involution on $\mathcal{Y}$ that acts trivially on $H^2(\mathcal{Y}, \mathbb{Z})$. Such a cohomologically trivial involution is remarkable because it cannot occur on a K3 surface by the Torelli theorem. In fact, complex Enriques surfaces with cohomologically trivial involution have been classified by Mukai and Namikawa [16] (later corrected by Mukai [15]).

THEOREM 9. *Let Y be a complex Enriques surface with a cohomologically trivial involution. Then $Y \in \mathcal{Y}$.*

4.3. Relation with Kummer Surfaces

The Barth–Peters family admits a Shioda–Inose structure (cf. Section 7), but it will be even more convenient for our purposes to pursue a different approach leading to Kummer surfaces. Namely, we will study the family $\mathcal{X}$ by applying a suitable symplectic involution such that the quotient family consists of Kummer surfaces of product type.

In order to relate the family $\mathcal{X}$ directly to some Kummer surfaces, we consider an alternative elliptic fibration. We proceed by identifying suitable divisors of Kodaira type (cf. Section 2.3). Presently we extract two singular fibers of type I_4^* from the curves visible in the elliptic fibration (7). One divisor of Kodaira type I_4^* is supported on the III^* fiber at 0 extended by zero section and identity components of III^* at ∞ and one I_2 fiber (say at $t = a$). The perpendicular curves (components of the III^* at $t = \infty$ plus 2-torsion section, far simple component of III^* at $t = 0$ and of I_2 at $t = b$) form another divisor of type I_4^*. This leaves two double components of the III^* fibers that serve as sections of the new fibration (zero and 2-torsion sections). All these (-2)-curves are sketched in Figure 1.

Explicitly, this elliptic fibration is extracted by the parameter $u = x/(t^2(t-a))$ in (7). We obtain the Weierstrass form (in t, y)

$$\mathcal{X}\colon\ y^2 = t(t^2 + u(1 + u - au^2)t - bu^4). \tag{8}$$

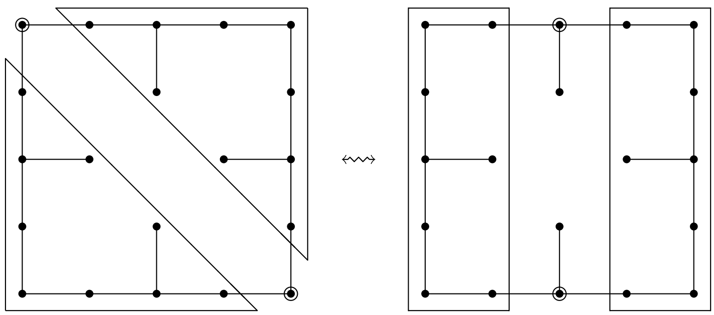

Figure 1 Divisors of type I_4^* versus III^*s and A_1s

Here, translation by the 2-torsion section $(0,0)$ defines a Nikulin involution. After desingularization, the quotient results in a family $\mathcal{X}'$ of K3 surfaces with two singular fibers of type I_2^* and four fibers of type I_2:

$$\mathcal{X}'\colon\ y^2 = t(t^2 - 2u(1+u-au^2)t + u^2((1+u-au^2)^2 + 4bu^2)). \qquad (9)$$

These elliptic surfaces have generally $\mathrm{MW} = (\mathbb{Z}/2\mathbb{Z})^2$ over $k\big(\sqrt{-b}\big)$ (given explicitly in Section 4.4).

4.4. Kummer Structure

By the classification of Oguiso [19], a general Kummer surface of product type $\mathrm{Km}(E \times E')$ admits an elliptic fibration with singular fibers and MW as described previously. Thus we compare 2-dimensional families of K3 surfaces: $\mathcal{X}'$ and Kummer surfaces of product type. Yet by Proposition 3 it follows from the discriminant form (or from Oguiso's classification) that

$$\mathrm{NS} = U + 2D_4(-1) + E_8(-1).$$

Because this lattice admits a unique embedding into the K3 lattice up to isometries (cf. Theorem 4), K3 surfaces with this Néron–Severi lattice form a unique 2-dimensional family. In particular, the family $\mathcal{X}'$ and the Kummer family of product type coincide. We proceed by working out the relation in detail.

Given $\mathcal{X}'$ over $k(a,b)$, there exist elliptic curves E and E' such that $\mathcal{X}' \cong \mathrm{Km}(E \times E')$. In order to find the elliptic curves, we exhibit an alternative elliptic fibration on $\mathcal{X}'$ with two fibers of type IV^*. This will allow us to obtain information about the j-invariants of the elliptic curves from the coefficients of the Weierstrass form by [8] (cf. [22]).

We identify two disjoint divisors of Kodaira type IV^* in the model (9) as depicted in Figure 2: on the one hand, the first five components of an I_2^* fiber (say at ∞) extended by zero section O and the 2-torsion section $R = (0,0)$ (which is distinguished by the fact that it meets all reducible fibers at nonidentity components); on the other hand, the last five components of the other I_2^* fiber extended by the other 2-torsion sections. These divisors induce an elliptic fibration on $\mathcal{X}'$ with two singular fibers of type IV^*. Here we have plenty of sections for the new fibration given by the remaining original fiber components.

To write down the fibration explicitly, it is convenient to translate x so that one of the other 2-torsion sections becomes $(0,0)$. For this purpose, we write $b = -c^2$. Then the conjugate 2-torsion sections have t-coordinate $u + u^2 \pm 2cu^2 - au^3$. The translation $t \mapsto t + u + u^2 + 2cu^2 - au^3$ gives the Weierstrass form

$$\mathcal{X}'\colon\ y^2 = t(t + 4cu^2)(t + u + u^2 + 2cu^2 - au^3).$$

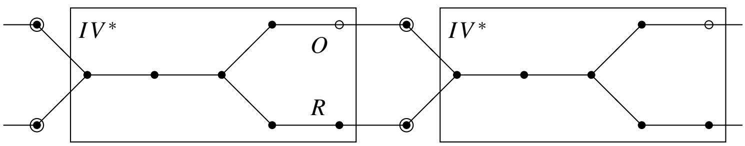

Figure 2 Divisors of type IV^* versus I_2^*s and 2-torsion sections

The last factor of the RHS encodes the distinguished 2-torsion section R. The foregoing divisors of Kodaira type IV^* are extracted at $v = 0, \infty$ by the affine parameter

$$v = \frac{y}{t + u + u^2 + 2cu^2 - au^3}.$$

Indeed, solving for y, we obtain the following family of cubics in $\mathbb{A}^2$ with coordinates t, u and parameter v:

$$\mathcal{X}' \colon (t + u + u^2 + 2cu^2 - au^3)v^2 = t(t + 4cu^2).$$

This model makes visible the quadratic base change from the rational elliptic surface that is given by the cubic pencil with $w = v^2$. Standard formulas give the following Weierstrass form in the usual coordinates x, y with elliptic parameter v and moduli a, b recovered from $b = -c^2$:

$$\begin{aligned}\mathcal{X}' \colon y^2 = x^3 &- \frac{16}{3}v^4(1 - 12b + 3a)x \\ &+ \frac{16}{27}v^4(8v^2 + 288v^2b + 36av^2 - 432b + 27a^2v^4).\end{aligned}$$

4.5. *Elliptic Curves*

Recall that the two families coincide; that is, there are elliptic curves E, E' such that $\mathcal{X}' = \operatorname{Km}(E \times E')$. Here we can compute the j-invariants j, j' as follows. The variable change

$$v \mapsto 2(-b)^{1/4}v/\sqrt{a}, x \mapsto 16(b/a)^{2/3}x$$

leads to the standard normal form

$$y^2 = x^3 - 3Av^4x - v^4(v^4 + 2Bv^2 + 1),$$

where A, B are algebraic expressions in a, b. By the work of Inose [8] (cf. [22]), $jj' = 12^6A^3$ and $(j - 12^3)(j' - 12^3) = 12^6B^2$ (so that A and B are products of Weber functions). In the present situation, one obtains

$$jj' = -\frac{4096(3a - 12b + 1)^3}{a^2b} \tag{10}$$

and

$$(j - 12^3)(j' - 12^3) = -\frac{1024(9a + 72b + 2)^2}{a^2b}. \tag{11}$$

Thus we can express the elliptic curves E, E' in terms of the moduli a, b. Note that we lost the symmetry in a, b when extracting the two I_4^* fibers on $\mathcal{X}$. Algebraically in the preceding formulas, this can be accounted for as follows: if j gives a solution of the system (10), (11) for the ordered pair (a, b), then $mj + l$ encodes a solution of the system for the ordered pair (b, a) with

$$m = \frac{a(64a^2 + 16a + 16ba + b^2)}{b(64b^2 + 16b + 16ba + a^2)},$$
$$l = \frac{8(156b^2a - 4b^3 + 16a + 128a^2 + 80ba - b^2 + 256a^3 - 192ba^2)}{b^2a}$$
$$+ m\frac{8(156ba^2 - 4a^3 + 16b + 128b^2 + 80ba - a^2 + 256b^3 - 192b^2a)}{a^2b}.$$

4.6. Conclusion

The Hodge structure on $\mathcal{X}$ is given by the pair (E, E') as described in Section 4.5.

For instance, if E, E' are isogenous but without complex multiplication (CM), then any X as before will have Picard number $\rho(X) = 19$. Note, however, that it is not clear from these computations how we can choose a, b so that X attains a chosen transcendental lattice of rank 2 or 3. This problem will be overcome in Section 5 for the cases related to Theorem 1 by geometric means.

4.7. Relations with Physics and Picard–Fuchs Equations

An Enriques Calabi–Yau threefold is the smooth quotient $(S \times E)/(\tau \times \iota_E)$, where S is a K3 surface admitting an Enriques involution τ, E is an elliptic curve, and ι_E is the hyperelliptic involution on E. These particular threefolds are intensively studied in the context of mirror symmetry and string theory in part because they are their own mirror. In a certain sense this property depends on the corresponding property for the K3 surfaces: the family of K3 surfaces admitting an Enriques involution is its own mirror within the framework of mirror symmetry of polarized K3 surfaces.

It is immediate to check with the Künneth formula that $h^{2,1}((S \times E)/(\tau \times \iota_E)) = 11$. Hence the family of Enriques Calabi–Yau threefolds is 11-dimensional. Note that the dimension of the family of the Enriques Calabi–Yau threefolds is the sum of the dimensions of the family of the K3 surfaces and of the elliptic curve involved in the construction. Thus all the deformations of the threefolds are induced by deformations of the K3 surface and of the elliptic curve.

To gain specific insight into Enriques Calabi–Yau threefolds, recently certain subfamilies have been studied extensively. In particular, the Barth–Peters family $\mathcal{X}$ has been studied from this viewpoint in [10].

In order to describe the mirror map for the resulting families of Calabi–Yau threefolds, the Picard–Fuchs equation of the family of K3 surfaces $\mathcal{X}$ is computed in [10, (6.26)]. Since the Barth–Peters family is a 2-dimensional family, one expects the Picard–Fuchs equation to be a partial differential equation of order 4. However, in this particular case the Picard–Fuchs equation splits into a system of two partial differential equations of order 2 that can be solved separately.

Our construction provides a geometric interpretation of this result through Kummer structures. Indeed, we have proved (in Section 4.6) that the variation of the Hodge structures of $\mathcal{X}$ depends only on the variation of the Hodge structures of two nonisogenous elliptic curves (and that the Picard–Fuchs equation of a family of elliptic curves is a second-order differential equation). In addition, one obtains the same Picard–Fuchs equations for several other families of K3 surfaces that are related by rational dominant maps between the generic members (such as $\mathcal{Y}$ in Section 7).

Naturally this property carries over to subfamilies. Along these lines, one can find the Picard–Fuchs equations for the families related to Theorem 1 and 2 (see Sections 3.2 and 5.3) through the results from [4].

5. Geometric Construction

This section may be considered the geometric heart of the paper because we construct explicitly the K3 surfaces in Theorem 1: here we exhibit in a purely geometric way the surfaces X_N that were introduced in Section 3 from the point of view of the lattices.

5.1. Outline

Given N-isogenous elliptic curves E, E', we consider a particular elliptic fibration on the Kummer surface $\mathrm{Km}(E \times E')$. If $N > 1$, then the graph of the isogeny induces an additional section. This section can be traced through two related elliptic fibrations until we reach the fibration from Section 4.3. Then the quotient by a 2-torsion section takes us to a member W_N of the Barth–Peters family $\mathcal{X}$. In fact, there are two ways to go through this whole procedure. In each case we compute the transcendental lattice $T(W_N)$, and one case leads to Theorem 1 (cf. Section 5.10).

5.2. Abelian Surface

Let E, E' denote complex elliptic curves without CM. Assume that they are (cyclically) N-isogenous. Then the abelian surface $A = E \times E'$ has transcendental lattice $T(A) = U + \langle 2N \rangle$.

5.3. Kummer Surface

It follows that the Kummer surface $\mathrm{Km} = \mathrm{Km}(E \times E')$ has transcendental lattice $T(\mathrm{Km}) = U(2) + \langle 4N \rangle$. We consider three specific elliptic fibrations that also live on general Kummer surfaces of product type—that is, Kummer surfaces of a product of nonisogenous elliptic curves (as classified by Oguiso [19]). We write the fibrations in terms of the reducible fibers and torsion in MW in the nondegenerate case $N > 1$:

(i) $[II^*, 2I_0^*]$, $\mathrm{MW}_{\mathrm{tor}} = \{0\}$;
(ii) $[III^*, I_2^*, 3I_2]$, $\mathrm{MW}_{\mathrm{tor}} = \mathbb{Z}/2\mathbb{Z}$;
(iii) $[2I_2^*, 4I_2]$, $\mathrm{MW}_{\mathrm{tor}} = (\mathbb{Z}/2\mathbb{Z})^2$.

If E, E' were not isogenous, then these fibrations would have MW-rank 0 with NS fully generated by the given singular fibers and sections. The generic fibration of the third kind has already appeared in Section 4.3. It is also instructive to note that, whereas the first fibration is unique on Km up to Aut(Km), the second and third are generally not. In fact, by [19] there are generally six (resp. nine) inequivalent such fibrations. This property will only enter implicitly in our construction (cf. Remark 11 and Section 5.10).

5.4. First Elliptic Fibration

This fibration has played a central role in the study of singular K3 surfaces (cf. [22; 23]). Like the fibration with two IV^* fibers in Section 4.5, the coefficients of the Weierstrass form admit a simple algebraic expression in the j-invariants of the elliptic curves. As a further advantage, it is easy to determine the abstract shape of a section induced by an isogeny of the elliptic curves.

The N-isogeny between E and E' induces an additional divisor on the fibration just described. If $N = 1$, then this is a fiber component since one of the I_0^* fibers degenerates to type I_1^*. In the following, we consider only the case $N > 1$. Then the additional divisor can be represented by a section P on the first elliptic fibration. We employ the theory of Mordell–Weil lattices [21] to find information about the section P.

LEMMA 10. *The section P meets one fiber* (*resp., both fibers*) *of type I_0^* at a nonidentity component when the parity of N is odd* (*resp., even*).

Proof. Recall the trivial lattice $U + 2D_4(-1) + E_8(-1)$ generated by zero section and fiber components. Since the trivial lattice has discriminant -16 and Km has discriminant $16N$, the section P ought to have height N. We will use that P cannot meet both I_0^* fibers at their identity components. Otherwise, it would be orthogonal to the two copies of $D_4(-1)$ from the trivial lattice inside NS and thus the 2-length of NS would be at least 4, exceeding the rank of $T(\text{Km})$, which is 3. Hence the height of P is

$$h(P) = 4 + 2(P.O) - \begin{cases} 1 & \text{if } P \text{ meets one } D_4, \\ 2 & \text{if } P \text{ meets both } D_4\text{s}. \end{cases}$$

Since $h(P) = N$, it follows that the intersection behavior is predicted by the parity of N as claimed. □

From the lemma and the height of P, we also obtain the intersection number:

$$(P.O) = \begin{cases} (N-3)/2 & \text{if } N \text{ is odd}, \\ (N-2)/2 & \text{if } N \text{ is even}. \end{cases}$$

5.5. Second Elliptic Fibration

From the first elliptic fibration we will extract an elliptic fibration of the second kind as in Section 5.3. From here on, we concentrate on the case where $N > 1$ is odd.

We identify a divisor of Kodaira type I_2^* on Km as follows. Take the I_0^* fiber met by P in a nonidentity component minus exactly that component and extend by zero section and identity components of the other two reducible fibers. This induces an elliptic fibration on Km with the second components of II^* and the other

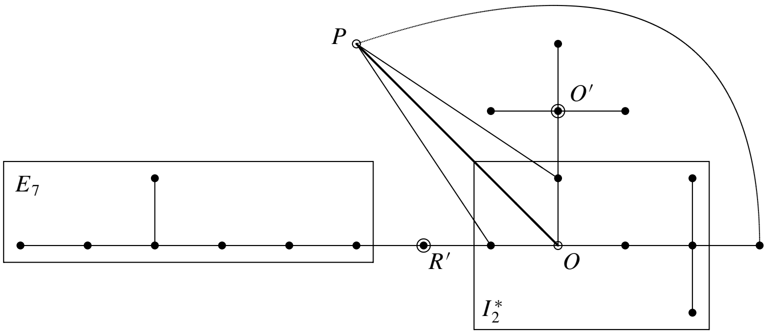

Figure 3 Divisors of type I_2^* and E_7

I_0^* as sections (zero and 2-torsion). Perpendicular to the new I_2^* fiber we find an E_7 coming from II^* and three A_1s coming from I_0^*. Thus we obtain exactly the second elliptic fibration from Section 5.3.

In Figure 3 we sketch these divisor classes. The figure shows the zero section O of the first fibration and the components of the reducible fibers. We also include the section P of the first fibration that intersects O with multiplicity $(N-3)/2$. For the second fibration, we mark the components of the two "big" singular fibers by boxes and the new sections O', R' by circles.

On the second fibration, P induces a multisection of degree $N-1$. In order to find the section P' associated to this multisection, we subtract suitable elements from the trivial lattice (fiber components and the zero section). For this, we fix the zero section O' and the 2-torsion section R' as depicted. In the present situation, we find the following intersection behavior of P' (which will also be sketched in Figure 4).

(i) P meets only the identity component of III^* (the one missing in the figure, which thus also meets O') with multiplicity $N-1$.

(ii) On the I_2 fibers, P meets the nonidentity components (also missing in the figure) with multiplicity $N-1$. Subtracting $(N-1)/2$ times the identity component C_0^i $(i=1,2,3)$, we obtain a divisor meeting only the identity component with multiplicity $N-1$.

(iii) On the I_2^* fiber, the section P meets the first double component with multiplicity $(N-3)/2$. Subtracting the identity component C_0^0 with the same multiplicity, we obtain a divisor that meets only the identity component (multiplicity $N-2$) and the near simple component (multiplicity 1).

By adding suitable multiples of O' and the general fiber F', we obtain a divisor D' with $D'^2=-2$ that meets each fiber in exactly one point:

$$D' = P - \frac{N-1}{2}(C_0^1 + C_0^2 + C_0^3) - \frac{N-3}{2}C_0^0 - (N-2)O' + \frac{N-1}{2}F'.$$

Since $D' \equiv P$ modulo the trivial lattice, D' represents a section P' meeting only the I_2^* fiber in a nonidentity component (near simple). Since $P'.O' = (N-3)/2$, we indeed find that P' has height N.

5.6. Third Elliptic Fibration

We continue with another elliptic fibration. We extract a new divisor of Kodaira type I_2^* along similar lines as in Section 5.5. Namely, we combine the first two simple and double components of the old I_2^* fiber with zero section and identity components of the III^* fiber and one of the I_2 fibers. In the orthogonal complement we find another D_6 (from III^*) and four A_1s (the far simple components of the original I_2^* and the nonidentity components of the two avoided I_2s). Then the original 2-torsion section R' (which will be omitted in the next figure to simplify the presentation) and the two remaining components of the original fibers of type I_2^* and III^* serve as sections (zero and twice 2-torsion). Thus we find the third fibration from Section 5.3, as depicted in Figure 4.

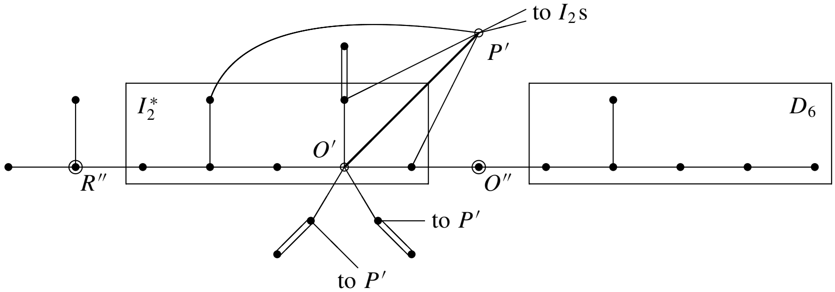

Figure 4 Divisors of type I_2^* and D_6

It remains to associate a section P'' to the multisection induced by P'. Here the multisection degree is N. We choose the zero section O'' as indicated in the figure and denote by R'' the 2-torsion section that is a component of the original I_2^* fiber. Along the same lines as in Section 5.5, one finds P'' meeting the I_2 fibers in identity components and the I_2^* fibers in the same far components that R'' meets. Since $P''.O'' = (N-1)/2$, P'' has height N as required.

REMARK 11. In this step we may need to extend the base field as we single out one of the three fibers of type I_2 (which correspond to the nonidentity simple components of the original I_0^* fibers where P meets the identity component). This extension is the reason why we cannot simply parameterize the family by $X_0^+(N)$ (see Section 6.5 for the case $N = 3$).

5.7. $\mathbb{Z}/2\mathbb{Z}$ Quotient

We want to divide out by the 2-torsion section that meets both I_2^* fibers in the near simple component and hence all I_2 fibers in nonidentity components. In terms of the group law, this section is $R' + R''$. The quotient results in a new K3 surface W_N that has an elliptic fibration with only two reducible fibers, each of type I_4^*, and 2-torsion in MW. In a few steps we shall see that W_N exactly realizes a surface X_N as in Theorem 1.

LEMMA 12. *W_N is a quadratic base change of a rational elliptic surface.*

Proof. The statement holds true for the previous fibration owing to the singular fibers and full 2-torsion. But then the sections respect the base change property, and so does the quotient. □

COROLLARY 13. *W_N is a member of the Barth–Peters family as in Section 4.3. We have*

$$T(W_N) = U(2) + \langle 2N\rangle, \qquad \mathrm{NS}(W_N) = U(2) + 2E_8(-1) + \langle -2N\rangle.$$

Proof. The section P'' on the third elliptic fibration on Km induces a section Q of height $2N$ on W_N. Here Q meets both I_4^*s in a far simple component. By construction, the 2-torsion section meets the same components, so their sum R is orthogonal to the two summands of $D_8(-1)$ in the trivial lattice corresponding to the I_4^* fibers. Thus we find the following sublattice L of $\mathrm{NS}(W_N)$:

$$L = U + (2D_8(-1) + \mathbb{Z}/2\mathbb{Z}) + \langle -2N\rangle. \tag{12}$$

Assume that $L \neq \mathrm{NS}(W_N)$. Then there is a divisible section in N; that is, either Q or R is divisible. But since they are related to Km by a 2-isogeny, these sections could only be 2-divisible. However, this is impossible in the present situation since it would result in a noninteger height $N/2$ while all correction terms in the height formula are integers (since singular fibers of type I_4^* have only integer correction terms).

We conclude that $L = \mathrm{NS}(W_N)$ and immediately find the claimed representations for $T(W_N)$ and $\mathrm{NS}(W_N)$. In particular this implies that $W_N \in \mathcal{X}$. More precisely, the surface W_N is a general member of the family $\mathcal{X}_N$ and so it is the surface called X_N in Section 3.2. □

5.8. Geometric Proof of Theorem 1

We claim that $X_N \in \mathcal{X}_N$ is a K3 surface, proving Theorem 1. Note that X_N admits a rational map of degree 2 to $\mathrm{Km}(E \times E')$ given by the 2-isogeny with the third elliptic fibration on $\mathrm{Km}(E \times E')$.

The Enriques involution τ on the Barth–Peters family descends to the special member X_N. On the previous fibration with two I_4^* fibers, it is given by deck transformation of the quadratic base change composed with translation by the 2-torsion section (cf. [7]). The invariant sublattice is contained in the trivial lattice. By (12), the section R is associated to a summand D in $\mathrm{NS}(X_N)$. Since $D^2 = -2N$ and N is odd, we find that $\pi^* \mathrm{Br}(X_N/\tau) = \{0\}$ by Theorem 5.

5.9. Complex Multiplication

Throughout this chapter we have assumed that the isogenous elliptic curves E, E' do not have CM. This assumption serves to rule out the special members with $\rho = 20$ (i.e., the singular K3 surfaces in the family). The assumption is not strictly necessary because it serves only two minor purposes: (i) to exclude those singular K3 surfaces where the involution fails to be fixed point free or the singular fibers degenerate (as seen in Section 6.6); and (ii) to ensure that the K3 surfaces are indeed not Kummer. In practice, it will always suffice for our purposes to exclude a finite number of CM points (see Proposition 15 for the family with $N = 3$).

5.10.

It is instructive to consider a second way to derive the second and third elliptic fibration from Section 5.3. Namely, when extracting the second elliptic fibration, we could opt to include the component met by P in the divisor of Kodaira type I_2^* (possibly at the cost of another extension of the base field). The overall construction goes through as before, but in the end the section P'' induced by P on the third elliptic fibration shows a different intersection behavior than before (even up to addition of 2-torsion sections). On the $\mathbb{Z}/2\mathbb{Z}$ quotient W_N we still obtain a section Q of height $2N$, but this time meeting only one I_4^* fiber at a far simple component. Thus one finds the transcendental lattice $T(W_N) = U + \langle 8N \rangle$.

6. Elliptic Fibrations and Moduli of the Family $\mathcal{X}_3$

In Sections 3 and 5, we considered generally complex K3 surfaces X_N where $\mathrm{NS}(X_N) \simeq U(2) + 2E_8(-1) + \langle -2N \rangle$ for odd $N > 1$. We derived an Enriques involution geometrically as well as lattice theoretically and showed that the Brauer group pulls back identically to zero. Here we consider one of these families in detail, the family $\mathcal{X}_3$ such that

$$\mathrm{NS}(\mathcal{X}_3) \simeq U(2) + 2E_8(-1) + \langle -6 \rangle. \tag{13}$$

For this family we give an explicit equation (defined over $\mathbb{Q}$) answering the problem posed in Section 2.6. We describe the Hodge structure (given by a pair of 3-isogenous elliptic curves E and E') and its specializations (related to the complex multiplication on the elliptic curves E and E'). We hope that its analysis will both illustrate our methods and give the reader an idea of how the constructions can be carried out explicitly.

6.1.

In Section 5.7 the surfaces X_N are constructed as quotients of known Kummer surfaces but without explicit equations. In order to find an explicit equation for the family $\mathcal{X}_3$, we will exhibit a convenient jacobian elliptic fibration on $\mathcal{X}_3$ (which is not among the ones coming from the geometric construction of Section 3.2). In the first instance, this amounts to writing $\mathrm{NS}(\mathcal{X}_3)$ as an orthogonal sum of the hyperbolic plane U and an even negative-definite lattice L. Preferably L is a root lattice, since then the Mordell–Weil group of the elliptic fibration is finite (cf. Section 2.3). We will proceed in two steps related to the isomorphisms

$$\mathrm{NS}(\mathcal{X}_3) \simeq U + D_8(-1) + E_8(-1) + \langle -6 \rangle \tag{14}$$

$$\simeq U + D_8(-1) + E_7(-1) + A_2(-1). \tag{15}$$

A direct computation shows that the lattices in (13), (14), and (15) have the same signature, the same discriminant group, and the same discriminant form. By Proposition 3 they are isometric. The elliptic fibration corresponding to the decomposition (15) of $\mathrm{NS}(\mathcal{X}_3)$ is especially convenient because it involves only U and root lattices. By Section 2.3, the latter correspond to reducible singular fibers of type I_4^*, III^*, I_3 (or IV a priori). In particular, the last elliptic fibration has no sections other than the zero section.

6.2. *The Elliptic Fibration* $[I_4^*, III^*, I_3]$ *on* $\mathcal{X}_3$

We explain how to find a model of the last elliptic fibration (15). Consider the quadratic twist at the nonreduced fibers that replaces them by fibers of type I_4, III. This results in a family of rational elliptic surfaces $\mathfrak{S}$ with configuration of singular fibers $[1, 1, 3, 4, III]$. Let k be any field of characteristic different from 2. In extended Weierstrass form, the family of rational elliptic surfaces can easily be parameterized over $k(r)$ as

$$\mathfrak{S}\colon\ y^2 = x^3 - t(r^2t - 1 - 2r)x^2 - 2(t+1)tr(rt-1)x - (t+1)^2t^2r^2.$$

As required, the given model has the following reducible singular fibers:

III	I_3	I_4
0	-1	∞

Note that the general member of the family $\mathfrak{S}$ has Mordell–Weil rank 2 by the Shioda–Tate formula [21, Cor. 5.3]. These sections are not preserved under the quadratic twist. The family $\mathcal{X}_3$ with elliptic fibration corresponding to the decomposition (15) is recovered by a quadratic twist at 0 and ∞ (i.e., the fibrations become isomorphic over $k(\sqrt{t}\,)$):

$$\mathcal{X}_3\colon\ y^2 = x^3 - t^2(r^2t - 1 - 2r)x^2 - 2(t+1)t^3r(rt-1)x - (t+1)^2t^5r^2. \quad (16)$$

Here a general member of the family $\mathcal{X}_3$ has $\rho = 19$ and NS given as before with $\mathrm{MW} = \{O\}$. Our next aim is to write down the Enriques involution explicitly and then give the anti-invariant divisor. Later we will study the parameterizing curve and special members of the family.

6.3. *The Elliptic Fibration* $[I_4^*, II^*]$ *on* $\mathcal{X}_3$

In order to find the elliptic fibration on $\mathcal{X}_3$ corresponding to (14), it suffices to determine a suitable divisor of Kodaira type II^*. In the present situation, this divisor is extracted from the fiber of type III^* extended by zero section and identity component of the fiber of type I_3. In $\mathrm{NS}(\mathcal{X}_3)$, this leaves the orthogonal summand $D_8(-1)$ formed by the nonidentity components of the I_4^* fiber. The two other components of the I_3 fiber serve as zero section on the one hand and section of height 6 on the other. We sketch these (-2)-curves in Figure 5.

In the terminology of [13], we shall work with the following elliptic parameter with respect to equation (16):

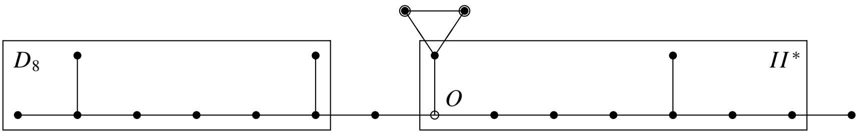

Figure 5 Divisors of type II^* and D_8

$$u = (x - rt^2(t+1)/2)/(t^3(t+1)).$$

After some variable transformations, one obtains the Weierstrass form

$$\mathcal{X}_3\colon\ y^2 = t^3 + 2u(r^3 - 8ru - 4u)t^2 + 16u^4(1 - 4r + 2r^2)t + 128ru^7 \quad (17)$$

with reducible fibers of type II^* at $u = \infty$ and I_4^* at $u = 0$. The section of height 6 is given in terms of its t-coordinate as

$$\big(-32u^5 + (64r^2 + 336r + 128)u^4 + (-32r^4 - 320r^3 - 720r^2 - 192r - 128)u^3 + 8r(6r^4 + 32r^3 + 21r^2 - 20r + 8)u^2 - 2r^3(12r^3 + 24r^2 - 27r + 8)u + r^5(2r-1)^2\big)\frac{\frac{1}{16}(r-2u)}{(r^2 - 2ru - r - 2u)^2}.$$

Thus we have indeed found the elliptic fibration corresponding to (14). From this, one can derive a double quartic model associated to the decomposition (13) of $\mathrm{NS}(\mathcal{X}_3)$ after adjoining a square root via $r = (1-q^2)/4$.

6.4. Enriques Involution and the Elliptic Fibration $[2III^*, 2I_2]$ *on* $\mathcal{X}_3$

To exhibit the specified Enriques involution on the family $\mathcal{X}_3$ explicitly, we use the Barth–Peters family for which we have worked out the Enriques involution in Section 4.2.

To find the Enriques involution on $\mathcal{X}_3$, it suffices to exhibit an elliptic fibration with two reducible fibers of type III^* and I_2 each and 2-torsion in MW. Then the Mordell–Weil lattice of a general member will have rank 1 and a generator of height 6. (The 2-torsion condition is crucial since the family $\mathcal{X}_3$ also admits an elliptic fibration with the same singular fibers but without torsion in MW, so in that case $\mathrm{MWL} = \langle 3/2 \rangle$.)

We work with the fibration on $\mathcal{X}_3$ corresponding to (14). In terms of the model in (17), the elliptic parameter $v = u/t^3$ extracts a divisor of type III^* from components of the I_4^* fiber extended by zero section and identity component of the II^* fiber. The adjacent fiber components form the new zero and 2-torsion section; the remaining fiber components form one divisor of type E_7 and two root lattices of type A_1. We sketch these rational curves in Figure 6, where we omit only the additional MW generator of height 6 for simplicity.

In suitable coordinates, we obtain the Weierstrass form

$$\mathcal{X}_3\colon\ y^2 = x(x^2 - 8t^2(1+2r)x + 2t^3(64r + 8t - 32rt + 16r^2t + t^2r^3)). \quad (18)$$

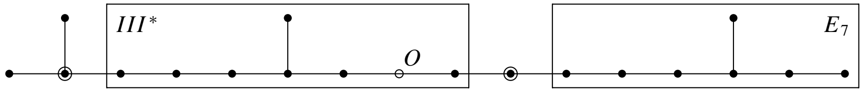

Figure 6 Divisors of type III^* and E_7 versus II^*s and I_4^*

The 2-torsion section is $(0,0)$ as before. The deck transformation $\imath$ for the quadratic base change f is given by

$$\imath\colon (x,y,t) \mapsto (\alpha^2 x/t^4, \alpha^3 y/t^6, \alpha/t), \quad \alpha = 64/r^2.$$

The quotient by the deck transformation $\imath$ is an extremal rational elliptic surface with $\mathrm{MW} = \mathbb{Z}/2\mathbb{Z}$. It follows that the section P of height 6 is anti-invariant for $\imath^*$. Thus it is induced by a section P' of height 3 on the quotient of $\mathcal{X}_3$ by the Nikulin involution $(-\mathrm{id}) \circ \imath$. Thanks to the low height and the presence of 2-torsion, the section P' is not hard to find. Here we give P only in terms of its x-coordinate:

$$\frac{1}{256}\,\frac{(t^2r^2 + 64 + 16rt - 32t)^2(rt-8)^2}{(rt+8)^2}.$$

As required, P and O intersect at exactly one of the ramification points of the quadratic base change, $t = -8/r$ (so that $P.O = 1$), whereas P does not meet any fiber at a nonidentity component. An anti-invariant divisor on $\mathcal{X}_3$ for the induced action of the Enriques involution τ (composition of $\imath$ and translation by the 2-torsion section $(0,0)$) is then given as

$$\varphi(P) = P - O - 3F, \qquad \varphi(P)^2 = -6. \tag{19}$$

This can be seen as follows. The Enriques lattice $U(2) + E_8(-2)$ embeds primitively into $\mathrm{NS}(\mathcal{X})$. On $\mathcal{X}_3$, where we have additional sections, this specializes to a primitive embedding into the trivial lattice of the given elliptic fibration. By definition, $\varphi(P)$ is orthogonal to the trivial lattice of the elliptic fibration (as in the theory of Mordell–Weil lattices). Hence it is anti-invariant for τ^*—a fact that can also be checked explicitly (as in [7]) with Mordell–Weil lattices.

6.5. *Moduli*

In order to determine the moduli curve of the family $\mathcal{X}_3$, we can argue using the Kummer structure of the fibration on $\mathcal{X}_3$ with two fibers of type I_4^* as in Sections 4.4 and 4.5. Thus we find a relation to a product of elliptic curves. Since the Picard number is generically 19, these elliptic curves ought to be isogenous. We will relate them to the modular curve $X^*(6) = X_0(6)/\langle w_2, w_3\rangle$, where we divide out $X_0(6)$ by all Fricke involutions. This curve parameterizes elliptic curves over biquadratic extensions of $\mathbb{Q}$ with prescribed isogenies to their Galois conjugates. Details (mostly in the context of $\mathbb{Q}$-curves) can be found in [20]. There a Hauptmodul a for $X^*(6)$ is fixed.

THEOREM 14. *The family $\mathcal{X}_3$ is parameterized by $X^*(6)$. The parameter r is related to the Hauptmodul a of $X^*(6)$ by $a = -2(r+2)/(4r-1)$.*

Proof. For the elliptic curves parameterized by $X^*(6)$, a Weierstrass form is given in [20]. In particular, we obtain the j-invariants of these elliptic curves. Since the field of definition is $\mathbb{Q}(\sqrt{a}, \sqrt{2a+1})$, it is actually more convenient to write

$$a = a(t) = \left(\frac{2t}{2t^2-1}\right)^2, \tag{20}$$

which makes both square roots rational in t. Then the j-invariants are represented by

$$j(t) = \frac{6912(5t^3+6t^2-2)^3t^3}{(2t-1)(t+1)^2(2t+1)^3(t-1)^6} \tag{21}$$

up to conjugation. We will show that these j-invariants coincide with those coming from the Kummer structure on $\mathcal{X}_3$. This suffices to prove the theorem.

Arguing as for the full Barth–Peters family in Sections 4.4 and 4.5 but specialized to $\mathcal{X}_3$, we find the following relations in terms of the parameter $q = \sqrt{1-4r}$:

$$j \cdot j' = 4096\frac{(q-3)^3(25q^3+15q^2+3q-51)^3}{(q+1)^8(q-1)^4},$$

$$j + j' = 128\frac{\begin{array}{c}(125q^6+800q^5-715q^4-3400q^3\\ +7511q^2-5464q+1399)(q-3)^3\end{array}}{(q-1)^3(q+1)^6}.$$

We now proceed in three steps. First we employ the modularity methods of point counting and lifting from [5] to find numerically members over $\mathbb{Q}$ of the family $\mathcal{X}_3$ with $\rho = 20$. We find CM values of r as given in Table 1. Then we try to match these values with the CM points of $X^*(6)$. This leads exactly to the given relation between the parameters r and a. Note that this relation is still conjectural, but it gives $q = 3/\sqrt{2a+1}$. Next we insert (20) for a, which leads to

$$q = \pm\frac{3(2t^2-1)}{1+2t^2}.$$

The negative sign choice gives precisely the j-invariant (21) and its conjugate by $t \mapsto -t$ as solutions to the system of equations for j, j' coming from the Kummer structure. Thus $\mathcal{X}_3$ is in fact parameterized by $X^*(6)$, and the conjectural relation between r and a holds true as claimed. □

6.6. Specializations

In [7], an explicit singular K3 surface X over $\mathbb{Q}$ was exhibited with an Enriques involution τ over $\mathbb{Q}$ and a τ^*-anti-invariant divisor D over $\mathbb{Q}(\sqrt{-3})$ with $D^2 = -6$. This surface is given abstractly by the transcendental lattice $T(X) = \langle 4\rangle + \langle 6\rangle$. In the family $\mathcal{X}_3$, the surface can be located by degenerating the two fibers of type I_2 of the elliptic fibration (18) to one fiber of type I_4. Actually there are two ways to achieve this degeneration: by specializing $r = 1/2$ and $r = 1/4$. We will now distinguish these two cases.

In the first case, the section P degenerates as well in the following sense: its x-coordinate attains a double root at the I_4 fiber at $t = 16$. In consequence, the

Table 1 CM points of $\mathcal{X}$ over $\mathbb{Q}$

d	-12	-15	-20	-24	-36	-48	-60	-72	-84
r^{-1}	$-1/2$	$5/8$	2	4	-2	$25/4$	$-49/8$	12	-14
d	-120	-132	-168	-228	-312	-372	-408	-708	
r^{-1}	40	-50	112	-338	1300	-3038	4900	-140450	

height drops to $h(P) = 5$. Thus the discriminant of the specialization is -20. The transcendental lattice of the special member has the following quadratic form and specialization embedding:

$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \hookrightarrow U(2) + \langle 6 \rangle.$$

If $r = 1/4$, then the two I_2 fibers are merged without the section P degenerating. In terms of the elliptic fibration corresponding to the decomposition (15), this is explained by the degeneration $D_8 \hookrightarrow D_9$. Hence the special member X has

$$\mathrm{NS}(X) = U + A_2(-1) + E_7(-1) + D_9(-1)$$

with transcendental lattice $T(X) = \langle 4 \rangle + \langle 6 \rangle$ as in [7]. In the model (18), the splitting field of the I_4 fiber is $\mathbb{Q}(\sqrt{-3})$. One can easily work out an isomorphism over $\mathbb{Q}(\sqrt{-3})$ with the model in [7]. Note that for the model described here, the invariant subspace of $\mathrm{NS}(X)$ under τ^* is fully Galois invariant.

Through the rational CM points and cusps of the modular curve $X^*(6)$, we find all other specializations over $\mathbb{Q}$ with $\rho = 20$. Together with the previous two specializations and all corresponding discriminants d, we collect the CM points (or rather their inverses) in Table 1.

PROPOSITION 15. *Let X be a special member of $\mathcal{X}_3$ at $r \neq 0$. Then the Enriques involution on $\mathcal{X}_3$ specializes without fixed points to X if and only if $r \neq -2$. Moreover, if $r \neq -2$ then there is a divisor D on X such that $\tau^* D = -D$ in* $\mathrm{NS}(X)$ *and $D^2 = -6$. In particular,* $\pi^*(\mathrm{Br}(X/\tau)) = \{0\}$.

Proof. For the specialization at $r = -2$, there is an additional singular fiber of type I_2 in both elliptic fibrations (15), (18). At the same time, the former fibration attains a 2-torsion section whereas for the latter fibration the section P becomes 2-divisible. By construction, the additional I_2 fiber is necessarily fixed under the deck transformation described previously (i.e., the base change f ramifies there). Since the 2-torsion section meets the identity component of the additional I_2 fiber, the Enriques involution on $\mathcal{X}$ does not specialize to a fixed point–free involution at $r = -2$.

On all other K3 specializations, the fixed fibers of the deck transformation are either smooth or of type I_4 as before. Hence the Enriques involution stays fixed point free under specialization. This proves the first claim.

For the divisor D, we can take $D = \varphi(P) = P - O - 3F$ as in (19) in all non-degenerate cases. The same divisor works also at $r = 1/4$ because P meets the degenerate I_4 fiber at the identity component (and the height is unchanged).

In contrast, at $r = 1/2$, the height of P degenerates to 5 because P meets the component Θ_2 of the I_4 fiber (numbered cyclically). Thus we still have $P.O = 1$, but the section $P' = [-P + (0,0)]$ meets the I_4 fiber at the identity component (as opposed to generically intersecting both I_2 fibers at the *non*identity components). Hence the intersection number $P'.O$ degenerates from 3 to 2 at $r = 1/2$. We claim that the following modification of (19) suffices to satisfy the conditions of the proposition:

$$D := P - O - 3F + \Theta_1 + \Theta_2.$$

The intersection number $D^2 = -6$ is easily verified. For the anti-invariance, we note that τ rotates the I_4 fiber. Hence

$$\tau^* D = P' - (0,0) - 3F + \underbrace{\Theta_3 + \Theta_0}_{=F-\Theta_1-\Theta_2}.$$

To prove that $D + \tau^* D = 0$ in $\mathrm{NS}(X)$, we then only need to verify that the sum is orthogonal to the trivial lattice (fiber components and zero section) and that it gives zero in $\mathrm{MW}(X)$.

With the divisor D at hand, the final claim of the proposition follows directly from Theorem 5. □

7. Kummer Surfaces

Our results about Enriques involutions and Brauer groups so far have exclusively concerned K3 surfaces that are generally not Kummer. Here we want to extend this approach to Kummer surfaces that turn up very naturally for the Barth–Peters family in the realm of Shioda–Inose structures.

7.1. Shioda–Inose Structure on the Barth–Peters Family

Recall that the Barth–Peters family $\mathcal{X}$ admits an elliptic fibration with I_{16} fiber and 2-torsion section. Translation by this section induces a Morrison–Nikulin involution $\jmath$ on $\mathcal{X}$, so that the desingularization of $\mathcal{X}/\jmath$ gives a family of Kummer surfaces that we denote by $\mathcal{Y}$. By standard formulas, we obtained as induced elliptic fibration

$$\mathcal{Y}\colon\ y^2 = x(x^2 - 2a(t)x + (a^2(t) - 4)). \tag{22}$$

Generally this has a fiber of type I_8 at ∞ and eight fibers of type I_2. There is full 2-torsion consisting of the sections $(0,0)$, $(a \pm 2, 0)$. A general member $Y \in \mathcal{Y}$ has transcendental lattice $T(Y) = U(2) + U(4)$.

LEMMA 16. *Let Y be a member of the family $\mathcal{Y}$ with $\rho(Y) = 18$. Then* $\mathrm{NS}(Y)$ *is generated by torsion sections, fiber components, and a section of height* $1/2$. *In particular,* $\mathrm{MWL}(Y) = [1/2]$. *Pulling back the infinite section to the quotient* $X \in \mathcal{X}$, *we obtain a* MWL *generator of* X.

Proof. By construction, Y is a base change of a rational elliptic surface. Namely, this property carries over directly from $\mathcal{X}$ since $a(t)$ is in fact quadratic in t. Because of the singular fibers (I_4 and 4 times I_2), the rational elliptic surface has Mordell–Weil rank 1. The generator is a section of height $1/4$, so the pull-back has height $1/2$ on Y. Comparing discriminants, we verify that this infinite section together with torsion sections and fiber components generates $\mathrm{NS}(Y)$. The final claim follows directly from the heights. □

7.2. Involutions

Since the family $\mathcal{Y}$ is a base change of a family of rational elliptic surfaces, we have in addition to the translations by 2-torsion sections and the hyperelliptic involution the deck transformation $\imath$ acting as a nonsymplectic involution on $\mathcal{Y}$. As in [7], this allows us to derive Enriques involutions on $\mathcal{Y}$ (outside some subfamily of codimension 1). Since the 2-torsion sections are both invariant and anti-invariant for $\imath^*$, the composition of their translation with the deck transformation $\imath$ defines an involution τ on $\mathcal{Y}$.

LEMMA 17. *Generally the involution τ is an Enriques involution if and only if the 2-torsion section involved is not* $(0,0)$.

Proof. We start by studying the fixed locus of the deck transformation $\imath$. Here $\imath$ fixes the fiber of type I_8 at $\tau = \infty$ and the fiber at $\tau = 0$. The latter fiber is smooth outside a subfamily of codimension 1. Since 2-torsion sections are always disjoint from the zero section, the composition is fixed point free if and only if the 2-torsion section meets a different component of the I_8 fiber than the zero section. This exactly rules out the given section. □

In short: outside a subfamily of codimension 1, Lemma 17 gives two Enriques involutions on $\mathcal{Y}$.

7.3. Specializations

We continue this approach by specializing the family $\mathcal{X}$ to the subfamily $\mathcal{X}_N$ comprising the surfaces X_N from Section 3.2 for some fixed $N \in \mathbb{N}$ and then switch to $\mathcal{Y}$.

As before, the Morrison–Nikulin involution $\jmath$ exhibits a Shioda–Inose structure. This time it relates to abelian surfaces A with $T(A) = U(2) + \langle 2N\rangle$. Thus the desingularization Y_N of $X_N/\jmath$ coincides with the Kummer surface $\mathrm{Km}(A)$ with $T(Y_N) = U(4) + \langle 4N\rangle$.

LEMMA 18. *Let Y_N be as before with $\rho(Y) = 19$.*

(i) *If $N > 1$, then the induced elliptic fibration on Y_N has nondegenerate singular fibers and* $\mathrm{MW}(Y_N) = (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}^2$ *with* $\mathrm{MWL}(Y_N) \simeq \begin{bmatrix} 1/2 & 0 \\ 0 & N \end{bmatrix}$.

(ii) *If $N = 1$, then the induced elliptic fibration on Y_N has singular fibers $I_8 + I_4 + 6I_2$ and* $\mathrm{MW}(Y_N) = (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}$ *with* $\mathrm{MWL}(Y_N) = [1/2]$.

Proof. We will treat only the case $N > 1$. The case $N = 1$ is proved analogously.

By pulling back from X_N, we induce a sublattice $\mathrm{MWL}(X_N)[2] \simeq \left[\begin{smallmatrix} 2 & 0 \\ 0 & 4N \end{smallmatrix}\right]$ of $\mathrm{MWL}(Y_N)$. Together with torsion sections and fiber components, this generates a sublattice of $\mathrm{NS}(Y_N)$ of rank 19 and discriminant $2^{10}N$. Since $T(Y_N)$ has discriminant $64N$, the index is 4 and must be accounted for completely by $\mathrm{MWL}(Y_N)$. But there we can only have 2-divisibility owing to the 2-isogeny between Y_N and X_N. Hence $\mathrm{MWL}(Y_N) = \mathrm{MWL}(X_N)[1/2]$. □

7.4. *Enriques Involutions*

As in Section 7.2, the deck transformation ι composed with translation by either 2-torsion section defines an involution τ on Y_N. In the following we refer to the Weierstrass form (22) specialized to Y_N for the natural elliptic fibration.

LEMMA 19. *The involution τ on Y_N fails to be an Enriques involution exactly in the following two cases*:

(i) *the 2-torsion section defining τ is* $(0, 0)$;
(ii) $N = 1$ *and the 2-torsion section defining τ is* $(a \pm 2, 0)$ *for the sign such that* $t \nmid (a \pm 2)$.

Proof. The argument from Lemma 17 rules out the first alternative. Recall that the deck transformation ι fixes the fiber at $t = 0$. If this fiber is smooth, then the same argument shows that τ has no fixed points for the 2-torsion sections $(a \pm 2, 0)$. Presently, this fiber is singular (type I_4) exactly in the degenerate case $N = 1$. Then, in order to induce a fixed point–free action on the fiber, the 2-torsion section has to meet a nonidentity component of the I_4 fiber. This is exactly the case $t \mid (a \pm 2)$. □

7.5. *Brauer Group*

Eventually we want to compute how the Brauer group pulls back from the Enriques quotients of Y and Y_N. Here's the result.

THEOREM 20. *Consider the K3 surfaces Y and Y_N with Enriques involution τ as specified in Lemmas 17 and 19.*

(i) *If N is even, then* $\pi^* \mathrm{Br}(Y_N/\tau) = \mathbb{Z}/2\mathbb{Z}$. *The same holds true for Y.*
(ii) *If N is odd, then* $\pi^* \mathrm{Br}(Y_N/\tau) = \{0\}$.

In particular, this theorem implies Theorem 2. Note that Y_N admits a rational map of degree 4 to the Kummer surface of $E \times E'$; this is the composition of the two 2-isogenies $Y_N \dashrightarrow X_N$ and $X_N \dashrightarrow \mathrm{Km}(E \times E')$ that we have exhibited before.

The proof of Theorem 20 is given in Sections 7.8 and 7.9. First we set up some notation regarding the given elliptic fibrations on Y and Y_N.

7.6. Setup

On Y, we number the components of the I_8 fiber cyclically $\Theta_0, \dots, \Theta_7$ so that Θ_0 meets the zero section O. We number the I_2 fibers from 1 to 8. In the following, we refer to their nonidentity components simply by their respective numbers. We have the sections $U = (0, 0)$ and $V, W = (a \pm 2, 0)$. Moreover, there is a section P of height $1/2$ obtained from the quotient rational elliptic surface Y/ι by pull-back. Up to rearranging the fibers (and adding a 2-torsion section to P) we can assume the following intersection pattern, where we indicate only the I_2 fibers that are met at nonidentity components.

		I_2s							
	I_8	1	2	3	4	5	6	7	8
U	Θ_0	1	2	3	4	5	6	7	8
V	Θ_4	1	2	3	4				
W	Θ_4					5	6	7	8
P	Θ_2	1	2			5	6		

From the Mordell–Weil pairing, it follows that all these sections are orthogonal on Y. The deck transformation ι acts trivially on the sections of Y and on the I_8 fiber while permuting the I_2 fibers as (12)(34)(56)(78).

A $\mathbb{Z}$-basis of $\mathrm{NS}(Y)$ can be obtained by omitting W and the nonidentity components 3 and 8, say. There are 18 divisors remaining: 13 nonidentity components and Θ_0 as well as the four sections O, U, V, P. The Gram matrix comprising their intersection numbers has full rank and determinant -64. Thus the specified divisors form a $\mathbb{Z}$-basis of $\mathrm{NS}(Y)$ as claimed.

In terms of this $\mathbb{Z}$-basis, it is easy to implement the Enriques involutions τ from Lemma 17. For each Enriques involution, one finds that

$$\mathrm{NS}(Y)^{\iota^*=-1} \cong E_8(-2). \tag{23}$$

In particular, all anti-invariant divisors D have $D^2 \equiv 0 \bmod 4$. Hence

$$\pi^* \operatorname{Br}(Y/\tau) = \mathbb{Z}/2\mathbb{Z}$$

by Theorem 5. This proves Theorem 20 for Y. We now turn to the subfamilies Y_N.

7.7. Lattice Enhancement

In this and the next section, we assume $N > 1$ (see Section 7.9 for the case $N = 1$). By Lemma 18, the subfamilies Y_N attain an additional section Q of height N. We determine the fiber components met by Q through the lattice enhancement construction described in Section 2.2.

LEMMA 21. *Up to renumbering, Y_N has a section Q of height $N > 1$ that meets only the following singular fibers nontrivially.*

N odd	1	2			5	6	7	8
N even	1			4		6	7	

Proof. The surface Y_N is obtained via specializing Y as in Section 2.2 by choosing the vector v to be $v_N := (1, -N, 0, 0) \in U(2) + U(4) \simeq T(Y)$. This is consistent with the specialization of X to X_N. Then $\mathrm{NS}(Y_N)$ is an overlattice of index 2 of $\mathrm{NS}(Y) + \langle v_N \rangle$. To find a generator of the full Néron–Severi lattice, we fix a basis within the discriminant group of $\mathrm{NS}(Y)$ that corresponds to the summand $U(2)$ of $T(Y)$ (the orthogonal summand $U(4)$ is not affected by the specialization).

In the present situation, this basis can be given as

$$m_1 = \frac{(1467)}{2}, \qquad m_2 = \frac{(2458)}{2},$$

where (1467) denotes the sum of the divisors $1, 4, 6, 7$. Then one can easily check that $D = (v_N + m_1 + Nm_2)/2$ is in $H^2(X, \mathbb{Z})$. This gives the missing generator of $\mathrm{NS}(Y_N)$.

It remains to find a section Q corresponding to D. For this we add fiber components and sections to D in such a way that the resulting divisor Q has self-intersection $Q^2 = -2$ and meets every fiber in exactly one component:

$$Q = \begin{cases} \frac{v_N - (125678)}{2} + O + \frac{N+3}{2} F & N \text{ odd}, \\ \frac{v_N - (1467)}{2} + O + \frac{N+2}{2} F & N \text{ even}. \end{cases} \tag{24}$$

In particular we read off the intersection behavior claimed in the lemma, and one verifies the given height. □

7.8. Proof of Theorem 20 for $N > 1$

It is immediate how to extend the action of the Enriques involution from Y to Y_N: on $\mathrm{NS}(Y)$ (and its image in $\mathrm{NS}(Y_N)$) it is known, and on v_N, τ acts as -1 since v_N specializes from $T(Y)$, which sits in the anti-invariant part. This determines the action of τ^* on $\mathrm{NS}(Y_N)$ completely. For instance, consider the case where $N > 1$ is odd and τ composes ι with translation by W. Then one derives

$$\tau^* Q = -Q - (12) + O + W + (N+1)F. \tag{25}$$

Independent of the parity and the 2-torsion section involved in τ, we know that the section $[2Q]$ meets all fibers at their identity components. In fact, from the description in (24) one derives

$$[2Q] = v_N + O + 2NF.$$

The orthogonal projection φ with respect to the hyperbolic plane $U = \langle O, F \rangle$ gives exactly the divisor

$$\varphi([2Q]) = [2Q] - O - 2NF = v_N.$$

That is, in $\mathrm{MW}(Y_N)$ the section $[2Q]$ corresponds exactly to v_N. By definition, this divisor is orthogonal to the whole image of $\mathrm{NS}(Y)$ in $\mathrm{NS}(Y_N)$ and is anti-invariant for τ^*. Using (23), we thus find the following sublattice of the anti-invariant part of $\mathrm{NS}(Y_N)$:

$$\mathrm{NS}(Y_N)^{\tau^*=-1} \hookleftarrow \mathrm{im}(\mathrm{NS}(Y)^{\tau^*=-1}) + \langle \varphi([2Q]) \rangle \cong E_8(-2) + \langle -4N \rangle.$$

The crucial question now is whether the inclusion is actually an equality or whether we have a proper sublattice of index 2. The index cannot be bigger since it amounts to deciding whether the section Q itself contributes to the anti-invariant part or only its multiple $[2Q]$. The following lemma answers this question.

LEMMA 22. (i) *If N is even, then* $\mathrm{NS}(Y_N)^{\tau^*=-1} \cong E_8(-2) + \langle -4N \rangle$.

(ii) *If $N > 1$ is odd, then* $\mathrm{NS}(Y_N)^{\tau^*=-1}$ *is an overlattice of* $E_8(-2) + \langle -4N \rangle$ *of index 2. It is generated by an anti-invariant divisor for* τ^* *of self-intersection* $-N-3$ *or* $-N-5$.

Proof. To prove the first case, we used a computer program to express the $\mathbb{Q}$-basis of $\mathrm{NS}(Y_N)^{\tau^*=-1}$ in terms of the specified $\mathbb{Z}$-basis of $\mathrm{NS}(Y_N)$. Then it is easily verified that the given lattice is not 2-divisible.

To prove the second case, it suffices to exhibit an anti-invariant divisor $D \in \mathrm{NS}(Y_N)$ for τ^* for each given intersection number. Since either $-N-3$ or $-N-5$ is not congruent to zero modulo 4, the respective divisor cannot be contained in $\mathrm{im}(\mathrm{NS}(Y)^{\tau^*=-1}) + \langle \varphi([2Q]) \rangle \cong E_8(-2) + \langle -4N \rangle$.

Here we only give these divisors for the case where τ is ι composed with translation by W. The Enriques involution involving V can be dealt with similarly.

Consider the following divisor classes on Y_N:

$$D_1 = Q + (1) - O - \frac{N+1}{2}F \quad \text{with} \quad D_1^2 = -N-3, \tau^* D_1 = -D_1.$$

$$D_2 = Q - (4) - V - \frac{N-1}{2}F \quad \text{with} \quad D_2^2 = -N-5, \tau^* D_2 = -D_2.$$

Let us check that these divisors are anti-invariant for τ^*. For instance,

$$\tau^* D_1 = \tau^* Q + (2) - W - \frac{N+1}{2}F.$$

By (25), one immediately finds $D_1 + \tau^* D_1 = 0$. A simple computation gives $D_1^2 = -N-3$. Similarly, one finds $D_2^2 = -N-5$ and

$$D_2 + \tau^* D_2 = O + W - U - V - (1234) + 2F.$$

One directly checks that this divisor is perpendicular to the trivial lattice of Y_N (fiber components and zero section). Moreover, $D_2 + \tau^* D_2$ induces the zero section in $\mathrm{MW}(Y_N)$. Hence $D_2 + \tau^* D_2 = 0$ in $\mathrm{NS}(Y_N)$. □

Theorem 20 follows from Lemma 22 as a direct application of Theorem 5. Recall that this implies Theorem 2 for $N > 1$.

7.9. *Proof of Theorem 20 for $N = 1$*

By Lemma 18, Y_1 admits an elliptic fibration with singular fibers $I_8 + I_4 + 6I_2$ and $\mathrm{MW}(Y_1) = (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}$. The surface Y_1 is obtained as a specialization of Y as in Section 2.2 by choosing the vector $v_1 := (1, -1, 0, 0) \in U(2) + U(4) \simeq T(Y)$ and hence the transcendental lattice of Y_1 is $\langle 4 \rangle + U(4)$. Geometrically this specialization consists of merging two fibers of type I_2 of the given elliptic fibration on Y in order to obtain a fiber of type I_4 on Y_1. As usual its components are numbered $C_0, \dots, C_3$. We recall that Y admits a $2:1$ map to a rational elliptic surface. Since this base change extends naturally to Y_1, the specialization is ramified at an I_2 fiber of the rational elliptic surface, and we are merging two fibers on Y that are exchanged by the deck transformation ι. We shall assume that these two fibers are the seventh and the eighth fiber (with notation as in Section 7.6). The sections of Y specialize to sections of Y_1, so we obtain the following intersection pattern.

			I_2s					
	I_8	I_4	1	2	3	4	5	6
U	Θ_0	C_2	1	2	3	4	5	6
V	Θ_4	C_0	1	2	3	4		
W	Θ_4	C_2					5	6
P	Θ_2	C_0	1	2			5	6

In this setup, the composition of the deck transformation ι with the translation by the 2-torsion section W is an Enriques involution (see Lemma 19). Consider the divisor

$$D := \Theta_4 + \Theta_5 + \Theta_6 + \Theta_7 + C_2 + C_3 - (24) - V + W.$$

One easily checks that D is τ^*-anti-invariant and $D^2 = -6$. By Theorem 5, this concludes the proof of Theorem 20 in case $N = 1$. Thus we have completed the proof of Theorem 2.

7.10. A Rigid Example

We conclude this paper with a brief description of the unpublished example from [6] mentioned in the Introduction. This example comes up naturally here as a specialization of the Kummer surface Y_1 described in the previous paragraph. Let us specialize (as in Section 2.2) the surface Y_1 by choosing the vector v to be $v := (1, 1, -1) \in \langle 4 \rangle + U(4) \simeq T(Y_1)$. The surface obtained has transcendental lattice isomorphic to $\left[\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right]$. Hence it is the Kummer surface, $\mathrm{Km}(E_i \times E_i)$, of the product of E_i (the elliptic curve with an automorphism of order 4) with itself. Geometrically this specialization consists of merging two further fibers of type I_2 with the fiber of type I_4. After this specialization, the given elliptic fibration of Y_1 attains singular fibers $2I_8 + 4I_2$. As before, we must choose the fibers of type I_2 that we merge with the fiber of type I_4 in such a way that the $2:1$ map from Y_1 to a rational elliptic surface is preserved. This allows only the I_2 fibers $5, 6$. The rational elliptic surface is forced to degenerate as well, attaining a second fiber of type I_4. Above this fiber, the K3 surface has the degenerate fiber of type I_8. Numbering its fiber components cyclically $D_0, \dots, D_8$, we obtain the following intersections.

			I_2s			
	I_8	I_8	1	2	3	4
U	Θ_0	D_4	1	2	3	4
V	Θ_4	D_0	1	2	3	4
W	Θ_4	D_4				
P	Θ_2	D_2	1	2		

The infinite section P on Y_1 becomes a 4-torsion section on $\mathrm{Km}(E_i \times E_i)$ (induced by the rational elliptic surface underneath), and W is exactly twice P. Hence the Mordell–Weil group of the elliptic fibration on $\mathrm{Km}(E_i \times E_i)$ is $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The Enriques involution τ on Y_1 (composition of the deck transformation with the translation by W) specializes without fixed points to $\mathrm{Km}(E_i \times E_i)$. The following divisor is anti-invariant for τ^*:

$$D := O - U + \Theta_4 + \Theta_5 + \Theta_6 + \Theta_7 + D_4 + D_5 + D_6 + D_7 - (13).$$

Since $D^2 = -10$, the Brauer group of $\mathrm{Km}(E_i \times E_i)/\tau$ pulls back to zero by Theorem 5.

We note that this elliptic fibration on $\mathrm{Km}(E_i \times E_i)$ as well as the Enriques involution τ can be defined over $\mathbb{Q}$. As a specialization of (22), it admits the Weierstrass form

$$\mathrm{Km}(E_i \times E_i)\colon\ y^2 = x(x - t^4)(x - t^4 + 4). \tag{26}$$

Here $\iota(t) = -t$ and $W = (t^4, 0)$.

7.11.

Naturally the surface $\mathrm{Km}(E_i \times E_i)$ is the quotient by a Morrison–Nikulin involution of a surface S specializing from $\mathcal{X}_1$. Here S is obtained from (26) by the 2-isogeny induced by the 2-torsion section $(0, 0)$. In accordance with Sections 3 and 5, the K3 surface S admits an elliptic fibration with singular fibers $I_{16}+I_4+4I_1$ and $\mathrm{MW} = \mathbb{Z}/4\mathbb{Z}$. The Morrison–Nikulin involution is the translation by the 2-torsion section. Abstractly, S is given as desingularization of $(E_i \times E_i)/\langle \alpha \times \alpha^3 \rangle$, where α is an order-4 automorphism of E_i.

7.12

For each N, we constructed two different 1-dimensional families of K3 surfaces with an Enriques involution: the families $\mathcal{X}_N$ and the families $\mathcal{Y}_N$, which are related by a 2-isogeny for fixed N. The families show a nice interplay at specializations with $\rho = 20$. For instance, the surface $\mathrm{Km}(E_i \times E_i)$ is a specialization both of $\mathcal{Y}_1$ and of $\mathcal{X}_2$. Note, however, that the induced Enriques involutions may differ (or degenerate), as we show next.

We have constructed $\mathrm{Km}(E_i \times E_i)$ as a specialization of $\mathcal{Y}_1$. The induced Enriques involution τ had the anti-invariant divisor D with $D^2 = -10$. On the other hand, $\mathrm{Km}(E_i \times E_i)$ arises as a specialization of $\mathcal{X}_2$ as in Section 2.2 if we choose the vector v to be $v := (0, 1, -1) \in \langle 4 \rangle + U(2)$. The Enriques involution on $\mathcal{X}_2$ induces an Enriques involution τ_2 on $\mathrm{Km}(E_i \times E_i)$. In this case, one finds that there is no anti-invariant divisor D on $\mathrm{Km}(E_i \times E_i)$ such that $D^2 \not\equiv 0 \bmod 4$. In consequence, the Enriques involutions on $\mathrm{Km}(E_i \times E_i)$ induced from $\mathcal{Y}_1$ and from $\mathcal{X}_2$ are not conjugate in the automorphism group of $\mathrm{Km}(E_i \times E_i)$.

Alternatively, one can argue using the automorphism groups of the Enriques quotients. It follows immediately from the construction that the Enriques surface $\mathrm{Km}(E_i \times E_i)/\tau$ admits an elliptic fibration with two double fibers of type I_4 while $\mathrm{Km}(E_i \times E_i)/\tau_2$ admits an elliptic fibration with one double fiber of type I_8. By Kondo's classification (see especially [12, Table 2]), these Enriques surfaces have different (finite) automorphism groups.

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A. Garbagnati
Dipartimento di Matematica
Università di Milano
via Saldini 50
I-20133 Milano
Italy

alice.garbagnati@unimi.it

M. Schütt
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
Germany

schuett@math.uni-hannover.de